A collocation method for high frequency scattering by convex polygons

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S. Arden [1], S. N. Chandler-Wilde, S. Langdon [*,2]

Department of Mathematics, University of Reading, Whiteknights, PO Box 220, Berkshire RG6 6AX, UK

Abstract

We consider the problem of scattering of a time-harmonic acoustic incident plane wave by a sound soft convex polygon. Standard boundary or finite element methods, with a piecewise polynomial approximation space, have a computational cost that grows linearly with respect to the frequency of the incident wave. Recently Chandler-Wilde and Langdon proposed a novel Galerkin boundary element method for this problem for which, by incorporating the products of plane wave basis functions with piecewise polynomials supported on a graded mesh into the approximation space, they were able to demonstrate that the number of degrees of freedom required to achieve a prescribed level of accuracy grows only logarithmically with respect to the frequency. Here we propose a related collocation method, using the same approximation space, for which we demonstrate via numerical experiments a convergence rate identical to that achieved with the Galerkin scheme, but with a substantially reduced computational cost.

Key words: collocation method, high frequency scattering, oscillatory integrals
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* Corresponding author.
Email addresses: stevearden@lycos.com (S. Arden), s.n.chandler-wilde@reading.ac.uk (S. N. Chandler-Wilde), s.langdon@reading.ac.uk (S. Langdon).
URL: www.maths.rdg.ac.uk/∼sms03sl (S. Langdon).
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1 Introduction

Consider the problem of scattering of a time-harmonic acoustic incident plane wave u^i by a sound soft convex polygon Ω. The total acoustic field u satisfies

$$\Delta u(\mathbf{x}) + k^2 u(\mathbf{x}) = 0, \quad \mathbf{x} \in D := \mathbb{R}^2 \backslash \bar{\Omega}, \tag{1}$$

$$u(\mathbf{x}) = 0, \quad \mathbf{x} \in \Gamma := \partial\Omega, \tag{2}$$

where the wavenumber $k > 0$ is proportional to the frequency of the incident wave, together with the Sommerfeld radiation condition

$$\lim_{r\to\infty} r^{1/2} \left(\frac{\partial u^s}{\partial r} - \mathrm{i}k u^s \right) = 0, \tag{3}$$

on the scattered field $u^s := u - u^i$, where $r := |\mathbf{x}|$ and the limit holds uniformly in all directions $\mathbf{x}/|\mathbf{x}|$. Existence and uniqueness of a solution $u \in C(\overline{D}) \cap C^2(D)$ to (1)–(3) follows from classical results; see [8] for details.

Using Green's theorem we can represent $u(\mathbf{x})$, $\mathbf{x} \in D$, as a combination of single and double layer potentials, and with the double layer potential disappearing due to (2) we have [11, theorem 3.12]

$$u(\mathbf{x}) = u^i(\mathbf{x}) - \int_\Gamma \Phi(\mathbf{x}, \mathbf{y}) \frac{\partial u}{\partial \mathbf{n}}(\mathbf{y}) \, \mathrm{d}s(\mathbf{y}), \quad \mathbf{x} \in D. \tag{4}$$

Here $\Phi(\mathbf{x}, \mathbf{y}) := (\mathrm{i}/4) H_0^{(1)}(k|\mathbf{x} - \mathbf{y}|)$ is the standard fundamental solution for the Helmholtz equation and $\mathbf{n}$ is the normal direction directed out of Ω. Thus our problem reduces to finding the complementary boundary data $\partial u/\partial \mathbf{n} \in L^2(\Gamma)$, and to do this we solve the well known second kind integral equation

$$(I + K) \frac{\partial u}{\partial \mathbf{n}} = f, \quad \text{on } \Gamma \backslash \{S\}, \tag{5}$$

where S is the set of corners of Ω, $f := 2\partial u^i/\partial \mathbf{n} + 2\mathrm{i}\eta u^i$, and for $v \in L^2(\Gamma)$

$$Kv(\mathbf{x}) := 2 \int_\Gamma \left(\frac{\partial \Phi(\mathbf{x}, \mathbf{y})}{\partial \mathbf{n}(\mathbf{x})} + \mathrm{i}\eta \Phi(\mathbf{x}, \mathbf{y}) \right) v(\mathbf{y}) \, \mathrm{d}s(\mathbf{y}),$$

where η is a coupling parameter, with $\eta \in \mathbb{R} \backslash \{0\}$ ensuring that (5) has a unique solution (see [8] for details).

The kernel, right hand side, and solution of (5) all oscillate rapidly when k is large, and thus it is well known that the computational cost of solving (5) by standard schemes, with piecewise polynomial approximation spaces, grows at least linearly with respect to the frequency k (see e.g. [8,21] and the references therein). However, by removing the high frequency asymptotics and solving a modified integral equation whose solution approaches zero almost everywhere as $k \to \infty$, it is possible to devise numerical schemes for solving integral equations such as (5) with computational costs that grow at a sublinear rate as k increases (see e.g. [1,6,8,19]).

In particular, in [8] Chandler-Wilde and Langdon recently proposed a novel Galerkin boundary element method for solving (5) for which it was demonstrated via both a rigorous error analysis and numerical simulations that the number of degrees of freedom required to solve (5) (and thus (1)–(3)) to a prescribed level of accuracy grows only logarithmically with respect to k. This appears to be the best result to date for problems of scattering by bounded obstacles, and was achieved by removing the leading order high frequency asymptotic behaviour from (5) and using a consideration of a related set of half plane problems to demonstrate that for $s \in [0, L]$, (where $\mathbf{x}(s)$, $s \in [0, L]$, parametrises Γ)

$$\frac{1}{k}\frac{\partial u}{\partial \mathbf{n}}(\mathbf{x}(s)) = \text{known leading order terms} + \mathrm{e}^{\mathrm{i}ks} v_+(s) + \mathrm{e}^{-\mathrm{i}ks} v_-(s), \qquad (6)$$

with $v_\pm$ and all its derivatives highly peaked near the corners of the polygon, and rapidly decaying away from the corners. The oscillatory nature of $\partial u/\partial \mathbf{n}$ is thus represented exactly in (6) by the known leading order terms and the terms $\mathrm{e}^{\pm \mathrm{i}ks}$, and to approximate $\partial u/\partial \mathbf{n}$ all that is required is to approximate the smooth functions $v_\pm$. These functions decay sufficiently quickly that the number of degrees of freedom required to maintain the accuracy of their best L^2 approximation from a space of piecewise polynomials supported on a graded mesh, with a higher concentration of mesh points closer to the corners of the polygon, grows only logarithmically with respect to k as $k \to \infty$.

The question then arises of how we might go about selecting our best L^2 approximation to $v_\pm$ from the approximation space. In [8] a Galerkin scheme is used, for which both stability and convergence are proved. However, the implementation of this scheme requires the evaluation of many highly oscillatory double integrals, which can become computationally expensive at high frequencies. Although there has been some recent work on the efficient evaluation of highly oscillatory double integrals (see e.g. [13,15]) many questions remain unanswered. By contrast, several integration schemes have recently been proposed in the literature specifically for the evaluation of highly oscillatory single integrals, and many of these new schemes have the property that their performance actually improves as the integrand becomes more oscilla-

tory (see for example [6,14,16,17] and the references therein). Using this as our motivation in the current paper, here we investigate the application of a collocation method for the solution of (5). We use the same approximation space, and thus we might anticipate achieving a similar sublinear convergence rate with respect to k as that achieved by the Galerkin scheme in [8], but the collocation scheme has the advantage that its implementation requires only the evaluation of highly oscillatory single integrals.

We begin in §2 by defining the approximation space more precisely, introducing our collocation method, and making some remarks about its conditioning, stability and convergence properties. We proceed in §3 with a full description of how the scheme is implemented, including a discussion of how we can evaluate the highly oscillatory single integrals which arise. In §4 we present some numerical results, demonstrating that the collocation method appears to converge to the same solution as the Galerkin scheme, for which a full error analysis has been carried out in [8], but with a significant reduction in computational cost. Finally in §5 we present some conclusions.

2 The boundary element method

We begin by defining some notation, as in [8]. We write the boundary of the polygon as $\Gamma = \cup_{j=1}^{n} \Gamma_j$, where Γ_j, $j = 1, \ldots, n$ are the n sides of the polygon, ordered so that Γ_j, $j = 1, \ldots, n_s$ are in shadow, and Γ_j, $j = n_s + 1, \ldots, n$ are illuminated. We denote the corners of the polygon by $P_j := (p_j, q_j)$, $j = 1, \ldots, n$, and we set $P_{n+1} = P_1$, so that for $j = 1, \ldots, n$, Γ_j is the line joining P_j with P_{j+1}. We denote the length of Γ_j by $L_j := |P_{j+1} - P_j|$, the external angle at each vertice P_j by $\Omega_j \in (\pi, 2\pi)$, the normal derivative to the line Γ_j by $\mathbf{n}_j := (n_{j1}, n_{j2})$, and the angle of the incident plane wave, as measured anticlockwise from the downward vertical, by $\theta \in [0, \pi/2]$. Writing $\mathbf{x} = (x_1, x_2)$ we then have

$$u^i(\mathbf{x}) = \mathrm{e}^{\mathrm{i}k(x_1 \sin\theta - x_2 \cos\theta)} = \mathrm{e}^{\mathrm{i}k\mathbf{x}.\mathbf{d}}, \tag{7}$$

where $\mathbf{d} := (\sin\theta, -\cos\theta)$. Defining further for j=1, $\ldots, n$,

$$a_j := \frac{p_{j+1} - p_j}{L_j}, \quad b_j := \frac{q_{j+1} - q_j}{L_j}, \quad c_j := p_j - a_j \tilde{L}_{j-1}, \quad d_j := q_j - b_j \tilde{L}_{j-1},$$

where $\tilde{L}_j := \sum_{m=1}^{j} L_m$, and noting that $n_{j1} = b_j$, $n_{j2} = -a_j$, we can rewrite (5) in parametrised form as

$$\phi(s) + \int_0^L K(s,t)\phi(t)\,\mathrm{d}t = f(s), \quad s \in [0, L], \tag{8}$$

where $\phi(s) := \frac{1}{k}\frac{\partial u}{\partial \mathbf{n}}(\mathbf{x}(s))$, $L := \tilde{L}_n$, and for $\mathbf{x}(s) \in \Gamma_l$, $\mathbf{y}(t) \in \Gamma_j$, $l, j = 1, \ldots, n$,

$$K(s,t) := -\frac{1}{2}\left[\eta H_0^{(1)}(kR) + \mathrm{i}k\left[(a_l b_j - b_l a_j)t + b_l(c_l - c_j) - a_l(d_l - d_j)\right]\frac{H_1^{(1)}(kR)}{R}\right], \tag{9}$$

with

$$R = R(s,t) := \sqrt{(a_l s - a_j t + c_l - c_j)^2 + (b_l s - b_j t + d_l - d_j)^2},$$

and

$$f(s) := 2\mathrm{i}[b_l \sin\theta + a_l \cos\theta + (\eta/k)]\mathrm{e}^{\mathrm{i}k((a_l s + c_l)\sin\theta - (b_l s + d_l)\cos\theta)}. \tag{10}$$

Recalling (6), the first step in the design of our numerical scheme is to separate off the leading order behaviour, namely the contribution from the incident field (see e.g. [6,8]). For $s \in \left(\tilde{L}_{l-1}, \tilde{L}_l\right)$, $l = 1, \ldots, n$, we define

$$\Psi(s) := \begin{cases} \frac{2}{k}\frac{\partial u^i}{\partial \mathbf{n}}(\mathbf{x}(s)) = 2\mathrm{i}(b_l \sin\theta + a_l \cos\theta)\mathrm{e}^{\mathrm{i}k[(a_l s + c_l)\sin\theta - (b_l s + d_l)\cos\theta]}, & l > n_s, \\ 0, & l \le n_s, \end{cases}$$

and

$$\varphi(s) := \phi(s) - \Psi(s). \tag{11}$$

Substituting into (8) we have

$$\varphi(s) + K\varphi(s) = F(s), \quad s \in [0, L], \tag{12}$$

where

$$K\psi(s) := 2\int_0^L K(s,t)\psi(t)\,\mathrm{d}t, \quad F(s) := 2f(s) - \Psi(s) - 2\int_0^L K(s,t)\Psi(t)\,\mathrm{d}t.$$

This is the integral equation we will solve numerically, with existence and boundedness for $(I+K)^{-1}$ following immediately from [8, theorem 2.5].

We now define more precisely our approximation space $V_{N,\nu}$. We begin by defining the graded mesh we will use, which is the same as in [8].

Definition 1 *For $A > \lambda$, $N = 2, 3, \ldots$, the mesh*

$$\Lambda_{N,A,\lambda,q} := \{y_0, \ldots, y_{N+\hat{N}_{A,\lambda,q}}\}$$

consists of the points

$$y_i = \lambda \left(\frac{i}{N}\right)^q, \quad i = 0, \ldots, N,$$

together with the points

$$y_{N+j} := \lambda \left(\frac{A}{\lambda}\right)^{j/\hat{N}_{A,\lambda,q}}, \quad j = 1, \ldots, \hat{N}_{A,\lambda,q}, \tag{13}$$

where $\hat{N}_{A,\lambda,q} = \lceil N^* \rceil$, the smallest integer greater than or equal to N^*, with

$$N^* = \frac{-\log(A/\lambda)}{q \log(1 - 1/N)}. \tag{14}$$

For $j = 1, \ldots, n$, we define $q_j := (2\nu + 3)/(2\pi/\Omega_j - 1)$, and the two meshes

$$\Gamma_j^+ := \tilde{L}_{j-1} + \Lambda_{N,L_j,\lambda,q_j}, \quad \Gamma_j^- := \tilde{L}_j - \Lambda_{N,L_j,\lambda,q_{j+1}}.$$

Letting $\mathrm{e}_\pm(s) := \mathrm{e}^{\pm \mathrm{i}ks}$, $s \in [0, L]$, we then define

$$V_{\Gamma_j^+,\nu} := \{\sigma \mathrm{e}_+ : \sigma \in \Pi_{\Gamma_j^+,\nu}\}, \quad V_{\Gamma_j^-,\nu} := \{\sigma \mathrm{e}_- : \sigma \in \Pi_{\Gamma_j^-,\nu}\},$$

for $j = 1, \ldots, n$, where

$$\begin{aligned}
\Pi_{\Gamma_j^+,\nu} := \{&\sigma \in L^2(0, L) : \sigma|_{(\tilde{L}_{j-1}+y_{m-1}, \tilde{L}_{j-1}+y_m)} \text{ is a polynomial of degree } \leq \nu, \\
&\text{for } m = 1, \ldots, N + \hat{N}_{L_j,\lambda,q_j}, \text{ and } \sigma|_{(0,\tilde{L}_{j-1})\cup(\tilde{L}_j,L)} = 0\}, \\
\Pi_{\Gamma_j^-,\nu} := \{&\sigma \in L^2(0, L) : \sigma|_{(\tilde{L}_j-\tilde{y}_m, \tilde{L}_j+\tilde{y}_{m-1})} \text{ is a polynomial of degree } \leq \nu, \\
&\text{for } m = 1, \ldots, N + \hat{N}_{L_j,\lambda,q_{j+1}}, \text{ and } \sigma|_{(0,\tilde{L}_{j-1})\cup(\tilde{L}_j,L)} = 0\},
\end{aligned}$$

with the points of the mesh $\Lambda_{N,L_j,\lambda,q_j}$ given by $y_0,\ldots,y_{N+\hat{N}_{L_j,\lambda,q_j}}$, and the points of the mesh $\Lambda_{N,L_j,\lambda,q_{j+1}}$ given by $\tilde{y}_0,\ldots,\tilde{y}_{N+\hat{N}_{L_j,\lambda,q_{j+1}}}$.

Our approximation space $V_{N,\nu}$ is then the linear span of

$$\bigcup_{j=1,\ldots,n}\{V_{\Gamma_j^+,\nu}\cup V_{\Gamma_j^-,\nu}\}.$$

Defining P_{NG} to be the operator of orthogonal projection from L^2 onto the approximation space $V_{N,\nu}$, a rigorous error analysis [8, theorem 5.4] demonstrates that

$$\|\varphi-P_{NG}\varphi\|_{2,(0,L)}\leq C_\nu\sup_{\mathbf{x}\in D}|u(\mathbf{x})|\frac{n^{1/2}(1+\log^{1/2}(k\max_{j=1,\ldots,n}L_j))}{k^{1/2}N^{\nu+1}}. \quad (15)$$

Moreover defining the Galerkin method approximation $\varphi_{NG}\in V_{N,\nu}$ by

$$(I+P_{NG}K)\varphi_{NG}=P_{NG}F, \quad (16)$$

it is also shown in [8, theorem 5.5] that

$$\|\varphi-\varphi_{NG}\|_{2,(0,L)}\leq C_\nu C_s\sup_{\mathbf{x}\in D}|u(\mathbf{x})|\frac{n^{1/2}(1+\log^{1/2}(k\max_{j=1,\ldots,n}L_j))}{k^{1/2}N^{\nu+1}}. \quad (17)$$

where $C_s:=\|(I+P_{NG}K)^{-1}\|_{2,(0,L)}$ is bounded, for N sufficiently large.

Here, instead of projecting orthogonally onto the approximation space we instead use the interpolatory projection P_{NC} from L^2 onto the same approximation space $V_{N,\nu}$, and solve

$$(I+P_{NC}K)\varphi_{NC}=P_{NC}F. \quad (18)$$

For this scheme we are unable to prove an estimate of the form (17), as we discuss below. However, from (15) we know that the error in the *best* approximation of φ in $V_{N,\Gamma}$ depends only logarithically on k. Although we cannot guarantee with the collocation scheme that this best approximation will be attained, there exists some hope that a similar estimate might hold when P_{NG} is replaced by P_{NC}.

In order to focus on some of the difficulties involved in the implementation of (18) we consider from now on only the case $\nu=0$. Writing φ_N as a linear

combination of the basis functions of $V_{N,0}$, we have

$$\varphi_N(s) := \sum_{j=1}^{M_N} c_j \rho_j(s), \tag{19}$$

where ρ_j is the jth basis function and M_N is the dimension of $V_{N,0}$. For $p = 1, \ldots, n$, where n is the number of sides of the polygon, we define $n_p^{\pm}$ to be the number of elements of $\Gamma_p^{\pm}$, so

$$n_p^+ := N + \hat{N}_{L_p,\lambda,q_p}, \quad n_p^- := N + \hat{N}_{L_p,\lambda,q_{p+1}},$$

and we denote the elements of $\Gamma_p^{\pm}$ by $s_{p,l}^{\pm}$, for $l = 1, \ldots, n_p^{\pm}$. Denoting further the total number of elements supported on Γ_p by $\hat{p} := \sum_{i=1}^{p-1} n_p^+ + n_p^-$, we then have for $p = 1, \ldots, n$,

$$\rho_{\hat{p}+j}(s) := \begin{cases} \mathrm{e}^{\mathrm{i}k(s-x_{p,j}^+)}\chi_{[s_{p,j-1}^+, s_{p,j}^+)}(s), & j = 1, \ldots, n_p^+, \\ \mathrm{e}^{-\mathrm{i}k(s-x_{p,j}^-)}\chi_{[s_{p,j-1}^-, s_{p,j}^-)}(s), & j = n_p^+ + 1, \ldots, n_p^+ + n_p^-, \end{cases} \tag{20}$$

where $\chi_{[y_1,y_2)}$ denotes the characteristic function of the interval $[y_1, y_2)$, and

$$x_{p,j}^{\pm} = \frac{s_{p,j}^{\pm} + s_{p,j-1}^{\pm}}{2}, \quad j = 1, \ldots, n_p^{\pm}, \tag{21}$$

are the collocation points. Substituting (19) into (18) then leads to a linear system of the form (where $M_N := \sum_{p=1}^{n} n_p^+ + n_p^-$)

$$\sum_{j=1}^{M_N} c_j \left[\rho_j(x_{p,m}^{\pm}) + K\rho_j(x_{p,m}^{\pm})\right] = F(x_{p,m}^{\pm}), \text{ for } p = 1, \ldots, n,\ m = 1, \ldots, n_p^{\pm}. \tag{22}$$

Since we have two overlapping meshes, an immediate difficulty presents itself. If $x_{p,j}^+ = x_{p,m}^-$ for any $p = 1, \ldots, n$, $j, m = 1, \ldots, n_p^{\pm}$ then the system (22) will be singular, and (18) will have no solution. Moreover, if $|x_j^+ - x_m^-| < \epsilon$, for any j, m, where ϵ is sufficiently small, then the system will be ill conditioned. To avoid this scenario, one approach would be to do away with the overlapping meshes. For example, if we took $A = L_j/2$ in the definition of the mesh, putting a mesh on $[0, L_j/2]$ and a symmetric mesh on $[L_j/2, L_j]$ on each side of the polygon, with two basis functions $\mathrm{e}^{\mathrm{i}ks}$ and $\mathrm{e}^{-\mathrm{i}ks}$ on each mesh interval, then we could force $|x_j^+ - x_m^-| > \epsilon$ for any $\epsilon < \min|y_{j+1} - y_j|$ through an appropriate choice of two collocation points on each interval. However, this approach is unsuitable for two reasons

(1) On the very short intervals near the corners of the polygon, $\mathrm{e}^{\mathrm{i}ks}$ and $\mathrm{e}^{-\mathrm{i}ks}$

will almost match, leading to ill-conditioned systems (see also [9] where a related problem was solved using a mesh of this type).

(2) This approach leads to a much larger number of degrees of freedom than is necessary, with v_- being approximated by far more basis functions than necessary on Γ_j near P_j, and v_+ being approximated by far more basis functions than necessary on Γ_j near P_{j+1}.

Instead we use the mesh described above, as for the Galerkin method of [8]. In general, it is hard to say much about the spacing of the collocation points, and hence about the conditioning of the linear system (22) for a general polygon. However, considering for simplicity the side Γ_1 we remark that the collocation points $x^+_{1,j}$ will be very dense on $[0,\lambda]$, and sparse on $(\lambda, L_1]$, whilst the collocation points $x^-_{1,j}$ will be very dense on $[L_1-\lambda, L_1]$, and sparse on $[0, L_1-\lambda)$. So, provided there are no collocation points $x^-_{1,j}$ in $[0,\lambda]$ or $x^+_{1,j}$ in $[L_1-\lambda, L_1]$, then there is a better chance for the system to be well conditioned. Considering the points of $x^+_{1,j}$, we require $s^+_{1,n^+_p-1} < L_1 - 2\lambda$, and recalling (13) this will be true provided

$$\hat{N}_{L_1,\lambda,q_0} < \frac{-\log(L_1/\lambda)}{\log(1-2\lambda/L_1)}.$$

Supposing that L_1 is k wavelengths long, i.e. $L_1 = k\lambda$, we would require $\hat{N}_{L_1,\lambda,q_0} < -\log k/\log(1-2/k)$, and recalling (14) this holds if

$$N < q_0 k/2. \tag{23}$$

Since the estimate (15) suggests that N need only grow logarithmically with respect to k as $k \to \infty$ in order to maintain accuracy, this is not a severe restriction. However, for fixed k (23) suggests that we cannot take $N \to \infty$ without encountering severe conditioning problems. This makes the derivation of a conventional asymptotic error estimate rather difficult (see e.g. [4]). Collocation schemes have though been applied very successfully to (1)–(3) in the past, although not particularly with regard to the case that k is large. But the success of these schemes (see e.g. [10,7,12] or [5, Chapter 8]) suggests that provided the collocation points are sufficiently separated, the scheme should converge in the same manner as the Galerkin scheme. Thus we rely in our implementation on an examination of the mesh to ensure that the conditioning is not too bad, and we run numerical examples in §4 to demonstrate convergence to the same solution as the Galerkin scheme.

3 Implementation

The Galerkin approximation (16) leads to a linear system of the form

$$\sum_{j=1}^{N_G} c_j \left[(\rho_j, \rho_m) + (K\rho_j, \rho_m) \right] = (f, \rho_m), \qquad \text{for } m = 1, 2, \ldots, N_G.$$

Recalling (9), (10), this leaves many double integrals of the form

$$(K\rho_j, \rho_m) = \int\limits_{\mathrm{supp}\rho_m} \int\limits_{\mathrm{supp}\rho_j} \left(\frac{\partial \Phi}{\partial \mathbf{n}} + i\eta\Phi \right) \rho_j(s)\, \rho_m(t)\, \mathrm{d}s\, \mathrm{d}t, \tag{24}$$

to evaluate (see [8] and also [18] for details). This is a double integral over the support of each of the basis functions of an oscillatory function, since the term $(\partial\Phi/\partial\mathbf{n} + \mathrm{i}\eta\Phi)$ is oscillatory as are the basis functions ρ_j and ρ_m. Using the Riemann-Lebesgue Lemma, and as described in [16], in principal at least an integral should become easier to evaluate as it becomes more oscillatory, as due to cancellation of oscillating terms the exact value will tend to zero more quickly as the oscillations increase. However, using this information to construct an accurate numerical scheme for highly oscillatory integrals of the form (24) is a difficult task, and most schemes presented recently in the literature for the evaluation of highly oscillatory integrals focus on one-dimensional integrals.

However, for the linear system (22) the single integrals

$$K\rho_j(s_m) = 2 \int\limits_{y_j}^{y_{j+1}} K(s_m, t) \mathrm{e}^{\pm \mathrm{i}k(t-s_j)}\, \mathrm{d}t, \tag{25}$$

are a little easier to evaluate, where here s_m, $m = 1, \ldots, M_N$ represent the collocation points and $[y_j, y_{j+1}]$ the support of ρ_j.

If the collocation point lies on the same side as the support of the basis function then

$$K(s_m, t) = -\frac{\eta}{4} H_0^{(1)}(k\,|s_m - t|), \tag{26}$$

and using the identity [20, equation (12.31)]

$$H_0^{(1)}(s) = -\frac{2i}{\pi} \int\limits_0^{\infty} \frac{e^{(i-t)s}}{t^{\frac{1}{2}}(t-2i)^{\frac{1}{2}}} dt, \qquad s > 0, \tag{27}$$

we can write (25) as

$$\frac{i\eta}{2\pi e^{\mathrm{i}ks_j}} \int_0^\infty \frac{I(r)}{r^{\frac{1}{2}}\,(r-2\mathrm{i})^{\frac{1}{2}}}\,\mathrm{d}r, \tag{28}$$

where

$$I(r) := \int_{y_j}^{y_{j+1}} \mathrm{e}^{(\mathrm{i}-r)k|s_m-t|+\sigma_j \mathrm{i}kt}\,\mathrm{d}t, \tag{29}$$

with $\sigma_j = \pm 1$. It is shown in [3] that

$$I(r) = \begin{cases} \frac{e^{k(r-i)s_m}\left(e^{-ky_j\left(r-i\left(1+\sigma_j\right)\right)} - e^{-ky_{j+1}\left(r-i\left(1+\sigma_j\right)\right)}\right)}{k(r-i(1+\sigma_j))}, & s_m < y_j, \\ \frac{e^{-k(r-i)s_m}\left(-e^{ky_j\left(r+i\left(\sigma_j-1\right)\right)} + e^{ky_{j+1}\left(r+i\left(\sigma_j-1\right)\right)}\right)}{k(r+i(\sigma_j-1))}, & s_m > y_{j+1}, \\ \frac{e^{kis_m\sigma_j} - e^{rk\left(y_j-s_m\right)+ik\left(s_m+y_j\left(\sigma_j-1\right)\right)}}{ik((\sigma_j-1))} & \\ \quad + \frac{e^{iks_m\sigma_j} - e^{rk\left(s_m-y_{j+1}\right)+ik\left(y_{j+1}\left(1+\sigma_j\right)-s_m\right)}}{ik(r-(1+\sigma_j))}, & y_j < s_m < y_{j+1}, \end{cases}$$

and then to evaluate (29) we make the substitution $r = s^2/(1-s^2)$, to reduce the interval of integration to $[0,1]$ and eliminate the singularity at $r = 0$, allowing us to use standard Gaussian quadrature, as the remaining integral is not oscillatory.

The second and more difficult case we need to consider is that where the basis function is supported on a different side from the collocation point. In this case we must evaluate integrals of the form

$$J := \int_a^b \left[H_0^1(k\sqrt{s^2+c^2}) + \frac{\mathrm{i}sH_1^1(k\sqrt{s^2+c^2})}{\sqrt{s^2+c^2}}\right] \mathrm{e}^{\pm \mathrm{i}ks}\,\mathrm{d}s,$$

where $a, b, c \in \mathbb{R}$. Defining

$$G(s) := \left[H_0^1(k\sqrt{s^2+c^2}) + \frac{\mathrm{i}sH_1^1(k\sqrt{s^2+c^2})}{\sqrt{s^2+c^2}}\right] \mathrm{e}^{-\mathrm{i}k\sqrt{s^2+c^2}},$$

it follows from standard properties of Hankel functions (see e.g. [2]) that $G(s)$ is slowly oscillating compared to $\mathrm{e}^{\mathrm{i}k(\sqrt{s^2+c^2}\pm s)}$. We thus consider the evaluation

of

$$J_+ := \int_a^b G(s)\mathrm{e}^{\mathrm{i}k(s+\sqrt{s^2+c^2})}\,\mathrm{d}s,$$

with the method for the evaluation of $J_- := \int_a^b G(s)\mathrm{e}^{\mathrm{i}k(-s+\sqrt{s^2+c^2})}\,\mathrm{d}s$ following analogously. Making the substitution $t = s + \sqrt{s^2+c^2}$ we have

$$J_+ = \int_{a+\sqrt{a^2+c^2}}^{b+\sqrt{b^2+c^2}} G\left(\frac{t^2-c^2}{2t}\right)\frac{\sqrt{t^2+c^2}}{2t^2}\mathrm{e}^{\mathrm{i}kt}\,\mathrm{d}t, \tag{30}$$

and methods for evaluating this type of integral are well established. In particular, Iserles shows in [16] that using Filon quadrature to evaluate (30) gives an error of the same order as the underlying quadrature scheme divided by k^2 - the error decreases as $k \to \infty$.

We remark that the evaluation of $\int_{L_{n_s}}^{L} K\left(s_m, t\right)\frac{\partial u^i}{\partial \mathbf{n}}\left(t\right)dt$ on the right hand side of (22) is carried out by a combination of the above two procedures.

4 Numerical results

For the Galerkin method described in [8] we have the error estimate (17). Although no such estimate has been proved for the collocation scheme described here, we hope to demonstrate via numerical examples that a similar result might be applicable.

As a numerical example we consider the problem of scattering by a square of side length 2π, with the angle of incidence $\pi/4$ as measured anticlockwise from the downward vertical. In calculating the errors we need an "exact" solution, and this is computed using the Galerkin scheme (for which we have proved convergence) using a large number of degrees of freedom. We remark that our test problem is the same as that considered in [8].

Table 1 demonstrates the results obtained using the collocation method for increasing values of k and N. For each k, we show the N values, the total number of degrees of freedom M_N, the relative error $\left\|\varphi - \varphi_{NC}\right\|_2 / \left\|\varphi\right\|_2$ and the estimated order of convergence

$$EOC := -\frac{1}{M}\sum_{j=1}^{M}\log_2\frac{\left\|\varphi-\varphi_{2^jNC}\right\|_2}{\left\|\varphi-\varphi_{2^{j-1}NC}\right\|_2} \geq -\frac{\log_2 C}{M} + 1.$$

k	N	M_N	$\|\varphi - \varphi_{NC}\|_2 \,/\, \|\varphi\|_2$	EOC
10	4	48	4.7335×10^{-1}	0.8
	8	96	2.6980×10^{-1}	1.0
	16	192	1.2670×10^{-1}	0.9
	32	376	6.8440×10^{-2}	1.0
	64	752	3.3034×10^{-2}	
20	4	48	7.1085×10^{-1}	1.2
	8	104	3.0762×10^{-1}	1.0
	16	200	1.7872×10^{-1}	1.2
	32	392	5.5728×10^{-2}	1.0
	64	792	4.1295×10^{-2}	
40	4	56	5.4597×10^{-1}	0.7
	8	104	3.4089×10^{-1}	0.3
	16	208	3.6095×10^{-1}	0.3
	32	416	2.8317×10^{-1}	1.0
	64	824	3.7158×10^{-2}	
80	4	56	4.6096×10^{-1}	1.0
	8	112	2.3333×10^{-1}	0.8
	16	216	1.5975×10^{-1}	0.6
	32	432	1.4203×10^{-1}	0.9
	64	864	4.4374×10^{-2}	
160	4	56	4.4455×10^{-1}	-0.1
	8	112	4.6445×10^{-1}	0.5
	16	224	2.3456×10^{-1}	0.7
	32	456	9.3327×10^{-2}	0.8
	64	904	4.8153×10^{-2}	

Table 1
Relative L_2 errors, $k = 10, 20, 40, 80, 160$. $N = 2, 4, 8, 16, 32, 64$

The results appear to suggest that for each value of k the solution is converging to the same solution as that achieved by the Galerkin scheme, for which we have proved convergence to the true solution of the integral equation, and at roughly the same rate (i.e. $EOC \approx 1$, as it would be if the estimate (17) held for the collocation scheme as well). Moreover, the relative error remains

roughly constant for fixed N as k increases, suggesting that, as for the Galerkin scheme, the number of degrees of freedom required to achieve a prescribed level of accuracy grows only logarithmically with respect to the frequency. Further numerical results can be found in [3].

5 Conclusions

We have proposed and implemented a new collocation method for solving problems of high frequency scattering by convex polygons. We use the same approximation space as for the Galerkin method in [8], and our numerical results appear to suggest that we achieve the same convergence rate, namely that the number of degrees of freedom required to achieve a prescribed level of accuracy grows only logarithmically with respect to the frequency. Moreover, the collocation method exhibits a significant reduction in the computational time compared to the Galerkin scheme.

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