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Extreme Functionals and Stone-Weierstrass Theory of Inner Ideals in JB*-Triples

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DECLARATION

I confirm that this is my own work and the use of all material from other sources has been properly and fully acknowledged.

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Abstract

Let I and J be norm closed inner ideals of a JB*-triple. The main theorem of the thesis states that I and J are equal precisely when $\partial_e(I_1^*) = \partial_e(J_1^*)$. Moreover, we prove that $I \subset J$ exactly when $\partial_e(I_1^*) \subset \partial_e(J_1^*)$. Thus, JB*-triple inner ideals are determined by extreme dual ball points.

The tool used to reach this conclusion is what we term the Inner Stone-Weierstrass Theorem for JB*-triples; we show that for norm closed inner ideals I and J of a JB*-triple, where $I \subset J$, we may conclude that $I = J$ if $\partial_e(I_1^*) = \partial_e(J_1^*)$. Our excuse for this terminology is that the equality of the extreme dual ball points implies a Stone-Weierstrass separation condition, that is, that I separates $\partial_e(J_1^*) \cup \{0\}$.

To create this tool, we first exploit structure space techniques to make a reduction to JC*-triples. Progressing further, we employ the Zelmanovian techniques of [MoRod1][MoRod2] to manufacture a composition series that allows further reduction to universally reversible JC*-algebras. We prove the Inner Stone-Weierstrass Theorem for C*-algebras and, by developing enveloping C*-algebra techniques, we proceed to obtain the required extension to all universally reversible JC*-algebras. Subsequently, by stages, we prove the final version of the Inner Stone-Weierstrass Theorem for all JB*-triples.

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Contents

Introduction

Historically, JB*-triples originate in the study of an algebraic characterisation of bounded symmetric domains in complex Banach spaces [Kaup]. Examples of JB*-triples include C*-algebras and JB*-algebras.

Given a norm closed inner ideal I of a JB*-triple A, it is well-known that each functional in $\partial_e(I_1^*)$ has a unique extension to a functional in the dual ball of A, and that in fact this extension is an extreme point. In addition, there is a bijective correspondence between $\partial_e(A_1^*)$ and the minimal tripotents of A^{**} [FrRu4]. Therefore, by identifying each functional $\rho \in \partial_e(I_1^*)$ with its extension, we may write

$$\partial_e(A_1^*) = \bigcup \partial_e(I_1^*),$$

as the union ranges over all norm closed inner ideals I of A.

The purpose of this thesis is to investigate to what extent the inner ideal structure of a JB*-triple is determined by these extreme functionals. We show that two norm closed inner ideals I and J of a JB*-triple A are equal precisely when $\partial_e(I_1^*) = \partial_e(J_1^*)$, and furthermore, that $I \subset J$ if and only if $\partial_e(I_1^*) \subset \partial_e(J_1^*)$.

Predominantly, however, the thesis is concerned with deriving the tool used to reach these determinations, that is, with establishing the following.

Let A be a JB-triple. Let I and J be norm closed inner ideals of A with $I \subset J$. Suppose that $\partial_e(I_1^*) = \partial_e(J_1^*)$. Then I and J are equal.*

We have chosen to use the terminology *Inner Stone-Weierstrass Theorem* to refer to this result; this will be explained later. It is immediately clear that as every norm closed inner ideal is itself a JB*-triple, it is sufficient to consider only a single inner ideal, in other words, to take $A = J$ in the above.

Essentially, the route we take to prove the Inner Stone-Weierstrass Theorem is to progressively establish the analogous result for various triple structures; for C*-algebras and universally reversible JC*-algebras in Chapter Four, and then for JC*-triples and JB*-triples in Chapter Five, with each stage intrinsically reliant upon the preceding one. A pivotal step, and one of possible independent interest, is the formation of a particular composition series that allows the extension of the universally reversible JC*-algebra version to its JC*-triple counterpart. Thus, we prove:

If A be a JC-triple such that all Cartan factor representations of A have rank greater than two, then A has a composition series of norm closed ideals $(J_\lambda)_{0\leq\lambda\leq\alpha}$ such that $J_{\lambda+1}/J_\lambda$, is isomorphic to an inner ideal in a universally reversible JC*-algebra.*

It should be understood that the opening two chapters comprise of well-known material, presented to support subsequent chapters. The emphasis here is on clarity and brevity. In Chapter One we supply the principal background to Jordan algebras, whereas Chapter Two details JB*-triples. We include known results relating to the focus of the thesis, that is JB*-triple inner ideals, alongside some original, wholly technical lemmas regarding these structures. Given our choice of terminology, we state brief details of the Stone-Weierstrass Theorem and its many generalisations.

The third chapter marks the beginning of the main body of original work. In essence, our aim is to undertake the groundwork required for what follows, focussing upon certain aspects of inner ideals in JW*-triples. In particular, by making extensive use of Horn's decomposition theory, [Ho2], we demonstrate a strong correspondence between the underlying generic type of homogenous JW*-triples and that of their weak* closed inner ideals. In this endeavour we also exploit the powerful notion of the centroid of a JB*-triple [DinTi][EdRü9].

Moving on, we investigate the inner ideal structure of universally reversible JW*-algebras. Our starting point is the work of [EdRüVa2]. For continuous JW*-algebras, or those isomorphic to von Neumann algebras, M say, the authors proved that each inner ideal is of the form $eM\phi(e)$, where ϕ is the canonical involution of the enveloping von Neumann algebra. It was our intention to form a definitive resolution using this template, however our extension is valid only for those algebras without symplectic part. Although this exception to some extent impedes our work, the restricted resolution is sufficient for the needs of the thesis.

In Chapter Four we prove the Inner Stone-Weierstrass Theorem for universally reversible JC*-algebras:

If A is a universally reversible JC-algebra with norm closed inner ideals I and J, with I contained in J and such that $\partial_e(I_1^*) = \partial_e(J_1^*)$, then $I = J$.*

We begin with a series of technicalities regarding the atomic part of a JB*-triple, that are used extensively throughout. We show that if A is a JB*-triple with a norm closed inner ideal I, then if $\partial_e(I_1^*) = \partial_e(A_1^*)$ we can conclude that I^{**} and A^{**} have equal atomic part, denoted by A^{**}_{at}. Furthermore, we prove that $I^{**}_{at} = A^{**}_{at}$ precisely when I separates $\partial_e(A_1^*) \cup \{0\}$. The latter is a Stone-Weierstrass separation condition. Thus our choice of nomenclature.

Throughout we make essential use of structure space techniques and of the universal enveloping C*-algebra $C^*(A)$ of a universally reversible JC*-algebra A. Especially, we investigate norm closed inner ideals of A via their extensions in $C^*(A)$, through methods inspired by results in [EdRüVa2]. At an intermediate stage we prove the Inner Stone-Weierstrass Theorem for C*-algebras.

Chapter Five culminates with our proof that norm closed inner ideals in a JB*-triple are completely determined by their extreme functionals, thereby achieving our main aim. Most of the work here is targeted at the fashioning of the key tool. Namely, the Inner Stone-Weierstrass Theorem for JB*-triples. A reduction to JC*-triples is readily made via structure space techniques. Indispensible use is made of the Zelmanovian techniques of [MoRod1][MoRod2] to manufacture a certain composition series, see page ii, that allows further crucial reduction to universally reversible JC*-algebras. We are then able to employ the work of Chapter Four to attain our goal.

References are given in terms of an alphanumeric code. Each author's name is represented by a series of two or three letters. A single digit is used to differentiate between different references with the same author(s).

Chapter 1

Preliminaries of Jordan Algebras

1.1 Introduction

In this chapter we lay out the principal preliminaries on Jordan algebras which we will use. Our aim is to aid the reader by providing a simple, unified presentation of definitions and results that are readily available in the main literature.

In the main, results detailed in this chapter are those which will be referred to more than once. Those pertaining to a particular technicality will be presented when applied.

For the fundamentals of functional analysis the reader is referred to [Con] and [Rud2]. For C*-algebras we have used [Dix] and [Ped1]. Further background to operator algebras can be found in [KaRi1] and [KaRi2]. Most of the particulars relating to Jordan algebras that appear uncited are taken from [HaSt].

We begin with a basic description of Jordan algebras, including their ideal structures. From here we define JB-algebras, which underpin much of our work on Jordan triple systems. Then, to mimic the relationship between C*-algebras and W*-algebras, we next introduce JBW-algebras.

Particular attention is paid to the concepts of type I decomposition and factor representations. These are recurrent themes within the thesis and therefore merit a clear treatment. In a similar vein we highlight the notions of universal reversibility and the universal enveloping C*-algebra.

We conclude by introducing JB*-algebras, a key part of what follows. Here a brief definition will suffice since, for the most part, necessary remarks can be derived from our exposition of JB-algebras.

We use standard notation, thus given a Banach space X, X^* denotes the dual space. We will habitually regard X as being contained in the second dual via the canonical embedding. In the same way, X^* is contained in X^{***}. The transpose of the embedding $X \mapsto X^{**}$ is the weak* continuous map $P : X^{***} \to X^*$, $(\rho \to \rho_{|X^*})$. If X is the dual of some Banach space Y then this map is $P : Y^{**} \to Y$, the weak* continuous projection.

We shall make frequent and tacit use of the following well-known result. Let X and E be Banach spaces, where E is the dual of some Banach space Y. Let $\pi : X \to E$ be a bounded linear map such that $\pi(X)$ is weak* dense in E. Then there is a unique weak* continuous extension, $\hat{\pi} : X^{**} \to E$, and, moreover, $\hat{\pi}(X^{**}) = E$.

Finally, as is usual, we will use $\mathbf{N}, \mathbf{R}$ and $\mathbf{C}$ to denote the natural, real and complex numbers, and $\mathbf{H}$ and $\mathbf{O}$ to denote the quaternions and the octonions respectively.

1.2 Jordan Algebras

We now describe a specific class of algebras, namely Jordan algebras.

1.2.1 Let A be an algebra over the field $\mathbf{F} = \mathbf{R}$ or $\mathbf{C}$, with product

$$\circ : A \times A \to A$$

denoted by $a \circ b$, for $a, b \in A$. A is said to be a *Jordan algebra* if it satisfies the following two properties:

(i) $a \circ b = b \circ a$ for all $a, b \in A$.

(ii) $(a \circ b) \circ a^2 = a \circ (b \circ a^2)$ for all $a, b \in A$.

1.2.2 Let A be an associative algebra. A new product, $\circ$, called the *special Jordan product*, on A is defined by

$$a \circ b = \frac{1}{2}(ab + ba),$$

where ab denotes the usual product. The product $\circ$ is bilinear and commutative. Let A^J denote the algebra A endowed with this product $\circ$. Under the special Jordan product the conditions (1.2.1(i)) and (1.2.1(ii)) given earlier hold, so that A^J is a Jordan algebra. A *special Jordan algebra* is a Jordan algebra that is isomorphic to a subalgebra of A^J, for some associative algebra A.

1.2.3 There exist Jordan algebras which are not special, i.e. that are not Jordan subalgebras of associative algebras with the special Jordan product. These are called *exceptional Jordan algebras.* A classical example is the 27 dimensional octonionic real Jordan algebra $M_3(\mathbf{O})_{sa}$, which we will denote by N_3^8.

1.2.4 We now define two fundamental operators on Jordan algebras. Let A be a Jordan algebra. For all elements a of A define the *multiplication operator* $T_a : A \longrightarrow A$ by $T_a(b) = a \circ b$. The condition (1.2.1(ii)) is equivalent to the condition that T_a and T_{a^2} commute for all a in A. Given $a \in A$, we will write $U_a = 2(T_a)^2 - T_{a^2}$.

1.2.5 Two elements a and b in a Jordan algebra A are said to *operator commute* if $T_a T_b = T_b T_a$. By the *centre* of a Jordan algebra A is meant the set

$$Z(A) = \{a \in A : \forall b \in A \quad T_a T_b = T_b T_a\},$$

that is, the set of elements of A which operator commute with every other element of A. This is an associative subalgebra of A.

1.2.6 A linear subspace I of a Jordan algebra A is said to be a *Jordan ideal*, or simply an *ideal*, of A, if $T_a(I) \subset I$ for all $a \in A$; and to be a *quadratic ideal* of A if $U_a(A) \subset I$ for all $a \in I$.

Every ideal in a Jordan algebra A is a quadratic ideal, and the quotient of A by an ideal is again a Jordan algebra.

1.3 JC-Algebras and JB-Algebras

1.3.1 We now define the class of algebras known as JB-algebras and their concrete version, JC-algebras. These are examples of real Jordan algebras with an additional norm structure. It will later be seen that JB-algebras have strong links to C*-algebras, (1.4), sharing many ideas and techniques. JC-algebras were first studied by Topping [Top1] and Størmer [Stø1], whilst the study of JB-algebras was begun later by, amongst others, Alfsen, Shultz and Størmer [AlShSt].

1.3.2 A *JB-algebra* is a real Jordan algebra A with a norm $\|.\|$ such that

(J1) $\|a \circ b\| \leq \|a\|\|b\|$ for all $a, b \in A$.

(J2) $\|a^2\| = \|a\|^2$ for all $a \in A$.

(J3) $\|a^2\| \leq \|a^2 + b^2\|$ for all $a, b \in A$.

The condition (J1) follows from (J2) and (J3), and (J3) has an equivalent form [AlShSt, p16],

(J3)$'$ $\|a^2 - b^2\| \leq max\{\|a^2\|, \|b^2\|\}$ for all $a, b \in A$.

A deep result is that if J is a norm closed ideal in a JB-algebra A, then A/J is also a JB-algebra [HaSt, 3.4.2].

1.3.3 Let H be a complex Hilbert space and, as is usual, let $B(H)$ be the algebra of all bounded linear operators on H. Note that if H is of finite dimension n then $B(H)$ is $M_n(\mathbf{C})$. For any H we see that $B(H)_{sa}$ is a special Jordan algebra equipped with operator norm. Every norm closed Jordan subalgebra of $B(H)_{sa}$ is a JB-algebra. A *JC-algebra* is defined to be a JB-algebra that is isometrically isomorphic to one of this form. There exist JB-algebras which are not JC-algebras, for example N_3^8.

1.3.4 A JC-subalgebra A of $B(H)_{sa}$, where H is a complex Hilbert space, is said to be *reversible* if $a_1...a_n + a_n...a_1$ lies in A whenever $a_1, ..., a_n$ do.

1.4 Spectral Theory and Order Structure

As for C*-algebras, spectral theory is a vital tool for JB-algebras. Here we record a spectral theory for both general associative JB-algebras and for those generated by a single element.

1.4.1 A Jordan algebra A is said to be *associative* or *abelian* if $A = Z(A)$. Let X be a locally compact Hausdorff space. Then the self adjoint part of $C_0(X)$ is an associative JC-algebra under pointwise multiplication and supremum norm. Conversely we have the following.

Theorem 1.4.2 ([AlShSt, 2.3])

Let A be an associative JB-algebra. Then there exists a locally compact Hausdorff space X such that A is isometrically isomorphic to $(C_0(X))_{sa}$. Moreover A is unital if and only if X is compact.

1.4.3 Let A be a JB-algebra with $a_1, ..., a_n \in A$ and let $C(a_1...a_n)$ denote the JB-subalgebra of A generated by $a_1, ..., a_n$. Then if $a_1, ..., a_n$ operator commute $C(a_1...a_n)$ is abelian and hence is isometric to the self adjoint part of some abelian C*-algebra. In particular, via C*-algebra theory, for any $a \in A$ we have the following isometric isomorphism

$$\begin{aligned} C_0(\sigma(a)) &\cong C(a) \\ f &\mapsto f(a), \end{aligned}$$

where $\sigma(a) = \{\lambda \in \mathbf{R} : a - \lambda 1 \text{ is not invertible in } C(1, a)\}$. Here, if A is non-unital 1 is the identity element of the unitisation of A. Note that in Jordan algebra terms a is said to be *invertible* with inverse b if $a \circ b = 1$ and $a^2 \circ b = a \circ b$.

1.4.4 Let A be a JB-algebra and let $a \in A$. Then a is said to be *positive*, written $a \geq 0$, if $\sigma(a) \subset [0, \infty)$. For a and b in A we write $a \geq b$ if and only if $a - b \geq 0$. Importantly we have,

$$A = A_+ - A_+,$$

where, as is usual, A_+ denotes the positive part of A [HaSt, §3.3].

1.5 States

1.5.1 Let A be a JB-algebra. A functional $\rho \in A^*$ is *positive* if $\rho(a) \geq 0$ for all $a \in A_+$. In which case we write $\rho \geq 0$.

The set of *quasi states* of A,

$$Q(A) = \{\rho \in A^* : \rho \geq 0, \|\rho\| \leq 1\},$$

is weak* compact and convex. The set of states of A is the convex set

$$S(A) = \{\rho \in A^* : \rho \geq 0, \|\rho\| = 1\}.$$

The non-zero extreme points of $Q(A)$ are states of A called the *pure states* of A, the set of which is denoted by $P(A)$. Furthermore, if A has an identity element, denoted by 1, then

$$S(A) = \{\rho \in A^* : \rho(1) = \|\rho\| = 1\}$$

and is weak* compact as well as being convex. Consequently, in that case we have $P(A) = \partial_e(S(A))$, that is, the set of extreme points of $S(A)$.

1.6 JW-Algebras and JBW-Algebras

It is natural to consider a subclass of JB-algebras which are in some sense the Jordan analogue of W*-algebras. We begin with the concrete version, JW-algebras.

1.6.1 Let H be a complex Hilbert space and consider $B(H)$ as a von Neumann algebra with the weak topology. A real Jordan subalgebra M of $B(H)_{sa}$ is said to be a *JW-algebra* if it is a weakly closed.

1.6.2 Before we can formally define JBW-algebras we need a few preliminary definitions. Let M be a JB-algebra. Then M is said to be *monotone complete* if each bounded increasing net (x_λ) in M has least upper bound x in M. A bounded linear functional ρ of M is said to be *normal* if for every such net (x_λ) we have $\rho(x_\lambda) \longrightarrow \rho(x)$. A set of functionals Γ is said to be *separating* if for any non-zero x in M there exists a functional ρ in Γ such that $\rho(x) \neq 0$.

1.6.3 A JB-algebra M is said to be a *JBW-algebra* if it is monotone complete with a separating set of positive normal bounded linear functionals. In line with Sakai's definition of a W*-algebra, there is an alternative and more elegant definition.

Theorem 1.6.4 ([Sh1, Theorem 2.3])

Let M be a JB-algebra. Then M is a JBW-algebra if and only if M is a Banach dual space.

In this case the predual is unique and consists of the normal linear functionals on M. It is denoted by M_*.

1.6.5 Every JW-algebra is a JBW-algebra. All JBW-algebras are unital [HaSt, 4.1.7].

1.6.6 The *normal states* of a JBW-algebra M are the weak* continuous linear functionals ρ on M satisfying $\rho(1) = \|\rho\| = 1$, where 1 denotes the unit of M.

1.6.7 The relationship between JB-algebras and JBW-algebras, mimicing that of C*-algebras and W*-algebras, is a powerful tool. Its strength is particularly evident when any JB-algebra is viewed, via the canonical injection, as a subset of its second dual. This is demonstrated by the next theorem.

Theorem 1.6.8 ([Sh1] [HaSt, 4.4.3, 4.7.5])

Let A be a JB-algebra. Then

(a) A^{**} *is a JBW-algebra;*

(b) the product of A^{**} *extends the usual product on* A *and is separately weak* continuous;*

(c) the weak continuous extension of each state of* A *to* A^{**} *is normal;*

(d) A^{**} *is the monotone completion of* A*, i.e.* A^{**} *is the smallest monotone closed subalgebra of* A^{**} *containing* A*;*

(e) if A *is a JC-algebra then* A^{**} *is a JW-algebra;*

(f) if A *is a JBW-algebra and a JC-algebra then it is a JW-algebra.*

In this manner it is common to regard a JB-algebra A as a subalgebra of A^{**} and to identify states of A with normal states of its bidual.

1.7 Projections

1.7.1 Let M be a JBW-algebra. The idempotents of M are called *projections*. Given an element a in M let $W(a)$ denote the weak* closure of $C(a)$. Then $W(a)$ is an abelian JBW-algebra with a unit, denoted by $r(a)$ and which is called the *range projection* of a in M [AlShSt] [HaSt, 4.1.10, 4.2.6]. The JBW-algebra $W(a)$, is isometrically isomorphic to the self adjoint part of a W*-algebra. Thus, M is amply endowed with projections. Through the order structure inherited from M, the set of all projections in M is a complete lattice and the set of all central projections is a complete sublattice. For each projection e of M, $c(e)$ denotes the least central projection of M majorising e, termed the *central support* of M.

1.7.2 If e is a projection in the JBW-algebra M then $U_e(M)$ is a JBW-subalgebra of M with unit e. Moreover, as e ranges over all projections, the $U_e(M)$ are precisely the weak* closed quadratic ideals of M [Ed, 2.3]. The weak* closed ideals of M are of the form $z \circ M$, where z is a central projection [HaSt, 4.3.6]. In particular, $c(e) \circ M$ is the weak* closed ideal of M generated by a projection e.

A non-zero projection e of M is said to be *minimal* if it does not majorise any other non-zero projection. Thus, e is a minimal projection in M if and only if $U_e(M) = \mathbf{R}e$. M is said to be a *factor* if $Z(M) = \mathbf{R}1$. Thus, M is a factor if and only if it has no non-trivial central projections.

1.7.3 Let ρ be a normal state on a JBW-algebra M. The *support* of ρ, denoted by $s(\rho)$, is defined to be the least projection e of M such that $\rho(e) = 1$, whereas the *central support* of ρ, $c(\rho)$, is the least central projection z such that $\rho(z) = 1$. We have $c(\rho) = c(s(\rho))$.

When A is a JB-algebra and ρ is a state of A, and therefore a normal state of A^{**}, by $s(\rho)$ and $c(\rho)$ we mean the support and central support, respectively, in A^{**}.

1.7.4 Using [AlSh1, 1.13], [AlSh2] and [AlShSt], the map $\rho \mapsto s(\rho)$ describes a bijection from the set of pure normal states of a JBW-algebra M to the set of minimal projections of M. The inverse map is given for a minimal projection e, by $e \mapsto \rho_e$, where $\rho_e(x)e = \{exe\}$ for all x in M. In terms of a JB-algebra A, this translates to a bijection between the set of pure states of A and set of minimal projections of A^{**}.

1.8 Type Decomposition of a JBW-Algebra

1.8.1 The notion of type classification in JB-algebras and JBW-algebras is, in essence, a translation of the respective concepts for C*-algebras and W*-algebras. As a type I W*-algebra is not necessarily type I as a C*-algebra, it is the convention to use the term postliminal to refer to those C*-algebras originally classed as type I. In consequence, the concept of postliminality is also used for JB-algebras. For details regarding postliminal JB-algebras see section (1.11).

Our particular interest is the type decomposition of JBW-algebras (actually JBW*-algebras), a characterisation in terms of abelian projections, mimicing that for W*-algebras. Such a decomposition will be invaluable in our later work.

Recall that a projection e in a JBW-algebra M is said to be *abelian* if $U_e(M)$ is an abelian JBW-algebra.

1.8.2 Let M be a JBW-algebra. Then M is said to be

(i) *type I* if there exists an abelian projection p in M with $c(p) = 1$;

(ii) *continuous* if M has no non-zero abelian projections.

Furthermore, M is said to be *type* I_n, where $n < \infty$, if there exist n abelian projections $p_1, .., p_n$ in M with $c(p_1) = ... = c(p_n) = 1$ and $\sum p_i = 1$. We define M to be *type I finite* if it is an ℓ^∞-sum of such type I_n JBW-algebras. If M is type I without type I finite part then it is said to be of type I_∞.

1.8.3 Let e be a minimal projection in a JBW-algebra M. Then $c(e) \circ M$, the weak* closed ideal generated by e, is a type I factor, N say. The norm closed ideal in N, and hence in M, generated by its minimal projections is called the *elementary ideal* of N and is denoted by $K(N)$. Further details of type I factors are provided in sections (1.12) and (1.13).

1.8.4 Let p and q be projections in JBW-algebra M. Then p and q are said to be *exchanged by a symmetry*, denoted by $p \underset{1}{\sim} q$, if $p = U_s(q)$ for some symmetry s in M.

We note the following technical result, of use in Chapters Three and Four.

Lemma 1.8.5 ([HaSt, 5.3.2])

Let M be a JBW-algebra with projections p and q. Then

(a) if p is abelian and $q \leq p$ then $q = c(q)p$;

(b) if p and q are abelian with $c(p)$=$c(q)$ then $p \underset{1}{\sim} q$;

(c) if M is type I then there exists an abelian projection r in M such that $r \leq q$ and $c(q)$=$c(r)$.

In particular, it follows from (a) that in a JBW-factor every abelian projection is minimal.

1.8.6 Projections p and q in a JBW-algebra are said to be *equivalent*, denoted by $p \sim q$, if there exists a finite sequence $s_1, ..., s_n$ of symmetries such that $q = U_{s_1}...U_{s_n}(p)$. If $p \sim q$ then $c(p) = c(q)$.

Theorem 1.8.7 ([Top1, Theorem 9])

Let M be a type I finite JBW-algebra. Then

(a) if $(e_i)$and (f_i) are families of orthogonal projections in M such that $e_i \sim f_i$ for all i then $\sum e_i \sim \sum f_i$;

(b) two equivalent projections e and f in M are exchanged by a symmetry.

1.9 Type I Factor Representations

1.9.1 Let A be a JB-algebra. A *type I factor representation* of A is a Jordan homomorphism, $\pi : A \longrightarrow M$, where M is a type I factor and $\pi(A)$ is weak* dense in M.

Two type I factor representations $\pi_1 : A \to M_1$ and $\pi_2 : A \to M_2$ are said to be *equivalent* if there exists an isometric isomorphism, $\psi : M_1 \longrightarrow M_2$, such that $\psi\pi_1 = \pi_2$.

Each $\rho \in P(A)$ induces the type I factor representation given below [AlShSt, 5.6,8.7].

$$\begin{aligned} \pi_\rho : A &\longrightarrow A_\rho = c(\rho) \circ A^{**} \\ a &\longmapsto c(\rho) \circ a \end{aligned}$$

Every type I factor representation of A is equivalent to π_ρ for some pure state ρ of A [AlHaSh, 2.2].

1.10 Exceptional Decomposition

A JB-algebra A is said to be *purely exceptional* if every factor representation is onto the exceptional JB-algebra N_3^8.

Theorem 1.10.1 ([AlShSt, 9.5])
Let A be a JB-algebra. Then there exists a unique ideal J of A such that

(a) J is purely exceptional;

(b) A/J is a JC-algebra.

For JBW-algebras we have a much stronger result.

Theorem 1.10.2 ([Sh1, 3.9])
Let M be a JBW-algebra. Then M has unique decomposition as the direct sum $M = M_{sp} \oplus M_{ex}$, where M_{sp} is a JW-algebra and M_{ex} is a purely exceptional JBW-algebra isomorphic to $C(X, N_3^8)$, for some compact hyperstonean space X.

1.11 Postliminal JB-algebras

1.11.1 Let A be a JB-algebra and let $x \in A_+$. Then x is called an *abelian* element of A if the Jordan algebra $U_x(A)$ is associative.

1.11.2 In line with the definition for C*-algebras, a JB-algebra A is said to be *postliminal* if each JB-quotient by a norm closed ideal contains a non-zero abelian element. A is said to be *antiliminal* if it contains no non-zero abelian elements. Finally, A is said to be *liminal* if for each type I factor representation π of A onto factor M we have $\pi(A) = K(M)$, where $K(M)$ is the elementary ideal.

Proposition 1.11.3 ([Bu2, 3.9])

Let A be a JB-algebra. Then there exists a Jordan ideal J of A such that

(a) J is postliminal;

(b) A/J is antiliminal.

1.11.4 A *composition series* of JB-algebra A is a strictly increasing family of closed ideals (I_λ), indexed by an ordinal segment $[0, \alpha\ ]$, such that

(i) $I_0 = 0$ and $I_\alpha = A$;

(ii) for each limit ordinal γ

$$I_\gamma = \overline{\bigcup_{\lambda<\gamma} I_\lambda},$$

where bar denotes norm closure;

(iii) each successive quotient is of the form $I_{\lambda+1}/I_\lambda$.

This definition is a similar to that for C*-algebras, with Jordan ideals replacing ideals. We note the following result, which is a Jordan analogue to that in [Kap] [Ped1, 6.2.6].

Theorem 1.11.5 ([Bu1, 2.3.2])

Let A be a JB-algebra. Then the following are equivalent.

(a) A is postliminal.

(b) A has a composition series with liminal successive quotients.

(c) A has a composition series with postliminal successive quotients.

1.12 Spin Factors

Lemma 1.12.1 ([HaSt, 6.1.3])

Let H be a real Hilbert space of dimension at least two. Let $A = H \oplus \mathbf{R}1$ with norm and product defined by:

$$\begin{aligned} \|a + \lambda 1\| &= \|a\|_2 + |\lambda| \\ (a + \lambda 1) \circ (b + \mu 1) &= (\mu a + \lambda b) + (\langle a, b\rangle + \lambda\mu)1 \quad \textit{for all } a, b \in H \ \lambda, \mu \in \mathbf{R}. \end{aligned}$$

Then A is a JW-algebra factor.

1.12.2 The JW-algebras that occur in the above lemma are the *spin factors*. They are uniquely determined by the orthonormal dimension of the underlying Hilbert space H. If H has orthonormal dimension λ, where λ is a cardinal number, we write $U_\lambda = H \oplus \mathbf{R}1$ to denote the spin factor of (1.12.1). The spin factor U_λ arises concretely in the following way.

Let $\{s_i\}$ be a family, of cardinality λ, of anti-commuting symmetries in a Von Neumann algebra W. Let U denote the norm closed linear span of 1 and the $\{s_i\}$. The norm closed linear span of the $\{s_i\}$ is a real Hilbert space K, in the operator norm, with $\{s_i\}$ as a complete orthonormal set, so that H and K are linearly isometric and U is a JW-algebra Jordan isomorphic to U_λ.

1.12.3 Let U be a spin factor. Then if U has a representation as a reversible JC-algebra it is isomorphic to U_2, U_3 or U_5. Conversely U_2 and U_3 are reversible in every representation, referred to as being *universally reversible*, while U_5 has both reversible and irreversible representations [Han2, §2]. We will see in the next section that spin factors constitute the only hindrance to an algebra being universally reversible.

1.13 The Universal Enveloping C*-Algebra

1.13.1 Throughout the thesis, by an *involution* on a C*-algebra we shall always mean a *-antiautomorphism of order two. Let B be a C*-algebra with involution α. We will habitually use the notation B^α to denote the set of points of B fixed by α.

The following structure is crucial to our later results, particularly to those in Chapters Three and Four. For further details refer to [HaSt, §7]. Throughout this section let A be a JC-algebra.

1.13.2 A *universal enveloping C*-algebra* of A is a pair $(C^*(A), \psi)$, where $C^*(A)$ is a C*-algebra and ψ is an injective Jordan homomorphism

$$\psi : A \longrightarrow C^*(A)_{sa}$$

such that

(i) $C^*(A)$ is the C*-algebra generated by $\psi(A)$;

(ii) A has the *universal extension property.* That is, each Jordan homomorphism $\pi : \psi(A) \to C_{sa}$, where C is any C*-algebra, extends uniquely to a *-homomorphism $\hat{\pi} : C^*(A) \to C$.

1.13.3 Each JC-algebra A has a universal enveloping C*-algebra [HaSt, §7]. In a sense there is only one such C*-algebra, for if (D, θ) is another universal enveloping C*-algebra then the defining properties imply that there exists a surjective isomorphism, $\pi : C^*(A) \longrightarrow D$, such that $\pi\psi = \theta$.

It is standard practice to refer to the universal enveloping C*-algebra as $C^*(A)$ rather than $(C^*(A), \psi)$, supressing the injection ψ by identifying A with $\psi(A)$.

1.13.4 Consider the inclusion of A in the opposite algebra $C^*(A)^{op}$ of $C^*(A)$. The universal extension property then gives rise to an involution,

$$\phi : C^*(A) \to C^*(A),$$

called the *canonical involution* of $C^*(A)$, which fixes each element of A. Let $C^*(A)^\phi = \{a \in C^*(A) : \phi(a) = a\}$. We have $A \subset C^*(A)^\phi_{sa}$.

1.13.5 Let M be a JW-algebra. Then the *universal enveloping W*-algebra* of M, denoted by $W^*(M)$, is defined in a similar way to its C*-algebra counterpart, replacing all morphisms with normal morphisms and C*-algebras with W*-algebras. Hence we have the inclusion

$$M \subset W^*(M)^\phi_{sa}.$$

If M is a spin factor then there is equality in the above inclusion if and only if $M = U_2$ or U_3. In fact, in the wider JW-algebra case, the remaining spin factors are the only obstacle to equality. This is formalised in the next proposition.

Proposition 1.13.6 ([Han2, §2])

(a) Let A be a JC-algebra. Then the following are equivalent.

(i) A is universally reversible.

(ii) $A = C^*(A)^{\phi}_{sa}$.

(iii) A has no spin factor representations other than those onto U_2 or U_3.

(b) Let M be a JW-algebra. Then the following are equivalent.

(i) M is universally reversible.

(ii) $M = W^*(M)^{\phi}_{sa}$.

(iii) The type I_2 part of M is isomorphic to $C(X, U_2) \oplus C(Y, U_3)$, where X and Y are compact hyperstonean spaces.

1.13.7 Let H be a complex Hilbert space and let $v : H \to H$ be a conjugate linear isometry. Define an involution $\alpha : B(H) \to B(H)$ by $x \mapsto vx^*v^*$. Evidently α is of order two if and only if $v^2 = \lambda 1$ for some $\lambda \in \mathbf{C}$ with $|\lambda| = 1$. Since v commutes with v^2 we see that v commutes with λ. In which case, as v is conjugate linear, λ is real. It follows that $v^2 = \pm 1$.

Such an involution α is said to be a *real flip* if $v^2 = 1$, or a *quaternionic flip* if $v^2 = -1$. A conjugate linear isometry v is called a *conjugation* if $v^2 = 1$ and a *unit quaternion* if $v^2 = -1$.

Suppose that α is a real flip. Then $H^v = \{h \in H : vh = h\}$ is a real Hilbert space with $H = H^v \oplus iH^v$. Let $x \in B(H)$. Then $\alpha(x) = x^*$ precisely when $vx = xv$, so that, restricting to H^v, we observe that $B(H)^{\alpha}_{sa} \cong B(H^v)_{sa}$.

On the other hand, suppose that α is a quaternionic flip. Let $k = iv$ so that $\{1, i, v, k\}$ generate the quaternions. To form a quaternionic Hilbert space $H_{\mathbf{H}}$ define a quaternionic inner product, denoted by $< ., . >_{\mathbf{H}}$,

$$< a, b >_{\mathbf{H}} = Re < a, b > -iRe < ia, b > -vRe < va, b > -kRe < ka, b >$$

for all $a, b \in H$. It is apparent that any element x in $B(H)$ satisfies $\alpha(x) = x^*$ if and only if $vx = xv$ and $kx = xk$. Thus elements of $B(H)^{\alpha}_{sa}$ are self adjoint $\mathbf{H}-$linear operators, that is, $B(H)^{\alpha}_{sa} \cong B(H_{\mathbf{H}})_{sa}$.
This discussion forms the basis of the next theorem.

Theorem 1.13.8 ([AlHaSh, 3.1])

Let M be a universally reversible JW-algebra factor of type I_n $(2 \leq n \leq \infty)$. Then there exists a complex Hilbert space H such that M is isomorphic to one of the following.

(a) $B(H)^{\alpha}_{sa} \cong B(H_{\mathbf{R}})_{sa}$, *where α is a real flip on $B(H)$ and $H_{\mathbf{R}}$ is a real Hilbert space.*

(b) $B(H)_{sa}$.

(c) $B(H)^{\beta}_{sa} \cong B(H_{\mathbf{H}})_{sa}$, *where β is a quaternionic flip on $B(H)$ and $H_{\mathbf{H}}$ is a quaternionic Hilbert space.*

1.13.9 A non-abelian type I JW-algebra factor M is said to be *real, complex* or *quaternionic* if M is isomorphic to $B(K)_{sa}$ where K is a real, complex or quaternionic Hilbert space. Hence the factors in the above theorem are respectively real, complex and quaternionic. By convention, the trivial factor $\mathbf{R}$ is referred to as a real factor.

1.14 JB*-Algebras and JC*-Algebras

Finally we describe the algebraic complexification of JB-algebras and JC-algebras, namely JB*-algebras and JC*-algebras.

1.14.1 A *JB*-algebra*, originally termed a *Jordan C*-algebra*, is a complex Banach space A which is a complex Jordan algebra with involution * and that satisfies the following conditions.

(JB1) $\|x \circ y\| \leq \|x\|\|y\|$ for all $x, y \in A$.

(JB2) $\|x^*\| = \|x\|$ for all $x \in A$.

(JB3) $\|U_x(x^*)\| = \|x\|^3$ for all $x \in A$.

A *JBW*-algebra* is a JB*-algebra with a Banach predual.

1.14.2 The set of all self adjoint elements of a JB*-algebra is a JB-algebra and conversely the algebraic complexification of a JB-algebra can be given a norm so that it is a JB*-algebra [Wr, 2.8]. We define *JC*-algebras* to be the complexifications of JC-algebras.

1.14.3 Any state of a JB*-algebra A restricts to one of the JB-algebra A_{sa}. Conversely any state of A_{sa} extends uniquely to one of A. Thus results regarding the states of JB-algebras can be translated to those of JB*-algebras and vice versa. The type decomposition of a JBW*-algebra also follows naturally from that of its self adjoint part. In this way, although the chapters that follow concern JB*-algebras, the material on JB-algebras will be sufficient.

Chapter 2

Preliminaries of Jordan Triple Systems

2.1 Introduction

Taking a similar role to Chapter One, this chapter presents the principal background to Jordan triple systems. Material has been selected from a wide variety of literature, perhaps the most consistently used of which are [FrRu4] and [FrRu5], with the intention to support later work by providing relevant information in a clear and concise manner.

We start by describing general Jordan triple systems, before moving on to define JB*-triples. We naturally include a summary of well-known results pertaining to the focus of the thesis, that is the inner ideals of JB*-triples. In particular, we present carefully chosen results from the wide ranging work of Edwards and Ruttimann [EdRü1]-[EdRü8]. We also establish some technical lemmas regarding inner ideals, which do not warrant deep examination, but which are required in subsequent chapters.

The notion of type decomposition, specifically that provided by Horn [Ho2], dominates later work, and is therefore given a thorough exposition here. We go on to give details of atomic decomposition, and of Friedman and Russo's Gelfand Naimark Theorem for JB*-triples. To conclude, we supply a succinct presentation of the Stone-Weierstrass Theorem and its many generalisations.

2.2 Jordan *-Triple Systems over C

2.2.1 A *Jordan *-triple system over* **C** is a complex vector space A with a triple product $A \times A \times A \to A$, denoted by $(a, b, c) \longmapsto \{abc\}$, such that

(i) $\{...\}$ is linear in a and c, conjugate linear in b;

(ii) $\{abc\} = \{cba\}$ for all $a, b, c \in A$;

(iii) $\{ab\{xyz\}\} = \{\{abx\}yz\} + \{xy\{abz\}\} - \{x\{bay\}z\}$.

The identity (iii) is often known as the *main identity*.
Henceforth, by a Jordan *-triple system we shall mean a Jordan *-triple system over **C**.

2.2.2 Let A be a Jordan triple system. Then a subspace B of A is said to be a *subtriple* if $\{BBB\} \subset B$. A subspace I of A is said to be an *ideal* if $\{AAI\} + \{AIA\} \subset I$, and an *inner ideal* if $\{IAI\} \subset I$.

2.2.3 For a pair of elements x and y in a Jordan triple A define a linear operator on A by $D(x, y)(z) = \{xyz\}$, and a conjugate linear operator on A by $Q(x)(z) = \{xzx\}$. Using the commutator notation, $[X, Y] = XY - YX$, we can rewrite the main identity (iii) as follows.

(iii)′ $[D(a, b), D(x, y)] = D(\{abx\}, y) - D(x, \{bay\})$.

2.2.4 We state a series of well-known identities satisfied by a Jordan triple system. (P1-P3) are referred to as *polarisation identities*.

(I) $\{yb\{xax\}\} = 2\{\{ybx\}ax\} - \{x\{bya\}x\}$.

(II) $\{xa\{xbx\}\} = \{x\{axb\}x\} = \{\{xax\}bx\}$.

(III) $\{ya\{xax\}\} = \{y\{axa\}x\}$.

(IV) $\{\{xax\}b\{xax\}\} = \{x\{a\{xbx\}a\}x\}$.

(V) $\{\{xax\}a\{xay\}\} = \{xa\{\{xax\}ay\}\}$.

(VI) $2\{ya\{xaz\}\} = \{y\{axa\}z\} + \{y\{aza\}x\}$.

(P1) $4\{xyz\} = \sum_0^3 i^k\{x + i^k y \quad x + i^k y \quad z\}$.

(P2) $4\{zxz\} = \sum_0^3 (-1)^k\{x + i^k z \quad x + i^k z \quad x + i^k z\}$.

(P3) $2\{xyz\} = \{x + z \quad y \quad x + z\} - \{xyx\} - \{zyz\}$.

2.2.5 Let A be a Jordan *-triple system. An element e in A is said to be a *tripotent* if $e = \{eee\}$. For such a tripotent e, let $D = D(e, e)$ and $Q = Q(e)$. We have the following identities.

(i) $Q^3 = Q$.

(ii) $DQ = QD = Q$.

(iii) $Q^2 = 2D^2 - D$.

(iv) $D(2D - I)(D - I) = 0$.

The associated *Peirce projections* P_0, P_1 and P_2 are defined as follows.

$$\begin{aligned} P_2 &= Q^2 = D(2D - I) \\ P_1 &= 2(D - Q^2) = 4D(I - D) \\ P_0 &= I - 2D + Q^2 = (I - D)(I - 2D). \end{aligned}$$

The Peirce projections are mutually orthogonal linear projections on A with $P_0 + P_1 + P_2 = I$. Thus, $A = P_0(A) \oplus P_1(A) \oplus P_2(A)$. Furthermore, since

$$(D - I)P_2 = (D - \frac{I}{2})P_1 = (D - 0)P_0 = 0,$$

it follows that $P_i(A) = ker(D - \frac{iI}{2})$, for $i = 0, 1, 2$. That is, for each $i \in \{0, 1, 2\}$ the Peirce i-space is the $\frac{i}{2}$-eigenspace of D.

2.2.6 Let A be a Jordan *-triple system and let $A_i = P_i(A)$, for $i = 0, 1, 2$. The following rules, often called *Peirce rules*, hold.

(i) $\{A_i A_j A_k\} \subset A_{i-j+k}$ if $i - j + k \in \{0, 1, 2\}$.

(ii) $\{A_i A_j A_k\} = 0$ if $i - j + k \notin \{0, 1, 2\}$.

(iii) $\{A A_2 A_0\} = \{A A_0 A_2\} = 0$.

Each A_i (i=0, 1, 2) is a subtriple of A, and A_0 and A_2 are inner ideals of A. When it is necessary to emphasise which tripotent e is being considered we will use the notations $P_i = P_i^e$ and $A_i = A_i(e)$.

2.3 Jordan *-Triple Systems and Jordan * Algebras

2.3.1 Let A be a Jordan *-triple system and let $a \in A$. Then we can define an algebra, denoted by $A^{(a)}$ and called the *a-homotope of* A, by defining a product $x \circ y = \{xay\}$, for all x and y in A. Using the symmetric property of the triple product and identity (V), we have that, for all $x, y \in A$,

$$\begin{aligned} x \circ y &= y \circ x; \\ x^2 \circ (x \circ y) = \{\{xax\}a\{xay\}\} &= \{xa\{\{xax\}ay\}\} = x \circ (x^2 \circ y). \end{aligned}$$

Hence $A^{(a)}$ is a complex Jordan algebra.

More specifically, the following proposition demonstrates that if we take a tripotent e in A then the triple system is locally a complex Jordan * algebra, in the sense that the Peirce 2-space $A_2(e)$ is a complex Jordan * algebra.

Proposition 2.3.2

*Let A be a Jordan *-triple system. Let e be a tripotent in A. Then $A_2(e)$ is a complex Jordan * algebra with identity e, and with product and involution given by: $x \circ y = \{xey\}$, $x^{\#} = \{exe\}$.*

On the other hand, every complex Jordan * algebra is a Jordan triple system with triple product given by

$$\{xyz\} = (x \circ y^*) \circ z + x \circ (y^* \circ z) - (x \circ z) \circ y^*. \tag{2.1}$$

2.4 Ordering the Tripotents of a Triple System

2.4.1 Let e and f be two tripotents of a Jordan triple system A. Then e and f are said to be *orthogonal*, denoted $e \perp f$, if they satisfy any of the following equivalent conditions.

(i) {eef}=0.

(ii) {ffe}=0.

(iii) D(e,f)=0.

(iv) D(f,e)=0.

Define $f \leq e$ if and only if $e-f$ is a tripotent orthogonal to f. This induces a natural partial ordering amongst the set of tripotents of A. We have another set of equivalent conditions.

(i) $f \leq e$.

(ii) f={fef}.

(iii) f={ffe}.

(iv) D(e-f,f)=0.

(v) D(f, e-f)=0.

(vi) f is a projection in the Jordan * algebra $A_2(e)$.

2.5 JB*-Triples and JBW*-Triples

We now introduce the structure within which our work is set, that is, the JB*-triple, a category which includes C*-algebras and JB*-algebras.

2.5.1 A *JB*-triple* is a complex Banach space A which is a Jordan *-triple system such that for all a in A

(i) $\|\{aaa\}\| = \|a\|^3$;

(ii) D(a,a) is an hermitian operator on A with non-negative spectrum.

Naturally every norm closed subtriple of a JB*-triple is itself a JB*-triple.

2.5.2 A *JBW*-triple* is a JB*-triple M with a Banach space predual, denoted by M_*. This predual is necessarily unique and the triple product on M is separately $\sigma(M,M_*)$ continuous [BaTi, 2.1]. It follows that a weak* closed subtriple of a JBW*-triple is itself a JBW*-triple. A result of Dineen, [Din], shows that if A is a JB*-triple then A^{**} is a JBW*-triple containing A, via the canonical embedding, as a weak* dense JB*-triple .

2.5.3 Via (2.3.2), if A is a JB*-triple with tripotent e then $P_2^e(A)$ is a JB*-algebra with unit e, and is a JW*-algebra if A is a JBW*-triple [FrRu4, p74].

2.6 Isometries and Isomorphisms

2.6.1 Let $\pi : A \longrightarrow B$ be a linear map between two Jordan *-triple systems A and B. Then π is said to be a *triple homomorphism* if

$$\pi(\{xyz\}) = \{\pi(x)\pi(y)\pi(z)\} \quad \text{for all } x, y, z \in A.$$

By polarisation, it follows that if $\pi : A \longrightarrow B$ is a linear map such that $\pi(\{xxx\}) = \{\pi(x)\pi(x)\pi(x)\}$ for all x in A, then π is a triple homomorphism. Triple isomorphisms between JB*-triples correspond precisely to surjective linear isometries.

Theorem 2.6.2 ([Kaup, 5.5][Ho1, 2.4])
Let A and B be JB-triples and let $\pi : A \longrightarrow B$. Then π is a surjective linear isometry if and only if π is a triple isomorphism.*

This was shown to hold for JC*-triples by Harris [Har1, Theorem 4].

2.7 Examples

(i) A C*-algebra is a JB*-triple with triple product defined by

$$\{xyz\} = \frac{1}{2}(xy^*z + zy^*x).$$

The tripotents of a C*-algebra are precisely its partial isometries.

In a similar manner a W*-algebra is a JBW*-triple.

JB*-triples also arise as the range of contractive projections on C*-algebras [FrRu2].

(ii) More generally, every JB*-algebra is a JB*-triple via the triple product, (2.1) of page 26.

Likewise, JBW*-algebras are JBW*-triples.

(iii) A *JC*-triple* is a JB*-triple linearly isometric, that is, triple isomorphic, to a norm closed subtriple of a C*-algebra. These were first studied under the name J*-algebras [Har2].

(iv) A *JW*-triple* is a JC*-triple with a predual.

2.8 Tripotents in JB*-Triples

2.8.1 Let e be a non-zero tripotent in a JB*-triple A. Then e is said to be:-

(i) *minimal* if $A_2(e) = \mathbf{C}e$;

(ii) *complete*, if $A_0(e) = \{0\}$;

(iii) *unitary* if $A_2(e) = A$;

(iv) *abelian* if $A_2(e)$ is an associative JB*-algebra (hence, an abelian C*-algebra).

Two tripotents e and f are said to be *collinear* if $e \in A_1(f)$ and $f \in A_1(e)$. In general, a JB*-triple may not have any non-zero tripotents. However, from the Krein-Milman Theorem and the following proposition, every JBW*-triple has many (complete) tripotents.

Theorem 2.8.2 ([KaUp, 3.5] [Har1, Theorem 11])
Let A be a JB-triple. Then $\partial_e(A_1)$ equals the set (possibly empty) of complete tripotents of A.*

2.8.3 The polar decomposition of functionals in the predual of a JBW*-triple, due to Friedman and Russo, is fundamental.

Theorem 2.8.4 ([FrRu4, Proposition 2]) *[Polar Decomposition]*
Let M be a JBW-triple and let $\rho \in M_*$. Then there exists a unique tripotent u in M such that $\rho = \rho \circ P_2^u$ and ρ is faithful and positive on $M_2(u)$.*

If $\rho \in M_*$, where M be a JBW*-triple, the unique tripotent u appearing in (2.8.4) is said to be the *support tripotent* of ρ and is denoted by $s(\rho)$.
Another key result of Friedman and Russo gives a precise link between minimal tripotents in a JBW*-triple and its predual ball extreme points.

Theorem 2.8.5 ([FrRu4, Proposition 4])

Let M be a JBW-triple. There is a bijective correspondence between the elements of $\partial_e(M_{*,1})$ and the minimal tripotents of M given by $\rho \mapsto s(\rho)$. Furthermore, if $\rho \in \partial_e(M_{*,1})$ with $u = s(\rho)$, then $P_2^u(x) = \rho(x)u$ for all $x \in M$.*

2.9 The Functional Calculus of a JB*-Triple

2.9.1 Let A be a JB*-triple and let $x \in A$. Consider the JB*-subtriple of A generated by x, denoted by A_x. There is a surjective isometry, thus a triple isomorphism, $\phi : A_x \longrightarrow C$, where C is a commutative C*-algebra that is generated by $\phi(x)$, and $\phi(x)$ is positive [Kaup, §1]. Bitransposing gives the isometry $\hat{\phi} : (A_x)^{**} \longrightarrow C^{**}$. Let $u(x) \in (A_x)^{**}$ be the unitary tripotent in $(A_x)^{**}$ defined by $u(x) = \hat{\phi}^{-1}(1)$, where 1 is the identity of the W*-algebra C^{**}. This is unambiguous as such a tripotent $u(x)$ is independent of both $\hat{\phi}$ and C [BuChZa2, §2].

2.9.2 One consequence of the functional calculus, repeatedly used in this thesis, is the existence of cube roots. That is, for each element x in a JB*-triple A there exists an element y in A_x, the "cube root" of x, such that $x = \{yyy\}$.

2.9.3 Next consider the inclusion π of a JB*-subtriple A into a JBW*-triple M, and its weak* continuous extension, $\hat{\pi} : A^{**} \longrightarrow \overline{\pi(A)}$, where bar denotes weak* closure in M. As $u(x)$ is unitary in $(A_x)^{**}$, we have

$$\overline{A_x} = \hat{\pi}(A_x^{**}) = \hat{\pi}((A_x^{**})_2(u(x))) = (\overline{A_x})_2(\hat{\pi}(u(x))).$$

The tripotent $\hat{\pi}(u(x))$, the *range tripotent of x in M*, is denoted by $r(x)$.

2.9.4 One result of the preceding discussion is that if M is a JBW*-triple with $x \in M$, and $\overline{M_x}$ denotes the JBW*-subtriple of M generated by x, then $\overline{M_x}$ is an abelian W*-subalgebra of $M_2(r(x))$. In particular, $x \in M_2(r(x))_+$. Since there exists a complete tripotent in M majorising $r(x)$, [Ho1, 3.12], we have the following.

Lemma 2.9.5

If $x \in M$, where M is a JBW-triple, then $x \in M_2(u)_+$ for some complete tripotent u of M.*

2.10 Norm Closed Inner Ideals, Ideals and Quotients

Crucially, JB*-triples are stable under appropriate quotients.

Theorem 2.10.1 ([Kaup, p523][FrRu5, p146])

Let A be a JB-triple with norm closed ideal J. In the quotient norm A/J is (a) a JB*-triple; (b) a JC*-triple if A is a JC*-triple.*

We remark that (2.10.1(b)) anticipates the Gelfand-Naimark Theorem discussed in (2.12).

2.10.2 Norm closed inner ideals and norm closed ideals in JB*-triples have geometric characterisations. In the following, (a) was obtained in [EdRü4] and (b) in [BaTi].

Theorem 2.10.3

Let A be a JB-triple.*

(a) A JB-subtriple I of A is an inner ideal of A if and only if each $\rho \in I^*$ has unique norm preserving extension in A^*.*

(b) A norm closed subspace J of A is an ideal of A if and only if J is an M-ideal of A.

2.10.4 Let A be a JB*-triple. Elements a and b in A satisfy $D(a,b) = 0$ if and only if $D(b,a) = 0$ [EdRü6, 3.1]. In which case, a and b are said to be *orthogonal*, written $a \perp b$. For any subset B of A, the annihilator $B^\perp$ of B in A, which is defined to be the set

$$B^\perp = \{a \in A : a \perp b \text{ for all } b \in B\},$$

is a norm closed inner ideal of A [EdRü6, 3.2]; it is weak* closed if A is a JBW*-triple.

2.10.5 For norm closed ideals in JB*-triples, the defining algebraic condition can be relaxed. In fact, (see [BuCh], [DinTi], [Har2]), a norm closed subspace I of A is an ideal of A if and only if it satisfies any one of the following equivalent conditions:

$$(i) \quad \{AAI\} \subset I \quad ; \quad (ii) \quad \{AIA\} \subset I \quad ; \quad (iii) \quad \{AII\} \subset I.$$

2.10.6 Let I and J be norm closed ideals of a JB*-triple A. By definition, $\{IAI^\perp\} \subset I \cap I^\perp = \{0\}$. Via this remark, the previous paragraph and the fundamental identity, we see that

$$\{AI\{AI^\perp I^\perp\}\} \subset \{\{AIA\}I^\perp I^\perp\} + \{A\{IAI^\perp\}I^\perp\} = \{0\},$$

so that $\{AI^\perp I^\perp\} \subset I^\perp$, implying that $I^\perp$ is a (norm closed) ideal of A.

I and J are said to be *orthogonal* if $D(a, b) = 0$ for all $a \in I$ and $b \in J$. By functional calculus, if $x \in I \cap J$ then $x = \{yyy\}$ for some $y \in I \cap J$. It follows that $\{AIJ\} = I \cap J$. In particular, $I \cap J = \{0\}$ if and only if I and J are orthogonal. We shall often write $I \perp J$ to indicate this latter situation.
We further remark that if I and J are orthogonal then $I + J$ is the ℓ^∞-sum of I and J.

2.10.7 A key structural theorem is Horn's bijective correspondence between weak* closed ideals of a JBW*-triple M and those of the JBW*-algebra $M_2(e)$, when e is complete tripotent of M. At this point we introduce some notation used throughout the thesis. Let A be a JB*-triple with norm closed inner ideal I. We shall let $\mathcal{T}(I)$ and $\mathcal{T}^w(I)$ denote, respectively, the norm closed and the weak* closed ideals of A generated by I.

Theorem 2.10.8 ([Ho1, 4.2])
Let M be a JBW-triple with complete tripotent e.*

(a) *There is a bijection between the weak* closed ideals of M and the weak* closed ideals of $M_2(e)$ given by $J \mapsto J \cap M_2(e)$. The inverse map is given by $I \mapsto \mathcal{T}^w(I)$.*

(b) *Let I be a weak* closed ideal of $M_2(e)$ and z be the unique central projection of $M_2(e)$ such that $I = z \circ M_2(e)$. We have*

$$\mathcal{T}^w(I) = M_2(z) + M_1(z) = P_2^z + P_1^z,$$

and $\mathcal{T}^w(I)^\perp = P_0^z(M)$. Moreover, $\mathcal{T}^w(I)_2(z) = M_2(z) = I_2(z)$.

(c) *If I is a weak* closed ideal of $M_2(e)$ with $I = \sum I_i$, for weak* closed ideals I_i of $M_2(e)$, then $\mathcal{T}^w(I) = \sum \mathcal{T}^w(I_i)$.*

(d) *M is the ℓ^∞-sum $J \oplus J^\perp$, for some weak* closed ideal J of M.*

In the above context, $J^{\perp}$ is often referred to as the *complementary* ideal of the weak* closed ideal J of M. Also, we shall see, (2.10.21), that part (c) generalises to any weak* closed ideal in a JBW*-triple.

2.10.9 Let A be a JB*-triple, let J be a norm closed ideal of A and let I be a norm closed inner ideal of A. Every minimal tripotent of the weak* closed inner ideal I^{**} of A^{**} is again minimal in A^{**}. The corresponding statement is true for the weak* closed ideal J^{**} of A^{**}. On the other hand, by (2.10.8(d)), a minimal tripotent of A^{**} lies in J^{**} or in its complementary ideal. These remarks, together with (2.8.5), have the following consequences.

(a) Each $\rho \in \partial_e(I_1^*)$ has a unique extension in $\partial_e(A_1^*)$.

(b) Each of the following maps is a bijection.

   (i) $\{\rho \in \partial_e(A_1^*) : s(\rho) \in I^{**}\} \to \partial_e(I_1^*) \quad (\rho \mapsto \rho_{|I})$.

   (ii) $\{\rho \in \partial_e(A_1^*) : \rho(J) \neq 0\} \to \partial_e(J_1^*) \quad (\rho \mapsto \rho_{|J})$.

   (iii) $\{\rho \in \partial_e(A_1^*) : \rho(J) = 0\} \to \partial_e((A/J)_1^*) \quad (\rho \mapsto \overline{\rho})$, where $\overline{\rho}$ denotes the functional given by $\overline{\rho}(x + J) = \rho(x)$, whenever ρ lies in the stated domain.

When confusion seems unlikely we will tend to identify the respective domain and codomain in the correspondences (i),(ii), and (iii). It is in this sense that we write

$$\partial_e(A_1^*) = \bigcup \partial_e(I_1^*),$$

where the union ranges over all norm closed inner ideals I of A.

2.10.10 Let B be a subtriple of a JBW*-triple M. Then B is said to be *complemented* in M if $M = B \oplus Ker B$ [LoNe], where

$$Ker B = \{a \in M : \{BaB\} = 0\}.$$

A projection $P : M \to M$ is said to be a *structural* projection if

$$P\{aP(b)a\} = \{P(a)bP(a)\}$$

for all $a, b \in M$.

These notions of complementation and structural projections were introduced into JBW*-triples, and JB*-triples, by Edwards and Rüttimann [EdRü7] with significant effect.

Theorem 2.10.11 ([EdRü7, 4.5, 4.8] [EdMcRü, 5.5, 5.6])
Let M be a JBW-triple. Then*

(a) all structural projections on M are contractive and weak continuous;*

(b) a subtriple of M is complemented if and only if it is a weak closed inner ideal;*

(c) the map, $P \mapsto P(M)$, is a bijection from the set of structural projections of M onto the set of weak closed inner ideals of M.*

If J is a weak* closed ideal of a JBW*-triple M, so that $M = J \oplus J^{\perp}$, the natural projection, $P : M \to M$, is the unique structural projection associated with J. If u is a tripotent in M then P_2^u is the structural projection onto $M_2(u)$, and $Ker M_2(u) = M_1(u) + M_0(u)$.

2.10.12 Edwards' and Rüttimann's description of the weak* closed inner ideals of a W*-algebra will prove to be vital. We let $\mathcal{CP}(W)$ denote the set of pairs of centrally equivalent projections of a W*-algebra W.

Theorem 2.10.13 ([EdRü1, 4.1])

Let W be a W-algebra. Then the map $(e, f) \mapsto eWf$, is an order preserving bijection from $\mathcal{CP}(W)$ onto the set of weak* closed inner ideals of W.*

The norm closed inner ideals of a C*-algebra are precisely the intersections of closed left and right ideals [EdRü3, 2.6]. A similar result holds the weak* closed inner ideals of W*-algebras [EdRü1, 3.16].

2.10.14 We end this section with a few technical lemmas regarding triple ideals and inner ideals that are required in subsequent chapters.

Lemma 2.10.15

Let M be a JBW-triple. Let I be a weak* closed inner ideal of M and let J be a weak* closed ideal of M. Let $P : M \longrightarrow J$ be the natural projection. Then*

(a) $I = I \cap J \oplus I \cap J^{\perp}$;

(b) $I \cap J^{\perp} = (I \cap J)^{\perp} \cap I$;

(c) $P(I) = I \cap J$.

Proof

(a) Let $x \in I$. Then, by functional calculus, $x = \{yyy\}$ for some $y \in I$. Using the ideal decomposition of M, there exists $a \in J$ and $b \in J^{\perp}$ such that $y = a + b$. Thus

$$x = \{yyy\} = \{yay\} + \{yby\} \in I \cap J \oplus I \cap J^{\perp}.$$

The converse is clear.

(b) Since $I \cap J$ is a weak* closed ideal of I we see that $I = I \cap J \oplus (I \cap J)^{\perp} \cap I$. Comparison with (a) gives the result.

(c) As P is the identity on J and vanishes on $J^{\perp}$ we have

$$P(I \cap J) = I \cap J \quad \text{and} \quad P(I \cap J^{\perp}) = 0.$$

The conclusion now follows via part (a). $\square$

Lemma 2.10.16

Let M be a JBW-triple. Let (J_α) be a family of weak* closed ideals of M such that M is the ℓ^∞-sum, $\sum J_\alpha$. If I is a weak* closed inner ideal of M then $I = \sum (J_\alpha \cap I)$.*

Proof

Consider the natural projections $P_\alpha : M \longrightarrow J_\alpha$. We have that $\sum P_\alpha$ is the identity map on M and, by (2.10.15(c)), we have $P_\alpha(I) = I \cap J_\alpha$ for each α, from which the result follows. $\square$

Lemma 2.10.17

Let A be a JB-triple. Let I be a norm closed inner ideal of A and let J be a norm closed ideal of A. Then*

(a) $I \cap J = \{IJI\}$;

(b) $(I \cap J)^{**} = I^{**} \cap J^{**}$.

Proof

(a) By definition, the right hand side is contained in $I \cap J$. On the other hand, as before (see (2.10.6)), given $x \in I \cap J$ there exists $y \in I \cap J$ such that $x = \{yyy\}$, which lies in the right hand side.

(b) Since $I \cap J \subset I^{**} \cap J^{**}$, which is weak* closed, clearly

$$(I \cap J)^{**} \subset I^{**} \cap J^{**}.$$

Conversely, using (a) and the separate weak* continuity of the triple product, we see that

$$I^{**} \cap J^{**} = \{I^{**} J^{**} I^{**}\} \subset \overline{\{IJI\}} = \overline{(I \cap J)} = (I \cap J)^{**},$$

where the bar denotes weak* closure. □

Lemma 2.10.18

Let N be a JBW-subtriple of a JBW*-triple M. Let J be a weak* closed ideal of M. Then $N + J$ is a JBW*-subtriple of M.*

Proof

Given $x \in N$ and $y \in J$, $\{x + y\ x + y\ x + y\}$ is the sum of $\{xxx\}$, which lies in N, and terms of the form $\{abc\}$ where a, b or c equals y, all of which belong to J. Thus $N + J$ is a subtriple of M.

Consider the natural projection $P : M \to J^{\perp}$. Since P is a weak* continuous homomorphism, $P(N)$ is a JBW*-subtriple of $J^{\perp}$ and, in particular, is weak* closed. Therefore $N + J = P^{-1}(P(N))$ is weak* closed, as required. □

Lemma 2.10.19

Let A be a JB-triple with norm closed inner ideal I. Let J be a norm closed ideal of A such that $A = I + J$. Then $(J^{**})^{\perp} \subset I^{**}$.*

Proof

We have $A = I + J \subset I^{**} + J^{**}$. Hence, by (2.10.18), $A^{**} \subset I^{**} + J^{**}$, giving equality. Thus, if $P : A^{**} \to (J^{**})^{\perp}$ is the natural projection, we have

$$(J^{**})^{\perp} = P(A^{**}) = P(I^{**}) + P(J^{**}) = P(I^{**}) = I^{**} \cap (J^{**})^{\perp}.$$

Here the final equality is given by (2.10.15(c)). It follows that $(J^{**})^{\perp} \subset I^{**}$, as claimed. $\square$

Recall that given a norm closed inner ideal I of a JB*-triple A, $\mathcal{T}(I)$ denotes the norm closed ideal of A generated by I.

Lemma 2.10.20

Let A be a JB-triple and let I and J be norm closed inner ideals of A. Then the following are equivalent.*

(a) $\{IAJ\} = 0$.

(b) $\{\mathcal{T}(I)\, A\, \mathcal{T}(J)\} = 0$.

(c) $\mathcal{T}(I) \cap \mathcal{T}(J) = 0$.

Proof

By functional calculus, for two norm closed ideals L and M of A we have $L \cap M = \{LAM\} = \{LMA\} = \{MLA\}$. It follows that (b) holds if and only if (c) holds.

Clearly if $\{\mathcal{T}(I)\, A\, \mathcal{T}(J)\} = 0$ then $\{IAJ\} = 0$.

Finally suppose that $\{IAJ\} = 0$. Then, by [BuChStZa, 2.1], A has two norm closed inner ideals L and M such that $I \subset L$, $J \subset M$ and $\{LAM\} = 0$. It is now immediate, by definition, that (b) holds. $\square$

Lemma 2.10.21

Let A be a JB-triple and let I be a norm closed inner ideal of A. Let J and K be two norm closed ideals of I such that $J \perp K$. Then $\mathcal{T}(J) \perp \mathcal{T}(K)$. Furthermore, if A is a JBW*-triple and I is a weak* closed inner ideal, then $\mathcal{T}^w(J) \perp \mathcal{T}^w(K)$.*

Proof

By the main identity

$$\begin{aligned}\{JAK\} &= \{JA\{KKK\}\} = \{\{JAK\}KK\} + \{KK\{JAK\}\} - \{K\{AJK\}K\} \\ &= 2\{\{JAK\}KK\} \subset \{IKK\} \subset K.\end{aligned}$$

Similarly $\{JAK\} \subset J$ so that $\{JAK\} \subset J \cap K = 0$. Finally, by (2.10.20), $\mathcal{T}(J) \cap \mathcal{T}(K) = 0$ and so $\mathcal{T}(J) \perp \mathcal{T}(K)$.

The remaining statement follows from the separate weak* continuity of the triple product. $\square$

2.11 Types of JBW*-Triples

2.11.1 The type classification of a JBW*-triple is a natural, though not obvious, analogue of that of a JBW*-algebra. Details of the latter can be found in [HaSt]. The study of type I JBW*-triples was initiated in [Ho1] and subsequently pursued in considerable detail in [Ho2] and [Ho3]. The structure of continuous JBW*-triples was investigated to resolution in [HoNe].

2.11.2 Let M be a JBW*-triple. Then M is said to be a *type I JBW*-triple* if every weak* closed ideal of M contains an abelian tripotent. Equivalently, M is type I if it contains a complete tripotent e such that $M_2(e)$ is a type I JBW*-algebra [Ho1, 4.14]. M is said to be *type* I_1 if and only if it contains a complete abelian tripotent. If M contains no non-zero abelian tripotents it is said to be *continuous*. We have the decomposition into the ℓ^∞-sum of weak* closed ideals, $M = N \oplus K$, where N is type I and K is continuous [Ho1, 4.13]. The continuous JBW*-triple K itself decomposes as the ℓ^∞-sum $K \cong W^\alpha \oplus pV$, where W and V are continuous von Neumann algebras, $\alpha : W \longrightarrow W$ is an involution, and p is a projection in V [HoNe, 1.20].

A finer type II and type III classification of continuous JBW*-triples is also given in [HoNe].

2.11.3 A JBW*-triple M is said to be a *factor* if it contains no non-trivial weak* closed ideals. The type I factors are precisely those that contain a minimal tripotent and, by [Ho2, 1.8], they are the *Cartan factors*, briefly described in the following.

It what follows we let H and K be complex Hilbert spaces of respective orthonormal dimensions n and m, where n and m are, possibly infinite, cardinals. Let $j : H \longrightarrow H$ be a conjugation.

(a) Rectangular, $R_{n,m}$: $M = B(H,K)$. If $n \leq m$, realising H as a closed subspace of K, let p be the orthogonal projection onto H. Then

$$M = B(pK, K) \cong B(K)p$$

and, taking any involution, $\psi : B(K) \to B(K)$, we also have that $M \cong \psi(p)B(K) \cong B(K, \psi(p)K) \cong B(K,H)$. Thus $R_{m,n} \cong R_{n,m}$, and the rectangular Cartan factors are the weak* closed left (or right) ideals of type I von Neumann factors. In the special case of $n = 1$, then $M = K$ and the triple product can be realised as follows.

$$\{xyz\} = \frac{1}{2}\big(< x, y > z + < z, y > x\big)$$

If $1 \leq n, m < \infty$, then $M = M_{n,m}(\mathbf{C})$.

(b) Hermitian, $S_n(\mathbf{C})$: $M = \{x \in B(H) : x = jx^*j\}$.
The map $\alpha : B(H) \to B(H)$ given by $x \mapsto jx^*j$ is a real flip, (see (1.13.7)), giving $M = B(H)^\alpha$, a JW*-algebra factor of type I. Moreover, if $1 \leq n < \infty$ then M is isomorphic to the $n \times n$ symmetric matrices (hence, the notation $S_n(\mathbf{C})$).

(c) Symplectic, $A_n(\mathbf{C})$ $(2 \le n \le \infty)$: $M = \{x \in B(H) : x = -jx^*j\}$.

If n is even and finite, or is infinite, then there is a unit quaternion $v : B(H) \to B(H)$, (see(1.13.7)), and the induced map β given by $\beta(x) = -vx^*v$, $(\beta : B(H) \to B(H))$, is a quaternionic flip. In which case, $M \cong B(H)^\beta$, via $x \mapsto -vjx$, again a JW*-algebra factor of type I. When $2 \le n < \infty$, M is identified with the $n \times n$ antisymmetric matrices.

(d) Complex spin factors: $M = V_\lambda = U_\lambda \oplus iU_\lambda$, the complexification of the real spin factor U_λ defined in (1.12). Henceforth, we refer to the V_λ as *the* spin factors and the U_λ as the *real spin factors.*

The previous four kinds of Cartan factors are JC*-triples with the following overlappings: $M_{1,3}(\mathbf{C}) \cong A_3(\mathbf{C})$, $S_2(\mathbf{C}) \cong V_2$, $M_2(\mathbf{C}) \cong V_3$, $A_4(\mathbf{C}) \cong V_5$ and $M_1(\mathbf{C}) \cong S_1(\mathbf{C}) \cong A_2(\mathbf{C}) \cong \mathbf{C}$. The hermitian or symplectic factors not of the form $S_1(\mathbf{C})$, $A_2(\mathbf{C})$ or $A_3(\mathbf{C})$ will be referred to as the non-trivial ones of their kind.

(e) The remaining two Cartan factors are the *exceptional factors*, $B_{1,2}$, the 1×2 matrices over the complex octonions, and M_3^8, the complexification of the real JB-algebra N_3^8. We remark that $B_{1,2}$ can be realised as a subtriple of M_3^8 via the map

$$(x_1, x_2) \mapsto \begin{pmatrix} 0 & x_1 & x_2 \\ \overline{x_1} & 0 & 0 \\ \overline{x_2} & 0 & 0 \end{pmatrix}.$$

We refer to the various classes of Cartan factors as the *generic types.*

2.11.4 Let M be a Cartan factor.

(a) The *rank* of M is the cardinality of a maximal orthogonal family of minimal tripotents in M. Let $1 \leq n < \infty$. Then $R_{n,m}$ has rank n, whenever $n \leq m \leq \infty$. Also, $S_n(\mathbf{C})$, $A_{2n}(\mathbf{C})$ and $A_{2n+1}(\mathbf{C})$ have rank n. All spin factors have rank two, as does $B_{1,2}$, whilst M_3^8 has rank three.

(b) The *elementary ideal* of M, denoted by $K(M)$, is the norm closed ideal of M generated by its minimal tripotents. The ideal $K(M)$ is the smallest non-zero norm closed ideal of M in the norm closed linear span of the minimal tripotents of M, and satisfies $K(M)^{**} = M$. We have $K(M) = M$ if and only if M has finite rank [BuChZa1][BuChZa2].

2.11.5 Let A be a JB*-triple. Then A is said to be

(a) *postliminal* if $K(M) \subset \pi(A)$, for every Cartan factor representation $\pi : A \rightarrow M$;

(b) *liminal* if $\pi(A) = K(M)$, for every Cartan factor representation $\pi : A \rightarrow M$.

The postliminal JB*-triples are exactly those whose bidual is a type I JBW*-triple [BuChZa1, 3.3].

2.11.6 Let M be a JBW*-triple. Given $\rho \in \partial_e(M_{*,1})$, let C_ρ denote the weak* closed ideal of M generated by the minimal tripotent $s(\rho)$ of M. It follows from (2.10.8(d)) and [Ho1, 1.8], that C_ρ is a Cartan factor for each $\rho \in \partial_e(M_{*,1})$. Let $\rho, \tau \in \partial_e(M_{*,1})$. Since $C_\rho \cap C_\tau$ is a weak* closed ideal in the factors C_ρ and C_τ, we have $C_\rho \perp C_\tau$ or $C_\rho = C_\tau$.

As ρ ranges over $\partial_e(M_{*,1})$, let M_{at} denote the ℓ^∞-sum of the distinct C_ρ's that arise. Then M_{at} is the smallest weak* closed ideal of M containing all minimal tripotents of M, whereas $M_{at}^\perp$ contains no minimal tripotents. M_{at} is called the *atomic part* of M and M is said to be an *atomic* JBW*-triple if $M = M_{at}$.

Via (2.10.8(d)) we arrive at the following decompositions.

Theorem 2.11.7 ([FrRu5, Proposition 2, Theorem D])

Let M be a JBW-triple. Then*

(a) $M = M_{at} \oplus M_{at}^\perp$;

(b) *M is atomic if and only if it is an ℓ^∞-sum of Cartan factors.*

2.11.8 By the type I classification of JBW*-triples due to Horn [Ho2], type I JBW*-triples are precisely l^∞ sums of either of the following forms.

(a) $A\bar{\otimes}C$, where A is an abelian von Neumann algebra and C is a Cartan factor realised as a JW*-subtriple of $B(H)$, for some Hilbert space H. Here bar denotes weak* closure inside the usual von Neumann tensor product $A \otimes B(H)$. We note that if C is finite dimensional then $A\bar{\otimes}C = A \otimes C$.

(b) $A \otimes C$, where A is an abelian von Neumann algebra and C is an exceptional Cartan factor.

2.11.9 A JBW*-triple is said to be *hermitian* (respectively, *symplectic*, *rectangular*, or of *spin type*), if it is of the form of an ℓ^∞-sum $\sum A_i \otimes C_i$, where each A_i is an abelian von Neumann algebra and each C_i is an hermitian (respectively, *symplectic*, *rectangular*, *spin type*) Cartan factor.

2.11.10 Finally, we note that the weak* closed inner ideals of type I JBW*-triples are themselves of type I [BuPe, 4.2].

2.12 Atomic and Cartan Factor Representations

2.12.1 Let A be a JB*-triple. By a *Cartan factor representation* of A we mean a (triple) homomorphism, $\pi : A \to M$, where M is a Cartan factor and $\pi(A)$ is weak* dense in M. The *rank* of such a Cartan factor representation is defined to be the rank of M.

2.12.2 Given $\rho \in \partial_e(A_1^*)$, where A is a JB*-triple, let C_ρ be the weak* closed Cartan factor ideal of A^{**} generated by $s(\rho)$, (see (2.11.6)), and let $P_\rho : A^{**} \to C_\rho$ be the natural projection. Let $\pi_\rho : A \to C_\rho$ denote the restriction of P_ρ to A. Let bar denote weak* closure. Since P_ρ is a weak* continuous homomorphism and as $C_\rho = P_\rho(A^{**}) \subset \overline{\pi_\rho(A)} \subset C_\rho$, so that $\pi_\rho(A)$ is weak* dense in C_ρ, we have that $\pi_\rho : A \to C_\rho$ is a Cartan factor representation of A and P_ρ is its unique weak* continuous extension.

Further, putting $u = s(\rho)$, since $A_2^{**}(u)$ is contained in C_ρ we have that

$$P_2^u P_\rho = P_\rho P_2^u = P_2^u,$$

so that $\rho = \rho \circ P_\rho$.

Moreover, given $\rho, \tau \in \partial_e(A_1^*)$ we have the following equivalent statements,

$$(a) \quad s(\tau) \in C_\rho; \qquad (b) \quad C_\tau = C_\rho; \qquad (c) \quad P_\tau = P_\rho;$$

each of which implies that $\pi_\tau = \pi_\rho$.

Now let $P_{at} : A^{**} \to A_{at}^{**}$ be the natural projection. Then

$$P_{at} = \sum P_\rho : A^{**} \to \sum C_\rho \qquad (a \mapsto \sum P_\rho(a)),$$

where the sum is an ℓ^∞-sum over the distinct C_ρ's arising.

Correspondingly, the restriction to A is the map

$$\pi_{at} = \sum \pi_\rho : A \to \Big(\sum C_\rho\Big)_\infty \qquad (a \mapsto \sum \pi_\rho(a)),$$

called the *atomic representation* of A. If $a \in A$ with $\pi_{at}(a) = 0$ then, by these remarks, $\tau(a) = 0$ for all $\tau \in \partial_e(A_1^*)$, giving $a = 0$. Thus, π_{at} is faithful and gives rise to the following Gelfand Naimark Theorems of Friedman and Russo.

Theorem 2.12.3 ([FrRu5, Theorem1])
Every JB-triple is isometrically isomorphic to a JB*-subtriple of an l^∞-sum of Cartan factors.*

Corollary 2.12.4 ([FrRu5, Corollary 1])
Every JB-triple is isometrically isomorphic to a subtriple of*

$$B(H) \oplus C(X, M_3^8),$$

where X is some compact hyperstonean space and H is a complex Hilbert space.

Corollary 2.12.5 ([FrRu5, Corollary 2])
Every JB-triple is isometrically isomorphic to a subtriple of a JB*-algebra.*

The next theorem is one major consequence of the Gelfand Naimark theorem for JB*-triples.

Theorem 2.12.6 ([FrRu5, Theorem 2])
Every JB-triple A contains a unique norm closed ideal J such that A/J is isomorphic to a JC*-triple and J is purely exceptional in the sense that every homomorphism of J into a JC*-triple is zero.*

2.12.7 Essentially, a Cartan factor representation is induced by extreme points in the way described in (2.12.2) [BuChZa2, 3.2].

2.13 The Stone-Weierstrass Theorem

2.13.1 The Stone-Weierstrass Theorem, a generalisation of the Weierstrass approximation Theorem, was proved by M.H. Stone in 1937. It is the driving force behind much of the thesis. We state a non-unital version.

Theorem 2.13.2 (Stone-Weierstrass)

Let X be a locally compact Hausdorff space and let A be a closed subalgebra of $C_0(X)$ such that

(a) *for each x in X there exists $f \in A$ such that $f(x) \neq 0$;*

(b) *A separates the points of X;*

(c) *if $f \in A$ then $\bar{f} \in A$.*

Then $A = C_0(X)$.

2.13.3 In the context of C*-algebras a more elegant statement is possible. Let A be a commutative C*-algebra. Then the Gelfand map is an isometric *-isomorphism of A onto $C_0(P(A))$, where $P(A)$ is given the weak* topology. Conversely, via the usual evaluation map, every locally compact Hausdorff space X can be identified with $P(C_0(X))$. This leads to the following reformulation of (2.13.2).

Theorem 2.13.4

Let A be a commutative C-algebra. Let B be a C*-subalgebra of A such that B separates the points of $P(A) \cup \{0\}$. Then $B = A$.*

2.13.5 At the time of writing, the extension of (2.13.4) to all C*-algebras, the *Stone-Weierstrass Conjecture*, remains an open problem. Progress has been made towards generalising (2.13.4), for instance, by weakening the constraint of commutativity, (Kaplansky successfully proved the result for postiliminal

C*-algebras [Kap]), or by enlarging of the set of functionals under consideration.

2.13.6 As JB*-triples and JB*-algebras are generalisations of C*-algebras, it is natural to consider a meaningful version of the Stone-Weierstrass Theorem for these structures. Due to the lack of positivity and thus of pure states in JB*-triples, pure states are replaced by the extreme points of the dual ball. Subtriples take the role given to subalgebras. In this way, the JB*-triple version of the Stone-Weierstrass conjecture is as follows.

Conjecture 2.13.7

Let A be a JB-triple and let I be a subtriple of A such that I separates $\partial_e(A_1^*) \cup \{0\}$. Then A=I.*

The JB*-triple counterpart of (2.13.4) has been shown to hold [FrRu3, 3.4]. Sheppard has obtained extensions of the Stone-Weierstrass Theorem, in particular, to postliminal JB-algebras and postliminal JB*-triples [Shep2, 4.11], [Shep3, 5.5]. We also note the following, which we use later.

Theorem 2.13.8 ([Shep3, 3.3])

Let M be a Cartan factor. Let I be a weak closed inner ideal of M such that $\rho(I) \neq 0$ for all $\rho \in \partial_e(M_{*,1})$. Then $I = M$.*

2.13.9 Broadly, the Stone-Weierstrass Theorem can be viewed as the key to using extreme functionals as a dual object. It is in this context that it is applied within this work, specifically in relation to the inner ideal structure of a JB*-triple.

Chapter 3

Inner Ideals in Type I JW*-Triples

3.1 Introduction

Our objective in this chapter is to undertake the groundwork that is necessary for subsequent chapters. The material presented here deals with certain aspects of the inner ideal structure of JW*-triples, in particular those of type I, and represents the first major collection of original work.

Taking [EdRü9] and [DinTi] as our inspiration, we examine the centroid of a JB*-triple and utilise the centroid to investigate the "generic" type of JW*-triples and their inner ideals. Specifically, we consider a homogeneous summand of a type I JW*-triple, that is $M \cong A \otimes C$, where A is an abelian von Neumann algebra and C is a Cartan factor of rank at least two, and establish that the generic type of M is almost completely determined by that of a weak* inner ideal I; the proviso here is that I has no type I_1 part. By this we mean that, under these prescriptions, M is hermitian precisely when I is hermitian, etcetera. Through summation, this is then extended to type I JW*-triples N, under the condition that neither N or its inner ideal, K say, have type I_1 part, and that N is generated as a weak* closed ideal by K.

An attempt is then made to describe the weak* inner ideals of particular JW*-triples, namely universally reversible JW*-algebras. Our motivation is the role these structures occupy in many of the arguments of the next chapter. Initially, our intention was to extend [EdRüVa2] to formulate a general resolution, that is, to prove that every weak* inner ideal of a universally reversible JW*-algebra M has a unique representation in the form $eM\phi(e)$, where e is a projection in $W^*(M)$ with ϕ-invariant central support. This proved not to be possible, the symplectic part forming an obstacle. So the restriction is made that M has no non-zero symplectic part. This is sufficient for our subsequent needs.

Finally, an account of Cartan factor representation theory is offered, as this represents a vital tool in what follows.

3.2 The Centroid of a JB*-Triple

3.2.1 Let A be a JB*-triple. The *centroid* of A is the set of $T \in B(A)$ satisfying $T(\{a\ b\ c\}) = \{Ta\ b\ c\}$ for all $a, b, c \in A$, and will be denoted by $C_e(A)$. Equivalently, for $T \in B(A)$, the condition that $T \in C_e(A)$ is characterised, separately, by each of the following conditions.

$$(i) T(\{a\,b\,c\}) = \{a\,b\,Tc\};\ (ii) TD(a,b) = D(a,b)T;\ (iii) TD(a,a) = D(a,a)T.$$

The centroid of a JB*-triple was introduced and studied in [DinTi] and developed further in [EdRü9] and [EdLoRü].

3.2.2 It is immediate from the definition, that $C_e(A)$ is a Banach subalgebra of $B(A)$. Further, given $S, T \in C_e(A)$ and $a, b, c \in A$, we have

$$ST\{a\ b\ c\} = S\{Ta\ b\ c\} = \{Ta\ b\ Sc\} = TS\{a\ b\ c\}.$$

Via functional calculus ("cube roots"), it follows that $ST = TS$. Hence $C_e(A)$ is a commutative Banach subalgebra of $B(A)$.

Let $T \in C_e(A)$. By [DinTi, 2.6], there is a unique $T^\# \in C_e(A)$ satisfying $T^\#\{a\ b\ c\} = \{a\ Tb\ c\}$, for all $a, b, c \in A$. The map $T \mapsto T^\#$ is a conjugate linear involution on $C_e(A)$. Moreover, given $T \in C_e(A)$ and $x \in A$, choose $y \in A$ such that $x = \{y\ y\ y\}$. Then

$$\|T^\# x\| = \|\{y\ Ty\ y\}\| \leq \|T\|\|y\|^3 = \|T\|\|x\|,$$

and $\|TT^\# x\| = \|\{y\ (T^\# Ty)\ y\}\| \leq \|T^\# T\|\|x\|$, giving $\|T\| = \|T^\#\|$ and $\|TT^\#\| = \|T\|^2$. Hence, $C_e(A)$ is an abelian C*-algebra with involution $T \mapsto T^\#$.

We remark that $C_e(A)$ is equal to the (Banach space) centralizer of A [DinTi, 2.8] and can be shown to be isomorphic to the bounded continuous functions on $Prim(A)$ [Be, 3.13].

Lemma 3.2.3

Let $T \in C_e(A)$, where A is a JB-triple. Let I be a norm closed inner ideal of A and let u be a tripotent of A. Then*

(a) $T(I) \subset I$ and $T(I)$ is an inner ideal of A;

(b) $kerT$ is a norm closed ideal of A;

(c) if $T(u) = 0$ then $A_2(u) + A_1(u) \subset kerT$.

Proof

(a) We have $T(I) = T(\{I\ I\ I\}) = \{I\ T^{\#}(I)\ I\} \subset I$, so

$$\{T(I)\ A\ T(I)\} = T^2(\{I\ A\ I\}) \subset T^2(I) \subset T(I).$$

(b) Since $T(\{kerT\ A\ A\}) = \{T(kerT)\ A\ A\} = 0$, $\{kerT\ A\ A\}$ is contained in $kerT$. Thus, being norm closed, $kerT$ is an ideal of A by (2.10.5).

(c) This follows from (b), as $A_2(u) = P_2^u(A), A_1(u) = 2(D(u,u) - P_2^u)(A)$ both lie in the ideal generated by u. $\square$

3.2.4 Henceforth we will concentrate on JBW*-triples. Recall that if $a \in M$, where M is a JBW*-algebra, then $T_a : M \to M$ denotes the map given by $T_a(x) = a \circ x$. If M is a JBW*-algebra, then $C_e(M) = \{T_a : a \in Z(M)\}$ [DinTi, 3.5], so that $C_e(M) \cong Z(M)$ via $T \mapsto T(1)$.

3.2.5 Let M be a JBW*-triple. Let $P : M \to M$ be an M-projection. Then M is the ℓ^∞-sum of the orthogonal weak* closed ideals $P(M)$ and $(I-P)(M)$. We have, for $a, b, c \in M$, $P\{a\ b\ c\} = \{Pa\ Pb\ Pc\} = \{Pa\ b\ c\} = \{a\ Pb\ c\}$, so that $P^{\#} = P \in C_e(M)$.

In the following, part (a) is a variation of [EdRü9, 3.4(ii)] and part (b) is a direct consequence of [EdRü9, 3.8].

Proposition 3.2.6

Let M be a JBW-triple and let u be a complete tripotent of M. Let I be a weak* closed inner ideal of M and let J be the weak* closed ideal of M generated by I. Then*

*(a) the map, $\psi : C_e(M) \to Z(M_2(u))$, given by $\psi(T) = T(u)$ is a surjective *-isomorphism;*

*(b) the restriction map $C_e(J) \to C_e(I)$ is a surjective *-isomorphism.*

Proof (of (a))

Let $T, S \in C_e(M)$. The restriction of T to the inner ideal $M_2(u)$ lies in the centroid of the JBW*-algebra $M_2(u)$, so that $T(u) \in Z(M_2(u))$ by the result of [DinTi] mentioned in (3.2.4). Furthermore,

$$ST(u) = ST\{u\ u\ u\} = \{S(u)\ u\ T(u)\} = S(u) \circ T(u),$$

and $T^{\#}(u) = T^{\#}\{u\ u\ u\} = \{u\ T(u)\ u\} = (T(u))^*$. Therefore, ψ is a *-homomorphism.

Since $T(u) = 0$ implies by (3.2.3(c)) that $M = M_2(u) + M_1(u) \subset kerT$, and thus that $T = 0$, ψ is injective. Finally, let z be a projection in $Z(M_2(u))$. Let J be the weak* closed ideal of M generated by z, and let $P : M \to J$ be the natural (M-)projection. We have $P \in C_e(M)$, (3.2.5), and, by [Ho1, 4.2] and its proof, $P = P_2^z + P_1^z$, so that $z = P(u) \in \psi(C_e(M))$. Hence, $Z(M_2(u)) \subset \psi(C_e(M))$, implying that ψ is surjective. □

We can use (3.2.6) to give a description of the centroid of a JBW*-triple that extends [DinTi, 3.5].

Corollary 3.2.7

Let M be a JBW-triple with complete tripotent u. For each $a \in Z(M_2(u))$ put $T_a = D(a, u)(2 - P_2^u)$. Then $C_e(M) = \{T_a : a \in Z(M_2(u))\}$. The map $Z(M_2(u)) \to C_e(M)$, $(a \mapsto T_a)$, is the inverse *-isomorphism of that given in (3.2.6).*

Proof

Let $S \in C_e(M)$. Then, by (3.2.6), there exists a unique $a \in Z(M_2(u))$ such that $S(u) = a$. Since $P_0^u = I - 2D(u,u) + P_2^u = 0$, for each $x \in M$ we have $x = 2\{uux\} - P_2^u(x)$, so that

$$\begin{aligned} S(x) &= 2\{S(u)\ u\ x\} - S(P_2^u(x)) = 2D(a,u)(x) - D(a,u)P_2^u(x) \\ &= D(a,u)(2 - P_2^u)(x). \end{aligned}$$

Thus $S = D(a,u)(2 - P_2^u)$.

To see the final statement, let $a \in Z(M_2(u))$. By (3.2.6), there is a unique $T \in C_e(M)$ such that $a = T(u)$. The above argument now implies that $T = D(a,u)(2 - P_2^u)$. □

A variation of [EdLoRü, 6.6] is next.

Corollary 3.2.8

Let M be a JBW-triple and let $T \in C_e(M)$. Then $T = T^2$ if and only if T is an M-projection.*

Proof

As stated in (3.2.5), every M-projection lies in the centroid.

Conversely, let $T = T^2$. Since $C_e(M)$ is an abelian C*-algebra, (actually a W*-algebra), T is self adjoint and so is a projection of $C_e(M)$. Thus, by (3.2.6) and (3.2.7), given a complete tripotent u of M there is a central projection z of $M_2(u)$ such that

$$T = D(z,u)(2 - P_2^u) = D(z,z)(2 - P_2^u) = 2D(z,z) - P_2^z = P_2^z + P_1^z,$$

that is, the M-projection onto the weak* closed ideal of M generated by z.□

3.2.9 Consider a JBW*-triple $M = A\overline{\otimes}C$, where A is an abelian von Neumann algebra and C is a Cartan factor. Let u be a complete tripotent of C. Then $1 \otimes u$ is a complete tripotent of M and we have

$$M_2(1 \otimes u) = A\overline{\otimes}C_2(u).$$

Since $C_2(u)$ is a type I JBW-algebra factor, it follows that

$$Z(M_2(1 \otimes u)) = A \otimes u.$$

Let $T \in C_e(M)$. Then, by the above together with (3.2.7), there is a unique $a \in A$ such that

$$T = D(a \otimes u, 1 \otimes u)(2I - P_2^{1\otimes u}) = D(a \otimes u, 1 \otimes u)(2I - I \otimes P_2^u).$$

Calculating, we therefore find that $T = T_a \otimes D(u, u)(2I - P_2^u) = T_a \otimes I$, that is, there is a unique $a \in A$ such that $T(x \otimes y) = ax \otimes y$.
Conversely, every function of this form lies in $C_e(M)$.
Parts (a) and (b) of the following statement are immediate from the above remarks, as is part (c), in conjunction with (3.2.6).

Proposition 3.2.10

Let $M = A\overline{\otimes}C$, where A is an abelian von Neumann algebra and C is a Cartan factor. Then

*(a) $a \mapsto T_a \otimes I$, is a *-isomorphism from A onto $C_e(M)$;*

(b) each weak closed ideal of M is of the form $zA\overline{\otimes}C$ for some unique projection z of A;*

*(c) $Z(M_2(u))$ is *-isomorphic to A, for every complete tripotent u in M.*

3.3 The Type of an Inner Ideal

3.3.1 In this section, in tensor product notation of the form $A\overline{\otimes}C$, the left hand side will always represent an abelian von Neumann algebra. Recall Horn's type I structure theorem [Ho2, 1.7].

Theorem 3.3.2
If M is a type I JBW-triple then $M \cong \left(\sum A_i\overline{\otimes}C_i\right)_\infty$, for (up to isomorphism) distinct Cartan factors C_i. Moreover such a decomposition is unique.*

3.3.3 The uniqueness above is not stated in [Ho2] but is implicit and can be seen as follows. Suppose there is a surjective linear isometry

$$\pi : \left(\sum A_i\overline{\otimes}C_i\right)_\infty \to \left(\sum B_j\overline{\otimes}D_j\right)_\infty,$$

where the C_i are distinct (up to isomorphism) Cartan factors, and similarly for the D_j. Fix i_0. By (3.2.10(b)) there exist projections $z_j \in B_j$ such that $\pi(A_{i_0}\overline{\otimes}C_{i_0}) = \sum z_j B_j\overline{\otimes}D_j$. Pick a j_0 such that $z_{j_0} \neq 0$. Thus, again by (3.2.10(b)), there is a non-zero weak* closed ideal J of A_{i_0} such that $\pi(J\overline{\otimes}C_{i_0}) = z_{j_0}B_{j_0}\overline{\otimes}D_{j_0}$, so that $C_{i_0} \cong D_{j_0}$, by [Ho3, §4]. By the uniqueness of the D_j's, this implies that $z_j = 0$ for all $j \neq j_0$. Thus,

$$\pi(A_{i_0}\overline{\otimes}C_{i_0}) = z_{j_0}B_{j_0}\overline{\otimes}D_{j_0},$$

giving, $\left(\sum_{i\neq i_0} A_i\overline{\otimes}C_i\right)_\infty \cong \left(\sum_{j\neq j_0} B_j\overline{\otimes}D_j\right)_\infty + (1-z_{j_0})B_{j_0}\overline{\otimes}D_{j_0}$. Now, the above argument, starting with the j_0-summand on the right, contradicts the uniqueness of the C_i if $z_{j_o} \neq 1$. That is, there is a unique j_0 such that

$$\pi(A_{i_0}\overline{\otimes}C_{i_0}) = B_{j_0}\overline{\otimes}D_{j_0},$$

in which case $A_{i_0} \cong B_{j_0}$, by taking centroids, and $C_{i_0} \cong D_{j_0}$ as above. In addition, by symmetry the indexing sets are cardinally equivalent.

3.3.4 We remark that up to (Jordan) *-isomorphism there is one and only one

(a) exceptional JBW*-algebra factor;

(b) spin factor of given dimension λ (possibly infinite);

(c) type I_n JW*-algebra factor of given Cartan type and rank n, where n is a cardinal greater than or equal to three.

Thus, if C and D are linearly isometric (i.e. triple isomorphic) JBW*-algebra type I factors then, since C and D must be of the same Cartan type, rank and dimension, we must have that C and D are *-isomorphic (as Jordan algebras).

More generally, we have the following.

Proposition 3.3.5

Linearly isometric type I JBW-algebras are *-isomorphic.*

Proof

Suppose that $A\overline{\otimes}C \cong B\overline{\otimes}D$, as triples, where A and B are abelian von Neumann algebras and C and D are type I JBW*-algebra factors. By remarks in (3.3.3), we have $A \cong B$ and $C \cong D$ as triples. By the Banach-Stone Theorem [DuSch, p442] in the first case, and by (3.3.4) in the second case, there exist *-isomorphisms $\pi_1 : A \to B$ and $\pi_2 : C \to D$. If C, and hence D, are exceptional, and so are finite dimensional, the natural map

$$\pi_1 \otimes \pi_2 : A \otimes C \to B \otimes D$$

gives the required *-isomorphism.

Otherwise, C and D are JW*-algebras and π_2 extends to a *-isomorphism $\hat{\pi}_2 : W^*(C) \to W^*(D)$ (of W*-algebras). This induces a *-isomorphism between von Neumann algebras [KaRi2],

$$\pi_1 \overline{\otimes} \hat{\pi}_2 : A \overline{\otimes} W^*(C) \to B \overline{\otimes} W^*(D),$$

which, by restriction, sends $A\overline{\otimes}C$ onto $B\overline{\otimes}D$.

The general case is now immediate from the homogeneous decomposition of type I JBW*-algebras together with (3.3.3). □

Lemma 3.3.6

Let M be a JBW-triple with complete tripotent u. Suppose that, for ideals I_i of $M_2(u)$, $M_2(u)$ is the ℓ^∞-sum $\sum I_i$. Put $u = \sum u_i$, where $u_i \in I_i$, for each i and let $J_i = \mathcal{T}^w(I_i)$, for each i. Then*

(a) $M = \big(\sum J_i\big)_\infty$;

(b) $M_2(u) = \sum (J_i)_2(u_i)$ and each u_i is a complete tripotent of J_i.

Proof

(a) This is immediate from (2.10.8(c)).

(b) For each i, u_i is the identity element of I_i (regarded as a JBW*-algebra) and $(J_i)_2(u_i) = (I_i)_2(u_i) = I_i$, by (2.10.8(b)). □

3.3.7 From this point on we concentrate on JW*-triples of type I. Let M be a JW*-triple with complete tripotent u such that $M_2(u) \cong A\overline{\otimes}C$, where C is a (special) Cartan factor; thus C is isomorphic to a type I JW*-algebra factor. As C varies, the possiblities for M, up to isomorphism, are as set out in the table below. Here n represents a cardinal number, possibly infinite, and we mean ℓ^∞-sum. We make tacit use of (3.2.10(c)) throughout.

	C	M	
[Ho2, 4.1]	V_n $(n \neq 3, 5)$	$A\overline{\otimes}V_n$	
[Ho2, 5.5]	$R_{n,n}$	$\sum A_m\overline{\otimes}R_{n,m}$, certain $m \geq n$	(includes V_3)
[Ho2, 6.1]	A_{2n}	$A\overline{\otimes}A_{2n} \oplus A'\overline{\otimes}A_{2n+1}$	(includes V_5)
[Ho2, 7.1]	S_n	$A\overline{\otimes}S_n$	(includes V_2).

3.3.8 If M is a type I JW*-triple with complete tripotent u, then via (2.11.10) and (3.3.2), $M_2(u) = \big(\sum I_i\big)_\infty$, where $I_i \cong A_i \otimes C_i$, for each i and where the C_i are distinct (special) Cartan factors. By (3.3.6) we have $M = \big(\sum J_i\big)_\infty$ where each J_i has a complete tripotent u_i such that $(J_i)_2(u_i) \cong A_i\overline{\otimes}C_i$.

3.3.9 Let M be a JW*-triple such that $M \cong A\overline{\otimes}C$, where C is a special Cartan factor. Let u be a complete tripotent of M. By (3.3.7) and (3.3.8), C varies the possibilities for $M_2(u)$, up to isomorphism, as follows. Here, as before, n is a possibly infinite cardinal.

C	$M_2(u)$
V_n	$A\overline{\otimes}V_n$
$R_{n,m}$ $(n \leq m)$	$A\overline{\otimes}R_{n,n}$
A_{2n}	$A\overline{\otimes}A_{2n}$
A_{2n+1}	$A\overline{\otimes}A_{2n}$
S_n	$A\overline{\otimes}S_n$.

Since the complete tripotent u was unspecified, it follows from the table that $M_2(u) \cong M_2(v)$ for any other complete tripotent v of M. Since $1 \otimes w$ is a complete tripotent of $A\overline{\otimes}C$, whenever w is a complete tripotent of C, we therefore have $M_2(u) \cong A\overline{\otimes}C_2(w)$, for every complete tripotent w of C.

Proposition 3.3.10

Let M be a type I JW-triple. Let u and v be complete tripotents of M. Then $M_2(u) \cong M_2(v)$.*

Proof

We have $M = \left(\sum J_i\right)_\infty$, where each $J_i \cong A_i \otimes C_i$ for a certain abelian von Neumann algebras A_i and Cartan factors C_i. We also have $u = \sum u_i$ and $v = \sum v_i$, where u_i and v_i are complete tripotents of J_i, for each i. By (3.3.9), $(J_i)_2(u_i) \cong (J_i)_2(v_i)$, for each i. Therefore,

$$M_2(u) = \sum (J_i)_2(u_i) \cong \sum (J_i)_2(v_i) = M_2(v). \qquad \square$$

3.3.11 Let M be a Cartan factor with a tripotent u. Then we say that u is of *rank* n if $M_2(u)$ is of rank n. Let C be an hermitian, a rectangular or symplectic Cartan factor of rank at least two. Let u be a complete tripotent of C of rank two. The structure of $C_2(u)$ relative to the type of C is as follows.

C :	hermitian;	rectangular;	symplectic.
$C_2(u)$:	V_2	V_3	V_5.

Lemma 3.3.12

Let M be a JW-triple such that $M = A\overline{\otimes}C$, where A is an abelian von Neumann algebra and C is an hermitian, a rectangular or symplectic Cartan factor of rank at least two. Let u be a tripotent of M such that $M_2(u)$ is a type I_2 JW*-algebra. Then $M_2(u) \cong zA \otimes V$, where z is a central projection of A and V is a spin factor such that*

(a) M is hermitian if and only if $V = V_2$;

(b) M is rectangular if and only if $V = V_3$;

(c) M is symplectic if and only if $V = V_5$.

Proof

Choose a tripotent v of M such that $u \leq v$. By (3.3.10) together with (3.3.5) there is a complete tripotent w of C and a *-isomorphism from $M_2(v)$ onto $A \otimes C_2(w)$. Therefore, by (3.3.9) we may pass to the above co-domain and suppose that C is a type I JW*-algebra factor and that u is a projection in $A\overline{\otimes}C$.

In which case, since uMu is type I_2, there exist projections $e_1, e_2 \in M$ such that $e_1 \sim e_2$ and $u = e_1 + e_2$. We have $c(e_1) = c(e_2) = c(u) = z \otimes 1$, for some central projection z in A.

Let f_1, f_2 be orthogonal minimal projections in C. Then $z \otimes f_1$ and $z \otimes f_2$ are abelian projections in M with central supports equal to $z \otimes 1$. By (1.8.5(b)), we have that $e_i \sim z \otimes f_i$, for $i = 1, 2$. Thus, since $sup(u, 1 \otimes f_1 + f_2)$ is finite [Top1, Theorem 12], (1.8.7) implies that there is a symmetry s in M such that $sus = s(e_1 + e_2) = z \otimes (f_1 + f_2)$. This gives

$$uMu \cong s(uMu)s = zA \otimes (f_1 + f_2)C(f_1 + f_2).$$

The desired conclusion is now immediate from (3.3.11). $\square$

3.3.13 We now show that the "generic type" of a type I JW*-triple (mostly) determines, and is determined by, that of a weak* closed inner ideal.

Theorem 3.3.14

Let M be a JW-triple such that $M \cong A\overline{\otimes}C$, where A is an abelian von Neumann algebra and C is a Cartan factor of rank at least two. Let I be a weak* closed inner ideal of M such that I has no type I_1 part. Then M is hermitian (respectively rectangular, symplectic) if and only if I is hermitian (respectively rectangular, symplectic).*

Proof

We have $I = \left(\sum I_i\right)_\infty$, where, for each i, $I_i \cong A_i\overline{\otimes}C_i$, with each A_i an abelian von Neumann algebra and C_i a Cartan factor of rank at least two. For each i choose a tripotent $u_i \in I_i$ such that $(I_i)_2(u_i)$ is type I_2. We have that $M_2(u_i) = (I_i)_2(u_i)$ is type I_2 for each i. By definition I is hermitian if and only if C is hermitian. Thus, the above equality, together with (3.3.12(a)), implies that M is hermitian if and only if I is hermitian. The remaining claims, those in parenthesis, are obtained in the same way using (3.3.12(b) and (c)). $\square$

Corollary 3.3.15

Let I be a weak closed inner ideal in a type I JW*-triple M such that neither I nor M have type I_1 part. If M is hermitian (respectively rectangular, symplectic) then I is hermitian (respectively rectangular, symplectic). The converse is true if I generates M as a weak* closed ideal.*

Proof

Now, M is the ℓ^∞-sum $\sum J_i$, where, for each i, $J_i \cong A_i \otimes C_i$ for some abelian von Neumann algebra A_i and Cartan factor C_i of rank at least two. By (2.10.16), we have $I = \left(\sum J_i \cap I\right)_\infty$.

If M is hermitian then each of the J_i is hermitian. In which case, each non-zero $J_i \cap I$ is hermitian by (3.3.14), implying that I is hermitian.

Conversely, suppose that I is hermitian and generates M as a weak* closed ideal. By (2.10.16), the latter condition implies that $J_i \cap I$ is non-zero for all i. Now (3.3.14) implies that each J_i, and therefore M, is hermitian.

Directly similar arguments, through (3.3.14), give the remaining cases. $\square$

3.3.16 We now turn to type I_1 JW*-triples, first recalling some particular features of type I rectangular JW*-triples.

(a) Let W be a von Neumann algebra with partial isometry u and let $l = uu^*$ and $r = u^*u$ so that $W_2(u) = lWr$. We have $lu = u = ur$ and a surjective linear isometry, hence a triple isomorphism, $lWr \xrightarrow{\sim} rWr$, $(x \mapsto u^*x)$.

(b) Let C be a rectangular Cartan factor. Up to isometry (see (2.11.3(a))) we may suppose that $C = B(H)e$, where H is a complex Hilbert space and e is a projection in $B(H)$. Given a tripotent u in C we have $u^*u \leq e$ so that with $l = uu^*$ and $r = u^*u$, part (a) gives

$$C_2(u) = lCr = lB(H)er = lB(H)r \cong rB(H)r.$$

(c) Let M be a type I rectangular JW*-triple. By part (b) (and [Ho2]), $M \cong \left(\sum A_i \overline{\otimes} B(H_i)e_i\right)_\infty = W(\sum 1_i \otimes e_i) = We$, where the A_i are abelian von Neumann algebras, the H_i are complex Hilbert spaces, and W is the von Neumann algebra, $W = \left(\sum A_i \overline{\otimes} B(H_i)\right)_\infty$, with $e = \sum 1_i \otimes e_i$.

We shall need the following two results in section (3.5).

Proposition 3.3.17

The following are equivalent for a JBW-triple M.*

(a) M is type I_1.

(b) $M \cong We$, for some abelian projection e in a type I von Neumann algebra W.

(c) $M_2(u)$ is abelian for all tripotents u in M.

Proof

(a) $\Rightarrow$ (b) This follows from (3.3.16(c)). If M is type I_1 then, in the notation of (3.3.16(c)), the projections e_i are minimal so that e is abelian.

(b) $\Rightarrow$ (c) This follows from (3.3.16(b)). If u is a tripotent in We, where e is an abelian projection in a type I von Neumann algebra W, (3.3.16(b)) implies that $(We)_2(u) \cong rWr$, where $r = u^*u \leq e$.

Finally, $(c) \Rightarrow (a)$ follows by definition. $\square$

Lemma 3.3.18

Let M be a type I_1 JW-triple. Then M^{**} is a type I_1 JW*-triple and every JW*-subtriple of M is type I_1.*

Proof

Let u be a complete abelian tripotent in M. Then $P_0^u(M) = 0$ and $P_2^u(M)$ is abelian. Taking weak* closures, in the $\sigma(M^{**}, M^*)$ topology, we therefore have $P_0^u(M^{**}) = 0$ and $P_2^u(M^{**})$ is abelian. Thus u is complete and abelian in M^{**} which is consequently type I_1.

Given a tripotent u in a JBW*-subtriple N of M we see that $N_2(u)$ is a JBW*-subalgebra of $M_2(u)$ and is therefore abelian. It follows that N is type I_1 by (3.3.17). $\square$

To conclude this section, we present the following two results, of use in the next chapter.

Proposition 3.3.19

Let M be a JBW-algebra with complete tripotent u such that $M_2(u)$ is isomorphic to a W*-algebra of type I. Then M is isomorphic to a W*-algebra of type I.*

Proof

By definition (see [Ho2, §5]) M is a type I rectangular JBW*-algebra. So, $M \cong We$ for some type I W*-algebra W and projection e in W (3.3.16(c)). Since M is a JBW*-algebra, We has a unitary tripotent v, giving

$$We = vv^*Wev^*v = vv^*Wv^*v \cong rWr,$$

where $r = v^*v$, the isometry being obtained as in (3.3.16(a)). □

Lemma 3.3.20

Let M be a JW-algebra. Let I be a weak* closed inner ideal of M such that $M = \mathcal{T}^w(I)$, the weak* closed ideal of M generated by I. Let I be rectangular without type I_1 part. Then M is *-isomorphic to a W*-algebra of type I.*

Proof

Suppose first that $I \cong A\overline{\otimes}R$, where A is an abelian von Neumann algebra and R is a Cartan factor of rank at least two. Since I is of type I, each tripotent of I is a sum of abelian tripotents of I which, of necessity, must be abelian in M. Therefore, I is contained in the type I part of M. So, since $M = \mathcal{T}^w(I)$, M must be equal to its type I part.

Now let $z \in Z(M)$ such that Mz is homogeneous type I. That is, let $Mz \cong B\overline{\otimes}F$, where F is a type I JW*-algebra factor, and B is an abelian von Neumann algebra. We note that Iz is a weak* closed inner ideal of Mz, and the latter is the weak* closed ideal generated by Iz.

The map $\psi : I \to Iz$ given by $\psi(x) = xz$ is a surjective triple homomorphism. Thus, $Iz \cong I/ker\psi$, which is *-isomorphic to a weak* closed ideal K of I. Therefore, by (3.2.10(b)), $Iz \cong D\overline{\otimes}R$, where D is an abelian von Neumann algebra.

Let u be a rank two tripotent in R and let v be the tripotent of Iz corresponding to $1 \otimes u$ in $D\overline{\otimes}R$. Then $(Mz)_2(v) = (Iz)_2(v) \cong D \otimes M_2(\mathbf{C})$. Therefore, by (3.3.12), F must be a type I von Neumann factor. Hence, Mz is *-isomorphic to a von Neumann algebra.

In the general case, I is the ℓ^∞-sum $\sum I_i$, where for each i, $I_i \cong A_i\overline{\otimes}R_i$, where the A_i are abelian von Neumann algebras and the R_i are rectangular factors of rank at least two. By the first part of the proof, the weak* closed ideal of M generated by I_i is *-isomorphic to a type I von Neumann algebra, for each i. The desired conclusion is now immediate from (2.10.21). $\square$

3.4 Inner Ideals in Universally Reversible JW*-Algebras

3.4.1 Let M be a JW*-algebra. Let ϕ denote the canonical involution on the enveloping W*-algebra, $W^*(M)$, of M. We have that $W^*(M)^\phi = M$ if and only if M is universally reversible. In which case, we let $\mathcal{P}(W^*(M), \phi)$ denote the set of projections e of $W^*(M)$ such that $c(e) \in Z(M)$. By [Aj, Theorem 8] M is continuous if and only if $W^*(M)$ is continuous. Moreover, if M is continuous or is (Jordan) *-isomorphic to a von Neumann algebra, then it is certainly universally reversible. Thus [EdRüVa2, 4.1, 4.2, 4.4] immediately yields the following.

Theorem 3.4.2

Let M be a universally reversible JW-algebra such that M is continuous or is *-isomorphic to a von Neumann algebra. Let ϕ be the canonical involution on $W^*(M)$. Then the map $e \mapsto eM\phi(e)$ is an order preserving bijection from $\mathcal{P}(W^*(M), \phi)$ to the set of weak* closed inner ideals of M.*

3.4.3 Theorem (3.4.2) will be required later. Certain extensions to weak* closed inner ideals in type I JW*-algebras without type I_2 part will also be needed. Our present objective is to progress sufficiently in this direction. In view of (3.4.2), in effect, it only remains to consider hermitian and symplectic JW*-algebras (of type I). Those of hermitian type present little difficulty. However, we have been unable to obtain a general resolution of weak* closed inner ideals of the form expressed in (3.4.2) because of the obstacle created by the symplectic JW*-algebras. Fortunately, this does not prevent us from reaching the ultimate goal of the thesis (or, more exactly, from proving the Inner Stone-Weierstrass theorem), although it does impede it. Nevertheless, an account of the pathological behaviour of symplectic JW*-algebras seems of sufficient interest to warrant inclusion.

3.4.4 Given a universally reversible JW*-algebra M and a projection $e \in M$, the canonical involution ϕ of $W^*(M)$ restricts to an involution on $eW^*(M)e$, with $(eW^*(M)e)^{\phi} = eMe$. Thus, it is necessary, generally, to consider involutions other than the possible canonical ones. It is convenient to make the following definition. A JW*-algebra will said to be *complex* if it is *-isomorphic to a von Neumann algebra.

3.4.5 Let α be an involution on a von Neumann algebra W. Then α is said to be *central* if it fixes each point of $Z(W)$, and is said to be *split* if $1 = z+\alpha(z)$ for some non-trivial central projection z in $Z(W)$. The properties of α are intimately connected to those of the JW*-algebra of α-fixed points, $W^{\alpha} = \{x \in W : \alpha(x) = x\}$. One elementary observation is that if e is a projection of W^{α}, then α is an involution on eWe with $(eWe)^{\alpha} = eW^{\alpha}e$. Another is as follows.

Lemma 3.4.6

Let α be a central involution on a von Neumann algebra W and let e be a projection in W^{α}. Then α is central on eWe.

Proof

This is immediate, since $Z(eWe) = eZ(W) = eZ(W^{\alpha}) = Z(eW^{\alpha}e)$. $\square$

Now, as an involution α on a von Neumann algebra W restricts to an involution on $Z(W)$, the following is immediate from [HaSt, 7.3.4 , 7.3.5] and (3.4.6).

Lemma 3.4.7

Let α be an involution on a von Neumann algebra W. Then α is either central, split or there is a non-trivial projection $z \in Z(W^{\alpha})$ such that α is central on Wz and split on $W(1-z)$.

Lemma 3.4.8

Let α be an involution on a von Neumann algebra W and let e be a projection in W. Then $\alpha(c(e)) = c(\alpha(e))$.

Proof

Since $e \leq c(e)$, $\alpha(e) \leq \alpha(c(e))$ and so $c(\alpha(e)) \leq \alpha(c(e))$. Through this principle, $c(e) = c(\alpha(\alpha(e))) \leq \alpha(c(\alpha(e)))$, giving $\alpha(c(e)) \leq c(\alpha(e))$ and hence equality. $\square$

We next state two key results of Gåsemyr [Gå]. The first is [Gå, 2.2(a), 2.8].

Theorem 3.4.9

Let α be an involution on a von Neumann algebra W. Then

*(a) the type I_2 part of W^α is *-isomorphic to $(A_1 \otimes V_2) \oplus (A_2 \otimes V_3) \oplus (A_3 \otimes V_5)$, where A_1, A_2 and A_3 are abelian von Neumann algebras;*

(b) if W^α has no abelian part, then it generates W.

3.4.10 As a JW*-algebra is universally reversible precisely if its type I_2 part is *-isomorphic to $(A_1 \otimes V_2) \oplus (A_2 \otimes V_3)$, where A_1 and A_2 are abelian von Neumann algebras (1.13.6), the next proposition is immediate from [Gå, 2.2(b)].

Proposition 3.4.11

*Let α be an involution on a von Neumann algebra W such that W^α is universally reversible without abelian part. Then there is a *-isomorphism $\gamma : W \to W^*(W^\alpha)$ such that $\alpha = \gamma^{-1}\phi\gamma$, where ϕ is the canonical involution on W^α.*

The next two results are complementary.

Lemma 3.4.12

Let α be an involution on a von Neumann algebra W. We have

(a) *if α is split then W^α is complex;*

(b) *if W^α is complex and has no abelian part then α is split.*

Proof

(a) Let z be a non-trivial projection in $Z(W)$ such that $\alpha(z) = 1 - z$, and consider the map $\pi : Wz \to W^\alpha$ given by $\pi(x) = x + \alpha(x)$. Since Wz and $\alpha(Wz) = (1 - z)W$ are orthogonal, a straightforward check shows that π is an injective Jordan *-homomorphism. Moreover, given $a \in W^\alpha$ we have that $a = az + a(1 - z) = \pi(az)$. So π is surjective.

(b) Suppose that the stated conditions hold. By [HaSt, 7.4.7] the canonical involution ϕ on W^α is split. Also, (3.4.11) implies that $\alpha = \gamma^{-1}\phi\gamma$ for some *-isomorphism $\gamma : W \to W^*(W^\alpha)$. Therefore, α is split. □

Lemma 3.4.13

Let α be an involution on a von Neumann algebra W. We have

(a) *α is not central if and only if there exists a non-trivial projection z in $Z(W)$ such that $z\alpha(z) = 0$;*

(b) *if W^α has no complex part, then α is central;*

(c) *if α is central and W^α is complex, then W^α is abelian;*

(d) *if W^α has no abelian part and α is central, then W^α has no complex part.*

Proof

(a) If α is not central then by (3.4.7) there is a non-trival projection p in $Z(W^\alpha)$ such that α is split on Wp, from which the required conclusion follows by definition. The converse is immediate from the definition.

(b) If α is not central, we can choose a non-trivial projection $z \in Z(W)$ such that $z\alpha(z) = 0$ by part (a). Then $Wz \cong (z + \alpha(z))W^\alpha$, by the proof (3.4.12(a)).

(c) Since α is invariant on the complement of the abelian part of W, this is immediate from (3.4.12(b)).

(d) Suppose that $W^\alpha z$ is complex for some unique projection z in W^α. If α is central and W^α has no abelian part then, applying (3.4.12(b)) to α on Wz, we conclude that $z = 0$. □

3.4.14 Let A be an abelian von Neumann algebra. Consider $M = A\overline{\otimes}B(H)^\alpha$, where α is a real quaternionic flip and where $dim(H) \geq 3$. The von Neumann algebra generated by M is $W = A\overline{\otimes}B(H)$, and $\phi = id\otimes\alpha$ is an involution on W such that $M = W^\phi$. By (3.4.11), we may suppose that $W = W^*(M)$ and that ϕ is the canonical involution. If α is a real flip the dimension condition can be relaxed to $dim(H) \geq 2$.

Let e be an abelian projection in M. Then $c(e) \in Z(M) = A \otimes 1$, so that $c(e) = z \otimes 1$, for some $z \in A$. Let f be a minimal projection in $B(H)^\alpha$. Then $z \otimes f$ is abelian in M and $c(z \otimes f) = z \otimes 1 = c(e)$. Therefore, by (1.8.5(b)), there is a symmetry s in M such that $ses = z \otimes f$.

(a) If α is a real flip, then f is minimal in $B(H)$ [Shep1, 3.2.3(i)]. Therefore, $e = s(z \otimes f)s$ is abelian in $W^*(M)$. Since ϕ is a central involution, it follows that $eW^*(M)e = Z(eW^*(M)e) = Z(eMe) = eMe$.

(b) Let α be a quaternionic flip. Then $f = p + \phi(p)$, for some minimal projection p in $B(H)$ [Shep1, 3.2.3(iii)]. Thus, with $q = z \otimes p$, we have that q is abelian in $W^*(M)$ and also that $ses = q + \phi(q)$.

Summing over homogeneous parts, we conclude:

Lemma 3.4.15

Let M be a universally reversible type I JW-algebra and let e be an abelian projection in M. We have*

(a) if M is hermitian then e is abelian in $W^*(M)$ and $eMe = eW^*(M)e$;

(b) if M is symplectic then $e = p + \phi(p)$, where p is an abelian projection of $W^*(M)$ and ϕ is the canonical involution.

Proposition 3.4.16

Let M be a universally reversible JW-algebra with no complex part. Let ϕ be the canonical involution on $W^*(M)$. Suppose that M has non-zero symplectic type I part. Then there exists a non-zero projection e in $W^*(M)$ such that*

(a) e is abelian in $W^*(M)$;

(b) $e + \phi(e)$ is abelian in M;

(c) $eM\phi(e) = 0$.

Proof

Let N be the symplectic type I part of M. By (3.4.15(b)) we can choose a non-zero projection e in $W^*(N)$ such that e is abelian in $W^*(N)$ and $e+\phi(e)$ is abelian in N. Then, since $W^*(N)$ is an ideal of $W^*(M)$, this projection e is abelian in $W^*(M)$ and $e + \phi(e)$ is abelian in M.

To affirm part (c) we observe that, by (3.4.13(b)), ϕ is central and so

$$\begin{aligned}(e+\phi(e))M(e+\phi(e)) &= Z\big((e+\phi(e))M(e+\phi(e))\big) = (e+\phi(e))Z(M) \\ &= (e+\phi(e))Z(W^*(M)).\end{aligned}$$

Therefore, as $e\phi(e)=0$,

$$eM\phi(e) = e\big((e+\phi(e))M(e+\phi(e))\big)\phi(e) = e(e+\phi(e))\phi(e)Z(W^*(M)) = 0.\square$$

Proposition 3.4.17

Let M be a universally reversible JW-algebra with no complex part. Let e be a non-zero projection in $W^*(M)$ such that $eM\phi(e)=0$. Then*

(a) $eW^*(M)e \cong (e+\phi(e))M(e+\phi(e))$;

(b) e is abelian in $W^*(M)$ and $e+\phi(e)$ is abelian in M;

(c) $Mc(e)$ is type I symplectic;

where ϕ is the canonical involution on $W^*(M)$.

Proof

Note that $\phi(e)Me = (eM\phi(e))^* = 0$ so in particular $e\phi(e)=0$.

(a) Let $N = eW^*(M)e + \phi(e)W^*(M)\phi(e)$. Then N is a W*-subalgebra of $W^*(M)$ and $\phi(N)=N$. We prove part (a) in two steps.

First, we claim that $(e+\phi(e))M(e+\phi(e)) = N^\phi$. Indeed, from our opening remark, it follows that

$$(e+\phi(e))M(e+\phi(e)) = eMe + \phi(e)M\phi(e) \subset N,$$

and thus, that $(e+\phi(e))M(e+\phi(e)) \subset N\cap M = N^\phi$. To see the converse let $a\in N^\phi$. Then $a = exe+\phi(e)y\phi(e)$ for some $x,y\in W^*(M)$. In fact, $eae = exe$ and $\phi(e)a\phi(e) = \phi(e)y\phi(e)$ so $a = eae+\phi(e)a\phi(e)$. Now, because $a = (e+\phi(e))a(e+\phi(e)) \in (e+\phi(e))M(e+\phi(e))$, our claim is proved.

Secondly, we claim that $N^\phi \cong eN$. This follows, through the proof of (3.4.12(a)), after we note that

$$e(exe + \phi(e)y\phi(e)) = exe = (exe + \phi(e)y\phi(e))e.$$

Now, through the two preceding statements, we can confirm that

$$eW^*(M)e = eN \cong N^\phi = (e + \phi(e))M(e + \phi(e)).$$

(b) Let $f = e + \phi(e)$. Then $(fW^*(M)f)^\phi = fMf$ and, as ϕ is central,

$$Z(fW^*(M)f) = fZ(W^*(M)) = fZ(M) = Z(fMf).$$

Therefore, ϕ is central on $fW^*(M)f$. By part (a) we see that

$$(fW^*(M)f)^\phi = fMf \cong eW^*(M)e,$$

which is a W*-algebra. In which case, by (3.4.13(c)), fMf is abelian and hence so is $eW^*(M)e$. It is now evident that e is abelian in $W^*(M)$ and $f = e + \phi(e)$ is abelian in M.

(c) Via (3.4.8) $c(e) = \phi(c(e)) = c(\phi(e))$. Since e is abelian in $W^*(M)$, (1.8.5(b)) implies that $e \sim \phi(e)$. Therefore

$$(e + \phi(e))W^*(M)(e + \phi(e))$$

is a type I_2 W*-algebra. In particular, $e+\phi(e)$ is not abelian in $W^*(M)$. As $e \leq e + \phi(e) \leq c(e)$, we have $c(e) = c(e + \phi(e))$. Then

$$Mc(e) = Mc(e + \phi(e))$$

is type I, because $e + \phi(e)$ is abelian in M. Now, suppose that there exists a non-zero central projection z majorised by $c(e)$ such that Mz is hermitian. Then, since $c(ez) = c(e)z = z \neq 0$, we note that $ez \neq 0$.

Let $f = e + \phi(e)$. Clearly fz is an abelian projection in $Mc(e)$. However, fz cannot be abelian in $W^*(M)c(e)$. Indeed, since $zfW^*(M)f$ is a weak* closed ideal of $fW^*(M)f$, which is a type I_2 W*-algebra, and as

$$zfW^*(M)f = fzW^*(M)c(e)fz,$$

it follows that $fzW^*(M)c(e)fz$ is of type I_2. Thus, in the light of (3.4.15(a)), the projection fz provides a contradiction and so the hermitian part of $Mc(e)$ is zero. Finally as, by assumption, M has no complex part, $Mc(e)$ must be symplectic. $\square$

Now (3.4.16) together with (3.4.17) gives the following.

Theorem 3.4.18

Let M be a universally reversible JW-algebra with no complex part. Then there exists a non-zero projection e in $W^*(M)$ such that $eM\phi(e) = 0$ if and only if M has non-zero symplectic type I part.*

Such a projection e satisfies the following, where ϕ is the canonical involution.

(a) $eW^*(M)e \cong (e + \phi(e))M(e + \phi(e))$.

(b) $(e + \phi(e))W^*(M)(e + \phi(e)) \cong A \otimes M_2(\mathbf{C})$, *for some abelian von Neumann algebra A.*

(c) $Mc(e)$ *is symplectic type I.*

We note the following corollary.

Corollary 3.4.19

Let M be any JW-algebra without symplectic part or complex part. Then every complete tripotent of M is unitary.*

Proof

Let u be a complete tripotent of M. Then

$$(1 - uu^*)M(1 - u^*u) = M_0(u) = \{0\}.$$

Put $e = (1 - uu^*)$. Then $\phi(e) = 1 - u^*u$, where ϕ is the canonical involution on $W^*(M)$, and hence $eM\phi(e) = 0$. Now, by hypothesis and (3.4.18), $e = 0$ and thus $\phi(e) = 0$. So, $uu^* = 1 = u^*u$. □

Lemma 3.4.20

Let u be an abelian tripotent in an hermitian JW-algebra M. Then u is an abelian tripotent of $W^*(M)$ and $M_2(u) = W^*(M)_2(u)$.*

Proof

We may suppose that M is homogeneous and is not abelian. Let v be a complete tripotent of M majorising u. Then $M_2(v)$ is hermitian and the canonical involution, ϕ, of $W^*(M)$ is an involution on $W^*(M)_2(v)$ with $(W^*(M)_2(v))^\phi = M_2(v)$. Hence, $W^*(M_2(v)) = W^*(M)_2(v)$. Since u is an abelian projection of $M_2(v)$, the result now follows from (3.4.15(a)). □

3.4.21 Tripotents u and v in a JB*-triple A are said to be *rigidly collinear* if $A_2(u) \subset A_1(v)$ and $A_2(v) \subset A_1(u)$.

Lemma 3.4.22

Let u and v be non-zero tripotents in an hermitian JW-algebra M. Then u and v are not rigidly collinear.*

Proof

Suppose that u and v are rigidly collinear and let $e = uu^*$ and $f = vv^*$. Let ϕ be the canonical involution of $W^*(M)$. We have $P_2^v(x) = fx\phi(f)$ and $P_1^u(x) = ex\phi(1 - e) + (1 - e)x\phi(e)$, for each $x \in M$. Collinearity implies that $v = ev + v\phi(e)$, giving $f = ef + v\phi(e)v^*$. In particular, $ef = fe$. Let $p = (1-e)f$. Since, by assumption, $M_2(v) \subset M_1(u)$, we deduce that

$$pM\phi(p) = (1-e)(fM\phi(f))\phi(1-e) \subset (1-e)(M_1(u))\phi(1-e) = \{0\}.$$

Therefore, because (3.4.18) now shows that $p = 0$, $f = ef$. Similarly, $e = fe = f$. Hence, $0 = v\phi(e)v^* = v\phi(f)v^* = vv^* = f$. Thus, $v = 0$ and likewise $u = 0$; a contradiction. □

Proposition 3.4.23

Let I be a weak closed inner ideal of an hermitian JW*-algebra M. Then there is a unitary tripotent u of I such that $I = M_2(u)$. Putting $e = uu^*$ (in $W^*(M)$), we have $I = eM\phi(e)$, where ϕ is the canonical involution of $W^*(M)$. Moreover, if I is type I_1 then it is abelian.*

Proof

Pick a complete tripotent u of I. If I has no type I_1 part then the first part of the statement is immediate from (3.3.15) and (3.4.19).

On the other hand, suppose that I is type I_1. If I is not abelian then, passing to a weak* closed ideal, in order to obtain a contradiction we may suppose that $I \cong A \otimes H$, where A is an abelian von Neumann algebra and H is a Hilbert space of dimension at least two. Choose orthonormal elements h_1 and h_2 of H. Then h_1 and h_2 are rigidly collinear. Now, since $P_i^{(1\otimes h_j)} = id \otimes P_i^{h_j}$, for $(i, j = 1, 2)$, $1 \otimes h_1$ and $1 \otimes h_2$ are rigidly collinear. Thus, the tripotents of I that correspond to $1 \otimes h_1$ and $1 \otimes h_2$ are rigidly collinear tripotents of M, contradicting (3.4.22). □

We have reached the following extension of (3.4.2).

Theorem 3.4.24

Let M be a universally reversible JW-algebra with no non-zero symplectic part. Then the map, $e \mapsto eM\phi(e)$, is an order preserving bijection from $\mathcal{P}(W^*(M), \phi)$ onto the set of weak* closed inner ideals of M, where ϕ is the canonical involution of $W^*(M)$.*

Proof

As ϕ is central on the hermitian part of M (by (3.4.13(b))), the given map is well-defined and surjective by (3.4.2) and (3.4.23).

In order to show injectivity we may use the fact that ϕ is a central involution. Let $I = eM\phi(e)$, where e is a projection in $W^*(M)$. Let I^{ew} denote the weak* closed inner ideal of $W^*(M)$ generated by I. We will show that $I^{ew} = eW^*(M)\phi(e)$. Let $R(I)$ and $L(I)$ denote the right and left ideals, respectively, generated by I in $W^*(M)$. Since ϕ fixes each point of I we have $\phi(R(I)) = L(I)$ and hence $\phi(\overline{R(I)}) = \overline{L(I)}$, where bar denotes weak* closure. By definition, there is some projection $f \leq e$ such that

$$\overline{R(I)} = fW^*(M) \subset eW^*(M).$$

Since $\overline{R(I)} \cap \overline{L(I)} = fW^*(M)\phi(f)$ is a weak* closed inner ideal of $W^*(M)$ containing I, we therefore have $I \subset I^{ew} \subset fW^*(M)\phi(f) \subset eW^*(M)\phi(e)$. Intersecting with M yields $I \subset fM\phi(f) \subset eM\phi(e) = I$, and therefore equality throughout. Hence, $(e-f)M\phi(e-f) = 0$, so that $e = f$ by (3.4.18).

By (2.10.13), $I^{ew} = pW^*(M)q$, where p and q are centrally equivalent projections of $W^*(M)$. As $\overline{R(I)} \subset pW^*(M)$ and $\overline{L(I)} \subset W^*(M)q$, we have $I^{ew} = pW^*(M)q = eW^*(M)\phi(e)$. Injectivity of the given map is now immediate from (2.10.13). $\square$

We will also require the next proposition.

Proposition 3.4.25

Let M be a universally reversible JW-algebra without type I_1 part. Let I be a weak* closed rectangular inner ideal of M without type I_1 part. Then $I = eM\phi(e)$, for some projection e in $W^*(M)$, where ϕ is the canonical involution of $W^*(M)$.*

Proof

Let Mz be the weak* closed ideal of M generated by I, where z is a central projection in M. The involution ϕ of $W^*(M)$, through restriction, gives rise to the canonical involution of $W^*(Mz) = W^*(M)z$. Further, since M is universally reversible, $Mz = W^*(M)^{\phi}z$ [HaSt, 7.3.3]. Now, through (3.3.20), Mz is *-isomorphic to a type I W*-algebra. Thus, via (3.4.2), there exists a projection e in $W^*(M)z$ such that $I = eMz\phi(e) = eM\phi(e)$, as required. $\square$

3.5 Cartan Factor Representation Theory

3.5.1 Cartan factor representation theory is a key technique used within the thesis, and thus warrants a clear account. The material given in this section, the majority of which is known, is influenced by that contained in [BuChZa1]. Simplistically, the Cartan factor representation structure of a JB*-triple A is (mostly) determined by the structure of its second dual. Specifically, if C is a finite dimensional Cartan factor, we shall show here that A has a Cartan factor representation onto C precisely when the bidual of A has a weak* closed ideal that is isomorphic to $C(X) \otimes C$, where X is some compact hyperstonean space.

Proposition 3.5.2

Let C be a Cartan factor and let A be a JB-triple. Then there exists a Cartan factor representation $\pi : A \longrightarrow C$ if and only if C is isometric to a weak* closed ideal of A^{**}_{at}.*

Proof

Suppose that such a Cartan factor representation $\pi : A \longrightarrow C$ exists. Then $A^{**} = C \oplus ker\hat{\pi}$, where $\hat{\pi} : A^{**} \longrightarrow C$ is the weak* continuous extension of π. Clearly $C \subset A^{**}_{at}$ and since $ker\hat{\pi}$ is a weak* closed ideal so is C [Ho1, §4]. Conversely suppose that C is a weak* closed ideal of A^{**}_{at} with natural projection $P : A^{**} \longrightarrow C$. Then the restriction of P to A is a Cartan factor representation of A. □

Proposition 3.5.3

Let C be a finite dimensional Cartan factor and X be a compact Hausdorff space. Let D be a Cartan factor such that $D \subset C(X) \otimes C$. Then D is isometric to a subfactor of C.

Proof

For each x in X define $\pi_x : C(X) \otimes C \to C$ to be the linear map satisfying $\pi_x(f \otimes a) = f(x)a$. Then π_x is a Cartan factor representation and moreover $\{\pi_x : x \in X\}$ is a faithful family of Cartan factor representations. The faithfulness condition follows from the fact that $C(X) \otimes C \cong C(X, C)$, via $f \otimes a \mapsto f(.)a$; if $b = \sum_1^n f_i \otimes a_i \neq 0$ then $g = \sum_i^n f_i(.)a_i \neq 0$ so that, for some $x \in X$, $\pi_x(b) = g(x) \neq 0$.

Suppose that D is non-zero and choose x in X such that $\pi_x(K(D)) \neq 0$. Such an x exists since the family is faithful, and since $K(D)$ is, by definition, non-zero. As $K(D)$ is simple we have $K(D) \cong \pi_x(K(D)) \subset C$. It follows that $K(D)$ is finite dimensional and is thus reflexive, so that

$$K(D) = K(D)^{**} = D,$$

and so D is isometric to a subfactor of C, as required. $\square$

Proposition 3.5.4 ([BuChZa1, 2.3])

Let C be a finite dimensional Cartan factor and let A be a JB-triple. Then all Cartan factor representations of A are onto C if and only if $A^{**} = C(X) \otimes C$ for some compact Haudsdorff space X.*

3.5.5 A JB*-triple A is said to be *type* C, for some finite dimensional Cartan factor C, if $A/ker\pi \cong C$, for all Cartan factor representations π of A, that is, if all Cartan factor representations of A are onto C. By convention, the zero triple is of every type.

We remark that by the previous theorem, if a JB*-triple A is type C, where C is a finite dimensional Cartan factor, then the bidual of A is isomorphic to $C(X) \otimes C$, for some compact hyperstonean space X.

Theorem 3.5.6 ([BuChZa2, 5.1])

Let A be a JB-triple with a Cartan factor representation with rank n, where $n < \infty$. Then either all Cartan factor representations of A have rank at most n, or A contains a non-zero ideal J such that*

(a) all Cartan factor representations of J have rank greater than n;

(b) all Cartan factor representations of A/J have rank at most n.

We have the following extension of (3.5.4).

Proposition 3.5.7

Let C be a finite dimensional Cartan factor and let A be a JB-triple. Then there exists a Cartan factor representation $\pi : A \longrightarrow C$ if and only if there exists a weak* closed ideal J of A^{**} such that $J \cong C(X) \otimes C$, for some compact hyperstonean space X.*

Proof

Let $\pi : A \longrightarrow C$ be a Cartan factor representation. Then, via (3.5.2), C is isometric to a weak* closed ideal of A^{**}_{at} so that the result follows.

Conversely, suppose that there exists a weak* closed ideal J of A^{**} such that $J \cong C(X) \otimes C$. Let $P : A^{**} \longrightarrow J$ be the natural weak* continuous projection. Then $J = \overline{P(A)}$, where bar denotes weak* closure. Since

$$P(A) \cong A/(A \cap ker P),$$

which is a quotient of A, it is enough to show that $P(A)$ has a Cartan factor representation onto C.

As $P(A) \subset J$, we have that $P(A)^{**} \subset J^{**} \cong C(Y) \otimes C$, for some compact hyperstonean space Y. Let D be a weak* closed Cartan factor ideal of $P(A)^{**}$. Then, by (3.5.3), D is isometric to a Cartan subfactor of C.

It is now immediate from (3.5.2) that all Cartan factor repesentations of $P(A)$ are onto subfactors of C.

Let $C_0, C_1, ..., C_n$ be the Cartan factors arising from the Cartan factor representations of $P(A)$. Then, using (3.5.6), there is a composition series of norm closed ideals of $P(A)$, $(J_i)_{0\leq i\leq n+1}$, with J_{i+1}/J_i homogenous type C_i, for distinct Cartan factors C_i. Thus, via (3.5.4), as remarked in (3.5.5),

$$P(A)^{**} \cong \sum \left(J_{i+1}/J_i\right)^{**} \cong \sum C(X_i) \otimes C_i,$$

where both sums are ℓ^∞-sums.

Finally, there is a weak* continuous triple homomorphism

$$\psi : P(A)^{**} \longrightarrow \overline{P(A)} = J \cong C(X) \otimes C,$$

and hence $P(A)^{**}$ has a weak* closed ideal isomorphic to $C(X) \otimes C$. Therefore $C_i = C$ for some i, and consequently $P(A)$ has a Cartan factor representation onto C. $\square$

3.5.8 We make the following aside. Let $J \cong C(X) \otimes C$ be a weak* closed ideal of the bidual of some JB*-triple, where C is a finite dimensional Cartan factor and X is some compact hyperstonean space. Let B be a JB*-triple contained in J. Using the argument contained in the proof of the previous proposition, (3.5.7), we see that every Cartan factor representation of B is necessarily onto a subfactor of C. We can go further and deduce that $B^{**} \cong \left(\sum C(X_i) \otimes C_i\right)_\infty$, for some compact hyperstonean spaces X_i, where the C_i denote these subfactors of C.

The next two propositions are counterparts of (3.5.4) and (3.5.7).

Proposition 3.5.9

Let A be a JB-triple. Then the following are equivalent.*

(a) A^{**} is type I_1.

(b) All Cartan factor representations of A are onto Hilbert spaces.

Proof

(a) $\Rightarrow$ (b) Assume (a) and let $\pi : A \longrightarrow C$ be a Cartan factor representation of A. Then C is isometric to a weak* closed ideal J of A^{**} (3.5.2). Let u be a non-zero tripotent in J. Choose a tripotent v of A^{**} such that u is a projection in $A_2^{**}(v)$. Since $A_2^{**}(v)$ is an abelian JW*-algebra (by (3.3.17(c))), and because $J_2(u)$ is a subfactor of it, we must have $J_2(u) = \mathbf{C}u$ so that u is a minimal tripotent of J. Thus all tripotents of J are minimal and hence J is a Hilbert space [DaFr, p308]. It is now evident that C is a Hilbert space.

(b) $\Rightarrow$ (a) Assume (b). Then the atomic part, M, of A^{**} is an ℓ^∞-sum of Hilbert spaces and so is type I_1 [Ho2, §2]. However, A embeds as a JB*-subtriple of M (2.12.3). So, A^{**} can be realised as a JBW*-subtriple of M^{**}, and it follows from (3.3.18) that A^{**} is type I_1. □

Proposition 3.5.10

The following are equivalent for a JB-triple A.*

(a) A has a Cartan factor representation onto a Hilbert space.

(b) A^{**} has non-zero type I_1 part.

Proof

(a) $\Rightarrow$ (b) This is clear.

(b) $\Rightarrow$ (a) Assume (b) and let $P : A^{**} \to J$ be the natural projection, where J denotes the type I_1 part of A^{**}. Then $P(A)^{**}$ is contained in J^{**} so that $P(A)^{**}$ is type I_1 by (3.3.18). Hence, all Cartan factor representations of $P(A)$ are onto Hilbert spaces (3.5.9). However, $P(A)$ is a quotient of A, giving (a). $\square$

Chapter 4

The Inner Stone-Weierstrass Theorem for Universally Reversible JC*-Algebras

4.1 Introduction

The ultimate aim of the thesis is to determine inner ideals in JB*-triples by extreme points of their dual balls. The means by which this objective is achieved is what we, from now on, choose to term the *Inner Stone-Weierstrass Theorem*:

Let A be a JB-triple with norm closed inner ideals I and J, such that $I \subset J$. Suppose that $\partial_e(I_1^*) = \partial_e(J_1^*)$. Then $I = J$.*

In this chapter, we establish the Inner Stone-Weierstrass Theorem for universally reversible JC*-algebras, a theorem which is exploited in Chapter Five to prove the full theorem for JB*-triples.

To validate our choice of terminology, we begin by explaining how the Inner Stone-Weierstrass Theorem for JB*-triples corresponds to a Stone-Weierstrass type result. Specifically, using the bijection between extreme points and minimal tripotents, (2.8.5), we show that in the statement given above, the condition that $\partial_e(I_1^*) = \partial_e(J_1^*)$ may be replaced by the condition that *I separates* $\partial_e(J_1^*) \cup \{0\}$, or equivalently, that $I_{at}^{**} = J_{at}^{**}$.

Let I and J be norm closed inner ideals in a universally reversible JC*-algebra A, such that $I \subset J$. To observe that I is equal to J, it is enough to prove the equality of the corresponding biduals in A^{**}. This JW*-algebra decomposes into a continuous part and a type I part, and the latter part in turn decomposes into summands of the form $B\overline{\otimes}C$, where C is a Cartan factor and B is an abelian von Neumann algebra [Ho2]. Making use of the Cartan factor representation theory of section (3.5), we find that the assumption that $I^{**}_{at} = J^{**}_{at}$ guarantees that the finite Cartan parts of I^{**} and J^{**}, that is those type I parts with summands formed with Cartan factors of finite dimension, are identical. Additionally, through structure theory, we observe that it can be assumed that no type I_1 parts are present.

It is left to show the coincidence of the remaining parts of I^{**} and J^{**}, both continuous and type I, when $I^{**}_{at} = J^{**}_{at}$. Taking [EdRüVa2] as inspiration (see (3.4)), these are found to be of the form $eA^{**}\phi(e)$, for some projection e in $C^*(A)^{**}$. Moreover, we show that an inner ideal of this form is determined by the weak* closed inner ideal that it generates in $C^*(A)^{**}$, in so far as it is precisely the intersection of this generated inner ideal and A^{**}. Consequently, we only need demonstrate that these generated inner ideals are identical, and in this manner our argument passes to the universal enveloping C*-algebra. With this in mind, we establish the Inner Stone-Weierstrass Theorem for C*-algebras. By exploiting this, we prove that, given the constraint on the bidual atomic parts, I and J necessarily generate the same norm closed inner ideal in the universal enveloping C*-algebra. Finally, by examining the second dual of these generated inner ideals we reach the desired conclusion.

4.2 Inner Ideals and the Atomic Part

4.2.1 This section takes the form of a series of necessary technical lemmas investigating the atomic part of the bidual of a norm closed inner ideal. Let A be a JB*-triple with norm closed inner ideal I. We shall prove that when $\partial_e(I_1^*) = \partial_e(A_1^*)$, the biduals of A and I have the same atomic part, or alternatively, that I separates $\partial_e(A_1^*) \cup \{0\}$, thereby validating our terminology. Furthermore, we will demonstrate that when $A_{at}^{**} = I_{at}^{**}$, we can conclude that A is equal to I, if either A is a C*-algebra or I is a triple ideal. Finally, utilising the exposition of Chapter Three (3.5), we will show that given this equality of atomic parts, I has a Cartan factor representation onto a specfic Cartan factor C precisely when A does.

Lemma 4.2.2

Let M be an atomic JBW-triple with weak* closed inner ideal I. Then I is atomic.*

Proof

As M is atomic it is an ℓ^∞-sum, $M = \sum C_\alpha$, where each C_α is a Cartan factor. Therefore, by (2.10.16), $I = \sum C_\alpha \cap I$, again an ℓ^∞-sum of Cartan factors. $\square$

Lemma 4.2.3

Let A be a JB-triple and let I be a norm closed inner ideal of A.*

Let $\rho \in \partial_e(A_1^*)$. Then

(a) $I_{at}^{**} = I^{**} \cap A_{at}^{**}$;

(b) $\rho(I) = 0$ *if and only if* $\rho(I_{at}^{**}) = 0$.

Proof

(a) Clearly $I_{at}^{**} \subset I^{**} \cap A_{at}^{**}$. Conversely, since $I^{**} \cap A_{at}^{**}$ is a weak* closed inner ideal of A_{at}^{**}, it follows by (4.2.2) that it is atomic. However it is also clearly a weak* closed inner ideal of I^{**} and so $I^{**} \cap A_{at}^{**} \subset I_{at}^{**}$.

(b) Let $\rho \in \partial_e(A_1^*)$ such that $\rho(I) = 0$. Then as $\rho(I_{at}^{**})$ is contained in $\rho(I^{**})$ and the latter is zero by weak* continuity of ρ, we see that $\rho(I_{at}^{**}) = 0$. Conversely suppose that $\rho(I_{at}^{**}) = 0$. By part (a) $I_{at}^{**} = I^{**} \cap A_{at}^{**}$, and so, via (2.10.15(a)), we have

$$\begin{aligned} I^{**} &= I^{**} \cap A_{at}^{**} \oplus I^{**} \cap (A_{at}^{**})^{\perp} \\ &= I_{at}^{**} \oplus I^{**} \cap (A_{at}^{**})^{\perp}. \end{aligned}$$

Therefore $\rho(I^{**}) = 0$, as required. $\square$

4.2.4 It is now possible to give the following equivalent conditions, that demonstrate, amongst other things, that the coincidence of the atomic parts of the biduals of a JB*-triple and its inner ideal can be represented in terms of a Stone-Weierstrass separation condition.

Theorem 4.2.5

Let A be a JB-triple and let I be a norm closed inner ideal of A. Then the following are equivalent.*

(a) $\rho(I) \neq 0$ *for all* $\rho \in \partial_e(A_1^*)$.

(b) $I_{at}^{**} = A_{at}^{**}$.

(c) For all $\rho \in \partial_e(A_1^*)$ *the restriction of* ρ *to* I *lies in* $\partial_e(I_1^*)$.

(d) The unique extension map from $\partial_e(I_1^*)$ *to* $\partial_e(A_1^*)$ *is a bijection.*

(e) I *separates* $\partial_e(A_1^*) \cup \{0\}$.

Proof

$(a) \Rightarrow (b)$ Assume condition (a) holds. We have an ℓ^∞-sum of Cartan factors, $A_{at}^{**} = \sum C_\alpha$. By (2.10.16) together with (4.2.3(a)), this gives

$$I_{at}^{**} = \sum I^{**} \cap C_\alpha.$$

Fix α and let $\rho \in \partial_e(C_{\alpha,*,1})$. We can regard ρ as an element of $\partial_e(A_1^*)$ via the identification made in (2.12.2) and note that $\rho(C_\beta) = 0$ for all $\beta \neq \alpha$. By (4.2.3(b)) $\rho(I_{at}^{**}) \neq 0$ and thus $\rho(I^{**} \cap C_\alpha) \neq 0$. Applying (2.13.8) to C_α now proves that $I^{**} \cap C_\alpha = C_\alpha$, so C_α is contained in I^{**}. This holds for all α and so $I_{at}^{**} = \sum C_\alpha = A_{at}^{**}$.

$(b) \Rightarrow (c)$ Suppose that $I_{at}^{**} = A_{at}^{**}$ and let $\rho \in \partial_e(A_1^*)$. Then $s(\rho) \in A_{at}^{**} = I_{at}^{**}$ and so the restriction is an element of $\partial_e(I_1^*)$ as desired.

$(c) \Rightarrow (d)$ Along with $(d) \Rightarrow (e)$, this is clear from the correspondence between dual ball extreme points and second dual minimal tripotents (2.8.5).

$(e) \Rightarrow (a)$ Suppose that there exists $\rho \in \partial_e(A_1^*)$ such that $\rho(I) = 0$. Then ρ agrees with the zero function on I. It follows that I does not separate $\partial_e(A_1^*) \cup \{0\}$. $\square$

4.2.6 The maximal norm closed left and right ideals of a C*-algebra are completely classified in terms of pure states.

Theorem 4.2.7 ([Ped1, 3.13.6])

Let A be a C-algebra and let $\rho \in P(A)$. Then*

(i) $L_\rho = \{x \in A : \rho(x^*x) = 0\}$ *is a maximal norm closed left ideal of A;*

(ii) $R_\rho = \{x \in A : \rho(xx^*) = 0\}$ *is a maximal norm closed right ideal of A;*

(iii) $L_\rho + R_\rho = ker\rho$;

(iv) $\rho(L_\rho) = \rho(R_\rho) = 0$;

(v) L_ρ *and* R_ρ *are precisely the maximal norm closed left and right ideals of A, respectively, as ρ ranges over $P(A)$.*

4.2.8 Let A be a C*-algebra with norm closed inner ideal I. From the classification provided above, in (4.2.7), it is simple to observe that I is equal to A if and only if it is not annihilated by any pure state of A. Indeed, if I is not equal to A then, as the intersection of closed left and right ideals, I is contained in L_ρ or R_ρ, and so in $ker\rho$, for some pure state ρ of A.

Proposition 4.2.9

Let A be a C-algebra and let I be a norm closed inner ideal of A such that $I_{at}^{**} = A_{at}^{**}$. Then $I = A$.*

Proof

Using the equivalent conditions of (4.2.5(a) and (b)), this is immediate from the preceding remark, (4.2.8). $\square$

Proposition 4.2.10

Let A be a JB-triple and let J be a norm closed ideal of A. Suppose that $J_{at}^{**} = A_{at}^{**}$. Then $J = A$.*

Proof

By (4.2.5(a)(b)), $\rho(J) \neq 0$ for all $\rho \in \partial_e(A_1^*)$. Suppose that J is not equal to A, so that there exists some non-zero $\rho \in \partial_e((A/J)_1^*)$. Then, via the bijection (2.10.9(b)(iii)) between $\partial_e((A/J)_1^*)$ and the extreme points of A_1^* vanishing on J, we obtain, in contradiction, $\hat{\rho} \in \partial_e(A_1^*)$ such that $\hat{\rho}(J) = 0$. $\square$

4.2.11 Let A be a JB*-triple with norm closed inner ideal I. Recall that $\mathcal{T}(I)$ denotes the norm closed ideal of A generated by I. Since $I_{at}^{**} \subset \mathcal{T}(I)_{at}^{**} \subset A_{at}^{**}$, we have the following corollary of (4.2.10).

Corollary 4.2.12

Let A be a JB-triple and let I be a norm closed inner ideal of A such that $I_{at}^{**} = A_{at}^{**}$. Then $\mathcal{T}(I) = A$. That is, A is generated as a norm closed ideal by I.*

4.2.13 Let I be a norm closed inner ideal of a JB*-triple A and suppose that A^{**} and I^{**} have equal atomic part. The role of the next proposition is, essentially, to illustrate how this supposition translates to the quotient of A by any norm closed ideal, J say, and also to the intersection, $I \cap J$. Importantly, we also show that this constraint is preserved by triple homomorphisms π, that is, if $I_{at}^{**} = A_{at}^{**}$ then $\pi(I)_{at}^{**} = \pi(A)_{at}^{**}$.

Proposition 4.2.14

Let A be a JB-triple and let J be a norm closed ideal of A. Let I be a norm closed inner ideal of A such that $I^{**}_{at} = A^{**}_{at}$. Then*

(a) $(J \cap I)^{**}_{at} = J^{**}_{at}$;

(b) $\big((I + J)/J\big)^{**}_{at} = (A/J)^{**}_{at}$;

(c) $\pi(I)^{**}_{at} = \pi(A)^{**}_{at}$, *for any triple homomorphism* π *of* A.

Proof

(a) Using (4.2.3(a)) and (2.10.17(b)) we see that

$$\begin{aligned}(J \cap I)^{**}_{at} &= (J \cap I)^{**} \cap A^{**}_{at} = J^{**} \cap I^{**} \cap A^{**}_{at} \\ &= J^{**} \cap I^{**}_{at} = J^{**} \cap A^{**}_{at} = J^{**}_{at}.\end{aligned}$$

(b) Using the equivalence given in (4.2.5(a)(b)), it will be enough to show that $\rho((I + J)/J) \neq 0$ for all $\rho \in \partial_e((A/J)^*_1)$.
Let $\rho \in \partial_e((A/J)^*_1)$. Then, via the bijection of (2.10.9(b)(iii)), we obtain $\hat{\rho} \in \partial_e(A^*_1)$, with $\hat{\rho}(a) = \rho(a + J)$. However, by assumption, $\hat{\rho}(I) \neq 0$ so that $\rho(I + J) \neq 0$ and thus $\rho((I + J)/J) \neq 0$.

(c) This is immediate from part (b), since $\pi(A) \cong A/ker\pi$, via the canonical isometry. □

4.2.15 Recall that if A is a JB*-triple, then A^{**}_{at} is an ℓ^{∞} direct sum of Cartan factors, $A^{**}_{at} = \sum C_{\alpha}$, say. Furthermore, if I is an inner ideal of A with $I^{**}_{at} = A^{**}_{at}$, then, as in the proof of (4.2.5(a)),

$$\sum C_{\alpha} = A^{**}_{at} = I^{**}_{at} = \sum I^{**} \cap C_{\alpha}.$$

Therefore, using (3.5.2), this condition on the atomic part of I^{**} guarantees that A and I have the same Cartan factor representations in the following sense.

Lemma 4.2.16

Let C be a Cartan factor. Let A be a JB-triple with norm closed inner ideal I. Suppose that $I^{**}_{at} = A^{**}_{at}$. Then I has a Cartan factor representation onto C if and only if A has a Cartan factor representation onto C.*

4.3 The Finite Cartan Part

4.3.1 We shall say that a JBW*-triple M is of *finite Cartan factor type* if it is an ℓ^{∞}-sum of the form $\sum A_i \otimes C_i$, where the A_i are abelian von Neumann algebras and the C_i are finite dimensional Cartan factors. For any JBW*-triple M, we define the *finite Cartan part* of M, denoted by M_f, to be the largest weak* closed ideal of finite Cartan factor type contained in M.

4.3.2 Let I be a norm closed inner ideal of a JB*-triple A such that the atomic parts of I^{**} and A^{**} coincide. We shall now show the equality of the finite Cartan factor parts of I^{**} and A^{**}. In particular, in conjunction with (4.2.5), this means that if K and J are norm closed inner ideals of A such that $K \subset J$ and $\partial_e(K^*_1) = \partial_e(J^*_1)$, then we can conclude that $K^{**}_f = J^{**}_f$. Hence, to prove the Inner Stone-Weierstrass theorem, that is, that under these conditions $K = J$, the outstanding problem is then to show the coincidence of the remaining parts.

Proposition 4.3.3

Let A be a JB-triple. Let I be a norm closed inner ideal of A such that $I^{**}_{at} = A^{**}_{at}$. Then I^{**} and A^{**} have the same finite Cartan part.*

Proof

We know that A^{**}_f is an ℓ^∞-sum of homogeneous type C weak* closed ideals, for certain distinct finite dimensional Cartan factors C. Let J be one such summand. Then $J \cong \mathcal{A} \otimes C$, where $\mathcal{A}$ is some abelian von Neumann algebra and C is a finite dimensional Cartan factor. Let C be of rank n.

By (3.5.7), because of the form of J, A has a Cartan factor representation onto C. If all Cartan factor representations of A have rank at most n then A is liminal, and so $I = A$ by [Shep3, 5.5] and we are done. Otherwise, by (3.5.6), A contains a non-zero norm closed ideal K, such that A/K is liminal and all Cartan factor representations of K have rank greater than n. Consequently, by [Shep3, 5.5] together with (4.2.14(b)),

$$(I + K)/K = A/K,$$

giving $I + K = A$, and so $(K^{**})^\perp$ is contained in I^{**} by (2.10.19).

Now, if $J \cap K^{**} \neq 0$ it must be a non-zero weak* closed ideal of K^{**} that is homogeneous type C, because it is also a weak* closed ideal of J. However, one more application of (3.5.7) demonstrates that K must have a Cartan factor representation onto C, that is, one of rank n, contradicting our assumption on K. Therefore J and K^{**} are orthogonal weak* closed ideals of the bidual of A and $J \subset (K^{**})^\perp \subset I^{**}$.

By the opening paragraph of our argument, it follows that A^{**}_f is contained in the second dual of I. Thus $I^{**}_f = A^{**}_f \cap I^{**} = A^{**}_f$. $\square$

4.4 The Inner Stone-Weierstrass Theorem for Universally Reversible JC*-Algebras

4.4.1 Let I and J be norm closed inner ideals in a universally reversible JC*-algebra A, such that $I \subset J$ and $\partial_e(I_1^*) = \partial_e(J_1^*)$. To prove that we may conclude that $I = J$, that is the Inner Stone-Weierstrass theorem, it is enough to demonstrate that the biduals of I and J coincide. Ensuing arguments are designed to reach this conclusion.

We have shown the equality of the finite Cartan parts of I^{**} and J^{**}, see (4.3.2), and so, as previously remarked, the outstanding problem is to determine the coincidence of the remaining bidual parts. To achieve this goal, our initial objective is to prove the theorem in the case when A is a C*-algebra, that is the Inner Stone-Weierstrass Theorem for C*-algebras. This is exploited to establish that I and J generate the same norm closed inner ideal in the universal enveloping C*-algebra $C^*(A)$, enabling us, finally, to deduce that $I = J$.

4.4.2 In order to prove the Inner Stone-Weierstrass Theorem for C*-algebras it is first necessary to recall the following relevant properties of open projections and hereditary C*-subalgebras. For further details see [Ped1, §1.5, 3.11.9].

Let A be a C*-algebra and let 1 denote the identity of A^{**}. Recall that a projection e in A^{**} is said to be *open* if $\overline{(eA^{**}e \cap A)} = eA^{**}e$, where bar denotes weak* closure. As ρ ranges over $P(A)$, the $1 - s(\rho)$ are the maximal open projections in A^{**} not equal to 1 [Ak, Proposition II.4] (the requirement that $1 \in A$ is well-known to be unnecessary).

The spaces $H_\rho = \{a \in A : \rho(a^*a) = \rho(aa^*) = 0\}$, are the maximal proper hereditary C*-subalgebras of A and we have

$$H_\rho^{**} = (1 - s(\rho))A^{**}(1 - s(\rho)).$$

We now determine the maximal hereditary C*-subalgebras of an hereditary C*-subalgebra of A. Let $B = eA^{**}e \cap A$ be an hereditary C*-subalgebra of A, where e is an open projection in the bidual of A. Then $B^{**} = eA^{**}e$ and has identity element e. By identifying each pure state of B with its unique extension to a pure state of A, we have

$$P(B) = \{\rho \in P(A) : \rho(e) = 1\} = \{\rho \in P(A) : s(\rho) \leq e\}.$$

Subsequently, through our opening remarks, the maximal proper hereditary C*-subalgebras H of B are exactly those for which

$$H^{**} = (e - s(\rho))B^{**}(e - s(\rho)) = (e - s(\rho))A^{**}(e - s(\rho)),$$

as ρ ranges over the pure states of B.

Lemma 4.4.3

Let A be a C-algebra and let e and p be open projections in A^{**} with $e \leq p$ and such that $e \neq p$. Then there exists a minimal projection g in A^{**} with $g \leq p - e$.*

Proof

Let $K = (eA^{**}e) \cap A$ and $B = (pA^{**}p) \cap A$. Since $e \leq p$ and $e \neq p$ it is evident that K is contained in B as a proper hereditary C*-subalgebra. Consequently, K is contained in some maximal proper hereditary C*-subalgebra of B. In which case, using the preceding remarks (4.4.2), there exists a minimal projection g in A^{**} such that $K^{**} = eA^{**}e \subset (p-g)A^{**}(p-g)$. Therefore $e \leq p - g$ and so $g \leq p - e$. □

We shall now prove the Inner Stone-Weierstrass Theorem for C*-algebras.

Theorem 4.4.4

Let A be a C^*-algebra and let I and J be norm closed inner ideals of A with $I \subset J$. Suppose that $\partial_e(I_1^*) = \partial_e(J_1^*)$. Then $I = J$.

Proof

Firstly, by (4.2.5), we may suppose that $I_{at}^{**} = J_{at}^{**}$. Using [EdRü1, 4.1], there exist open projections $e, f, p, q \in A^{**}$ such that $c(e) = c(f)$, $c(p) = c(q)$ and $I^{**} = eA^{**}f \subset J^{**} = pA^{**}q$. Therefore $e \leq p$ and $f \leq q$, by [EdRü1, 3.10].

Suppose that $e \neq p$. Then, by (4.4.3), there exists a minimal projection g in A^{**} such that $g \leq p - e$. In which case, $g \leq c(p-e) \leq c(p) = c(q)$. In particular, $c(g)c(q) \neq 0$ so that, by [Sak2, 1.10.7], $gA^{**}q \neq 0$.

Now, $g \in A_{at}^{**}$ and $g \leq p$ hence, via (4.2.3(a)),

$$gA^{**}q = p(gA^{**})q \subset J^{**} \cap A_{at}^{**} = J_{at}^{**} = I_{at}^{**}.$$

However, $ge = 0$ so that $gA^{**}q \subset gI_{at}^{**} \subset geA^{**}f = 0$.

We have shown, by contradiction, that $e = p$. Similarly $f = q$, so that $I^{**} = J^{**}$ and $I = J$, as required. $\square$

4.4.5 Let A be a JC*-algebra and let I be a norm closed inner ideal of A. In order to exploit the previous theorem for C*-algebras, (4.4.4), and ultimately prove the analogous result for JC*-algebras, we consider the embedding of A in its universal enveloping C*-algebra, $C^*(A)$. In the remainder of this chapter we employ the following notation. We shall use I^e to denote the norm closed inner ideal of $C^*(A)$ generated by I. We have the subsequent commutative diagram of canonical embeddings.

$$\begin{array}{ccccc} I & \hookrightarrow & A & \longrightarrow & A^{**} \\ \downarrow & & \downarrow & & \downarrow \\ I^e & \hookrightarrow & C^*(A) & \longrightarrow & C^*(A)^{**} = W^*(A^{**}) \end{array}$$

Here, the equality comes from [HaSt, 7.1.11].

In what follows we also re-use some earlier notation; if J is a weak* closed inner ideal in a JW*-algebra M, J^{ew} shall denote the weak* closed inner ideal of $W^*(M)$ generated by J.

Lemma 4.4.6

Let A be a JC-algebra with norm closed inner ideal I. Then $(I^{**})^{ew} = (I^e)^{**}$.*

Proof

By looking at the second duals, we have $I^{**} \subset (I^e)^{**}$, which is a weak* closed inner ideal of $C^*(A)^{**}$. Thus, by definition, $(I^{**})^{ew} \subset (I^e)^{**}$.

Conversely, I is contained in $(I^{**})^{ew} \cap C^*(A)$, a norm closed inner ideal of $C^*(A)$. So, again by definition, $I^e \subset (I^{**})^{ew} \cap C^*(A) \subset (I^{**})^{ew}$. The converse now follows by taking the weak* closure of I^e in $C^*(A)^{**}$. $\square$

Theorem 4.4.7 ([HaSt, §7],[Aj, §4])

Let M be a JW-algebra generating a W*-algebra W (in some $B(H)$).*

(a) *If M has no type I_2 part then M is type I, II or III if and only if W is type I, II or III, respectively.*

(b) *If W is of type I then M is of type I.*

Proposition 4.4.8

Let A be a JC-algebra with norm closed inner ideals I and J such that $I \subset J$. Suppose that $\partial_e(I_1^*) = \partial_e(J_1^*)$. Then $I^e = J^e$. That is, I and J generate the same norm closed inner ideal in $C^*(A)$.*

Proof

First note that, by (4.4.4) together with (4.2.5(b)), it is enough to prove that $\rho(I^e) \neq 0$ for all $\rho \in \partial_e((J^e)_1^*)$.

Let $\rho \in \partial_e((J^e)_1^*)$. Then $s(\rho) \in (J^e)^{**}$, and this is a minimal tripotent in $C^*(A)^{**}$. In which case, $s(\rho)$ generates a type I factor weak* closed ideal in $C^*(A)^{**}$, $C^*(A)^{**}z$, where z is some central projection in $C^*(A)^{**}$.

Clearly $A^{**}z$ is contained in $C^*(A)^{**}z$ as a JW*-subalgebra and generates it. Consequently, since the centre of $A^{**}z$ lies in the centre of $C^*(A)^{**}z$ [Top1, Proposition 1], $A^{**}z$ is a type I factor by (4.4.7). Now, $J^{**}z$, as a weak* closed inner ideal of $A^{**}z$, is a Cartan factor.

The map given by $x \mapsto xz$ is a surjective, weak* continuous, triple homomorphism between J^{**} and $J^{**}z$. Since $J^{**}z$ is a Cartan factor, this must restrict to an isomorphism on some Cartan factor summand of J^{**}, and hence on some Cartan factor summand of J^{**}_{at}. In particular, we must have $J^{**}_{at}z = J^{**}z$, giving $J^{**}z = J^{**}_{at}z = I^{**}_{at}z \subset I^{**}z$, so that $I^{**}z = J^{**}z$.

Next we consider the associated weak* closed inner ideals that are generated in the universal enveloping W*-algebra. We have

$$(I^{**})^{ew}z = (I^{**}z)^{ew} = (J^{**}z)^{ew} = (J^{**})^{ew}z,$$

so that, via (4.4.6), $(I^e)^{**}z = (J^e)^{**}z$.
Since $s(\rho) \leq z$, so that $\rho(z) = 1$, we have $\rho(az) = \rho(a)$ for all a in $C^*(A)^{**}$. Hence $\rho((I^e)^{**}) = \rho((I^e)^{**}z) = \rho((J^e)^{**}z) = \rho((J^e)^{**}) \neq 0$. So $\rho(I^e) \neq 0$, as required. □

4.4.9 Let I be a weak* closed inner ideal of a universally reversible JW*-algebra M. Furthermore, suppose that neither I nor M has non-zero type I_1 part. There exist central projections $z_1, z_2 \in M$ with $z_1 + z_2 = 1$, and such that $M = Mz_1 \oplus Mz_2$, where Mz_1 and Mz_2 are, respectively, the type I and continuous parts of M. Then I has a similar decomposition into type I and continuous parts, given by Iz_1 and Iz_2, respectively.
The canonical involution ϕ of $W^*(M)$, through restriction, gives rise to the canonical involution on $W^*(Mz_2) = W^*(M)z_2$. As M is universally reversible, $Mz_i = W^*(M)^\phi z_i$ $(i = 1, 2)$ [HaSt, 7.3.3]. So, by (3.4.2), the continuous part of I is of the form "$eM\phi(e)$".
The type I part of I can be further decomposed into the sum of its finite Cartan part and its complement, N, say. By construction, N is an ℓ^∞-sum of JW*-triples of the form $A\overline{\otimes}C$, where A is an abelian von Neumann algebra and C is a Cartan factor of infinite dimension. Since, except when C is rectangular, all such summands possess a unitary tripotent, we have $N = K \oplus R$, where K contains a unitary tripotent u, say, and where R is rectangular without type I_1 part. As a consequence, $K = uu^*Ku^*u = eM\phi(e)$, where e is the projection uu^* in $W^*(M)$. Through (3.4.25), R is also of the form "$eM\phi(e)$".

Taking this discussion as our motivation, in the final preparatory step of this section we show that inner ideals of the form "$eM\phi(e)$" are determined by the weak* closed inner ideals they generate in $W^*(M)$. In fact, for those universally reversible JW*-algebra without symplectic type I part, this result is contained within the proof of (3.4.24).

Lemma 4.4.10

Let M be a universally reversible JW-algebra and let e be a projection in $W^*(M)$. Let $I = eM\phi(e)$. Then $I = I^{ew} \cap M$.*

Proof

Since I is contained in $eW^*(M)\phi(e)$ we have, by definition,

$$I^{ew} \subset eW^*(M)\phi(e).$$

However $I = eW^*(M)\phi(e) \cap M$. Indeed given $x \in eW^*(M)\phi(e) \cap M$ we have $x = ea\phi(e)$ for some a in $W^*(M)$, so that

$$x = \phi(x) = e\phi(a)\phi(e) = e\Big(\frac{a + \phi(a)}{2}\Big)\phi(e) \in I.$$

The conclusion is now clear. $\square$

4.4.11 We are now ready to prove the Inner Stone-Weierstrass Theorem for universally reversible JC*-algebras. Subsequently, we shall extend this theorem to all JB*-triples (5.4.9), included amongst which are all JC*-algebras and JB*-algebras. We make repeated use of the elementary fact that if U, V and W are subspaces of the same linear space, and are such that $U \subset V$, $U + W = V + W$ and $U \cap W = V \cap W$, then $U = V$.

Theorem 4.4.12

Let A be a universally reversible JC-algebra. Let I and J be norm closed inner ideals of A such that $I \subset J$. Suppose that $\partial_e(I_1^*) = \partial_e(J_1^*)$. Then $I = J$.*

Proof

By (4.2.5), $I_{at}^{**} = J_{at}^{**}$. We claim that it is enough to prove the theorem in the case when A^{**}, I^{**} and J^{**} have no type I_1 part. To see this, note that by (3.5.6), A has a norm closed ideal K such that A/K is abelian and K has no one dimensional representations. By (4.2.14(b)), $(I+K)/K$ and $(J+K)/K$ have the same atomic second dual parts and are equal, by the Stone-Weierstrass Theorem for abelian JC*-triples [FrRu3, 3.4], or, more generally, [Shep3, 5.5]. Thus, if $I \cap K = J \cap K$, then $I = J$. Therefore, replacing I, J and A with $I \cap K$, $J \cap K$ and K, respectively, we may as well suppose that A^{**} has no non-zero type I_1 part, through (3.5.10).

Furthermore, by (3.5.6) and (3.5.10), there is a norm closed ideal $\hat{J}$ of J such that $(J/\hat{J})^{**}$ is type I_1, and J_1^{**} has no type I_1 part. By (4.2.14(a)(b)), we have

$$(I \cap \hat{J})_{at}^{**} = (\hat{J})_{at}^{**} \quad \text{and} \quad (I + \hat{J}/\hat{J})_{at}^{**} = (J/\hat{J})_{at}^{**}.$$

Since $J/\hat{J}$ is postliminal, it follows from [Shep3, 5.5] that $I + \hat{J} = J$. Thus $I = J$ if $I \cap \hat{J} = \hat{J}$. Hence, replacing I and J with $I \cap \hat{J}$ and $\hat{J}$, respectively, we may assume that I^{**} and J^{**} have no type I_1 part, proving our claim.

Carrying on, the bidual of each inner ideal has a decomposition into a finite Cartan part and a weak* closed ideal containing no part of finite Cartan type. By (4.3.3) I^{**} and J^{**} have the same finite Cartan part. Therefore let $I^{**} = M \oplus N$ and $J^{**} = M \oplus K$, where M denotes the common finite Cartan part and N and K denote the respective ideals containing no part of finite Cartan type.

Now we consider N. Using type decomposition and Cartan factor structure, we can create a further decomposition into a continuous part, and a type I part (containing no part of finite Cartan type). As discussed in (4.4.9), each summand is of the form $eA^{**}\phi(e)$, for some projection e in $C^*(A)^{**}$, and thus N also has this form. Since a similar argument holds for K, we have projections $e, f \in C^*(A)^{**}$ such that $N = eA^{**}\phi(e)$ and $K = fA^{**}\phi(f)$. Now, through (2.10.21) and (4.4.6), together with (4.4.8), we observe that

$$M^{ew} \oplus N^{ew} = (I^e)^{**} = (J^e)^{**} = M^{ew} \oplus K^{ew}.$$

Thus $N^{ew} = K^{ew}$. It follows from (4.4.10) that

$$N = N^{ew} \cap A^{**} = K^{ew} \cap A^{**} = K,$$

and therefore $I^{**} = J^{**}$, so that $I = J$, as required. $\square$

Chapter 5

Extreme Functionals and the Inner Ideal Structure of JB*-Triples

5.1 Introduction

It is in this chapter that we achieve the fundamental objective of the thesis, that is, we determine norm closed inner ideals of JB*-triples in terms of extreme functionals. To be precise, we prove that two norm closed inner ideals I and J of a JB*-triple coincide if and only if $\partial_e(I_1^*) = \partial_e(J_1^*)$. In addition, we observe that I is contained in J exactly when $\partial_e(I_1^*)$ is contained in $\partial_e(J_1^*)$. To fulfil this aim we use the Inner Stone-Weierstrass Theorem for JB*-triples, and as a consequence the majority of this chapter is concerned with deriving this result. For clarity, our global conjecture is:

Let A be a JB-triple with norm closed inner ideals I and J, with $I \subset J$ and such that $\partial_e(I_1^*) = \partial_e(J_1^*)$. Then I and J are equal.*

The JB*-triple version naturally supercedes those for C*-algebras and universally reversible JC*-algebras given in Chapter Four (4.4.4)(4.4.12). As each norm closed inner ideal is a JB*-triple in its own right, we may suppress one inner ideal, that is, in the above we may assume that $J = A$.

One technicality, in conjunction with Cartan factor representation theory (3.5.6), is vital to our argument. Namely, we demonstrate that to deduce the Inner Stone-Weierstrass Theorem for a JB*-triple A, it is sufficient to show it holds for an ideal J of A and for the quotient, A/J. Through this technicality, our attention can be focused upon JC*-triples whose Cartan factor representations all have rank greater than two. Indeed, the Inner Stone-Weierstrass Theorem for JB*-triples can be constructed from the JC*-triple version, using this argument, since every JB*-triple A has an exceptional postliminal ideal J such that A/J is a JC*-triple (2.12.6). The postliminal case is covered by the Stone-Weierstrass Theorem for postliminal JB*-triples provided by Sheppard, [Shep3, 5.5]. Furthermore, through (3.5.6), we can isolate a non-zero ideal K of A such that K/J is postliminal, and all Cartan factor representations of J have rank greater than two.

In broad terms, to prove the Inner Stone-Weierstrass theorem for these particular JC*-triples, we make use of the version for universally reversible JC*-algebras (4.4.12). We use a composition series argument typical of [BuChZa2]. More precisely, in the most important stage of this chapter, for a JC*-triple whose Cartan factor representations all have rank greater than two, we identify a composition series with successive quotients that are isomorphic to inner ideals in a universally reversible JC*-algebra. Such an isomorphism may be of independent interest. The crucial step is the identification of a non-zero ideal, through [MoRod1], with which to commence the series.

5.2 Clifford and Z-Hermitian Type JB*-Triples

5.2.1 Our remarks in this section are overwhelmingly influenced by the pioneering interpretation by Moreno-Galindo and Rodriguez-Palacios, given in [MoRod1] and [MoRod2], of the algebraic work of Zel'manov. The deep achievement of [MoRod1] was the classification of prime JB*-triples (together with their real versions). This was derived by the introduction of new techniques into JB*-triple theory. It is some of these techniques that we exploit in order to penetrate structure theory. We show, see (5.3.9), that if a JC*-triple has only Cartan factor representations of rank greater than two, then it has a composition series in which successive quotients can be realised as inner ideals in a universally reversible JC*-algebra. This is an important step towards the proof of the general Inner Stone-Weierstrass Theorem. It might also be of some independent interest.

For clarity and in order to explain terms, we have extracted the following synopsis from [MoRod1], where full details can be found. We first remark that in place of the term "hermitian" used in [MoRod1][MoRod2] we have employed the invented term *Z-hermitian*. Our excuse for this is that throughout this thesis we use the term hermitian in a way that conflicts with its meaning in [MoRod1][MoRod2].

5.2.2 Given a set of indeterminates X, $\mathcal{A}(X)$ denotes the free associative algebra generated by X and $ST(X)$ denotes the Jordan subtriple of $\mathcal{A}(X)$ generated by X. The Jordan triple system $ST(X)$ is the free Jordan triple system generated by X. A certain distinguished ideal, the *Zel'manov ideal*, G, of $ST(X)$ leads to the following definitions.

Let A be a JC*-triple. Then A is said to be *Z-hermitian* if $G(A) \neq 0$, that is, if for some triple polynomial $p(x_1, ..., x_n) \in G$ there exist $a_1, ..., a_n \in A$ such that $p(a_1, ..., a_n) \neq 0$. If $G(A) = 0$ then A is said to be *Clifford type*. It is immediate from the definition that if A is a JC*-triple of Clifford type then so is every JC*-subtriple and quotient of A.

Theorem 5.2.3 ([MoRod1, 6.1])

Let C be a special Cartan factor. Then

(a) C is of Clifford type if and only if C has rank at most two;

(b) C is of Z-hermitian type if and only if C has rank greater than two.

Proof

Part(a) is [MoRod1, 6.1]. Part (b) is immediate from (a). $\square$

Proposition 5.2.4

Let A be a JC-triple. Then*

(a) A is of Clifford type if and only if all Cartan factor representations of A have rank at most two;

(b) A is of Z-hermitian type if and only if A has a Cartan factor representation of rank greater than two.

Proof

(a) Let $\pi : A \to C$ be a Cartan factor representation. If A is of Clifford type then C is of Clifford type through the argument of the second paragraph of the proof of [MoRod1, 7.3]. Thus the rank of C is at most two (5.2.3(a)).

On the other hand, suppose that C has rank at most two, and so, through (5.2.3), is of Clifford type. Let $p(x_1, ..., x_n)$ lie in the Zel'manov ideal and let $a_1, ..., a_n \in A$. As C is of Clifford type, we have

$$\pi(p(a_1, ..., a_n)) = p(\pi(a_1), ..., \pi(a_n)) = 0.$$

Therefore, if all Cartan factor representations of A have rank at most two, we must have $p(a_1, ..., a_n) = 0$, and so $G(A) = 0$, giving that A is of Clifford type.

(b) This is immediate from part (a). □

5.3 A Composition Series Relating JC*-Triples to Universally Reversible JC*-Algebras

5.3.1 The purpose of this section is to show that every Z-hermitian JC*-triple, that is, a JC*-triple for which all Cartan factor representations have rank greater than two, has a composition series whose successive quotients can be realised as norm closed inner ideals in a universally reversible JC*-algebra. As in section (5.2), we again make essential use of the ideas and results of [MoRod1].

5.3.2 To begin with, we recall that a C*-algebra A has a *matricial decomposition*, $\{A_{ij}\}$, where $i, j \in \{1, 2\}$, if the A_{ij} are norm closed subspaces of A with linear direct sum A, and satisfying $A_{ij}A_{kl} \subset \delta_{jk}A_{il}$. An involution, $\alpha : A \to A$, is said to be *even swapping* if $\alpha(A_{11}) = A_{22}$ and $\alpha(A_{12}) = A_{12}$.

5.3.3 Paraphrasing the argument of [MoRod1, 5.5], if A is a JC*-subtriple of $B(H)$, for some complex Hilbert space H, take any involution, $\beta : A \to A$. For example, we might take β to be a real flip. Define a further involution,

$$\gamma : M_2(B(H)) \to M_2(B(H)); \qquad \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} \beta(d) & \beta(b) \\ \beta(c) & \beta(a) \end{pmatrix}\right],$$

and induce yet another, $\alpha : B \to B$,

$$\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} \oplus \begin{pmatrix} x & y \\ z & w \end{pmatrix} \mapsto \begin{pmatrix} \beta(w) & \beta(y) \\ \beta(z) & \beta(x) \end{pmatrix} \oplus \begin{pmatrix} \beta(d) & \beta(b) \\ \beta(c) & \beta(a) \end{pmatrix}\right],$$

where $B = M_2(B(H)) \oplus M_2(B(H))$.

Consider the isomorphism

$$\pi : B(H) \to B \qquad \left[x \mapsto \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & \beta(x) \\ 0 & 0 \end{pmatrix}\right].$$

Now, B has the obvious matricial decomposition, with respect to which α is even swapping and we have $\pi(A) \subset B^\alpha \cap B_{12}$. (That is, with

$$B_{12} = \left\{ \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} : a, x \in B(H) \right\}$$

and so on). When the JC*-triple A is Z-hermitian [MoRod1, 5.6] uses Zel'manov theory, (see (5.2)), to find an α-invariant C*-subalgebra of B to establish the following crucial refinement.

Theorem 5.3.4 ([MoRod1, 5.6])

Let A be a Z-hermitian JC-triple. Then there exists a C*-algebra B with a matricial decomposition $\{B_{ij}\}$ and an associated even swapping involution, α, such that*

(a) A contains a non-zero ideal $J \cong B^\alpha \cap B_{12}$;

(b) $B^\alpha \cap B_{12}$ generates B as a C-algebra.*

We now come to the point of this discussion.

Lemma 5.3.5

Let $\{A_{ij}\}$ be a matricial decomposition of a C-algebra A. Then A_{12} is a norm closed inner ideal of A.*

Proof

By definition, A_{12} is norm closed and the multiplication rules give

$$\begin{aligned} A_{12}AA_{12} &= A_{12}(A_{11}+A_{12}+A_{21}+A_{22})A_{12} \\ &\subset A_{12}A_{21}A_{12} \subset A_{12}. \end{aligned} \qquad \square$$

The following is slightly more general than we require.

Lemma 5.3.6

Let I be a norm closed inner ideal in a JC-triple A and let J be the norm closed ideal of A generated by I. Let every Cartan factor representation of I have rank greater than two. Then every Cartan factor representation of J has rank greater than two.*

Proof

Let $\pi : J \to C$ be a Cartan factor representation of J. Since I generates J, if π vanishes on I then it vanishes on J. Hence, the weak* closure D of $\pi(I)$ in C is a non-zero weak* closed inner ideal of C, and thus a Cartan factor. By hypothesis, D has rank greater than two. Therefore, C has rank greater than two. $\square$

Proposition 5.3.7

Let A be a JC-triple for which every Cartan factor representation has rank greater than two. Then A contains a non-zero norm closed ideal J that is isomorphic to a norm closed inner ideal in a universally reversible JC*-algebra.*

Proof

By (5.2.4) A is Z-hermitian. Therefore, by (5.3.4) and (5.3.5), there exists a C*-algebra B with an involution α and a norm closed inner ideal I, such that $B^\alpha \cap I$ generates B and is isomorphic to a norm closed ideal, J say, of A.

Now, since all Cartan factor representations of J must have rank greater than two, the same must be true of $B^\alpha \cap I$. The latter is a norm closed inner ideal of the JC*-algebra B^α and, by assumption, must generate it as a JC*-algebra. Since norm closed ideals of B^α are, in particular, JC*-subalgebras, it follows from (5.3.6) that all Cartan factor representations of B^α have rank greater than two. Hence, B^α has no Jordan algebra spin factor representations and is therefore universally reversible, by (1.13.6). $\square$

5.3.8 We can now prove the main result of this section. The implication is that most JC*-triple theory can be reduced to the study of norm closed inner ideals in universally reversible JC*-algebras.

Theorem 5.3.9

Let A be a JC-triple for which all Cartan factor representations of A have rank greater than two. Then A has a composition series of ideals, $(J_\lambda)_{0\leq\lambda\leq\alpha}$, such that $J_{\lambda+1}/J_\lambda$ is isomorphic to an inner ideal in a universally reversible JC*-algebra.*

Proof

By (5.3.7), A contains a non-zero norm closed ideal J_1 that is isomorphic to an inner ideal in some universally reversible JC*-algebra. Since all Cartan factor representations of A/J_1 must have rank greater than two also, it follows that there is a norm closed ideal J_2 of A, such that $J_1 \subset J_2$ and $J_1 \neq J_2$, with J_2/J_1 again of the required form. Proceeding by transfinite induction, the result follows. □

5.4 The Inner Stone-Weierstrass Theorem for JB*-Triples

5.4.1 We are now ready to establish the Inner Stone-Weierstrass Theorem for JB*-triples, that is, if A is a JB*-triple with norm closed inner ideals I and J, with I contained in J and such that the extreme dual ball points of I and J are identical, then I and J are equal. As already intimated in the introduction to this chapter, as a norm closed inner ideal is a JB*-triple in its own right it is sufficient to show that the aforementioned theorem holds when $J = A$. It only remains to bring together all the key techniques developed in the earlier sections of this chapter.

5.4.2 Before we begin this section in earnest, we establish a vital technicality that enables a useful reduction to be made. That is, we show that it will be enough to consider JC*-triples whose Cartan factor representations are all of rank greater than two. We first recall the conjecture we aim to affirm.

Conjecture 5.4.3

Let A be a JC-triple and let I be a norm closed inner ideal of A. Suppose that $\partial_e(I_1^*) = \partial_e(A_1^*)$. Then $I = A$.*

Proposition 5.4.4

Let A be a JB-triple and let J be a norm closed ideal of A. Then if the conjecture (5.4.3) holds for J and for A/J, it holds for A.*

Proof

Let I be a norm closed inner ideal of A such that $\partial_e(I_1^*) = \partial_e(A_1^*)$, so that $I_{at}^{**} = A_{at}^{**}$ by (4.2.5). Then $I \cap J$ is a norm closed inner ideal of J and, via (4.2.14(a)), $(I \cap J)_{at}^{**} = J_{at}^{**}$. In which case $I \cap J = J$, since, by assumption, the conjecture holds for J. Therefore J is contained in I.

Also $(I + J)/J$ is a norm closed inner ideal of A/J and, via (4.2.14(b)),

$$(A/J)_{at}^{**} = ((I + J)/J)_{at}^{**},$$

so $(I + J)/J = A/J$, since, by assumption, the conjecture holds for A/J. In conclusion

$$I/J = (I + J)/J = A/J,$$

and so $I = A$. $\square$

5.4.5 If all Cartan factor representations of a JC*-triple have rank at most two then the triple is postliminal. In this case, using Sheppard's Stone-Weierstrass Theorem for postliminal JB*-triples [Shep3, 5.5], our conjecture (5.4.3) holds. Accordingly, let A be a JC*-triple with a Cartan factor representation of rank greater than two. Then, via (3.5.6), A has a norm closed ideal J such that all Cartan factor representations of J are of rank greater than two and A/J is postliminal. Consequently, by using (5.4.4) in conjunction with [Shep3, 5.5], it remains to show that the conjecture (5.4.3) holds for a JC*-triple whose Cartan factor representations all have rank greater than two.

Theorem 5.4.6

Let A be a JC-triple and let I be a norm closed inner ideal of A. Suppose that $\partial_e(I_1^*) = \partial_e(A_1^*)$. Then $I = A$.*

Proof

As we have just remarked, (5.4.5), we may suppose that A is a JC*-triple whose Cartan factor representations all have rank greater than two. By (5.3.9), A has a composition series of closed ideals, $(J_\lambda)_{0\leq\lambda\leq\alpha}$, such that $J_{\lambda+1}/J_\lambda$ is isomorphic to an inner ideal in a universally reversible JC*-algebra. Suppose that $A \neq I$. Then there is a first ordinal β such that $J_\beta \not\subset I$. In particular, $J_\lambda \subset I$ for all $\lambda < \beta$. If β is a limit ordinal then, since I is norm closed,

$$J_\beta = \overline{(\bigcup_{\lambda<\beta} J_\lambda)} \subset I,$$

where bar denotes norm closure. Therefore, by contradiction, there exists $\beta' < \beta$ such that $\beta = \beta' + 1$. Then $J_{\beta'} \subset I$.

Hence, $(I \cap J_\beta)/J_{\beta'}$ is a norm closed inner ideal of $J_\beta/J_{\beta'}$ and, by (4.2.5) together with (4.2.14(a)(b)), we have

$$((I \cap J_\beta)/J_{\beta'})^{**}_{at} = (J_\beta/J_{\beta'})^{**}_{at}.$$

Thus, using the relevant Inner Stone-Weierstrass Theorem, (4.4.12), in conjunction with (4.2.5), we see that $I \cap J_\beta = J_\beta$, so that $J_\beta \subset I$, a contradiction. $\square$

5.4.7 Let A be a JB*-triple. By Friedman and Russo's result, (2.12.3), A has an exceptional ideal J such that A/J is a JC*-triple. Since J is postliminal, by (5.4.4) together with [Shep3, 5.5], it is enough to show that the conjecture holds for A/J, that is for a JC*-triple. Therefore the next result follows as a simple corollary of (5.4.6).

Theorem 5.4.8

Let A be a JB-triple and let I be a norm closed inner ideal of A. Suppose that $\partial_e(I_1^*) = \partial_e(A_1^*)$. Then $I = A$.*

Finally we reach our desired conclusion, the Inner Stone-Weierstrass Theorem for JB*-triples.

Theorem 5.4.9

Let A be a JB-triple. Let I and J be norm closed inner ideals of A with $I \subset J$. Suppose that $\partial_e(I_1^*) = \partial_e(J_1^*)$. Then I=J.*

5.5 Inner Ideals and Extreme Functionals

5.5.1 The ultimate goal of the thesis, and in particular of this chapter, is to show that norm closed inner ideals in JB*-triples are determined by their dual ball extreme points. The major instrument necessary to fulfil this aim has now been established, (5.4.9). It only remains to prove one final technicality. Let A be a JB*-triple with norm closed inner ideal I. It is clear that the atomic part of the bidual of I is contained within the atomic part of the bidual of A. We show that the same is true for the respective, complementary, non-atomic parts.

Lemma 5.5.2

Let A be a JB-triple and let I be a norm closed inner ideal of A. Then $(I_{at}^{**})^{\perp}$ is an inner ideal of $(A_{at}^{**})^{\perp}$.*

Proof

Since the annihilator of an inner ideal is always an inner ideal (2.10.4), we have only to show that $(I_{at}^{**})^{\perp}$ is contained in $(A_{at}^{**})^{\perp}$. By (2.10.15(a)),

$$(I_{at}^{**})^{\perp} = ((I_{at}^{**})^{\perp} \cap A_{at}^{**}) \oplus ((I_{at}^{**})^{\perp} \cap (A_{at}^{**})^{\perp}).$$

As the first summand is an atomic weak* closed ideal of the non-atomic part of I^{**}, it must vanish. Therefore, $(I_{at}^{**})^{\perp} = (I_{at}^{**})^{\perp} \cap (A_{at}^{**})^{\perp}$, which is clearly contained in $(A_{at}^{**})^{\perp}$. □

Theorem 5.5.3

Let A be a JB-triple and let I and J be norm closed inner ideals of A. Then I=J if and only if $\partial_e(I_1^*) = \partial_e(J_1^*)$.*

Proof

Suppose that $\partial_e(I_1^*) = \partial_e(J_1^*)$ so that $I_{at}^{**} = J_{at}^{**}$ by (4.2.5). Let $\mathcal{N}(I,J)$ denote the norm closed inner ideal in A generated by I and J, and henceforth let bar denote weak* closure. By atomic decomposition (2.11.7(a)), we have

$$I^{**} = I_{at}^{**} \oplus (I_{at}^{**})^{\perp} \quad \text{and} \quad J^{**} = J_{at}^{**} \oplus (J_{at}^{**})^{\perp} = I_{at}^{**} \oplus (J_{at}^{**})^{\perp}.$$

From (5.5.2) we see that $(I_{at}^{**})^{\perp}$ and $(J_{at}^{**})^{\perp}$ are inner ideals of $(A_{at}^{**})^{\perp}$, which is weak* closed. Let $K = \overline{\mathcal{N}((I_{at}^{**})^{\perp}, (J_{at}^{**})^{\perp})}$. Then K is a weak* closed inner ideal of $(A_{at}^{**})^{\perp}$. So it follows that $K_{at} = \{0\}$. Now let $L = \overline{\mathcal{N}(I^{**}, J^{**})}$, where $\mathcal{N}(I^{**}, J^{**})$ denotes the norm closed inner ideal of A^{**} generated by I^{**} and J^{**}. Since $I_{at}^{**} = J_{at}^{**} \subset A_{at}^{**}$ and $K \subset (A_{at}^{**})^{\perp}$, we have

$$\begin{aligned} L &= \overline{\mathcal{N}(I_{at}^{**} \oplus (I_{at}^{**})^{\perp}, J_{at}^{**} \oplus (J_{at}^{**})^{\perp})} \\ &= \overline{\mathcal{N}(I_{at}^{**}, J_{at}^{**})} \oplus \overline{\mathcal{N}((I_{at}^{**})^{\perp}, (J_{at}^{**})^{\perp})} \\ &= I_{at}^{**} \oplus K. \end{aligned}$$

In consequence, $L_{at} = I_{at}^{**}$ and $(L_{at})^{\perp} = K$.

Clearly, $\mathcal{N}(I,J) \subset \mathcal{N}(I^{**}, J^{**}) \subset \overline{\mathcal{N}(I^{**}, J^{**})} = L$ and so $\mathcal{N}(I,J)^{**} \subset L$. Since I is a norm closed inner ideal of $\mathcal{N}(I,J)$, it follows that

$$\mathcal{N}(I,J)_{at}^{**} \subset L_{at} = I_{at}^{**} \subset \mathcal{N}(I,J)_{at}^{**}.$$

In summary, we have $I \subset \mathcal{N}(I,J) \subset A$ with $I_{at}^{**} = \mathcal{N}(I,J)_{at}^{**}$. It is now immediate from the Inner Stone-Weierstrass Theorem, (5.4.9), in conjunction with (4.2.5), that J is contained in I. Similarly $I \subset J$, giving equality.

The converse is clear. $\square$

Theorem 5.5.4

Let A be a JB-triple and let I and J be two norm closed inner ideals of A. Then $I \subset J$ if and only if $\partial_e(I_1^*) \subset \partial_e(J_1^*)$.*

Proof

We use the notation of the previous theorem, in particular bar will denote weak* closure.

Suppose that $\partial_e(I_1^*) \subset \partial_e(J_1^*)$, so that $I_{at}^{**} \subset J_{at}^{**}$ by (4.2.5). As noted in the fourth paragraph of (5.5.3), (and in the notation of (5.5.3)),

$$\mathcal{N}(I, J)^{**} \subset \overline{\mathcal{N}(I^{**}, J^{**})},$$

and $J_{at}^{**} \subset \mathcal{N}(I, J)_{at}^{**} \subset \overline{\mathcal{N}(I_{at}^{**}, J_{at}^{**})}$, which is contained in J_{at}^{**}, since the atomic part of the bidual of I is contained in that of the bidual of J. Thus $\mathcal{N}(I, J)$ and J are norm closed inner ideals of A whose second duals have identical atomic parts. It follows, from (5.4.9), together with (4.2.5), that $I \subset \mathcal{N}(I, J) = J$, as claimed.

The converse is apparent. $\square$

5.5.5 To conclude, we consider a reformulation of the previous theorem. Let A be a JB*-triple with norm closed inner ideal I. Let $\rho \in A^*$ and let $\rho_{|_I}$ denote its restriction to I. By weak* continuity, we can identify ρ with its weak* continuous extension and, in this manner, identify the restriction of ρ to I^{**} with $\rho_{|_I}$. Let $I^{\#} = \{\rho \in A^* : \|\rho_{|_I}\| = \|\rho\|\}$, that is, the set of functionals of A with norm preserving restriction to I.

Corollary 5.5.6

Let A be a JB-triple. Let I and J be norm closed inner ideals of A. Then $I \subset J$ if and only if $I^{\#} \cap \partial_e(A_1^*) \subset J^{\#} \cap \partial_e(A_1^*)$.*

Proof

Let $\rho \in \partial_e(A_1^*)$. Then $\rho \in I^{\#}$ if and only if $s(\rho) \in I_{at}^{**}$. Indeed, if $s(\rho)$ is in I^{**} then $\|\rho_{|_I}\| \geq \rho(s(\rho)) = \|\rho\|$, so that $\|\rho\| = \|\rho_{|_I}\|$. On the other hand, suppose that $\rho \in I^{\#}$, so that $\|\rho\| = \|\rho_{|_I}\|$. Now, through (2.10.3(a)), $\rho_{|_I}$ has a unique norm preserving extension to a functional on A, so $\rho_{|_I} = \rho$ and hence $s(\rho) \in I^{**}$.

In this way, if $I^{\#} \cap \partial_e(A_1^*) \subset J^{\#} \cap \partial_e(A_1^*)$ we see that $I_{at}^{**} \subset J_{at}^{**}$. It is now evident from (5.5.4), in conjunction with (4.2.5), that I is contained in J.

The converse is clear. $\square$

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