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Preconditioners for inhomogenous anisotropic problems in spherical geometry

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Abstract

Inhomogenous Anisotropic Elliptic operators arise from the integration of the Navier-Stokes equations for a hydrostatic Boussinesq fluid on a sphere. Anisotropy may be defined as the variation of the property of a material with the direction in which it is measured. An Anisotropic Elliptic operator arises in the free-surface formulation of the Met Office Ocean model. A modified Helmholtz problem is iteratively solved using conjugate gradient with a diagonal preconditioner. The anisotropy in the corresponding discretised equations causes the convergence of the method to be slow, particularly in polar regions. Block diagonal and Alternating Direction Implicit preconditioners are considered here as alternatives and their impact on the pole problem and on the overall convergence are assessed.

1 Introduction

Most ocean models in use today are based on integrating the incompressible Navier-Stokes equations on a sphere. Complex topography is used at the ocean bottom and the ocean surface is either fixed or free to move with time. The ocean basins themselves typically contain irregularly shaped coastlines and islands which require the inclusion of specific boundary conditions into any solution algorithm.

The forms of the operators that arise in the spherical coordinate framework are anisotropic. An operator is anisotropic if its local properties vary with direction. As an example consider the constant coefficient partial differential equation

$$-\frac{\partial}{\partial x}\left(L_x\frac{\partial U}{\partial x}\right)-\frac{\partial}{\partial y}\left(L_y\frac{\partial U}{\partial y}\right)=\gamma(x,y) \tag{1}$$

where L_x and L_y are taken here to be constant. Note that the case $L_x = L_y = 1$ is just the Laplacian operator which, when discretised on a regular Cartesian grid, is known to be relatively easy to model. If we alter the coefficients though, making L_x much larger than L_y, the operator becomes poorly conditioned. This is an example of strong anisotropy.

In the ocean models we consider, the effects of anisotropy are seen in the latitudinally varying rates of convergence of the elliptic methods. Poor rates of convergence are observed in polar regions compared to equatorial and mid-latitude regions. In these cases L_x and L_y are non constant, with $L_x \approx L_y$ in equatorial regions, but $L_x >> L_y$ in polar regions. This is an example of inhomogenous anisotropy. In these cases an operator is inhomogenous if its properties change with location. A typical convergence result encountered with this type of operator is shown in Figure 1 This shows the variation in residual error latitudinally, after a convergence tolerance has been reached, for a numerical experiment with the free surface formulation of the Met Office ocean model. Significantly higher error values are observed in the polar regions; over an order of magnitude higher than the errors in equatorial and mid-latitude regions.

The aims of this paper are to discuss the effect the anisotropy of the elliptic operators has on the conditioning of model problem and on the speed of convergence of the Preconditioned Conjugate Gradient (PCG) method, and also to consider the use of alternative preconditioners to the diagonal (pointwise Jacobi) preconditioner currently used in the Met Office free surface ocean model. Block diagonal (Jacobi) and Alternating-Direction-Implicit (ADI) preconditioners are considered as alternatives and their impact on the convergence speeds and conditioning are investigated.

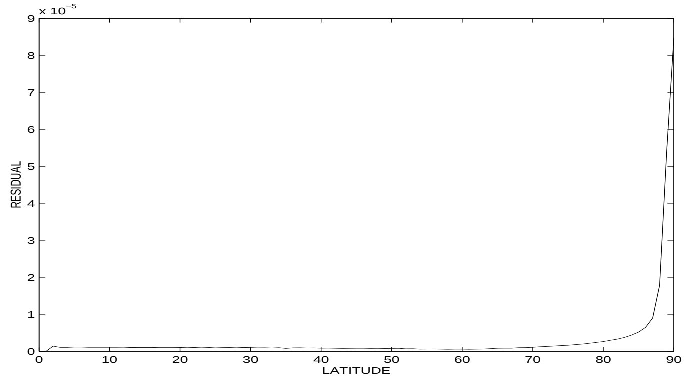

Figure 1: Latitudinal variance in convergence for experiment with free surface formulation of Met.Office ocean model (Northern Hemisphere only)

2 Met Office Free-surface Model

Most of the Ocean General Circulation models in use today, including the free-surface barotropic model used by the Met Office, are based on the Bryan-Cox-Semtner (henceforth BCS) model initially introduced by Bryan [2] in the late 1960's and later modified by Cox [3] and Semtner [9]. The BCS model solves the primitive equations, derived from the Navier-Stokes equations, in a spherical coordinate system using hydrostatic and Boussineq approximations. The implicit free-surface barotropic model was introduced by Dukowicz [4] and is summarised briefly here.

The barotropic, or vertically averaged, equations of state are given by

$$\begin{aligned} &\frac{\partial u}{\partial t} - fv = -g\frac{1}{acos\phi}\frac{\partial \eta}{\partial \lambda} + G^x, \\ &\frac{\partial v}{\partial t} + fu = -g\frac{1}{a}\frac{\partial \eta}{\partial \phi} + G^y, \\ &\frac{\partial \eta}{\partial t} + \frac{1}{acos\phi}\left[\frac{\partial Hu}{\partial \lambda} + \frac{\partial Hvcos\phi}{\partial \phi}\right] = 0, \end{aligned} \tag{2}$$

where λ and ϕ are longitude and latitude respectively, f is the coriolis parameter, g is the gravitational acceleration constant, $H = H(\lambda, \phi)$ is the total depth of the ocean, (u, v) are the barotropic velocity components and G^x, G^y represent baroclinic forcing. Dukowicz [4] considered the following general time discretisation of equation (2) :

$$\begin{aligned} &\frac{u^{n+1}-u^{n-1}}{2\tau} - fv^{\alpha'} = -g\frac{1}{acos\phi}\frac{\partial \eta^\alpha}{\partial \lambda} + G^{x,n}, \\ &\frac{v^{n+1}-v^{n-1}}{2\tau} + fu^{\alpha'} = -g\frac{1}{a}\frac{\partial \eta^\alpha}{\partial \phi} + G^{y,n}, \\ &\frac{\eta^{n+1}-\eta^n}{\tau} + \frac{1}{acos\phi}\left[\frac{\partial Hu^\theta}{\partial \lambda} + \frac{\partial Hv^\theta cos\phi}{\partial \phi}\right] = 0, \end{aligned} \tag{3}$$

with

$$\begin{aligned} &u^{\alpha'} = \alpha' u^{n+1} + (1 - \alpha' - \gamma')u^n + \gamma' u^{n-1}, \\ &\eta^\alpha = \alpha\eta^{n+1} + (1 - \alpha - \gamma)\eta^n + \gamma\eta^{n-1}, \\ &u^\theta = \theta u^{n+1} + (1 - \theta)u^n. \end{aligned} \tag{4}$$

where τ is the fixed timestep, n is the current time level and α, α', γ, γ' and θ are coefficients used to parameterise the time centering of the pressure gradient, Coriolis, and divergence terms. Eliminating u^{n+1} and v^{n+1} in (3) and rearranging we can obtain an implicit equation for η' which represents the change in free surface height η between two consecutive timesteps of the overall Met Office Unified model. The elliptic operator, which is solved at every timestep, is given by:

$$\frac{1}{acos\phi}\left[\frac{\partial}{\partial \lambda}\left(\frac{H}{acos\phi}\frac{\partial \eta'}{\partial \lambda}\right) + \frac{\partial}{\partial \phi}\left(\frac{Hcos\phi}{a}\frac{\partial \eta'}{\partial \phi}\right)\right] - \beta\eta' = S(\lambda, \phi), \tag{5}$$

where

$$\beta = \frac{1}{2\alpha\theta g\tau}, \tag{6}$$

3 Preconditioners for model problem

We consider a Limited Area, Northern Hemisphere model problem of the following form in our numerical experiments:

$$\begin{cases} \frac{1}{cos\phi}\left[\frac{\partial}{\partial\lambda}\left(\frac{1}{cos\phi}\frac{\partial U}{\partial\lambda}\right) + \frac{\partial}{\partial\phi}\left(cos\phi\frac{\partial U}{\partial\phi}\right)\right] - kU = \gamma(\lambda,\phi) \\ \lambda \in (0^oE, 30^oE), \quad \phi \in (10^oN, l) \\ U(0^oE,\phi) = 0, U(30^oE,\phi) = 0 \\ U(\lambda, 10^oN) = 0, U(\lambda, l) = 0 \\ l \in (40^oN, 89.5^oN). \end{cases} \tag{7}$$

where $k \geq 0$, and γ is known. In our experiments we investigated the effects of moving the northern boundary of the domain closer to the pole. We discretise the problem using a standard five-point discretisation scheme with a constant stepsize h in both directions and taking a natural ordering of the grid points. This gives rise to a matrix equation of the general form

$$A\mathbf{U} = \mathbf{b}, \tag{8}$$

where the variable $\mathbf{U}$ is a (unknown) column vector of the grid point values of the variable U and b is a (known) column vector representing boundary values and source terms. The system matrix A is a real, symmetric, $m \times m$ matrix representing the discretised model equations (where m is the number of grid points). It is also square, sparse, irreducible and diagonally dominant with strict diagonal dominance in at least one row. It is therefore irreducibly diagonally dominant and hence positive-definite ([10]).

The Met Office free-surface model currently uses a PCG method with a preconditioner containing only the diagonal elements of A, a_{ii}. The PCG method may be thought of as an acceleration method for the point Jacobi iterative method. Due to the block nature of A we may also consider the block Jacobi splitting as a preconditioner for the PCG method ([1], [5], [8]). In our experiments we also consider an ADI preconditioner ([10]) based on the splitting

$$A = H_\omega + V_\omega + \Sigma,$$

where

$$\begin{aligned} (H_\omega U)(\lambda_i,\phi_j) &= -U(\lambda_i + h, \phi_j) + 2U(\lambda_i,\phi_j) - U(\lambda_i - h, \phi_j) \\ (V_\omega U)(\lambda_i,\phi_j) &= -U(\lambda_i, \phi_j + h) + 2U(\lambda_i,\phi_j) - U(\lambda_i, \phi_j - h) \\ (\Sigma U)(\lambda_i,\phi_j) &= kU(\lambda_i,\phi_j) \end{aligned} \tag{9}$$

The matrices defined in 9 have the following properties : Σ is a non-negative diagonal $m \times m$ matrix and is hence non-negative definite. H and V are Stieltjes matrices and are diagonally dominant and positive definite (since they have positive diagonal entries and non-positive non-diagonal entries).

4 Numerical Experiments

Figure 2 shows the effect the increased anisotropy due to moving the northern boundary closer to the pole has on the eigenvalues of G_D, the point Jacobi iteration matrix. We observe the clustering of secondary eigenvalues of G_D near the, slightly larger, leading eigenvalue. This suggests that more eigenmodes will contribute significantly to the errors with increased anisotropy. Figure 3 shows the leading four eigenvectors of G_D. Whilst the lead eigenmode does not possess a significant signal in the polar regions the others do and it is these that become more significant with increased anisotropy and therefore will contribute much more to the residual errors in the method.
Tables 1 and 2 show the effect on the conditioning of the problem with the increased anisotropy due to moving the northern boundary closer to the pole, and the use of the different preconditioners (where G_P is the iteration matrix of the preconditioned system $P^{-1}A$ with $G_P = I - P^{-1}A$). We observe that the conditioning becomes over an order of magnitude larger by moving the boundary near to the pole. We also observe that the conditioning of the matrix with the Block preconditioner is better than with the diagonal preconditioner, with ADI preconditioning a further improvement on that. We would therefore expect ADI preconditioning and to a lesser extent Block preconditioning, to yield better convergence rates to the diagonal preconditioner, and this is indeed what is shown in Figure 4. This shows the residual errors after a fixed amount of CPU time, with ADI and Block preconditioning clearly superior.

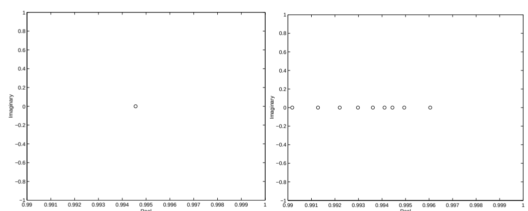

Figure 2: Leading eigenvalues of G_D for problem with northern boundary at 40^o and 89^o respectively, $h = 1^o$, $k = 0.01$.

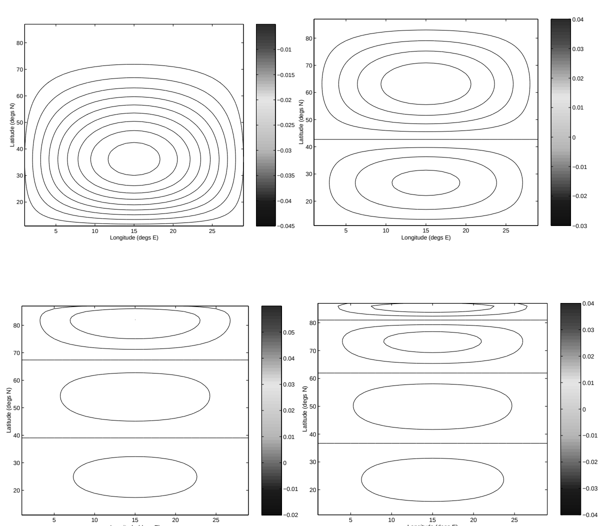

Figure 3: Leading four eigenvectors of G_D, $l = 88^o$, $h = 1^o$, $k = 0.01$.

	$\kappa(A)$		
Boundary	$h = \frac{1}{2}^o$	$h = 1^o$	$h = 2^o$
40^o	2.19×10^3	544.18	134.17
70^o	4.28×10^3	1.04×10^3	249.57
88^o	3.12×10^4	6.51×10^3	1.21×10^3
89^o	5.20×10^4	9.75×10^3	NA

Table 1: Variation of condition number with varying northern boundary, $k = 0.01$.

Preconditioner	$\kappa(P^{-1}A)$	$\rho(G_P)$
None	6.51×10^3	-
Diagonal	691.92	0.9960
Block	293.92	0.9901
ADI	139.31	0.9601

Table 2: Variation of condition number and spectral radii with preconditioners, 88^o Northern Boundary, $h = 1^o$, $k = 0.01$.

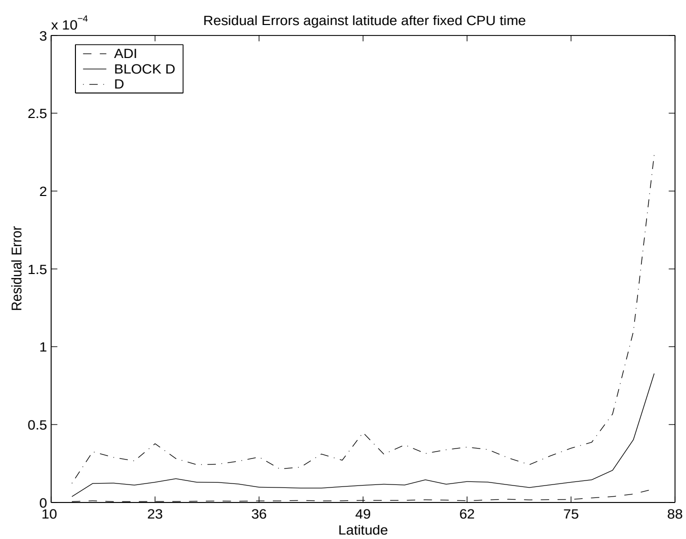

Figure 4: Latitudinal variance in convergence for Limited Area Helmholtz problem with Diagonal, Block and ADI preconditioners. Northern boundary at 88^o, $h = 1^o$, $k = 0.01$.

5 Conclusions

Our analysis of the eigenvectors and eigenvalues of the iteration matrix G_D of our iterative method leads us to conclude that the problem of larger residual errors in polar regions of our model is at least partly due to the increased importance, with increased anisotropy, of 'nearly' leading eigenvectors with significant values in polar regions. Our numerical experiments have shown that Block Jacobi and particularly ADI preconditioners can yield significant improvements with regards to addressing the issue of larger residual errors in polar regions and hence improve convergence speeds as a whole.

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