A GALERKIN BOUNDARY ELEMENT METHOD FOR HIGH FREQUENCY SCATTERING BY CONVEX POLYGONS

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Abstract. In this paper we consider the problem of time-harmonic acoustic scattering in two dimensions by convex polygons. Standard boundary or finite element methods for acoustic scattering problems have a computational cost that grows at least linearly with respect to the frequency of the incident wave. Here we present a novel Galerkin boundary element method, with an approximation space consisting of the products of plane waves with piecewise polynomials supported on a graded mesh, with smaller elements closer to the corners of the polygon. We demonstrate via both a rigorous error analysis and numerical results that the number of degrees of freedom required to achieve a prescribed level of accuracy grows only logarithmically with respect to the frequency. Our boundary element method is a discretisation of a well-known second kind combined-layer-potential integral equation. We provide a proof that this equation and its adjoint are well-posed in a Sobolev space setting for general Lipschitz domains.

Key words. Galerkin boundary element method, high frequency scattering, convex polygons, Helmholtz equation, large wave number, Lipschitz domains

AMS subject classifications. 35J05, 65R20

1. Introduction.

The scattering of time-harmonic acoustic waves by bounded obstacles is a classical problem that has received much attention in the literature over the years. Much effort has been put into the development of efficient numerical schemes, but an outstanding question yet to be fully resolved is how to achieve an accurate approximation to the scattered wave with a reasonable computational cost in the case that the scattering obstacle is large compared to the wavelength of the incident field.

The standard boundary or finite element method approach is to seek an approximation to the scattered field from a space of piecewise polynomial functions. However, due to the oscillatory nature of the solution, such an approach suffers from the limitation that a fixed number of degrees of freedom M are required per wavelength in order to achieve a good level of accuracy, with the accepted guideline in the engineering literature being to take $M = 10$ (see for example [44] and the references therein). A further difficulty, at least for the finite element method, is the presence of "pollution errors", phase errors in wave propagation across the domain, which can lead to even more severe restrictions on the value of M when the wavelength is short [9, 36].

Let L be a linear dimension of the scattering obstacle, and set $k = 2\pi/\lambda$, where λ is the wavelength of the incident wave, so that k is the wave number, proportional to the frequency of the incident wave. Then a consequence of fixing M is that the number of degrees of freedom will be proportional to $(kL)^d$, where $d = N$ in the case of the finite element method (FEM), $d = N - 1$ in the case of the boundary element method (BEM), and $N = 2$ or 3 is the number of space dimensions of the problem. Thus, as either the frequency of the incident wave or the size of the obstacle grows, so does the number of degrees of freedom, and hence the computational cost of the numerical scheme. As a result, the numerical solution of many realistic physical problems is intractable using current technologies. In fact, for some of the most

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powerful recent algorithms for three-dimensional scattering problems (e.g. [13, 21]), the largest obstacles for which numerical results have been reported have diameter not more than a few hundred times the wavelength.

For boundary element methods, the cost of setting up and solving the large linear systems which arise can be reduced substantially through a combination of preconditioned iterative methods [4, 22, 34] combined with fast matrix-vector multiply methods based on the fast multipole method [5, 26, 21] or the FFT [13]. However, this does nothing to reduce the growth in the number of degrees of freedom as kL increases (linear with respect to kL in 2D, quadratic in 3D). Thus computations become infeasible as $kL \to \infty$.

To achieve a dependence of the number of degrees of freedom on kL which is lower than $(kL)^d$, it seems essential to use an approximation space better able to replicate the behaviour of the scattered field at high frequencies than piecewise polynomials. To that end, much attention in the recent literature has focused on enriching the approximation space with oscillatory functions, specifically plane waves or Bessel functions.

A common approach (see e.g. [8, 16, 27, 35, 44]) is to form an approximation space consisting of standard finite element basis functions multiplied by plane waves travelling in a large number of directions, approximately uniformly distributed on the unit circle (in 2D) or sphere (in 3D). Theoretical analysis (e.g. [8]) and computational results (e.g. [44]) suggest that these methods converge rapidly as the number of plane wave directions increases, with a significant reduction in the number of degrees of freedom required per wavelength, compared to standard finite and boundary element methods. But the number of degrees of freedom is still proportional to $(kL)^d$, and serious conditioning problems occur when the number of plane wave directions is large.

A related idea is to attempt to identify the important wave propagation directions at high frequencies, and to incorporate the oscillatory part of this high frequency asymptotic behaviour into the approximation space. This is the idea behind the finite element method of [31] and the boundary element method of [33, 29]. This idea has been investigated most thoroughly in the case that the scattering obstacle is smooth and strictly convex. In this case the leading order oscillatory behaviour is particularly simple on the boundary of the scattering obstacle, so that this approach is perhaps particularly well-adapted for boundary element methods. If a direct integral equation formulation is used, in which the solution to be determined is the trace of the total field or its normal derivative on the boundary, the most important wave direction to include is that of the incident wave (see for example [1, 25, 12, 28]). This approach is equivalent to approximating the ratio of the total field to the incident field, with physical optics predicting that this ratio is approximately constant on the illuminated side and approximately zero on the shadow side of the obstacle at high frequencies.

In [1], Abboud et. al. consider the two dimensional problem of scattering by a smooth, strictly convex obstacle. They suggest that the ratio of the scattered field to the incident field can be approximated with error of order $N^{-\nu} + ((kL)^{1/3}/N)^{\nu+1}$ using a uniform mesh of piecewise polynomials of degree ν, so that the total number of degrees of freedom N only needs to be proportional to $(kL)^{1/3}$ in order to maintain a fixed level of accuracy. In fact this paper appears to be the first in which the dependence of the error estimates on the wave number k is indicated, and the requirement that the number of degrees of freedom is proportional to $(kL)^{1/3}$ is a big improvement over the usual requirement for proportionality to kL. This approach is coupled with a fast multipole method in [25], where impressive numerical results are reported for large scale 3D problems.

The same approach is combined with a mesh refinement concentrating degrees of freedom near the shadow boundary in [12]. The numerical results in [12] for scattering by a circle suggest that, with this mesh refinement, both the number of degrees of freedom and the total computational cost required to maintain a fixed level of accuracy remain constant as $kL \to \infty$. In [28] a numerical method in the spirit of [12] is proposed, namely a p-version boundary element method with a k-dependent mesh refinement in a transition region around the shadow boundary. A rigorous error analysis, which combines estimates using high frequency asymptotics of derivatives of the solution on the surface with careful numerical analysis, demonstrates that the approximation space is able to represent the oscillatory solution to any desired accuracy using a number of degrees of freedom which remains fixed as the wave number increases. This theoretical result is confirmed by numerical experiments using this approximation space as the basis of a Galerkin method. The method of [12] has recently been applied to deal with each of the multiple scatters which occur when a wave is incident on two, separated, smooth convex 2D obstacles [33]. Numerical experiments have also recently been presented in [29] where the convergence of this iterative approach to the multiple scattering problem is analysed.

In this paper, we consider specifically the problem of scattering by convex polygons. This is, in many respects, a more challenging problem than the smooth convex obstacle since the corners of the polygon give rise to strong diffracted rays which illuminate the shadow side of the obstacle much more strongly than the rays that creep into the shadow zone of a smooth convex obstacle. These creeping rays decay exponentially, so that it is enough to remove the oscillation of the incident field to obtain a sufficiently simple field to approximate by piecewise polynomials.

This approach does not suffice for a scatterer with corners. In brief, our algorithm for the convex polygon is as follows. From the geometrical theory of diffraction, one expects, on the sides of the polygon, incident, reflected and diffracted ray contributions. On each illuminated side, the leading order behaviour as $k \to \infty$ consists of the incident wave and a known reflected wave. The first stage in our algorithm is to separate this part of the solution explicitly. (On sides in shadow this step is omitted.) The remaining field on the boundary consists of waves which have been diffracted at the corners and which travel along the polygon sides. We approximate this remaining field by taking linear combinations of products of piecewise polynomials with plane waves, the plane waves travelling parallel to the polygon sides. A key ingredient in our algorithm is to design a graded mesh to go on each side of the polygon for the piecewise polynomial approximation. This mesh has larger elements away from the corners and a mesh grading near the corners depending on the internal angles, in such a way as to equidistribute the approximation error over the subintervals of the mesh, based on a careful study of the oscillatory behaviour of the solution.

Our algorithm and analysis are closely related to those in recent work of the authors [20, 39] on the specific problem of 2D acoustic scattering by an inhomogeneous, piecewise constant impedance plane. In [20, 39] a Galerkin boundary element method for this problem is proposed, in which the leading order high frequency behaviour as $k \to \infty$, consisting of the incident and reflected ray contributions, is first subtracted off. The remaining scattered wave, consisting of rays diffracted by discontinuities in impedance, is expressed as a sum of products of oscillatory and non-oscillatory functions, with the non-oscillatory functions being approximated by piecewise polynomials supported on a graded mesh, with larger elements away from discontinuities in impedance. For the method in [20] it was shown in that paper that the number of degrees of freedom needed to maintain accuracy as $k \to \infty$ grows only logarithmically with k. This result was improved in [39] where it was shown, via sharper regularity results and a modified mesh, that for a fixed number of degrees of freedom the error is bounded independently of k, the first such result supported by a rigorous error analysis for any scattering problem.

The major results of the paper are as follows. We begin in §2 by introducing the exterior Dirichlet scattering problem that we will solve numerically via a second kind boundary integral equation formulation. Our boundary integral equation is well known (e.g. [23]), obtained from Green's representation theorem. The boundary integral operator is a linear combination of a single-layer potential and its normal derivative, so that the integral equation is precisely the adjoint of the equation proposed independently for the exterior Dirichlet problem by Brakhage and Werner [11], Leis [40], and Panic [43]. However, as noted recently in [14], there exists no account of these standard formulations for Lipschitz domains (the treatment in [23] is for domains of class C^2). We remedy this gap in the literature in §2, showing that our operator is a bijection on the boundary Sobolev space $H^{s-1/2}(\Gamma)$ and the adjoint operator of [11] a bijection on $H^{s+1/2}(\Gamma)$, both for $|s| \leq 1/2$. Our starting points are known results on the (Laplace) double-layer potential operator on Lipschitz domains [47, 30] coupled with mapping properties of the single-layer potential operator [41]. Of course the results we obtain apply in particular to a polygonal domain in 2D.

The design of our numerical algorithm depends on a careful analysis of the oscillatory behaviour of the solution of the integral equation (which is the normal derivative of the total field on the boundary Γ). This is the content of §3 of the paper. In contrast e.g. to [28], where this information is obtained by difficult high frequency asymptotics, we adapt a technique from [20, 39], where explicit representations of the solution in a half-plane are obtained from Green's representation theorem. In the estimates we obtain of high order derivatives, we take care to obtain as precise information as possible, with a view to the future design of alternative numerical schemes, perhaps based on a p- or hp-boundary element method.

Section 4 of the paper contains, arguably, the most significant theoretical and practical results. In this section we design an approximation space for the normal derivative of the total field on Γ. As outlined above, on each side we represent this unknown as the sum of the leading order asymptotics (known explicitly, and zero on a side in shadow) plus an expression of the form $\exp(\mathrm{i}ks)V_+(s) + \exp(-\mathrm{i}ks)V_-(s)$, where s is arc-length distance along the side and $V_\pm(s)$ are piecewise polynomials. We show, as a main result of the paper, that the approximation space based on this representation has the property that the error in best approximation of the normal derivative of the total field is bounded by $C_\nu(1 + \log(kL))^{\nu+3/2} M_N^{-\nu-1}$, where M_N is the total number of degrees of freedom, L is the length of the perimeter, ν is the polynomial degree, and the constant C_ν depends only on ν and the corner angles of the polygon. This is a strong result, showing that the number of degrees of freedom need only increase like $\log^{3/2}(kL)$ as $kL \to \infty$ to maintain accuracy.

In §5 we analyse a Galerkin method, based on the approximation space of §4. We show that the same bound holds for our Galerkin method approximation to the solution of the integral equation, except that an additional stability constant is introduced. We do not attempt the (difficult) task of ascertaining the dependence of this stability constant on k. In §6 we present some numerical results which fully support our theoretical estimates, and we discuss, briefly, some numerical implementation issues, including conditioning and evaluation of the integrals that arise. We finish the paper with some concluding remarks and open problems.

We note that the Galerkin method is, of course, not the only way to select a numerical solution from a given approximation space. In [6] we present some results for a collocation method, based on the approximation space results in §4. The attraction of the Galerkin method we present in §5 is that we are able to establish stability, at least in the asymptotic limit of sufficient mesh refinement, which we do not know how to do for the collocation method.

2. The boundary value problem and integral equation formulation. Consider scattering of a time-harmonic acoustic plane wave u^i by a sound-soft convex polygon Υ, with boundary $\Gamma := \bigcup_{j=1}^{n} \Gamma_j$, where Γ_j, $j = 1, \ldots, n$ are the n sides of the polygon with j increasing anticlockwise, as shown in figure 2.1. We denote by

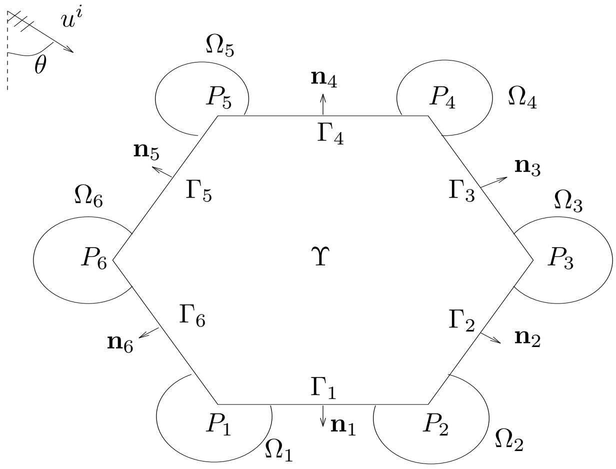

FIG. 2.1. *Our notation for the polygon.*

$P_j := (p_j, q_j)$, $j = 1, \ldots, n$, the vertices of the polygon, and we set $P_{n+1} = P_1$, so that, for $j = 1, \ldots, n$, Γ_j is the line joining P_j with P_{j+1}. We denote the length of Γ_j by $L_j := |P_{j+1} - P_j|$, the external angle at each vertex P_j by $\Omega_j \in (\pi, 2\pi)$, the unit normal perpendicular to Γ_j and pointing out of Υ by $\mathbf{n}_j := (n_{j1}, n_{j2})$, and the angle of incidence of the plane wave, as measured anticlockwise from the downward vertical, by $\theta \in [0, 2\pi)$. Writing $\mathbf{x} = (x_1, x_2)$ and $\mathbf{d} := (\sin\theta, -\cos\theta)$, we then have

$$u^i(\mathbf{x}) = \mathrm{e}^{\mathrm{i}k(x_1 \sin\theta - x_2 \cos\theta)} = \mathrm{e}^{\mathrm{i}k\mathbf{x}.\mathbf{d}}.$$

We will say that Γ_j is in shadow if $\mathbf{n}_j \cdot \mathbf{d} \geq 0$ and is illuminated if $\mathbf{n}_j \cdot \mathbf{d} < 0$. If n_s is the number of sides in shadow, it is convenient to choose the numbering so that sides $1, \ldots, n_s$ are in shadow, sides $n_s + 1, \ldots, n$ are illuminated.

We will formulate the boundary value problem we wish to solve for the total acoustic field u in a standard Sobolev space setting. For an open set $G \subset \mathbb{R}^N$, let $H^1(G) := \{v \in L^2(G) : \nabla v \in L^2(G)\}$ (∇v denoting here the weak gradient of v). We recall [41] that, if G is a Lipschitz domain then there is a well-defined trace operator, the unique bounded linear operator $\gamma : H^1(G) \to H^{1/2}(\partial G)$ which satisfies $\gamma v = v|_{\partial G}$ in the case when $v \in C^\infty(\bar{G}) := \{w|_{\bar{G}} : w \in C^\infty(\mathbb{R}^N)\}$. Let $H^1(G; \Delta) := \{v \in H^1(G) : \Delta v \in L^2(G)\}$ (Δ the Laplacian in a weak sense), a Hilbert space with the norm $\|v\|_{H^1(G;\Delta)} := \{\int_G [|v|^2 + |\nabla v|^2 + |\Delta v|^2] \mathrm{d}x\}^{1/2}$. If G is Lipschitz, then [41] there is also a well-defined normal derivative operator, the unique bounded linear operator $\partial_{\mathbf{n}} : H^1(G;\Delta) \to H^{-1/2}(\partial G)$ which satisfies

$$\partial_{\mathbf{n}} v = \frac{\partial v}{\partial \mathbf{n}} := \mathbf{n} \cdot \nabla v,$$

almost everywhere on Γ, when $v \in C^\infty(\bar{G})$. $H^1_{\mathrm{loc}}(G)$ denotes the set of measurable $v : G \to \mathbb{C}$ for which $\chi v \in H^1(G)$ for every compactly supported $\chi \in C^\infty(\bar{G})$.

The polygonal domain Υ is Lipschitz as is its exterior $\bar{D} := \mathbb{R}^2 \setminus \bar{\Upsilon}$. Let $\gamma_+ : H^1(D) \to H^{1/2}(\Gamma)$ and $\gamma_- : H^1(\Upsilon) \to H^{1/2}(\Gamma)$ denote the exterior and interior trace operators, respectively, and let $\partial^+_{\mathbf{n}} : H^1(D;\Delta) \to H^{-1/2}(\Gamma)$ and $\partial^-_{\mathbf{n}} : H^1(\Upsilon;\Delta) \to H^{-1/2}(\Gamma)$ denote the exterior and interior normal derivative operators, respectively, the unit normal vector $\mathbf{n}$ directed out of Υ. Then the boundary value problem we seek to solve is the following: given $k > 0$ (the wave number) find $u \in C^2(D) \cap H^1_{\mathrm{loc}}(D)$ such that

$$\Delta u + k^2 u = 0 \quad \text{in } D, \tag{2.1}$$

$$\gamma_+ u = 0 \quad \text{on } \Gamma, \tag{2.2}$$

and the scattered field, $u^s := u - u^i$, satisfies the Sommerfeld radiation condition

$$\lim_{r\to\infty} r^{1/2} \left(\frac{\partial u^s}{\partial r}(\mathbf{x}) - \mathrm{i}ku^s(\mathbf{x}) \right) = 0, \tag{2.3}$$

where $r = |\mathbf{x}|$ and the limit holds uniformly in all directions $\mathbf{x}/|\mathbf{x}|$.

THEOREM 2.1. *(see e.g. [41, theorem 9.11]). The boundary value problem (2.1)–(2.3) has exactly one solution.*

Suppose that $u \in C^2(D) \cap H^1_{\mathrm{loc}}(D)$ satisfies (2.1)–(2.3). Then, by standard elliptic regularity estimates [32, §8.11], $u \in C^\infty(\bar{D} \setminus \Gamma_C)$, where $\Gamma_C := \{P_1, \ldots, P_n\}$ is the set of corners of Γ. It is, moreover, possible to derive an explicit representation for u near the corners. For $j = 1, \ldots, n$, let $R_j := \min(L_{j-1}, L_j)$ (with $L_{-1} := L_N$). Let (r,θ) be polar coordinates local to a corner P_j, chosen so that $r = 0$ corresponds to the point P_j, the side Γ_{j-1} lies on the line $\theta = 0$, the side Γ_j lies on the line $\theta = \Omega_j$, and the part of $\bar{D}$ within distance R_j of P_j is the set of points with polar coordinates $\{(r,\theta) : 0 \le r < R_j,\ 0 \le \theta \le \Omega_j\}$. Choose R so that $R \le R_j$ and $\rho := kR < \pi/2$, and let G denote the set of points with polar coordinates $\{(r,\theta) : 0 \le r < R, 0 \le \theta \le \Omega_j\}$ (see figure 2.2). The following result, in which J_ν denotes the Bessel function of the first kind of order ν, follows by standard separation of variables arguments.

THEOREM 2.2 (representation near corners). *Let $g(\theta)$ denote the value of u at the point with polar coordinates (R,θ). Then, where (r,θ) denotes the polar coordinates of $\mathbf{x}$, it holds that*

$$u(\mathbf{x}) = \sum_{n=1}^{\infty} a_n J_{n\pi/\Omega_j}(kr) \sin\left(\frac{n\theta\pi}{\Omega_j}\right), \quad \mathbf{x} \in G, \tag{2.4}$$

where

$$a_n := \frac{2}{\Omega_j J_{n\pi/\Omega_j}(kR)} \int_0^{\Omega_j} g(\theta) \sin\left(\frac{n\theta\pi}{\Omega_j}\right) \mathrm{d}\theta, \quad n \in \mathbb{N}. \tag{2.5}$$

REMARK 2.3. *The condition $\rho = kR < \pi/2$ ensures that $J_{n\pi/\Omega_j}(kR) \neq 0$, $n \in \mathbb{N}$, in fact (see (3.12)) that $|a_n J_{n\pi/\Omega_j}(kr)| \le C(r/R)^{-n}$, where the constant C is*

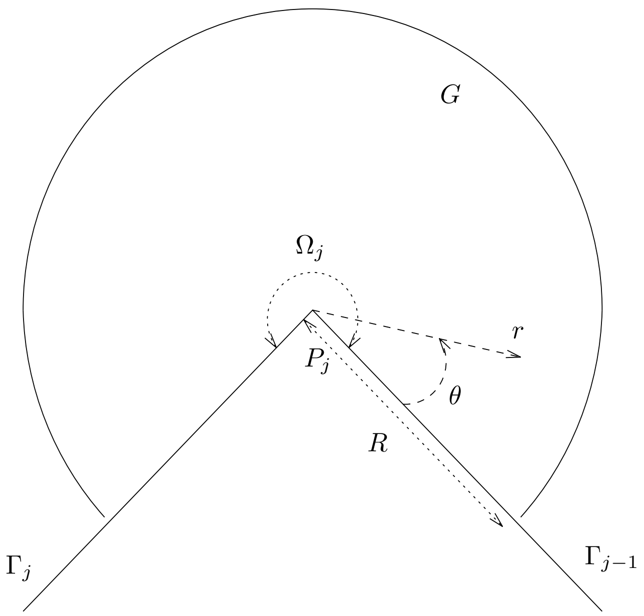

FIG. 2.2. *Neighbourhood of a corner.*

independent of n *and* $\mathbf{x}$*, so that the series (2.4) converges absolutely and uniformly in* G*. Thus* $u \in C(\bar{D})$*. Moreover, from this representation and the behaviour of the Bessel function* J_ν *(cf. theorem 3.3) it follows that, near the corner* P_j*,* $\nabla u(\mathbf{x})$ *has the standard singular behaviour that*

$$|\nabla u(\mathbf{x})| = \mathcal{O}\left(r^{\pi/\Omega_j - 1}\right) \text{ as } r \to 0. \tag{2.6}$$

From [24, theorem 3.12] and [41, theorems 7.15, 9.6] we see that, if u satisfies the boundary value problem (2.1)–(2.3), then a form of Green's representation theorem holds, namely

$$u(\mathbf{x}) = u^i(\mathbf{x}) - \int_\Gamma \Phi(\mathbf{x},\mathbf{y})\partial_{\mathbf{n}}^+ u(\mathbf{y})\,\mathrm{d}s(\mathbf{y}), \quad \mathbf{x} \in D, \tag{2.7}$$

where $\mathbf{n}$ is the normal direction directed out of Υ and $\Phi(\mathbf{x},\mathbf{y}) := (\mathrm{i}/4)H_0^{(1)}(k|\mathbf{x}-\mathbf{y}|)$ is the standard fundamental solution for the Helmholtz equation, with $H_0^{(1)}$ the Hankel function of the first kind of order zero. Note that, since $u \in C^\infty(\bar{D} \setminus \Gamma_C)$ and the bound (2.6) holds, we have in fact that $\partial_{\mathbf{n}}^+ u = \partial u/\partial \mathbf{n} \in L^2(\Gamma) \cap C^\infty(\Gamma \setminus \Gamma_C)$.

Starting from the representation (2.7) for u, we will obtain the boundary integral equation for $\partial u/\partial \mathbf{n}$ which we will solve numerically later in the paper. This integral equation formulation is expressed in terms of the standard single-layer potential operator ($\mathcal{S}$) and the adjoint of the double-layer potential operator ($\mathcal{T}$), defined, for $v \in L^2(\Gamma)$, by

$$\mathcal{S}v(\mathbf{x}) := 2\int_\Gamma \Phi(\mathbf{x},\mathbf{y})v(\mathbf{y})\,\mathrm{d}s(\mathbf{y}),\ \mathcal{T}v(\mathbf{x}) := 2\int_\Gamma \frac{\partial\Phi(\mathbf{x},\mathbf{y})}{\partial\mathbf{n}(\mathbf{x})}v(\mathbf{y})\,\mathrm{d}s(\mathbf{y}),\ \mathbf{x} \in \Gamma\setminus\Gamma_C. \tag{2.8}$$

We note that both $\mathcal{S}$ and $\mathcal{T}$ are bounded operators on $L^2(\Gamma)$. In fact, more generally [41], $\mathcal{S} : H^{s-1/2}(\Gamma) \to H^{s+1/2}(\Gamma)$ and $\mathcal{T} : H^{s-1/2}(\Gamma) \to H^{s-1/2}(\Gamma)$, for $|s| \leq 1/2$, and these mappings are bounded. We state the integral equation we will solve in the next theorem. Our proof of this theorem is based on that in [23] for domains of class C^2, modified to use more recent results on layer potentials on Lipschitz domains.

THEOREM 2.4. *If $u \in C^2(D) \cap H^1_{\mathrm{loc}}(D)$ satisfies the boundary value problem (2.1)–(2.3) then, for every $\eta \in \mathbb{R}$, $\partial^+_{\mathbf{n}} u = \frac{\partial u}{\partial \mathbf{n}} \in L^2(\Gamma)$ satisfies the integral equation*

$$(\mathcal{I} + \mathcal{K})\partial^+_{\mathbf{n}} u = f \quad \text{on } \Gamma, \tag{2.9}$$

where $\mathcal{I}$ is the identity operator, $\mathcal{K} := \mathcal{T} + \mathrm{i}\eta\mathcal{S}$, and

$$f(\mathbf{x}) := 2\frac{\partial u^i}{\partial \mathbf{n}}(\mathbf{x}) + 2\mathrm{i}\eta u^i(\mathbf{x}), \quad \mathbf{x} \in \Gamma \setminus \Gamma_C.$$

Conversely, if $v \in H^{-1/2}(\Gamma)$ satisfies $(\mathcal{I} + \mathcal{K})v = f$, for some $\eta \in \mathbb{R} \setminus \{0\}$, and u is defined in D by (2.7), with $\partial^+_{\mathbf{n}} u$ replaced by v, then $u \in C^2(D) \cap H^1_{\mathrm{loc}}(D)$ and satisfies the boundary value problem (2.1)–(2.3). Moreover, $\partial^+_{\mathbf{n}} u = v$.

Proof. Suppose first that $v \in H^{-1/2}(\Gamma)$ satisfies $(\mathcal{I} + \mathcal{K})v = f$ and define u by $u := u^i - Sv$ where

$$Sv(\mathbf{x}) := \int_\Gamma \Phi(\mathbf{x}, \mathbf{y}) v(\mathbf{y}) \, \mathrm{d}s(\mathbf{y}), \quad \mathbf{x} \in \mathbb{R}^2 \setminus \Gamma.$$

Then [41, theorem 6.11, chapter 9] $u \in C^2(\mathbb{R}^2 \setminus \Gamma) \cap H^1_{\mathrm{loc}}(\mathbb{R}^2)$ and satisfies (2.1) in $\mathbb{R}^2 \setminus \Gamma$ and (2.3). Thus u satisfies the boundary value problem as long as $\gamma_+ u = 0$. Now, standard results on boundary traces of the single-layer potential on Lipschitz domains [41] give us that

$$2\gamma_\pm Sv = \mathcal{S}v, \quad 2\partial^\pm_{\mathbf{n}}(Sv) = (\mp\mathcal{I} + \mathcal{T})v. \tag{2.10}$$

On the other hand, we have that $(\mathcal{I} + \mathcal{T} + \mathrm{i}\eta\mathcal{S})v = f$. Thus

$$2\partial^-_{\mathbf{n}} u = 2\frac{\partial u^i}{\partial \mathbf{n}} - (\mathcal{I} + \mathcal{T})v = \mathrm{i}\eta\mathcal{S}v - 2\mathrm{i}\eta\gamma_+ u^i = -2\mathrm{i}\eta\gamma_- u.$$

Applying Green's first identity [41, theorem 4.4] to $u \in H^1(\Upsilon; \Delta)$ we deduce that

$$-\eta \int_\Gamma |\gamma_- u|^2 \, \mathrm{d}s = \mathrm{Im} \int_\Gamma \partial^-_{\mathbf{n}} u \, \gamma_- \bar{u} \, \mathrm{d}s = 0.$$

Thus $\gamma_+ u = \gamma_- u = 0$, so that u satisfies the boundary value problem (2.1)–(2.3). Further, $\partial^-_{\mathbf{n}} u = 0$ and $\partial^+_{\mathbf{n}} u = v + \partial^-_{\mathbf{n}} u = v$.

Conversely, if u satisfies the boundary value problem, in which case $\partial^+_{\mathbf{n}} u = \frac{\partial u}{\partial \mathbf{n}} \in L^2(\Gamma) \subset H^{-1/2}(\Gamma)$ and (2.7) holds, then, applying the trace results (2.10), we deduce

$$2\gamma_+ u^i = \mathcal{S}\partial^+_{\mathbf{n}} u, \quad 2\frac{\partial u^i}{\partial \mathbf{n}} = (\mathcal{I} + \mathcal{T})\partial^+_{\mathbf{n}} u.$$

Hence equation (2.9) holds. ∎

The above theorem, together with theorem 2.1, implies that the integral equation (2.9) has exactly one solution in $H^{-1/2}(\Gamma)$, provided we choose $\eta \neq 0$.

REMARK 2.5. *The idea of taking a linear combination of first and second kind integral equations to obtain a uniquely solvable boundary integral equation equivalent* *to an exterior scattering problem for the Helmholtz equation dates back to Brakhage and Werner [11], Leis [40], and Panich [43] for the exterior Dirchlet problem and Burton and Miller [15] for the Neumann problem. In fact, the integral equation in [11, 40, 43] is precisely the adjoint of equation (2.9) (see the discussion and corollary 2.7 below). The above proof is based on that in [23]. But, while Colton and Kress [23] restrict attention to the case when* Γ *is sufficiently smooth (of class* C^2*), the proof given above is valid for arbitrary Lipschitz* Γ*, and in an arbitrary number of dimensions. (Note however that, for general Lipschitz* Γ*,* $\mathcal{T}v$*, for* $v \in H^{-1/2}(\Gamma)$*, must be understood as the sum of the normal derivatives of* Sv *on the two sides of* Γ *[41, Chapter 7]. This definition of* $\mathcal{T}v$ *is equivalent to that in (2.8) when* Γ *is Lyapunov in a neighbourhood of almost every point on* Γ*, e.g. if* Γ *is a polyhedron, and if* v *is sufficiently smooth, e.g.* $v \in L^\infty(\Gamma)$ *[41, theorem 7.4].)*

The following theorem, which shows that the operator $\mathcal{I} + \mathcal{K}$ is bijective on a range of Sobolev spaces, holds for a general Lipschitz boundary Γ (with $\mathcal{T}$ defined as in remark 2.5 in the general case), in any number of space dimensions ≥ 2.

THEOREM 2.6. *Let* $\mathcal{A} := \mathcal{I} + \mathcal{K}$ *and suppose that* $\eta \in \mathbb{R} \setminus \{0\}$*. Then, for* $|s| \leq 1/2$*, the bounded linear operator* $\mathcal{A} : H^{s-1/2}(\Gamma) \to H^{s-1/2}(\Gamma)$ *is bijective with bounded inverse* $\mathcal{A}^{-1}$*.*

Proof. It is enough to show this result for $s = \pm 1/2$; it then follows for all s by interpolation [41]. We note first that, since $H^1(\Gamma)$ is compactly embedded in $L^2(\Gamma)$, so that $L^2(\Gamma)$ is compactly embedded in $H^{-1}(\Gamma)$, and since $\mathcal{S}$ is a bounded operator from $H^{-1}(\Gamma)$ to $L^2(\Gamma)$, it follows that $\mathcal{S}$ is a compact operator on $H^{-1}(\Gamma)$ and $L^2(\Gamma)$. Let $\mathcal{T}_0$ denote the operator corresponding to $\mathcal{T}$ in the case $k = 0$; explicitly, in the case when Γ is a 2D polygon, $\mathcal{T}_0 v$, for $v \in L^2(\Gamma)$, is defined by (2.8) with $\Phi(\mathbf{x}, \mathbf{y})$ replaced by $\Phi_0(\mathbf{x}, \mathbf{y}) := -(2\pi)^{-1} \log |\mathbf{x} - \mathbf{y}|$. Then $\mathcal{T}_0 - \mathcal{T}$ is a bounded operator from $H^{-1}(\Gamma)$ to $L^2(\Gamma)$ and so a compact operator on $H^{-1}(\Gamma)$ and $L^2(\Gamma)$. (To see the boundedness of $\mathcal{T}_0 - \mathcal{T}$ it is perhaps easiest to show that the adjoint operator, $\mathcal{T}_0' - \mathcal{T}'$, is a bounded operator from $L^2(\Gamma)$ to $H^1(\Gamma)$, which follows since $D(\mathcal{T}_0' - \mathcal{T}')$ is a bounded operator on $L^2(\Gamma)$. Here D is the surface gradient operator, $\mathcal{T}'$ and $\mathcal{T}_0'$ are standard double-layer potential operators [41, theorem 6.17], in particular

$$T'v(\mathbf{x}) := \int_\Gamma \frac{\partial \Phi(\mathbf{x}, \mathbf{y})}{\partial \mathbf{n}(\mathbf{y})} v(\mathbf{y}) \mathbf{d}s(\mathbf{y}), \quad \mathbf{x} \in \Gamma,$$

and the boundedness of the integral operator $D(\mathcal{T}_0' - \mathcal{T}')$ follows since its kernel is continuous or weakly singular.) Thus $\mathcal{A}$, as an operator on $H^{s-1/2}(\Gamma)$, $s = \pm 1/2$, is a compact perturbation of $\mathcal{I} + \mathcal{T}_0$. But it is known that $\mathcal{I} + \mathcal{T}_0'$ is Fredholm of index zero on $H^{s+1/2}(\Gamma)$, for $|s| \leq 1/2$ (see [47, 30]), from which it follows from [41, theorem 6.17] that the adjoint operator $\mathcal{I} + \mathcal{T}_0'$ is Fredholm of index zero on $H^{s-1/2}(\Gamma)$, for $|s| \leq 1/2$. Thus $\mathcal{A}$ is Fredholm of index zero on $H^{s-1/2}(\Gamma)$, $s = \pm 1/2$. Since $\mathcal{A}$ is Fredholm with the same index on $H^{-1}(\Gamma)$ and $L^2(\Gamma)$, and $L^2(\Gamma)$ is dense in $H^{-1}(\Gamma)$, it follows from a standard result on Fredholm operators (see e.g. [45, §1]) that the null-space of $\mathcal{A}$, as an operator on $H^{-1}(\Gamma)$, is a subset of $L^2(\Gamma)$. But it follows from theorems 2.1 and 2.4 that $\mathcal{A}v = 0$ has no non-trivial solution in $H^{-1/2}(\Gamma) \supset L^2(\Gamma)$. Thus $\mathcal{A} : H^{s-1/2}(\Gamma) \to H^{s+1/2}(\Gamma)$ is invertible for $s = \pm 1/2$. ◻

We have observed in remark 2.5 that an alternative integral equation formulation for the exterior Dirichlet problem was introduced in [11, 40, 43]. In this formulation one seeks a solution to the exterior Dirichlet problem in the form of a combined single- and double-layer potential with some unknown density $\tilde{\phi}$ and arrives at the boundary integral equation $\mathcal{A}'\tilde{\phi} = 2\gamma_+ u^i$, where

$$\mathcal{A}' = \mathcal{I} + \mathcal{T}' + i\eta S$$

is the adjoint of $\mathcal{A}$ in the sense that the duality relation holds that $\langle A\phi, \psi\rangle_\Gamma = \langle \phi, A'\psi\rangle_\Gamma$, for $\phi \in H^{-1/2}(\Gamma)$, $\psi \in H^{1/2}(\Gamma)$, where $\langle \phi, \psi\rangle_\Gamma := \int_\Gamma \phi(y)\psi(y)\mathrm{d}s(y)$ [41, theorems 6.15, 6.17]. It is known that $\mathcal{A}'$ maps $H^{s+1/2}(\Gamma)$ to $H^{1/2}(\Gamma)$ and this mapping is bounded, for $|s| \leq 1/2$ [41]. This, the duality relation, and theorem 2.6 imply the invertibility of $\mathcal{A}'$. Precisely, we have the following result.

Corollary 2.7. *For $|s| \leq 1/2$ and $\eta \in \mathbb{R} \setminus \{0\}$, the mapping $\mathcal{A}' : H^{s+1/2}(\Gamma) \to H^{s+1/2}(\Gamma)$ is bijective with bounded inverse ${\mathcal{A}'}^{-1}$.*

In the remainder of the paper we will focus on the properties of $\mathcal{A}$ as an operator on $L^2(\Gamma)$. We remark that the result that $\mathcal{I} + \mathcal{T}_0'$ is Fredholm of index zero on $L^2(\Gamma)$ dates back to [46] in the case when Γ is a 2D polygon. Letting $\|\cdot\|_2$ denote the norm on $L^2(\Gamma)$, the technique in [46] (or see [17]) is to show that $\mathcal{T}_0' = \mathcal{T}_1' + \mathcal{T}_2'$, where $\|\mathcal{T}_1'\|_2 < 1$. Since taking adjoints preserves norms and compactness, and since $\mathcal{S}$ and $\mathcal{T} - \mathcal{T}_0$ are compact operators on $L^2(\Gamma)$, it holds in the case of a 2D polygon that $\mathcal{A} = \mathcal{I} + \mathcal{K} = \mathcal{I} + \mathcal{K}_1 + \mathcal{K}_2$, where $\|\mathcal{K}_1\|_2 < 1$ and $\mathcal{K}_2$ is a compact operator on $L^2(\Gamma)$.

Through the remainder of the paper we suppose that $\eta \in \mathbb{R}$ with $\eta \neq 0$, so that $\mathcal{A}$ is invertible, and let

$$C_S := \|\mathcal{A}^{-1}\|_2 = \|(\mathcal{I} + \mathcal{K})^{-1}\|_2. \tag{2.11}$$

We note that C_S is a function of k, η, and the geometry of Γ; in particular it depends on k in an unspecified way. We remark that recently it has been shown, in the case when Γ is a circle and the choice $\eta = k$ is made, that $C_S \leq 2$ [28]. There exist no rigorous estimates of C_S for more general obstacles, except that, recently [19], the analogous operator to $\mathcal{A}$ has been studied when Γ is an unbounded scattering surface, the graph of a bounded, Lipschitz continuous function which has Hölder continuous gradient. In this case Rellich lemma-type arguments have been applied to establish that $C_S \leq 5(1 + L)^2$, where L is the Lipschitz constant, when $\eta = k/2$.

3. Regularity results. In this section we aim to understand the behaviour of $\partial u/\partial \mathbf{n}$, the normal derivative of the total field on Γ, which is the unknown function in the integral equation (2.9). Precisely, we will obtain bounds on the surface tangential derivatives of $\partial u/\partial \mathbf{n}$ in which the dependence on the wave number is completely explicit. This will enable us in §4 to design a family of approximation spaces well-adapted to approximating $\partial u/\partial \mathbf{n}$.

To understand the behaviour of $\partial u/\partial \mathbf{n}$ near the corners P_j our technique will be to use the explicit representation (2.4). To understand the behaviour away from the corners we will need another representation for $\partial u/\partial \mathbf{n}$ which we now derive.

Our starting point is the observation that, if $U = \{\mathbf{x} = (x_1, x_2), x_1 \in \mathbb{R}, x_2 > 0\}$ is the upper half-plane, and $v \in C^2(U) \cap C(\bar{U})$ satisfies the Helmholtz equation in U and the Sommerfeld radiation condition, then [18, theorem 3.1]

$$v(\mathbf{x}) = 2\int_{\partial U} \frac{\partial \Phi(\mathbf{x}, \mathbf{y})}{\partial y_2} v(\mathbf{y})\,\mathrm{d}s(\mathbf{y}), \quad \mathbf{x} \in U. \tag{3.1}$$

The same formula holds [18] if v is a horizontally or upwards propagating plane wave, i.e. if $v(\mathbf{x}) = \mathrm{e}^{\mathrm{i}k\mathbf{x}.\mathbf{d}}$ with $\mathbf{d} = (d_1, d_2)$, $|\mathbf{d}| = 1$, and $d_2 \geq 0$.

To make use of this observation, we make the following construction. Extend the line Γ_j to infinity in both directions; the resulting infinite line comprises Γ_j and

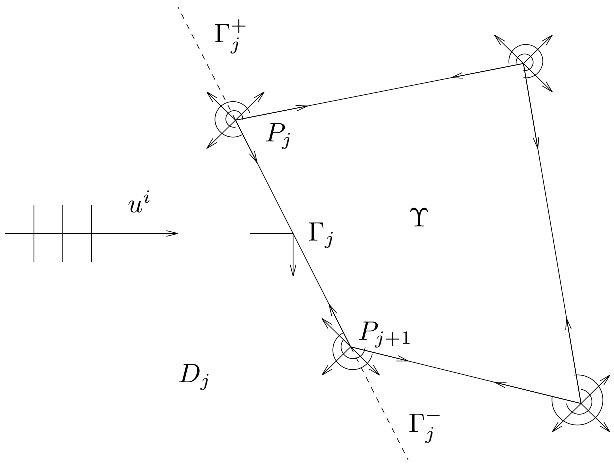

FIG. 3.1. *Extension of* Γ_j*, for derivation of regularity estimates.*

the half-lines Γ_j^+ and Γ_j^-, above P_j and below P_{j+1}, respectively, see figure 3.1. Let $D_j \subset D$ denote the half-plane on the opposite side of this line to Υ.

Now consider first the case when Γ_j is in shadow, by which we mean that $\mathbf{n}_j.\mathbf{d} \geq 0$. Then it follows from (3.1) that

$$u^s(\mathbf{x}) = 2\int_{\Gamma_j^+\cup\Gamma_j\cup\Gamma_j^-} \frac{\partial\Phi(\mathbf{x},\mathbf{y})}{\partial\mathbf{n}(\mathbf{y})} u^s(\mathbf{y})\,\mathrm{d}s(y), \quad \mathbf{x} \in D_j, \tag{3.2}$$

and also that

$$u^i(\mathbf{x}) = 2\int_{\Gamma_j^+\cup\Gamma_j\cup\Gamma_j^-} \frac{\partial\Phi(\mathbf{x},\mathbf{y})}{\partial\mathbf{n}(\mathbf{y})} u^i(\mathbf{y})\,\mathrm{d}s(y), \quad \mathbf{x} \in D_j. \tag{3.3}$$

Since $u = u^i + u^s$ and $u = 0$ on Γ, we deduce that

$$u(\mathbf{x}) = 2\int_{\Gamma_j^+\cup\Gamma_j^-} \frac{\partial\Phi(\mathbf{x},\mathbf{y})}{\partial\mathbf{n}(\mathbf{y})} u(\mathbf{y})\,\mathrm{d}s(y), \quad \mathbf{x} \in D_j.$$

In the case when Γ_j is illuminated ($\mathbf{n}_j.\mathbf{d} < 0$), (3.2) holds but (3.3) is replaced by

$$u^i(\mathbf{x}) = -2\int_{\Gamma_j^+\cup\Gamma_j\cup\Gamma_j^-} \frac{\partial\Phi(\mathbf{x},\mathbf{y})}{\partial\mathbf{n}(\mathbf{y})} u^i(\mathbf{y})\,\mathrm{d}s(y), \quad \mathbf{x} \in \mathbb{R}^2\backslash\bar{D}_j. \tag{3.4}$$

Now let $u^r(\mathbf{x}) := -u^i(\mathbf{x}')$, for $\mathbf{x} \in D_j$, where $\mathbf{x}'$ is the reflection of $\mathbf{x}$ in the line $\Gamma_j^+ \cup \Gamma_j \cup \Gamma_j^-$. (The physical interpretation of u^r is that it is the plane wave that would be reflected if Γ_j were infinitely long.) From (3.4), for $\mathbf{x} \in D_j$,

$$u^r(\mathbf{x}) = 2\int_{\Gamma_j^+\cup\Gamma_j\cup\Gamma_j^-} \frac{\partial\Phi(\mathbf{x}',\mathbf{y})}{\partial\mathbf{n}(\mathbf{y})} u^i(\mathbf{y})\,\mathrm{d}s(y) = -2\int_{\Gamma_j^+\cup\Gamma_j\cup\Gamma_j^-} \frac{\partial\Phi(\mathbf{x},\mathbf{y})}{\partial\mathbf{n}(\mathbf{y})} u^i(\mathbf{y})\,\mathrm{d}s(y),$$

and adding this to (3.2) we find that

$$u(\mathbf{x}) = u^i(\mathbf{x}) + u^r(\mathbf{x}) + 2\int_{\Gamma_j^+\cup\Gamma_j^-} \frac{\partial\Phi(\mathbf{x},\mathbf{y})}{\partial\mathbf{n}(\mathbf{y})} u(\mathbf{y})\,\mathrm{d}s(y), \quad \mathbf{x} \in D_j.$$

Thus on an illuminated side it holds that

$$\frac{\partial u}{\partial \mathbf{n}}(\mathbf{x}) = 2\frac{\partial u^i}{\partial \mathbf{n}}(\mathbf{x}) + 2\int_{\Gamma_j^+\cup\Gamma_j^-} \frac{\partial^2\Phi(\mathbf{x},\mathbf{y})}{\partial\mathbf{n}(\mathbf{x})\partial\mathbf{n}(\mathbf{y})} u(\mathbf{y})\,\mathrm{d}s(\mathbf{y}), \quad \mathbf{x}\in\Gamma_j. \tag{3.5}$$

The same expression, but without the term $2\frac{\partial u^i}{\partial \mathbf{n}}(\mathbf{x})$, holds when Γ_j is in shadow. The high frequency Kirchhoff or physical optics approximation to $\partial u/\partial\mathbf{n}$ is just $\partial u/\partial\mathbf{n} = 2\partial u^i/\partial\mathbf{n}$ on the illuminated sides and zero on the sides in shadow. Thus the integral in (3.5) is an explicit expression for the correction to the physical optics approximation.

The representation (3.5) is very useful in understanding the oscillatory nature of the solution on a typical side Γ_j. In particular we note that, in physical terms, the integral over Γ_j^+ can be interpreted as the normal derivative on Γ_j of the field due to dipoles distributed along Γ_j^+. The point is that the field due to each dipole has the same oscillatory behaviour $\mathrm{e}^{\mathrm{i}ks}$ on Γ_j. To exhibit this explicitly, we calculate, using standard properties of Bessel functions [2], that, for $\mathbf{x}\in\Gamma_j$, $\mathbf{y}\in\Gamma_j^\pm$, with $\mathbf{x}\neq\mathbf{y}$,

$$\frac{\partial^2\Phi(\mathbf{x},\mathbf{y})}{\partial\mathbf{n}(\mathbf{x})\partial\mathbf{n}(\mathbf{y})} = \frac{\mathrm{i}kH_1^{(1)}(k|\mathbf{x}-\mathbf{y}|)}{4|\mathbf{x}-\mathbf{y}|} = \frac{\mathrm{i}k^2}{4}\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{y}|}\mu(k|\mathbf{x}-\mathbf{y}|), \tag{3.6}$$

where $\mu(z) := \mathrm{e}^{-\mathrm{i}z}H_1^{(1)}(z)/z$, for $z>0$. The function $\mu(z)$ is singular at $z=0$ but increasingly smooth as $z\to\infty$, as quantified in the next theorem (cf. [20, lemma 2.5]).

THEOREM 3.1. *For every* $\epsilon>0$,

$$|\mu^{(m)}(z)| \le C_\epsilon (m+1)!\, z^{-3/2-m},$$

for $z\ge\epsilon$ *and* $m=0,1,\ldots$, *where*

$$C_\epsilon = \frac{2\sqrt[4]{5}(1+\epsilon^{-1/2})}{\pi}. \tag{3.7}$$

Proof. From [42, equation (12.31)], $\mu(z) = (-2\mathrm{i}/\pi)\int_0^\infty (t^2-2\mathrm{i}t)^{1/2}\mathrm{e}^{-zt}\,\mathrm{d}t$, for $\mathrm{Re}z>0$, where the branch of $(t^2-2\mathrm{i}t)^{1/2}$ is chosen so that $\mathrm{Re}(t^2-2\mathrm{i}t)^{1/2}\ge 0$. Thus

$$\mu^{(m)}(z) = (-1)^{m+1}\frac{2\mathrm{i}}{\pi}\int_0^\infty t^{m+1/2}(t-2\mathrm{i})^{1/2}\mathrm{e}^{-zt}\,\mathrm{d}t$$

and hence

$$|\mu^{(m)}(z)| \le \frac{2}{\pi}\int_0^\infty t^{m+1/2}(t^2+4)^{1/4}\mathrm{e}^{-zt}\,\mathrm{d}t.$$

Now, for $t\in[0,1]$, $(t^2+4)^{1/4}\le 5^{1/4}$ and, for $t\in[1,\infty)$, $(t^2+4)^{1/4}\le 5^{1/4}t^{1/2}$. So

$$\begin{aligned}\frac{\pi}{2\sqrt[4]{5}}|\mu^{(m)}(z)| &\le \int_0^\infty t^{m+1/2}\mathrm{e}^{-zt}\,\mathrm{d}t + \int_0^\infty t^{m+1}\mathrm{e}^{-zt}\,\mathrm{d}t \\ &= \Gamma(m+3/2)z^{-3/2-m} + \Gamma(m+2)z^{-2-m} \le (1+\epsilon^{-1/2})\Gamma(m+2)z^{-3/2-m},\end{aligned}$$

for $z\ge\epsilon$. ◻

To make use of the above result, let $\mathbf{x}(s)$ denote the point on Γ whose arc-length distance measured anticlockwise from P_1 is s. Explicitly,

$$\mathbf{x}(s) = P_j + \left(s-\tilde{L}_{j-1}\right)\left(\frac{P_{j+1}-P_j}{L_j}\right), \quad \text{for } s\in[\tilde{L}_{j-1},\tilde{L}_j], \quad j=1,\ldots,n,$$

where $\tilde{L}_0 := 0$ and, for $j = 1, \ldots, n$, $\tilde{L}_j := \sum_{m=1}^{j} L_m$ is the arc-length distance from P_1 to P_{j+1}. Define

$$\phi(s) := \frac{1}{k}\frac{\partial u}{\partial \mathbf{n}}(\mathbf{x}(s)), \quad \text{for } s \in [0, L], \tag{3.8}$$

where $L := \tilde{L}_n$, so that $\phi(s)$ is the unknown function of arc-length whose behaviour we seek to determine. Let

$$\Psi(s) := \begin{cases} \frac{2}{k}\frac{\partial u^i}{\partial \mathbf{n}}(\mathbf{x}(s)), & \text{if } s \in (\tilde{L}_{n_s}, L) \\ 0, & \text{if } s \in (0, \tilde{L}_{n_s}), \end{cases}$$

so that $\Psi(s)$ is the physical optics approximation to $\phi(s)$, and set $\psi_j(s) := u(\tilde{\mathbf{x}}_j(s))$, $s \in \mathbb{R}$, where $\tilde{\mathbf{x}}_j(s) \in \Gamma_j^+ \cup \Gamma_j \cup \Gamma_j^-$ is the point

$$\tilde{\mathbf{x}}_j(s) := P_j + \left(s - \tilde{L}_{j-1}\right)\left(\frac{P_{j+1} - P_j}{L_j}\right), \quad -\infty < s < \infty.$$

From (3.5) and (3.6) we have the explicit representation for ϕ on the side Γ_j, that

$$\phi(s) = \Psi(s) + \frac{\mathrm{i}}{2}[\mathrm{e}^{\mathrm{i}ks} v_j^+(s) + \mathrm{e}^{-\mathrm{i}ks} v_j^-(s)], \quad s \in [\tilde{L}_{j-1}, \tilde{L}_j], \quad j = 1, \ldots, n, \tag{3.9}$$

where

$$v_j^+(s) := k \int_{-\infty}^{\tilde{L}_{j-1}} \mu(k|s-t|)\mathrm{e}^{-\mathrm{i}kt}\psi_j(t)\,\mathrm{d}t, \quad s \in [\tilde{L}_{j-1}, \tilde{L}_j], \quad j = 1, \ldots, n,$$
$$v_j^-(s) := k \int_{\tilde{L}_j}^{\infty} \mu(k|s-t|)\mathrm{e}^{\mathrm{i}kt}\psi_j(t)\,\mathrm{d}t, \quad s \in [\tilde{L}_{j-1}, \tilde{L}_j], \quad j = 1, \ldots, n.$$

The terms $\mathrm{e}^{\mathrm{i}ks} v_j^+(s)$ and $\mathrm{e}^{-\mathrm{i}ks} v_j^-(s)$ in (3.9) are the integrals over Γ_j^+ and Γ_j^-, respectively, in equation (3.5), and can be thought of as the contributions to $\partial u/\partial \mathbf{n}$ on Γ_j due to the diffracted rays travelling from P_j to P_{j+1} and from P_{j+1} to P_j, respectively, including all multiply diffracted ray components.

So the equation we wish to solve is (2.9), and we have the explicit representation (3.9) for its solution. At first glance this may not appear to help us, since the unknown solution u appears (as ψ_j) on the right hand side of (3.9). However, (3.9) is extremely helpful in understanding how ϕ behaves since it explicitly separates out the oscillatory part of the solution. The functions $v_j^{\pm}$ are not oscillatory away from the corners, as the following theorem quantifies. In this theorem and hereafter we let

$$u_M := \sup_{\mathbf{x} \in D} |u(\mathbf{x})| < \infty \tag{3.10}$$

and note that $\|\psi_j\|_\infty \le u_M$, $j = 1, \ldots, n$.

Theorem 3.2 (solution behaviour away from corners). *For $\epsilon > 0$, $j = 1, \ldots, n$, and $m = 0, 1, \ldots$, it holds for $s \in [\tilde{L}_{j-1}, \tilde{L}_j]$ that*

$$|v_j^{+(m)}(s)| \le 2C_\epsilon m! u_M k^m (k(s - \tilde{L}_{j-1}))^{-1/2-m}, \quad k(s - \tilde{L}_{j-1}) \ge \epsilon,$$
$$|v_j^{-(m)}(s)| \le 2C_\epsilon m! u_M k^m (k(\tilde{L}_j - s))^{-1/2-m}, \quad k(\tilde{L}_j - s) \ge \epsilon,$$

where C_ϵ is given by (3.7).

Proof. From theorem 3.1, for $s \in [\tilde{L}_{j-1} + \epsilon/k, \tilde{L}_j]$,

$$\begin{aligned} |v_j^{+(m)}(s)| &= k^{m+1} \left| \int_{-\infty}^{\tilde{L}_{j-1}} \mu^{(m)}(k|s-t|) \mathrm{e}^{-\mathrm{i}kt} \psi_j(t) \, \mathrm{d}t \right| \\ &\le C_\epsilon (m+1)! k^{m+1} \|\psi_j\|_\infty \int_{-\infty}^{\tilde{L}_{j-1}} (k|s-t|)^{-3/2-m} \, \mathrm{d}t \\ &= C_\epsilon \frac{(m+1)!}{(m+1/2)} k^{-1/2} \|\psi_j\|_\infty (s - \tilde{L}_{j-1})^{-1/2-m} \\ &\le 2 C_\epsilon m! u_M k^m (k(s - \tilde{L}_{j-1}))^{-1/2-m}. \end{aligned}$$

The bound on $v_j^{-(m)}(s)$ is obtained similarly. □

The above theorem quantifies precisely the behaviour of $\partial u/\partial \mathbf{n}$ away from the corners. Complementing this bound, using theorem 2.2 we can study the behaviour of $\partial u/\partial \mathbf{n}$ near the corners. To state this result it is convenient to extend the definition of ϕ from $[0, L]$ to $\mathbb{R}$ by the periodicity condition $\phi(s + L) = \phi(s)$, $s \in \mathbb{R}$.

THEOREM 3.3 (solution behaviour near corners). *If $kR_j = \min(kL_{j-1}, kL_j) \ge \pi/4$, for $j = 1, \dots, n$, then, for $j = 1, \dots, n$ and $0 < k|s - \tilde{L}_{j-1}| \le \pi/12$, it holds that*

$$\left| \phi^{(m)}(s) \right| \le C u_M \sqrt{m + \frac{1}{2}} \, m! k^m (k|s - \tilde{L}_{j-1}|)^{-\alpha_j - m}, \quad m = 0, 1, \dots,$$

where

$$\alpha_j := 1 - \frac{\pi}{\Omega_j} \in (0, 1/2) \tag{3.11}$$

and $C = 72\sqrt{2}\, \pi^{-1} \, \mathrm{e}^{1/\mathrm{e} + \pi/6}$.

Proof. To analyse the behaviour of u using (2.4) we will use the representation for the Bessel function of order ν [2, (9.1.20)],

$$J_\nu(z) = \frac{2(z/2)^\nu}{\pi^{1/2} \Gamma(\nu + 1/2)} \int_0^1 (1 - t^2)^{\nu - 1/2} \cos(zt) \, \mathrm{d}t, \text{ for } \mathrm{Re} z > 0, \ \nu > -1/2,$$

where the branch of $(z/2)^\nu$ is chosen so that $(z/2)^\nu > 0$ for $z > 0$ and $(z/2)^\nu$ is analytic in $\mathrm{Re} z > 0$. This representation implies that

$$\cos z \le \frac{J_\nu(z) \pi^{1/2} \Gamma(\nu + 1/2)}{2(z/2)^\nu \int_0^1 (1 - t^2)^{\nu - 1/2} \, \mathrm{d}t} \le 1, \quad 0 \le z \le \pi/2. \tag{3.12}$$

Recalling the definitions of R and G before theorem 2.2 and the definition (2.5) of the coefficient a_n, we have that $\rho := kR < \pi/2$ and

$$|a_n| \le \frac{2u_M}{J_{n\pi/\Omega_j}(\rho)}. \tag{3.13}$$

Thus, for $0 < r < R$,

$$\left| a_n J_{n\pi/\Omega_j}(kr) \right| \le \frac{2u_M}{\cos \rho} \left(\frac{r}{R} \right)^{n\pi/\Omega_j}, \tag{3.14}$$

confirming that the series (2.4) converges for $0 \le r < R$. Further, the bound (3.14) justifies differentiating (2.4) term by term to get that, for $\mathbf{x} \in \Gamma_{j-1} \cap G$, $\frac{\partial u}{\partial \mathbf{n}}(\mathbf{x}) = kF(kr)$, where

$$F(z) := \frac{\pi}{\Omega_j z} \sum_{n=1}^{\infty} n a_n J_{n\pi/\Omega_j}(z), \quad \mathrm{Re} z > 0, \quad |z| < \rho. \tag{3.15}$$

Since $|\cos z| \le \mathrm{e}^{|\mathrm{Im} z|}$, $z \in \mathbb{C}$, so that $|\cos zt| \le \mathrm{e}^{|\mathrm{Im} z|}$ for $z \in \mathbb{C}$, $0 \le t \le 1$, we see from (3.13) that, for $\mathrm{Re} z > 0$,

$$\left| n a_n J_{n\pi/\Omega_j}(z) \right| \le \frac{2u_M n}{\cos \rho} \mathrm{e}^{|\mathrm{Im} z|} \left(\frac{|z|}{\rho} \right)^{n\pi/\Omega_j}. \tag{3.16}$$

So the series (3.15) is absolutely and uniformly convergent in $\mathrm{Re} z > 0$, $|z| < \rho_0$, for every $\rho_0 < \rho$, and F is analytic in $\mathrm{Re} z > 0$, $|z| < \rho$. Further, from (3.16), and since, for $0 \le \alpha < 1$, $\sum_{n=1}^{\infty} n\alpha^n = \alpha \frac{\mathrm{d}}{\mathrm{d}\alpha} \sum_{n=1}^{\infty} \alpha^n = \frac{\alpha}{(1-\alpha)^2}$, we see that, for $\mathrm{Re} z > 0$, $|z| < \rho$,

$$|F(z)| \le \frac{\pi}{\Omega_j |z|} \frac{2u_M}{\cos \rho} \frac{\mathrm{e}^{|\mathrm{Im} z|}}{(1 - |z/\rho|^{\pi/\Omega_j})^2} \left(\frac{|z|}{\rho} \right)^{\pi/\Omega_j}.$$

We can use this bound to obtain bounds on derivatives of F, and hence bounds on derivatives of $\partial u/\partial \mathbf{n}$. For $0 < t \le \rho/3$, $0 < \varepsilon < t$, from Cauchy's integral formula we have that

$$|F^{(m)}(t)| = \frac{m!}{2\pi} \left| \int_{\Gamma_\varepsilon} \frac{F(z)}{(z-t)^{m+1}} \, \mathrm{d}z \right|,$$

where Γ_ε is the circle of radius ε centred on t, which lies in $\mathrm{Re} z > 0$, $|z| < \rho$. Since

$$|F(z)| \le \frac{2\pi u_M \mathrm{e}^{|\mathrm{Im} z|} (t-\varepsilon)^{\pi/\Omega_j - 1}}{\Omega_j \rho^{\pi/\Omega_j} \cos \rho (1 - (2/3)^{\pi/\Omega_j})^2},$$

for $z \in \Gamma_\varepsilon$, we see that

$$|F^{(m)}(t)| \le \frac{2\pi u_M \mathrm{e}^t (t-\varepsilon)^{\pi/\Omega_j - 1} \varepsilon^{-m} m!}{\Omega_j \rho^{\pi/\Omega_j} \cos \rho (1 - (2/3)^{\pi/\Omega_j})^2}. \tag{3.17}$$

Now, for $\alpha > 0$, $\beta > 0$, $(t-\varepsilon)^{-\alpha} \varepsilon^{-\beta}$ is minimised on $(0, t)$ by the choice $\varepsilon = \beta t/(\alpha + \beta)$. Setting $\varepsilon = mt/(m + 1 - \pi/\Omega_j)$ in (3.17) we see that

$$|F^{(m)}(t)| \le \frac{2\pi u_M \mathrm{e}^t m! (m + 1 - \pi/\Omega_j)^{m+1-\pi/\Omega_j} t^{\pi/\Omega_j - 1 - m}}{\Omega_j \rho^{\pi/\Omega_j} \cos \rho (1 - (2/3)^{\pi/\Omega_j})^2 m^m (1 - \pi/\Omega_j)^{1 - \pi/\Omega_j}}.$$

Now

$$\frac{(m + 1 - \pi/\Omega_j)^{m+1-\pi/\Omega_j}}{m^m} \le \frac{(m + 1/2)^{m+1/2}}{m^m} = \left(1 + \frac{1}{2m}\right)^m \sqrt{m + \frac{1}{2}} \le \mathrm{e}^{1/2} \sqrt{m + \frac{1}{2}},$$

$$\frac{2\pi}{\Omega_j (1 - \pi/\Omega_j)^{1-\pi/\Omega_j} (1 - (2/3)^{\pi/\Omega_j})^2} \le \frac{18}{(1 - \pi/\Omega_j)^{1-\pi/\Omega_j}} \le 18 \mathrm{e}^{1/\mathrm{e}},$$

and hence

$$|F^{(m)}(t)| \le \frac{18 \mathrm{e}^{1/\mathrm{e} + 1/2 + t} \sqrt{m + 1/2}\, m! u_M}{\rho^{\pi/\Omega_j} \cos \rho} t^{\pi/\Omega_j - 1 - m}, \quad 0 < t \le \rho/3. \tag{3.18}$$

Since $\frac{\partial u}{\partial \mathbf{n}}(\mathbf{x}) = kF(kr)$, this implies that

$$\left|\frac{\partial^{(m)}}{\partial r^m}\left[\frac{\partial u}{\partial \mathbf{n}}(\mathbf{x})\right]\right| \leq \tilde{C} u_M\, k^{m+1}(kr)^{\pi/\Omega_j-1-m}, \quad 0 < r \leq R/3 < \frac{\pi}{6k},$$

where $\tilde{C} = (18\mathrm{e}^{1/\mathrm{e}+1/2+\pi/6}\sqrt{m+1/2}\,m!)/(\rho^{\pi/\Omega_j}\cos\rho)$. Choosing $\rho = \pi/4$ the result follows. ◻

From theorems 3.2 and 3.3, and equation (3.9), which gives that

$$v_j^{\pm}(s) = -2\mathrm{i}\mathrm{e}^{\mp \mathrm{i}ks}(\phi(s) - \Psi(s)) - \mathrm{e}^{\pm 2\mathrm{i}ks} v_j^{\mp}(s),$$

we deduce the following corollary, in which $\alpha_{n+1} := \alpha_1$.

COROLLARY 3.4. *Suppose that* $kR_j = \min(kL_{j-1}, kL_j) \geq \pi/4$, *for* $j = 1, \dots, n$. *Then, for* $m = 0, 1, \dots$, *there exists* $C_m > 0$, *dependent only on* m, *such that, if* $j \in \{1, \dots, n\}$, *then*

$$|v_j^{+(m)}(s)| \leq C_m u_M k^m (k(s - \tilde{L}_{j-1}))^{-\alpha_j - m}, \quad 0 < k(s - \tilde{L}_{j-1}) \leq \pi/12,$$
$$|v_j^{-(m)}(s)| \leq C_m u_M k^m (k(\tilde{L}_j - s))^{-\alpha_{j+1} - m}, \quad 0 < k(\tilde{L}_j - s) \leq \pi/12.$$

The following limiting case suggests that the bounds in theorem 3.2 and corollary 3.4 are optimal in their dependence on k, $s - \tilde{L}_{j-1}$, and $\tilde{L}_j - s$, in the sense that no sharper bound holds uniformly in the angle of incidence. Suppose that Υ lies in the right hand half-plane with P_1 located at the origin and $\mathbf{d}\cdot\mathbf{n}_1 = 0$, and consider the limit $\min(kL_0, kL_1) \to \infty$ and $\Omega_1 \to 2\pi$. In this limit $\alpha_1 \to 1/2$ and it is plausible that $u(\mathbf{x}) \to u_{\text{k.e.}}(\mathbf{x})$, where $u_{\text{k.e.}}$ is the solution to the following "knife edge" diffraction problem: where $\Gamma_{\text{k.e.}} := \{(x_1, 0) : x_1 \geq 0\}$, given the incident plane wave u^i, find the total field $u_{\text{k.e.}} \in C^2(\mathbb{R}^2 \setminus \Gamma_{\text{k.e.}}) \cap C(\mathbb{R}^2)$ such that $\Delta u_{\text{k.e.}} + k^2 u_{\text{k.e.}} = 0$ in $\mathbb{R}^2 \setminus \Gamma_{\text{k.e.}}$, $u_{\text{k.e.}} = 0$ on $\Gamma_{\text{k.e.}}$, and $u_{\text{k.e.}} - u^i$ has the correct radiating behaviour. The solution to this problem which satisfies the physically correct radiation condition is given by [10, equation (8.24)]. This solution implies that $\varphi(s) := \frac{1}{k}\frac{\partial u_{\text{k.e.}}}{\partial \mathbf{n}}((s,0)) = \pm \mathrm{e}^{\mathrm{i}ks} v(s)$, where the $+/-$ sign is taken on the upper/lower surface of the knife edge and $v(s) := \hat{c}(ks)^{-1/2}$, where $\hat{c} = \mathrm{e}^{-\mathrm{i}\pi/4}\sqrt{2/\pi}$. The function $v(s)$ and its derivatives satisfy the bounds on v_1^+ in theorem 3.2 and corollary 3.4 (with $\alpha_j = 1/2$), but do not satisfy any sharper bounds in terms of dependence on k or $s - \tilde{L}_{j-1}$.

4. The approximation space. Our aim now is to use the regularity results of §3 to design an optimal approximation space for the numerical solution of (2.9). We begin by rewriting (2.9) in parametric form. Defining, for $j = 1, \dots, n$,

$$a_j := \frac{p_{j+1} - p_j}{L_j}, \quad b_j := \frac{q_{j+1} - q_j}{L_j}, \quad c_j := p_j - a_j \tilde{L}_{j-1}, \quad d_j := q_j - b_j \tilde{L}_{j-1},$$

and noting that $n_{j1} = b_j$, $n_{j2} = -a_j$, we can rewrite (2.9) as

$$\phi(s) + \int_0^L \kappa(s,t)\phi(t)\,\mathrm{d}t = f(s), \quad s \in [0, L], \tag{4.1}$$

where, for $\mathbf{x}(s) \in \Gamma_l$, $\mathbf{y}(t) \in \Gamma_j$, i.e. for $s \in (\tilde{L}_{l-1}, \tilde{L}_l)$, $t \in (\tilde{L}_{j-1}, \tilde{L}_j)$,

$$\kappa(s,t) := -\frac{1}{2}\left[\eta H_0^{(1)}(kR) + \mathrm{i}k\left[(a_l b_j - b_l a_j)t + b_l(c_l - c_j) - a_l(d_l - d_j)\right]\frac{H_1^{(1)}(kR)}{R}\right],$$

with $R = R(s,t) := \sqrt{(a_l s - a_j t + c_l - c_j)^2 + (b_l s - b_j t + d_l - d_j)^2}$ and $f \in L^2(0,L)$ defined by

$$f(s) := 2\mathrm{i}[b_l \sin\theta + a_l \cos\theta + (\eta/k)]\mathrm{e}^{\mathrm{i}k((a_l s + c_l)\sin\theta - (b_l s + d_l)\cos\theta)}.$$

The first step in our numerical method is to separate off the explicitly known leading order behaviour, the physical optics approximation $\Psi(s)$. Thus we introduce a new unknown,

$$\varphi := \phi - \Psi \in L^2(0,L). \tag{4.2}$$

Substituting into (4.1) we have

$$\varphi + K\varphi = F, \tag{4.3}$$

where the integral operator $K : L^2(0,L) \to L^2(0,L)$ and $F \in L^2(0,L)$ are defined by

$$K\psi(s) := \int_0^L \kappa(s,t)\psi(t)\,\mathrm{d}t,\ 0 \le s \le L, \quad F := f - \Psi - K\Psi.$$

Equation (4.3) is the integral equation we will solve numerically. By theorem 2.6, (4.3) has a unique solution in $L^2(0,L)$ and $\|(I+K)^{-1}\|_2 = C_S$, C_S defined in (2.11).

We will design an approximation space to represent φ based on (3.9). The novelty of the scheme we propose is that on each side Γ_j, $j = 1,\ldots,n$, of the polygon, we approximate $v_j^{\pm}$ by conventional piecewise polynomials, rather than approximating φ itself. This makes sense since, as quantified by theorem 3.2, the functions $v_j^{\pm}$ are smooth (their higher-order derivatives are small) away from the corners P_j and P_{j+1}. To approximate $v_j^{\pm}$ we use piecewise polynomials of a fixed degree $\nu \ge 0$ on a graded mesh, the mesh grading adapted in an optimal way to the bounds of theorems 3.2 and 3.3. In [20] the 2D problem of scattering of a plane wave by a straight boundary of piecewise constant surface impedance was considered. We will construct a similar mesh on each side of the polygon as was used on each interval of constant impedance in [20], except that we use a different grading near the corners, with the grading near each corner dependent on the angle at that corner.

To construct this mesh we choose a constant $c^* > 0$ (we take $c^* = 1$ in the numerical examples in §6) and set $\lambda^* := c^*/k$. Next, for every $A > \lambda^*$, we define a composite graded mesh on $[0,A]$, with a polynomial grading on $[0,\lambda^*]$ and a geometric grading on $[\lambda^*, A]$ (note that the mesh on $[0,\lambda^*]$ is similar to that classically used near corners (e.g. [17], [7]) for solving Laplace's equation on polygonal domains).

Definition 4.1. *For $A > \lambda^*$, $N = 2,3,\ldots$, $\Lambda_{N,A,q} := \{y_0,\ldots,y_{N+N_{A,q}}\}$ is the mesh consisting of the points*

$$y_i = \lambda^*\left(\frac{i}{N}\right)^q,\ i = 0,\ldots,N,\ \text{and}\ y_{N+j} := \lambda^*\left(\frac{A}{\lambda^*}\right)^{j/N_{A,q}},\ j = 1,\ldots,N_{A,q}, \tag{4.4}$$

where $N_{A,q} := \lceil N^* \rceil$, i.e. $N_{A,q}$ is the smallest integer greater than or equal to N^*, and

$$N^* := \frac{-\log(A/\lambda^*)}{q\log(1-1/N)}.$$

This choice of N^* ensures a smooth transition between the two parts of the mesh. In particular, in the case that $N_{A,q} = N^*$, it holds that $y_{N+1}/y_N = y_N/y_{N-1}$, so that y_{N-1} and y_N are points in both the polynomial and the geometric parts of the mesh. By the mean value theorem, $-\log(1-1/N) = 1/(\xi N)$ for some $\xi \in (1-1/N, 1)$, and hence

$$N_{A,q} < \frac{N\log(kA/c^*)}{q} + 1. \tag{4.5}$$

For $a < b$ let $\|\cdot\|_{2,(a,b)}$ denote the norm on $L^2(a,b)$, $\|f\|_{2,(a,b)} := \{\int_a^b |f(s)|^2 \mathrm{d}s\}^{1/2}$. Similarly, for $f \in C[a,b]$, let $\|f\|_{\infty,(a,b)} := \sup_{a<s<b} |f(s)|$. For $A > \lambda^*$, $\nu \in \mathbb{N} \cup \{0\}$, $q \geq 1$, let $\Pi_{N,\nu} \subset L^2(0,A)$ denote the set of piecewise polynomials

$$\Pi_{N,\nu} := \{\sigma : \sigma|_{(y_{j-1},y_j)} \text{ is a polynomial of degree} \leq \nu, \text{ for } j = 1,\ldots,N+N_{A,q}\},$$

and let P_N^* be the orthogonal projection operator from $L^2(0,A)$ to $\Pi_{N,\nu}$, so that setting $p = P_N^* f$ minimises $\|f-p\|_{2,(0,A)}$ over all $p \in \Pi_{N,\nu}$.

THEOREM 4.2. *Suppose that $f \in C^\infty(0,\infty)$, $kA > c^*$, and $\alpha \in (0,1/2)$, and that for $m = 0,1,2,\ldots$ there exist constants $c_m > 0$ such that*

$$|f^{(m)}(s)| \leq \begin{cases} c_m k^m (ks)^{-\alpha-m}, & ks \leq 1, \\ c_m k^m (ks)^{-1/2-m}, & ks \geq 1. \end{cases} \tag{4.6}$$

Then, with the choice $q := (2\nu+3)/(1-2\alpha)$, there exists a constant C_ν, dependent only on c^*, ν, and α, such that, for $N = 2,3,\ldots$,

$$\|f - P_N^* f\|_{2,(0,A)} \leq \frac{C_\nu \tilde{c}_\nu (1+\log(kA/c^*))^{1/2}}{k^{1/2} N^{\nu+1}},$$

where $\tilde{c}_\nu := \max(c_0, c_{\nu+1})$.

Proof. Throughout the proof let C_ν denote a positive constant whose value depends on ν, c^*, and α, not necessarily the same at each occurrence. For $0 \leq a < b \leq A$, let $p_{a,b,\nu}$ denote the polynomial of degree $\leq \nu$ which is the best approximation to f in the L^2 norm on (a,b). Then it follows from Taylor's theorem that

$$\|f - p_{a,b,\nu}\|_{2,(a,b)} \leq C_\nu (b-a)^{\nu+3/2} \|f^{(\nu+1)}\|_{\infty,(a,b)}. \tag{4.7}$$

Now

$$\|f - P_N^* f\|_{2,(0,A)}^2 = \sum_{j=1}^{N+N_{A,q}} \int_{y_{j-1}}^{y_j} |f - P_N^* f|^2 \, \mathrm{d}s = \sum_{j=1}^{N+N_{A,q}} e_j, \tag{4.8}$$

where $e_j := \|f - p_{y_{j-1},y_j,\nu}\|_{2,(y_{j-1},y_j)}^2$. From the definition (4.4) we see that

$$e_1 \leq \int_0^{y_1} |f(s)|^2 \, \mathrm{d}s \leq c_0^2 k^{-2\alpha} \int_0^{\lambda^*/N^q} s^{-2\alpha} \, \mathrm{d}s \leq \frac{C_\nu c_0^2}{kN^{2\nu+3}}. \tag{4.9}$$

Using (4.7) we have, for $j = 2,3,\ldots,N+N_{A,q}$,

$$e_j \leq C_\nu (y_j - y_{j-1})^{2\nu+3} \|f^{(\nu+1)}\|_{\infty,(y_{j-1},y_j)}^2. \tag{4.10}$$

Further, for $j = 2,\ldots,N$,

$$y_j - y_{j-1} = \frac{c^*}{kN^q}\left[j^q - (j-1)^q\right] \leq \frac{c^* q j^{q-1}}{kN^q}, \tag{4.11}$$

and, using (4.6) and since $N/(j-1) \leq 2N/j$,

$$\|f^{(\nu+1)}\|_{\infty,(y_{j-1},y_j)} \leq c_{\nu+1}k^{-\alpha}y_{j-1}^{-\alpha-\nu-1} \leq c_{\nu+1}k^{\nu+1}\left(\frac{2N}{j}\right)^{q(\alpha+\nu+1)}. \tag{4.12}$$

Combining (4.10)-(4.12) we see that, for $j = 2,\ldots,N$,

$$e_j \leq \frac{C_\nu c_{\nu+1}^2}{kN^{2\nu+3}}. \tag{4.13}$$

For $j = N+1,\ldots,N_{A,q}$, recalling (4.4) and the choice of N^*, and then using (4.11),

$$y_j - y_{j-1} = y_{j-1}\left(\frac{y_j - y_{j-1}}{y_{j-1}}\right) \leq y_{j-1}\left(\frac{y_N - y_{N-1}}{y_{N-1}}\right) \leq y_{j-1}\frac{q}{N-1} \leq 2y_{j-1}\frac{q}{N}.$$

Also, from (4.6),

$$\|f^{(\nu+1)}\|_{\infty,(y_{j-1},y_j)} \leq c_{\nu+1}k^{-1/2}y_{j-1}^{-\nu-3/2}.$$

Using these bounds in (4.10), we see that the bound (4.13) holds also for $j = N+1,\ldots,N+N_{A,q}$. Combining (4.8), (4.9), and (4.13),

$$\|f - P_N^* f\|_{2,(0,A)}^2 \leq \frac{C_\nu \tilde{c}_\nu^2(N+N_{A,q})}{kN^{2\nu+3}} \leq \frac{C_\nu \tilde{c}_\nu^2(1+\log(kA/c^*))}{kN^{2\nu+3}},$$

using (4.5). Hence the result follows. □

We assume through the remainder of the paper that $c^* > 0$ is chosen so that

$$kL_j \geq c^*, \quad j = 1,\ldots,n. \tag{4.14}$$

For $j = 1,\ldots,n$, recalling (3.11), we define $q_j := (2\nu+3)/(1-2\alpha_j)$, and the two meshes

$$\Gamma_j^+ := \tilde{L}_{j-1} + \Lambda_{N,L_j,q_j}, \quad \Gamma_j^- := \tilde{L}_j - \Lambda_{N,L_j,q_{j+1}}.$$

Letting $\mathrm{e}_\pm(s) := \mathrm{e}^{\pm iks}$, $s \in [0,L]$, we then define

$$V_{\Gamma_j^+,\nu} := \{\sigma \mathrm{e}_+ : \sigma \in \Pi_{\Gamma_j^+,\nu}\}, \quad V_{\Gamma_j^-,\nu} := \{\sigma \mathrm{e}_- : \sigma \in \Pi_{\Gamma_j^-,\nu}\},$$

for $j = 1,\ldots,n$, where

$$\begin{aligned}\Pi_{\Gamma_j^+,\nu} := \{\sigma \in L^2(0,L) : \sigma|_{(\tilde{L}_{j-1}+y_{m-1},\tilde{L}_{j-1}+y_m)} &\text{ is a polynomial of degree } \leq \nu,\\ \text{for } m = 1,\ldots,N+N_{L_j,q_j}, &\text{ and } \sigma|_{(0,\tilde{L}_{j-1})\cup(\tilde{L}_j,L)} = 0\},\\ \Pi_{\Gamma_j^-,\nu} := \{\sigma \in L^2(0,L) : \sigma|_{(\tilde{L}_j-\tilde{y}_m,\tilde{L}_j+\tilde{y}_{m-1})} &\text{ is a polynomial of degree } \leq \nu,\\ \text{for } m = 1,\ldots,N+N_{L_j,q_{j+1}}, &\text{ and } \sigma|_{(0,\tilde{L}_{j-1})\cup(\tilde{L}_j,L)} = 0\},\end{aligned}$$

with $0 = y_0 < y_1 < \ldots < y_{N+N_{L_j,q_j}} = L_j$ the points of the mesh Λ_{N,L_j,q_j}, and $0 = \tilde{y}_0 < \tilde{y}_1 < \ldots < \tilde{y}_{N+N_{L_j,q_{j+1}}} = L_j$ the points of the mesh $\Lambda_{N,L_j,q_{j+1}}$. We define P_N^+ and P_N^- to be the orthogonal projection operators from $L^2(0,L)$ onto $\Pi_{\Gamma_+,\nu}$ and $\Pi_{\Gamma_-,\nu}$, respectively, where $\Pi_{\Gamma_\pm,\nu}$ denotes the linear span of $\bigcup_{j=1,\ldots,n}\Pi_{\Gamma_j^\pm,\nu}$. We also define the functions $v_\pm \in L^2(0,L)$ by

$$v_+(s) := v_j^+(s), \;\; v_-(s) := v_j^-(s), \quad \tilde{L}_{j-1} < s < \tilde{L}_j, \; j = 1,\ldots,n.$$

We then have the following error estimate, in which u_M is as defined in (3.10) and we abbreviate $\| \cdot \|_{2,(0,L)}$ by $\| \cdot \|_2$.

THEOREM 4.3. *There exists a constant $C_\nu > 0$, dependent only on c^*, ν, and Ω_1, Ω_2, ..., Ω_n, such that*

$$\|v_+ - P_N^+ v_+\|_2 \le C_\nu u_M \frac{n^{1/2}(1+\log(k\bar{L}/c^*))^{1/2}}{k^{1/2} N^{\nu+1}},$$

where $\bar{L} := (L_1 \dots L_n)^{1/n}$, with an identical bound holding on $\|v_- - P_N^- v_-\|_2$.

Proof. From theorem 3.2, corollary 3.4, and theorem 4.2,

$$\|v_+ - P_N^+ v_+\|_2^2 = \sum_{j=1}^n \|v_j^+ - P_N^+ v_j^+\|_{2,(\tilde{L}_{j-1},\tilde{L}_j)}^2 \le n \frac{C_\nu^2 u_M^2 (1+\log(k\bar{L}))}{kN^{2\nu+2}},$$

and the result follows. ∎

Our approximation space $V_{\Gamma,\nu}$ is the linear span of

$$\bigcup_{j=1,\dots,n} \{V_{\Gamma_j^+,\nu} \cup V_{\Gamma_j^-,\nu}\}.$$

The dimension of this approximation space, i.e. the number of degrees of freedom, is

$$M_N = 2(\nu+1) \sum_{j=1}^n (N + N_{L_j,q_j}) < 2(\nu+1) n N (1 + N^{-1} + \log(k\bar{L}/c^*)) \tag{4.15}$$

by (4.5). We define P_N to be the operator of orthogonal projection from $L^2(0,L)$ onto $V_{\Gamma,\nu}$. It remains to prove a bound on $\|\varphi - P_N\varphi\|_2$, showing that our mesh and approximation space are well adapted to approximating φ.

To use theorem 4.3 we note from (3.9) and (4.2) that $\varphi = \frac{\mathrm{i}}{2}(\mathrm{e}_+ v_+ + \mathrm{e}_- v_-)$. But $\mathrm{e}_+ P_N^+ v_+ + \mathrm{e}_- P_N^- v_- \in V_{\Gamma,\nu}$ and $P_N\varphi$ is the best approximation to φ in $V_{\Gamma,\nu}$. Applying theorem 4.3 we thus have that

$$\begin{aligned} \|\varphi - P_N\varphi\|_2 &\le \|\varphi - \frac{\mathrm{i}}{2}(e_+ P_N^+ v_+ + e_- P_N^- v_-)\|_2 \\ &= \frac{1}{2}\|e_+(v_+ - P_N^+ v_+) + e_-(v_- - P_N^-)\|_2 \\ &\le \|e_+\|_\infty \|v_+ - P_N^+ v_+\|_2 + \|e_-\|_\infty \|v_- - P_N^- v_-\|_2 \\ &\le C_\nu u_M \frac{n^{1/2}(1+\log^{1/2}(k\bar{L}))}{k^{1/2}N^{\nu+1}}. \end{aligned}$$

Combining this bound with (4.15) we obtain the following main result of the paper. We remind the reader that we are assuming throughout that (4.14) holds.

THEOREM 4.4. *There exist positive constants C_ν and C'_ν, depending only on c^*, ν, and Ω_1, Ω_2, ..., Ω_n, such that*

$$k^{1/2}\|\varphi - P_N\varphi\|_2 \le C_\nu u_M \frac{n^{1/2}(1+\log(k\bar{L}/c^*))^{1/2}}{N^{\nu+1}} \le C'_\nu u_M \frac{(1+\log(k\bar{L}/c^*))^{\nu+3/2}}{M_N^{\nu+1}}.$$

A comment on the factor $k^{1/2}$ on the left hand side is probably helpful. Reflecting that the solution of the physical problem must be independent of the unit of length measurement and that we are designing our numerical scheme to preserve this property, it is easy to see that the values of both $k^{1/2}\|\varphi\|_2$ and $k^{1/2}\|\varphi - P_N\varphi\|_2$ remain fixed as k changes, if we keep kL_j fixed for $j = 1, \dots, n$ (and also, of course, keep Ω_j, $j = 1, \dots, n$, c^*, and ν fixed). Thus inclusion of the factor $k^{1/2}$ ensures that the value of $k^{1/2}\|\varphi - P_N\varphi\|_2$ is independent of the unit of length measurement as are the bounds on the right hand side.

5. Galerkin method. Theorem 4.4 has shown that it is possible to approximate accurately the solution of the integral equation (4.3) with a number of degrees of freedom that grows only very modestly as the wave number increases. To select an approximation, φ_N, from the approximation space $V_{\Gamma,\nu}$ we use the Galerkin method. Let $(\cdot,\cdot)$ denote the usual inner product on $L^2(0,L)$, defined by $(\chi_1, \chi_2) := \int_0^L \chi_1(s)\bar{\chi}_2(s)\,\mathrm{d}s$, so that $\|\chi\|_2 = (\chi,\chi)^{1/2}$. Then our Galerkin method approximation $\varphi_N \in V_{\Gamma,\nu}$ is defined by

$$(\varphi_N, \rho) + (K\varphi_N, \rho) = (F, \rho), \quad \text{for all } \rho \in V_{\Gamma,\nu}; \tag{5.1}$$

equivalently

$$\varphi_N + P_N K \varphi_N = P_N F. \tag{5.2}$$

Our goal now is to show that (5.2) has a unique solution φ_N, to establish a bound on the error $\|\varphi - \varphi_N\|_2$ in this numerical method, and to relate this error to the best approximation error $\|\varphi - P_N\varphi\|_2$. We begin by establishing that $I + P_N K$ is invertible if N is large enough. We remind the reader (see the end of §2) that we are assuming that $\eta \in \mathbb{R}$, the coupling parameter in the integral equation, is chosen with $\eta \neq 0$ which ensures that $I + K$ is invertible.

THEOREM 5.1. *For all $v \in L^2(0,L)$, $\|P_N v - v\|_2 \to 0$ as $N \to \infty$.*

Proof. Since $\|P_N\|_2 = 1$ it is enough to show that $P_N v \to v$ in $L^2(0,L)$ for all $v \in C^\infty[0,L]$, a dense subset of $L^2(0,L)$. But this follows from theorem 4.2 and the definition of P_N. ∎

THEOREM 5.2. *There exists a constant $N^* \geq 2$, dependent only on Γ, k, and η, such that, for $N \geq N^*$, the operator $I + P_N K : L^2(0,L) \to L^2(0,L)$ is bijective with*

$$C_s := \sup_{N \geq N^*} \|(I + P_N K)^{-1}\|_2 < \infty, \tag{5.3}$$

so that (5.2) has exactly one solution for $N \geq N^*$.

Proof. Recalling the discussion at the end of §2, we note that it holds that $K = K_1 + K_2$, where $\|K_1\|_2 < 1$ and K_2 is a compact operator on $L^2(0,L)$. Since $\|P_N K_1\|_2 \leq \|K_1\|_2 < 1$, $I + P_N K_1$ is invertible and $\|(I + P_N K_1)^{-1}\|_2 \leq (1 - \|K_1\|_2)^{-1}$. Since K_2 is compact and $I + K$ is injective, it follows from theorem 5.1 and standard perturbation arguments for projection methods (e.g. [7, theorem 8.2.1], [17]) that $(I + P_N K)^{-1}$ exists and is uniformly bounded for all N sufficiently large. ∎

From (4.3) and (5.2) it follows that $\varphi - \varphi_N = (I + P_N K)^{-1}(\varphi - P_N\varphi)$, and hence

$$\|\varphi - \varphi_N\|_2 \leq \|(I + P_N K)^{-1}\|_2 \|\varphi - P_N\varphi\|_2. \tag{5.4}$$

Combining (5.3) and (5.4) with theorem 4.4 we obtain our final error estimate.

THEOREM 5.3. *There exist positive constants C_ν and C'_ν, depending only on c^*, ν, and $\Omega_1, \Omega_2, \dots, \Omega_n$, such that*

$$k^{1/2}\|\varphi - \varphi_N\|_2 \leq C_s C_\nu u_M \frac{n^{1/2}(1 + \log(k\bar{L}/c^*))^{1/2}}{N^{\nu+1}} \leq C_s C'_\nu u_M \frac{(1 + \log(k\bar{L}/c^*))^{\nu+3/2}}{M_N^{\nu+1}},$$

for $N \geq N^*$*, where* N^* *and* C_s *are as defined in theorem 5.2.*

Note that we will take $c^* = 1$ and $\eta = k$ in all our numerical calculations in the next section. Note also that, while the constants C_ν and C'_ν, from the best approximation theorem 4.4, depend only on c^*, ν and the corner angles of Γ, the numbers N^* and C_s depend additionally on k, L_1, L_2, $\dots$, L_n and η. We do not attempt the difficult task of elucidating this dependence in this paper. We note only that, very recently, for the boundary integral equation formulation (2.9) applied to scattering by a circle, Dominguez et al. [28] have shown that $\mathcal{I}+\mathcal{K}$ is elliptic if $\eta = k$, so that every Galerkin method is automatically stable; specifically, (5.3) holds for every N^* if P_N is the orthogonal projection from $L^2(0, L)$ onto the Galerkin approximation space. Further it is shown in [28] that $C_s = O(k^{1/3})$ as $k \to \infty$. Our numerical results in §6 will suggest the stronger result that, for our particular scheme and geometry, the bound of theorem 5.3 holds with a constant C_s independent of k.

Of course our aim in approximating φ by φ_N is to approximate $\partial^+_{\mathbf{n}} u$ and hence, via (2.7), the solution u of the scattering problem. Clearly, from (3.8) and (4.2), an approximation to $\partial u/\partial \mathbf{n}$ is

$$\frac{\partial u}{\partial \mathbf{n}}(\mathbf{x}(s)) \approx k(\Psi(s) + \varphi_N(s)), \quad 0 \leq s \leq L.$$

Using this approximation in (2.7), we conclude that

$$u(\mathbf{x}) \approx u_N(\mathbf{x}) := u^i(\mathbf{x}) - k \int_0^L \Phi(\mathbf{x}, \mathbf{x}(s))[\Psi(s) + \varphi_N(s)]\, \mathrm{d}s, \quad \mathbf{x} \in D. \tag{5.5}$$

Theorem 5.3 implies the following error estimate.

THEOREM 5.4. *There exist positive constants* C_ν *and* C'_ν*, depending only on* c^**,* ν*, and* Ω_1*,* Ω_2*,* $\dots$*,* Ω_n*, such that*

$$\frac{\sup_{\mathbf{x}\in D} |u(\mathbf{x}) - u_N(\mathbf{x})|}{\sup_{\mathbf{x}\in D} |u(\mathbf{x})|} \leq C_s C_\nu \frac{n(1 + \log(k\bar{L}/c^*))}{N^{\nu+1}} \leq C_s C'_\nu \frac{n^{1/2}(1 + \log(k\bar{L}/c^*))^{\nu+2}}{M_N^{\nu+1}},$$

for $N \geq N^*$*, where* N^* *and* C_s *are as defined in theorem 5.2.*

Proof. From (2.7) and (5.5),

$$\begin{aligned} |u(\mathbf{x}) - u_N(\mathbf{x})| &= k \left| \int_0^L \Phi(\mathbf{x}, \mathbf{x}(s))\, [\varphi(s) - \varphi_N(s)]\, \mathrm{d}s \right| \\ &\leq \frac{k}{4} \left\{ \int_0^L |H_0^{(1)}(k|\mathbf{x} - \mathbf{x}(s)|)|^2\, \mathrm{d}s \right\}^{1/2} \|\varphi - \varphi_N\|_2 \\ &\leq \frac{k}{4} \left\{ 2 \sum_{j=1}^n \int_0^{L_j/2} |H_0^{(1)}(kt)|^2\, \mathrm{d}t \right\}^{1/2} \|\varphi - \varphi_N\|_2 \\ &\leq C_\nu k^{1/2} n^{1/2} (1 + \log(k\bar{L}/c^*))^{1/2} \|\varphi - \varphi_N\|_2, \end{aligned}$$

where we have used that $|H_0^{(1)}(t)|$ is a monotonic decreasing function of t on $(0, \infty)$ and that $|H_0^{(1)}(t)| = O(t^{-1/2})$ as $t \to \infty$ (see e.g. [2]). The result now follows from theorem 5.3. ∎

6. Numerical results. There has been much work on the optimal choice of the parameter η in (2.9) (see e.g. [3, 37]). Here we choose $\eta = k$ as in [28]. We also set $c^* = 1$ and restrict attention to the case $\nu = 0$. For higher values of ν the implementation of the scheme is similar. Note that, with $c^* = 1$ and $\nu = 0$, there are approximately N degrees of freedom used to represent the solution on the intervals of length k^{-1} on each side adjacent to a corner.

The equation we wish to solve is (5.1) with $\nu = 0$. Writing φ_N as a linear combination of the basis functions of $V_{\Gamma,0}$, we have

$$\varphi_N(s) = \sum_{j=1}^{M_N} v_j \rho_j(s), \quad 0 \le s \le L,$$

where ρ_j is the jth basis function and M_N is the dimension of $V_{\Gamma,0}$. For $p = 1, \dots, n$, where n is the number of sides of the polygon, we define $n_p^\pm$ to be the number of points in the mesh $\Gamma_p^\pm$, so $n_p^+ = N + N_{L_p,q_p}$, $n_p^- = N + N_{L_p,q_{p+1}}$, and we denote the points of the mesh $\Gamma_p^\pm$ by $s_{p,l}^\pm$, for $l = 1, \dots, n_p^\pm$, with $s_{p,1}^\pm < \dots < s_{p,n_p^\pm}^\pm$. Setting $n_1 := 0$, $n_p := \sum_{j=1}^{p-1}(n_j^+ + n_j^-)$, for $p = 2, \dots, n-1$, we define, for $p = 1, \dots, n$,

$$\rho_{n_p+j}(s) := \begin{cases} \mathrm{e}^{\mathrm{i}ks}\chi_{(s_{p,j-1}^+, s_{p,j}^+)}(s) / \sqrt{s_{p,j}^+ - s_{p,j-1}^+}, & j = 1, \dots, n_p^+, \\ \mathrm{e}^{-\mathrm{i}ks}\chi_{(s_{p,j-1}^-, s_{p,j}^-)}(s) / \sqrt{s_{p,j}^- - s_{p,j-1}^-}, & j = n_p^+ + 1, \dots, n_p^+ + n_p^-, \end{cases}$$

where $\chi_{(y_1,y_2)}$ denotes the characteristic function of the interval (y_1, y_2). From (4.15), $M_N = \sum_{j=1}^n (n_j^+ + n_j^-) < 2nN(3/2 + \log(k\bar{L}/c^*))$.

Equation (5.1) with $\nu = 0$ is equivalent to the linear system

$$\sum_{j=1}^{M_N} v_j((\rho_j, \rho_m) + (K\rho_j, \rho_m)) = (F, \rho_m), \quad m = 1, \dots, M_N. \tag{6.1}$$

In order to set up this linear system we need to determine the integrals (ρ_j, ρ_m), $(K\rho_j, \rho_m)$ and (F, ρ_m). We note that many of the integrals $(K\rho_j, \rho_m)$ and (F, ρ_m) are highly oscillatory, in particular all these integrals become highly oscillatory in the limit as $k \to \infty$ with N fixed. The efficient calculation of these integrals is an aspect of the numerical scheme which requires further research. But note that explicit formulae for the analytic evaluation of some of these integrals, and a consideration of the quadrature techniques required to evaluate the rest of them numerically, are presented in [38].

Another important issue is the conditioning of the linear system. Standard analysis of the Galerkin method for second kind equations [7] implies that, where $M := [(\rho_j, \rho_m)]$ is the mass matrix (which is necessarily Hermitian and positive definite) and $A = [(\rho_j, \rho_m) + (K\rho_j, \rho_m)]$ is the whole matrix, it holds that $\mathrm{cond}_2 A \le C_s \mathrm{cond}_2 M$, where C_s is defined by (5.3). Thus theorem 5.2 implies that $\mathrm{cond}_2 A$ is bounded as $N \to \infty$ if the mass matrix is well-conditioned. Unfortunately, it appears that, as $N \to \infty$ with k fixed, M must ultimately become badly conditioned. However, the results below will show only moderate condition numbers of A even for large values of N (see table 6.1). More positively, in the limit as $k \to \infty$ with N fixed, $\mathrm{cond}_2 M \to 1$. To see this we observe that, if (ρ_j, ρ_m) is a non-zero off-diagonal element of the mass matrix (in which case the supports of ρ_j and ρ_m are overlapping subintervals of the meshes Γ_p^+ and Γ_p^-, for some side p), it holds that $|(\rho_j, \rho_m)| = \sin(ko)\sqrt{o/(kS_1S_2)}$, where S_1 and S_2 are the lengths of the two-subintervals, o the length of the overlap.

As a numerical example, we consider the problem of scattering by a square with sides of length 2π. In this case $n = 4$ and $\Omega_j = 3\pi/2$, $j = 1, 2, 3, 4$. The corners of the square are $P_1 := (0, 0)$, $P_2 := (2\pi, 0)$, $P_3 := (2\pi, 2\pi)$, $P_4 := (0, 2\pi)$, and the incident angle is $\theta = \pi/4$, so the incident field is directed towards P_4, with P_2 in shadow (as shown in figure 6.1, where the total acoustic field is plotted for $k = 10$).

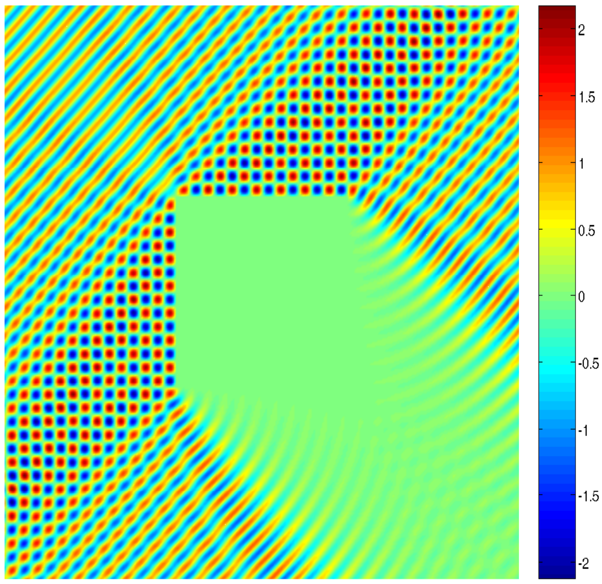

FIG. 6.1. *Total acoustic field, scattering by a square, $k = 10$. Incident field is directed from the top left corner towards the bottom right corner.*

In figure 6.2 we plot $|\varphi_N(s)|$ against s for $k = 10$ and $N = 4, 16, 64, 256$. As we expect, $|\varphi_N(s)|$ is highly peaked at the corners of the polygon, $s = 0$, 2π, 4π, 6π and 8π (which is the same corner as $s = 0$), where $\varphi(s)$ is infinite. Except at these corners, $|\varphi_N(s)|$ appears to be converging pointwise as N increases. (We do not plot $\varphi_N(s)$ itself which is highly oscillatory.)

In order to test the convergence of our scheme, we take the "exact" solution to be that computed with a large number of degrees of freedom, namely with $N = 256$. For $k = 5$ and $k = 320$ the relative L^2 errors $\|\varphi_N - \varphi_{256}\|_2/\|\varphi_{256}\|_2$ are shown in table 6.1 (all L^2 norms are computed by approximating by discrete L^2 norms, sampling at 100000 evenly spaced points around the boundary of the square). For this example, theorem 5.3 predicts that, for $N \geq N^*$, $\|\varphi - \varphi_N\|_2 \leq CN^{-1}$, where C is a constant. Thus theorem 5.3 predicts that, for $N > N^*$, the average rate of convergence,

$$EOC := \frac{\log(\|\varphi - \varphi_N\|_2/\|\varphi - \varphi_{N^*}\|_2)}{\log(N/N^*)} \geq 1 - \frac{\hat{C}}{\log(N/N^*)} \sim 1$$

as $N \to \infty$, where $\hat{C} := \log(\|\varphi - \varphi_N\|_2/C)$. This behaviour is clearly seen in the EOC values (defined with $N^* = 8$) in table 6.1, for both values of k. We also show in table 6.1 the 2-norm condition number, $\mathrm{cond}_2 A$, of the matrix $A = [(\rho_j, \rho_m) + (K\rho_j, \rho_m)]$

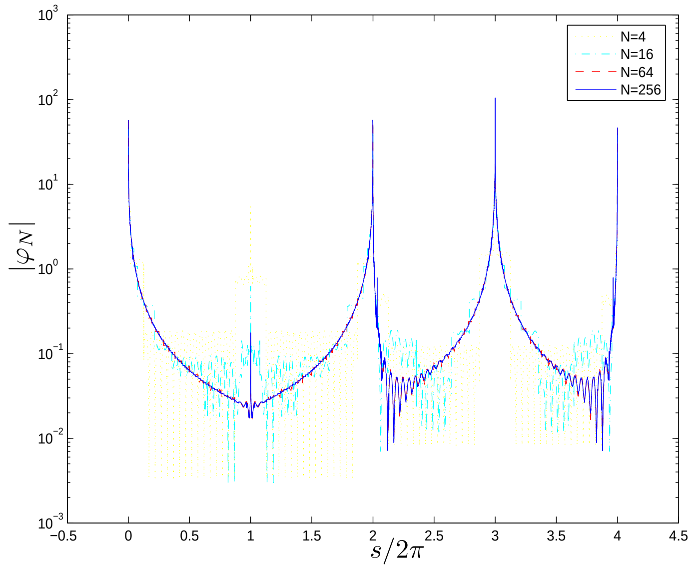

FIG. 6.2. $|\varphi_N(s)|$ *plotted against* s*, various* N*, for scattering by a square of side length ten wavelengths.*

k	N	M_N	$k^{1/2}\|\varphi_N - \varphi_{256}\|_2$	$\|\varphi_N - \varphi_{256}\|_2/\|\varphi_{256}\|_2$	*EOC*	$\text{cond}_2 A$
5	8	88	5.7339×10^{-1}	2.4426×10^{-1}		9.5×10^{0}
	16	176	3.7454×10^{-1}	1.5955×10^{-1}	0.6	4.6×10^{1}
	32	360	1.6176×10^{-1}	6.8909×10^{-2}	0.9	2.6×10^{1}
	64	712	7.7267×10^{-2}	3.2916×10^{-2}	1.0	2.4×10^{2}
	128	1416	3.3541×10^{-2}	1.4289×10^{-2}	1.0	1.5×10^{3}
320	8	120	7.0765×10^{-1}	3.6736×10^{-1}		2.4×10^{2}
	16	240	5.9792×10^{-1}	3.1040×10^{-1}	0.2	6.9×10^{2}
	32	472	1.9668×10^{-1}	1.0211×10^{-1}	0.9	8.1×10^{2}
	64	944	7.5808×10^{-2}	3.9354×10^{-2}	1.1	1.1×10^{3}
	128	1888	4.8814×10^{-2}	2.5341×10^{-2}	1.0	3.8×10^{3}

TABLE 6.1

Errors and relative L^2 *errors, various* N*,* $k = 5$ *and* $k = 320$*.*

for each example. Unlike methods where the approximation space is formed by multiplying standard finite element basis functions by many plane waves, travelling in a large number of directions [27, 44, 35], the condition number does not grow significantly as the number of degrees of freedom increases.

In table 6.2 we fix $N = 64$ and show $\|\varphi_{64} - \varphi_{256}\|_2/\|\varphi_{256}\|_2$ and $k^{1/2}\|\varphi_{64} - \varphi_{256}\|_2$ for increasing values of k. Both measures of errors remain approximately constant in magnitude as k increases. Recall that, keeping N fixed as k increases corresponds to keeping the number of degrees of freedom per wavelength fixed near each corner and increasing the total number of degrees of freedom, M_N, approximately in proportion to $\log(k\bar{L})$. Thus these results are supportive of the approximation error estimate of theorem 4.2 and the Galerkin error estimate, theorem 5.3, which suggest that increasing M_N proportional to $\log^{3/2}(k\bar{L})$ is enough to keep the error bounded. Note that the condition number of the coefficient matrix A only increases modestly as k

increases, and is approximately constant for $k \geq 40$.

k	M_N	$k^{1/2}\|\varphi_{64} - \varphi_{256}\|_2$	$\|\varphi_{64} - \varphi_{256}\|_2/\|\varphi_{256}\|_2$	$\mathrm{cond}_2 A$
5	712	7.7267×10^{-2}	3.2916×10^{-2}	2.4×10^{2}
10	752	6.6373×10^{-2}	2.8702×10^{-2}	8.4×10^{1}
20	792	3.8309×10^{-1}	1.6914×10^{-1}	5.1×10^{3}
40	824	1.3162×10^{-1}	5.9856×10^{-2}	1.2×10^{3}
80	864	7.4315×10^{-2}	3.4801×10^{-2}	2.7×10^{3}
160	904	7.0884×10^{-2}	3.4570×10^{-2}	1.4×10^{3}
320	944	7.5808×10^{-2}	3.9354×10^{-2}	1.1×10^{3}
640	984	6.4280×10^{-2}	3.5693×10^{-2}	1.5×10^{3}

TABLE 6.2
Errors and relative L^2 errors, various k, $N = 64$.

In table 6.3 we show numerical convergence of the total field $u_N(\mathbf{x})$ at the three points $\mathbf{x} = (-\pi, 3\pi)$ (illuminated), $\mathbf{x} = (3\pi, 3\pi)$, and $\mathbf{x} = (3\pi, -\pi)$ (shadow), for $k = 5$ and $k = 320$. The errors are consistent with the estimate of theorem 5.4. As might be expected for the computation of linear functionals of φ_N, the relative errors in table 6.3 are a lot smaller and converge to zero more rapidly than the relative errors in the computation of the boundary data in tables 6.1 and 6.2.

k	N	$\mathbf{x} = (-\pi, 3\pi)$	$\mathbf{x} = (3\pi, 3\pi)$	$\mathbf{x} = (3\pi, -\pi)$
5	4	1.9587×10^{-2}	1.0071×10^{-3}	1.5885×10^{-2}
	8	4.2629×10^{-3}	2.8031×10^{-3}	2.3215×10^{-3}
	16	3.6284×10^{-4}	3.1410×10^{-4}	1.3513×10^{-3}
	32	6.7523×10^{-5}	2.9803×10^{-5}	1.7939×10^{-5}
	64	1.2675×10^{-5}	5.9626×10^{-6}	4.6158×10^{-6}
320	4	2.2938×10^{-3}	2.9350×10^{-3}	2.0897×10^{-2}
	8	4.3176×10^{-3}	1.5157×10^{-3}	1.1652×10^{-2}
	16	3.3908×10^{-3}	9.6409×10^{-4}	9.3922×10^{-3}
	32	3.3898×10^{-4}	1.6984×10^{-4}	9.0526×10^{-4}
	64	1.0022×10^{-4}	9.6493×10^{-5}	2.6204×10^{-4}

TABLE 6.3
Relative errors, $|u_N(\mathbf{x}) - u_{256}(\mathbf{x})|/|u_{256}(\mathbf{x})|$, as a function of N, at three points $\mathbf{x}$.

7. Conclusions. In this paper we have described a novel Galerkin boundary integral equation method for solving problems of high frequency scattering by convex polygons. In §2 we have shown that the standard second kind boundary integral equations for the exterior Dirichlet problem for the Helmholtz equation are well-posed for general Lipschitz domains. We have understood very completely in §3 the oscillatory behaviour of the normal derivative of the field on the boundary of the polygon. We have then used this understanding to design an optimal graded mesh for approximation of the diffracted field by products of piecewise polynomials and plane waves. Our error analysis and supporting numerical results demonstrate that the number of degrees of freedom required to achieve a prescribed level of accuracy grows only logarithmically with respect to the wave number k as $k \to \infty$.

There are very many open problems in extending the results of this paper to more general scatterers. In this extension we expect that our mesh design and parts of our analysis will have relevance for representing certain components of the total field. For example, in the case of 2D convex curvilinear polygons, something close to the mesh grading we use may be appropriate on each side of the polygon, especially at higher frequencies when the waves diffracted by the corners become more localised near the corners. In the case of three-dimensional scattering by convex polyhedra it seems to us likely that the mesh we propose will be useful in representing the variation of edge scattered waves in the direction perpendicular to the edge.

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REFERENCES

[1] T. Abboud, J. C. Nédélec, and B. Zhou, *Méthodes des équations intégrales pour les hautes fréquences*, C.R. Acad. Sci. I-Math, 318 (1994), pp. 165–170.

[2] M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions*, Dover, 1972.

[3] S. Amini, *On the choice of the coupling parameter in boundary integral formulations of the exterior acoustic problem*, Applicable Analysis, 35 (1990), pp. 75–92.

[4] S. Amini and N. D. Maines, *Preconditioned Krylov subspace methods for boundary element solution of the Helmholtz equation*, Internat. J. Numer. Methods Engrg., 41 (1998), pp. 875–898.

[5] S. Amini and A. T. J. Profit, *Multi-level fast multipole Galerkin method for the boundary integral solution of the exterior Helmholtz equation*, in Current trends in scientific computing (Xi'an, 2002), vol. 329 of Contemp. Math., Amer. Math. Soc., Providence, RI, 2003, pp. 13–19.

[6] S. Arden, S. N. Chandler-Wilde and S. Langdon, *A collocation method for high frequency scattering by convex polygons*, to appear in J. Comp Appl. Math.

[7] K. E. Atkinson, *The numerical solution of integral equations of the second kind*, Cambridge University Press, 1997.

[8] I. Babuška and J. M. Melenk, *The partition of unity method*, Int. J. Numer. Meth. Eng., 40 (1997), pp. 727–758.

[9] I. Babuška and S. Sauter, *Is the pollution effect of the FEM avoidable for the Helmholtz equation considering high wave numbers?*, SIAM J. Numer. Anal., 34 (1997), pp. 2392–2423. Reprinted in *SIAM Rev.* **42** (3), 451-484, 2000.

[10] J. J. Bowman, T. B. A. Senior and P. L. E. Uslenghi, *Electromagnetic and Acoustic Scattering by Simple Shapes*, Noth-Holland, Amsterdam, 1969.

[11] H. Brakhage and P. Werner, *Uber das Dirichletsche Aussenraumproblem für die Helmholtzsche Schwingungsgleichung*, Arch. Math., 16 (1965), pp. 325–329.

[12] O. P. Bruno, C. A. Geuzaine, J. A. Monro Jr, and F. Reitich, *Prescribed error tolerances within fixed computational times for scattering problems of arbitrarily high frequency: the convex case*, Phil. Trans. R. Soc. Lond A, 362 (2004), pp. 629–645.

[13] O. P. Bruno and L. A. Kunyansky, *A fast, high-order algorithm for the solution of surface scattering problems: Basic implementation, tests and applications*, J. Comput. Phys., 169 (2001), pp. 80–110.

[14] A. Buffa and S. Sauter, *Stabilization of the acoustic single layer potential on non-smooth domains*, to appear in SIAM J. Sci. Comput.

[15] A. J. Burton and G. F. Miller, *The application of integral equation methods to the numerical solution of some exterior boundary-value problems*, Proc. R. Soc. Lond. A, 323 (1971), pp. 201–210.

[16] O. Cessenat and B. Després, *Using plane waves as base functions for solving time harmonic equations with the ultra weak variational formulation*, J. Comp. Acoust., 11 (2003), pp. 227–238.

[17] G. Chandler, *Galerkin's method for boundary integral equations on polygonal domains*, J. Austral. Math. Soc., Series B, 26 (1984), pp. 1–13.

[18] S. N. Chandler-Wilde, *Boundary value problems for the Helmholtz equation in a half-plane*, in Proc. 3rd Int. Conf. on Mathematical and Numerical Aspects of Wave Propagation, G. Cohen, ed., SIAM, Philadelphia, 1995.

[19] S. N. Chandler-Wilde, E. Heinemeyer, and R. Potthast, *A well-posed integral equation formulation for 3D rough surface scattering*, submitted for publication.

[20] S. N. Chandler-Wilde, S. Langdon, and L. Ritter, *A high-wave number boundary-element method for an acoustic scattering problem*, Phil. Trans. R. Soc. Lond. A, 362 (2004), pp. 647–671.

[21] W. C. Chew, J. M Song, T. J. Cui, S. Velarnparambil, M. L. Hastriter, B. Hu, *Review of large scale computing in electromagnetics with fast integral equation solvers*, CMES-Computer Modeling Engineering Sciences 5 (2004), pp. 361-372.

[22] S. H. Christiansen and J. C. Nédélec, *Preconditioners for the numerical solution of boundary integral equations from electromagnetism*, C.R. Acad. Sci. I-Math, 331 (2000), pp. 733–738.

[23] D. Colton and R. Kress, *Integral Equation Methods in Scattering Theory*, Wiley, 1983.

[24] ———, *Inverse Acoustic and Electromagnetic Scattering Theory*, Springer Verlag, 1992.

[25] E. Darrigrand, *Coupling of fast multipole method and microlocal discretization for the 3-D Helmholtz equation*, J. Comput. Phys., 181 (2002), pp. 126–154.

[26] E. Darve and P. Havé, *A fast multipole method for Maxwell equations stable at all frequencies*, Phil. Trans. R. Soc. Lond A, 362 (2004), pp. 603–628.

[27] A. de La Bourdonnaye and M. Tolentino, *Reducing the condition number for microlocal discretization problems*, Phil. Trans. R. Soc. Lond. A, 362 (2004), pp. 541–559.

[28] V. Domínguez, I. G. Graham, and V. P. Smyshlyaev, *A hybrid numerical-asymptotic boundary integral method for high-frequency acoustic scattering*, Preprint 1/2006, Bath Institute for Complex Systems, 2006.

[29] F. Ecevit, *Integral Equation Formulations of Electromagnetic and Acoustic Scattering Problems: High-frequency Asymptotic Expansions and Convergence of Multiple Scattering Iterations* , PhD Thesis, University of Minnesota, 2005.

[30] J. Elschner, *The double-layer potential operator over polyhedral domains I: solvability in weighted Sobolev spaces*, Appl. Anal., 45 (1992), pp. 117–134.

[31] E. Giladi and J. B. Keller, *A hybrid numerical asymptotic method for scattering problems*, J. Comput. Phys., 174 (2001), pp. 226–247.

[32] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, Berlin, 1983.

[33] C. Geuzaine C, O. Bruno and F. Reitich, *On the $O(1)$ solution of multiple-scattering problems*, IEEE Trans. Magnetics 41 (2005), pp. 1488-1491.

[34] P. J. Harris and K. Chen, *On efficient preconditioners for iterative solution of a Galerkin boundary element equation for the three-dimensional exterior Helmholtz problem*, J. Comput. Appl. M., 156 (2003), pp. 303–318.

[35] T. Huttunen, P. Monk, F. Collino, and J. P. Kaipio, *The ultra-weak variational formulation for elastic wave problems*, SIAM J. Sci. Comput., 25 (2004), pp. 1717–1742.

[36] F. Ihlenburg, *Finite Element Analysis of Acoustic Scattering*, Springer-Verlag, New York, 1998.

[37] R. Kress, *Minimizing the condition number of boundary integral operators in acoustic and electromagnetic scattering*, Q. Jl. Mech. Appl. Math., 38 (1985), pp. 323–341.

[38] S. Langdon and S. N. Chandler-Wilde, *Implementation of a boundary element method for high frequency scattering by convex polygons*, in Advances in Boundary Integral Methods (Proceedings of the 5th UK Conference on Boundary Integral Methods), K. Chen, ed., Liverpool University Press, 2005, pp. 2–11.

[39] ———, *A wavenumber independent boundary element method for an acoustic scattering problem*, SIAM J. Numer. Anal., 43 (2006), pp. 2450–2477.

[40] R. Leis, *Zur Dirichletschen Randwertaufgabe des Außenraumes der Schwingungsgleichung*, Math. Zeitschr., 90 (1965), pp. 205–211.

[41] W. McLean, *Strongly Elliptic Systems and Boundary Integral Equations*, Cambridge University Press, 2000.

[42] F. Oberhettinger and L. Badii, *Tables of Laplace Transforms*, Springer-Verlag, 1973.

[43] O. J. Panič, *On the solubility of exterior boundary value problems for the wave equation and for a system of Maxwell's equations [Russian]*, Uspeshi Mat. Nauk, 20 (1965), pp. 221–226.

[44] E. Perrey-Debain, O. Lagrouche, P. Bettess, and J. Trevelyan, *Plane-wave basis finite elements and boundary elements for three-dimensional wave scattering*, Phil. Trans. R. Soc. Lond. A, 362 (2004), pp. 561–577.

[45] S. Prössdorf and B. Silbermann, *Numerical Analysis for Integral and Related Operator Equations*, Birkhäuser, 1991.

[46] V. J. Šelepov, *The index of an integral operator of potential type in the space L_p*, Soviet Math. Dokl., 10 (1969), pp. 754–757.

[47] G. C. Verchota, *Layer potentials and regularity for the Dirichlet problem for Laplace's equation in Lipschitz domains*, J. Funct. Anal., 59 (1984), pp. 572-611.