High Frequency Boundary Element Methods for Scattering by Convex Polygons

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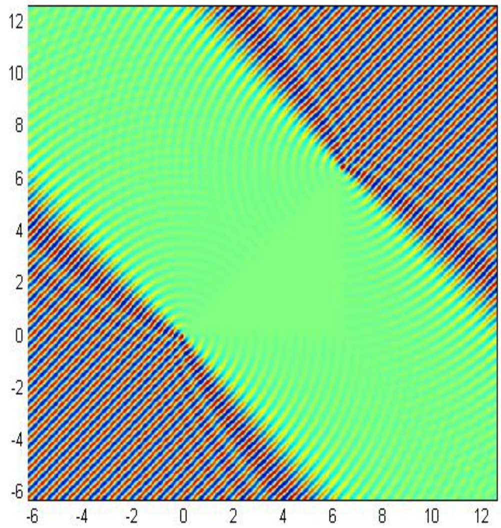
Declaration

I confirm that this work is my own and the use of all other material from other sources has been properly and fully acknowledged.

Abstract



We consider a numerical approximation to the scattering of a high-frequency plane wave by a sound soft convex polygon. By reformulating the domain problem to a boundary problem, we approximate the solution on the boundary by piecewise polynomials multiplied by waves. Using theory of the best approximation of polynomials we aim to show an error bound and how it varies with the number of mesh points and the polynomial degree. We discover that we can achieve exponential convergence, as well as seeing the optimum values and ratios of the parameters involved.

Contents

1 Introduction

1.1 Background

In engineering and physics there are many problems where the acoustic scattering of objects in a medium by a given wave require simulating, for example sonar and noise reduction. In a homogeneous medium the pressure $P(\mathbf{x}, t)$ satisfies the wave equation

$$\nabla^2 P - \frac{1}{c^2}\frac{\partial^2 P}{\partial t^2} = 0$$

where c is the wave speed in the medium. By considering only the time-harmonic case with angular frequency ω, the pressure can be written as

$$P(\mathbf{x}, t) = \mathrm{Re}\left(u(\mathbf{x})\mathrm{e}^{-\mathrm{i}\omega t}\right)$$

The function $u(\mathbf{x})$ is known as the *complex acoustic pressure*. By substituting this back into the wave equation we get the Helmholtz equation

$$\nabla^2 u + k^2 u = 0 \tag{1.1}$$

Here

$$k := \frac{\omega}{c} = \frac{2\pi f}{c} = \frac{2\pi}{\lambda}$$

where f is the frequency and λ is the wavelength of the incoming wave. A result of Green's second theorem is that if (1.1) is satisfied in a domain D with boundary ∂D, the solution $u(\mathbf{x})$ must satisfy the following integral equation for all $\mathbf{x} \in D$ with $\mathbf{x} \neq \mathbf{x}_0$,

$$u(\mathbf{x}) = G(\mathbf{x}_0, \mathbf{x}) + \int_{\partial D}\left[G(\mathbf{y}, \mathbf{x})\frac{\partial u}{\partial n}(\mathbf{y}) - u(\mathbf{y})\frac{\partial G(\mathbf{y}, \mathbf{x})}{\partial n(\mathbf{y})}\right] ds(\mathbf{y})$$

where

$$G(\mathbf{x}, \mathbf{x}_0) := -\frac{\mathrm{i}}{4}H_0^{(1)}(k|\mathbf{x} - \mathbf{x}_0|)$$

is a fundamental solution of the Helmholtz equation. The function $H_0^{(1)}$ is the Hankel function of the first kind of order zero, and its real and imaginary parts are Bessel functions. Note the expression $\mathbf{n}(\mathbf{y})$ denotes the normal direction at $\mathbf{y}$, and the expression

$$\frac{\partial G(\mathbf{y},\mathbf{x})}{\partial n(\mathbf{y})}$$

is the rate of increase of $G(\mathbf{y},\mathbf{x})$ as $\mathbf{y}$ moves off the boundary in the direction $\mathbf{n}(\mathbf{y})$. The integral is sometimes called a *Green's representation formula*, and has the crucial property that if we know the values of u and $\frac{\partial u}{\partial n}$ on ∂D, we have an explicit formula for computing the solution throughout the domain D, though not on ∂D.

The boundary condition considered for this dissertation is the 'sound-soft' condition $u = 0$ on the boundary of a convex polygon Ω, and $D = \mathrm{R}^2 \backslash \Omega$. This removes one of the terms of the integral equation, and additionally removes the complication that the integral equation does not hold when $\mathbf{x} \in \partial D$. Thus the problem now becomes to solve the following boundary integral equations

$$u(\mathbf{x}) = G(\mathbf{x}_0,\mathbf{x}) + \int_{\partial D} G(\mathbf{y},\mathbf{x})\frac{\partial u}{\partial n}(\mathbf{y})ds(\mathbf{y}), \quad \mathbf{x} \in D$$

$$0 = G(\mathbf{x}_0,\mathbf{x}) + \int_{\partial D} G(\mathbf{y},\mathbf{x})\frac{\partial u}{\partial n}(\mathbf{y})ds(\mathbf{y}), \quad \mathbf{x} \in \partial D$$

The second equation is an integral equation of the first kind for $\frac{\partial u}{\partial n}$, which once solved gives us $\frac{\partial u}{\partial n}$ on ∂D and thus enables us to calculate the value of $u(\mathbf{x})$ throughout D. Furthermore, we have reduced an unbounded 2D problem to a finite 1D problem.

There is one further condition on the solution u. We consider our incident field to be an acoustic plane wave of frequency k approaching in the direction $\mathbf{d} = (\sin\theta, -\cos\theta)$ where θ is the anticlockwise angle from the downward vertical. Thus $u^i(\mathbf{x}) = \mathrm{e}^{\mathrm{i}\mathbf{x}.\mathbf{d}}$, and we define the scattered field $u^s := u - u^i$ which must satisfy the Sommerfeld radiation condition

$$\lim_{r\to\infty} r^{\frac{1}{2}} \left(\frac{\partial u^s}{\partial r}(\mathbf{x}) - \mathrm{i}ku^s(\mathbf{x})\right) = 0$$

where $r = |\mathbf{x}|$ and the limit holds uniformly in all directions $\frac{\mathbf{x}}{|\mathbf{x}|}$.

This condition is equivalent to saying that from a great distance the waves scattered by the polygon are roughly like a a single source. This is analogous to waves on a pond, where the superposition of many sources of ripples of varying strengths looks similar to a single source when viewed from a distance. This results in almost radial waves of frequency k, and so $u^s \approx A\mathrm{e}^{\mathrm{i}kr}$, which leads to the term in brackets tending to zero. We note further that the only waves scattered by the polygon are those in the finite band of waves which actually hit the object, and thus a finite amount of wave energy is scattered. When this scattered energy is distributed uniformly across a circle of radius r, the modulus of $|u^s|$ should tend as $r^{\frac{1}{2}}$.

In [1] it is shown that the integral equation can be reformulated as

$$\frac{1}{2}\frac{\partial u}{\partial n} + \int_{\partial D} \left(\frac{\partial \Phi(\mathbf{x}, \mathbf{y})}{\partial n(\mathbf{x})} + \mathrm{i}\eta\Phi(\mathbf{x}, \mathbf{y}) \right) \frac{\partial u}{\partial n} ds(\mathbf{y}) = f(\mathbf{x})$$

where I is the identity operator, $f = \frac{\partial u^i}{\partial n}(\mathbf{x}) + \mathrm{i}\eta u^i(\mathbf{x})$, and $\Phi(\mathbf{x}, \mathbf{y}) = -G(\mathbf{x}, \mathbf{y})$. The parameter η ensures a unique and non-trivial solution, whereas the original problem could be solved by $u(\mathbf{x}) \equiv 0$ which is clearly not the solution for a scattered wave.

1.2 Motivation

To solve the boundary integral equation the edge is discretised into a mesh and we consider a basis function on each mesh cell. Using a uniform mesh to solve the boundary integral equation typically requires the number of mesh points to grow linearly with the wavenumber k, which results in a matrix system with $O(k^2)$ entries, each of which is an integral which must be accurately calculated. This results in a fast-growing computational cost and memory cost, and is not considered very effective.

In terms of accuracy, when the basis functions are piecewise constants the error is only proportional to n^{-1}, though for a mesh which is tighter near to the corners of the polygon and order p polynomials are used on each mesh cell, the error decreases proportional to $n^{-(1+p)}$ (see [2]). However this method still requires n to grow in proportion to k.

In [1] a modification to the method showed that if the leading order behaviour (i.e. the reflected wave on sides that are lit) is removed, with use of a graded mesh which is very tight near the corners and by using piecewise polynomials multiplied by plane waves as an approximation space, the number of mesh points for a required level of accuracy grows only with $\log k$.

If we parameterise the boundary of length L by the variable s moving in an anticlockwise direction around the polygon, the method used in [1] describes the behaviour of $\frac{\partial u}{\partial n}$ along the boundary, namely on each edge

$$\frac{1}{k}\frac{\partial u}{\partial n}(s) = \Psi(s) + \frac{\mathrm{i}}{2}\left[\mathrm{e}^{iks}v_+(s) + \mathrm{e}^{-iks}v_-(s)\right], \quad s \in [0, L]$$

where

$$\Psi(s) = \begin{cases} \frac{2}{k}\frac{\partial u^i}{\partial n}(\mathbf{x}(s)), & \text{if } \mathbf{x}(s) \text{ is illuminated} \\ 0 & \text{if } \mathbf{x}(s) \text{ is in shadow} \end{cases}$$

is the physical optics approximation to $\frac{\partial u}{\partial n}$, and $v_\pm(s)$ are polynomials on each edge. Theory shows that these polynomials and all their derivatives are highly peaked near the corners of the polygon and rapidly decay away from the corners. In particular, as s moves on a given edge, $v_+(s)$ is highly peaked on the first corner of that edge and decays towards the second. Likewise $v_-(s)$ is very small near to the first corner and becomes highly peaked at the second, so there is a seperate polynomial for each peak.

However an issue with the method as computed is that the approximation space is difference for every value of n, and there is no notion of subspace except in the limit as $N \to \infty$. If however the approximation space determined by n and p in some sense contained a number of approximation spaces determined by lesser values of n and p, then it could be seen that the approximation space tends to the solution space. Furthermore, standard results about normed spaces hold in such a situation (e.g. there exists a best approximation which minimises the error, the error term is orthogonal to the approximation space and the related equations to solve are relatively simple), and the convergence rate with respect to n and p can be calculated separately.

1.3 Overview

In section 2.1 we will consider the error in approximating over $[-1, 1]$ a function which is analytic in a domain around $[-1, 1]$. The best polynomial approximation will be taken using Chebyshev polynomials and the error calculation will be minimised.

In section 2.2 we will use the results from section 2.1 to find the minimum error for a problem on a geometric mesh, and seek to minimise the error bound with respect to the parameters.

In section 2.3 we consider a function which is singular at the origin and seek an error bound from a geometric mesh, using results from the previous two sections.

In section 3.1 we will review current theory about the normal derivative of the total field of a polygon scattering problem, and show how it relates to the work done in the previous sections.

In section 3.2 we take a program designed to solve the *hp* problem, and taking large values of the parameters as an 'exact' solution, we compare how the approximation error varies with the number of degrees of freedom. We then present the results.

In section 3.3 we analyse the results and compare the deductions, theory, and global progress in the field for an overall conclusion.

In section 3.4 we shall describe ideal avenues of further research to improve upon or verify ideas that have resulted from this project.

2 Part I: Theory of hp-approximants in 1-D

2.1 Best Polynomial Approximation of Analytic Functions in the L_2-Norm

Definition 2.1 *We denote by E_ρ, $\rho > 1$ the ellipse*

$$E_\rho := \left\{ z \in \mathbb{C} | \ |z-1| + |z+1| < \rho + \rho^{-1} \right\}$$

Definition 2.2 *The space of polynomials of degree $\leq p$ is denoted by P_p*

Definition 2.3 *The nth Chebyshev polynomial* $\cos(n \cos^{-1}(x))$ *is denoted by $T_n(x)$.*

Lemma 2.4 *Let $\rho > 1$ and $u(z)$ be analytic and bounded in E_ρ with bound M. Then the best approximation $u_p(x) \in P_p$ to $u(z)$ over [-1,1] satisfies*

$$\|u - u_p\|_{L_2[-1,1]} \leq M 2\sqrt{2} \frac{\rho^{-p}}{\rho - 1}$$

Proof. $T_n(x)$ has polynomial degree n, so a basis for P_p is $\{T_0(x), T_1(x) \ldots T_p(x)\}$. $T_n(x)$ is analytic in $E_\rho \forall n$ so we can express $u(z)$ in the form $u(x) = \sum_{n=0}^{\prime\infty} a_n T_n(x)$ for some complex constants a_n, and equivalently $u_p(x) = \sum_{n=0}^{\prime p} a_n T_n(x)$. So the L_2 error between the two functions is the 2-norm of the function

$$v(\theta) = \sum_{n=p+1}^{\infty} a_n \cos(n\theta) = (u - u_p)(x)$$

where $\theta = \arccos(x)$. The function $v(\theta)$ is analytic, 2π periodic and even, and thus can be expressed as a Fourier Cosine Series with coefficients

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} v(\theta) \cos(n\theta) d\theta.$$

Now in the complex plane, $x = \cos(\theta) = \frac{z+z^{-1}}{2}$, meaning x analytic in E_ρ is equivalent to z analytic in the annulus $\rho^{-1} < |z| < \rho$. So

$$a_n = \frac{1}{2\pi\imath} \oint_{|z|=1} (u - u_p) \left(\frac{z + z^{-1}}{2} \right) z^{n-1} dz +$$

$$\frac{1}{2\pi\imath} \oint_{|z|=1} (u - u_p) \left(\frac{z + z^{-1}}{2} \right) z^{-n-1} dz$$

and by Cauchy's Integral Theorem

$$
\begin{aligned}
&= \frac{1}{2\pi\imath}\oint_{|z|=r_1} (u-u_p)\left(\frac{z+z^{-1}}{2}\right) z^{n-1}dz + \\
&\frac{1}{2\pi\imath}\oint_{|z|=r_2} (u-u_p)\left(\frac{z+z^{-1}}{2}\right) z^{-n-1}dz
\end{aligned}
$$

for some $\rho^{-1} \leq r_1, r_2 \leq \rho$. Therefore

$$
\begin{aligned}
|a_n| \leq \frac{1}{2\pi}\oint_{|z|=r_1} &\left\|(u-u_p)\left(\frac{z+z^{-1}}{2}\right)\right\| \left\|z^{n-1}\right\| dz + \\
\frac{1}{2\pi}\oint_{|z|=r_2} &\left\|(u-u_p)\left(\frac{z+z^{-1}}{2}\right)\right\| \left\|z^{-n-1}\right\| dz \\
&\leq M\left(r_1^n + r_2^{-n}\right).
\end{aligned}
$$

This bound holds for all r_1, r_2 s.t. $\rho^{-1} \leq r_1, r_2 \leq \rho$, and is clearly minimised when $r_1 = \rho^{-1}$, $r_2 = \rho$, meaning $|a_n| \leq 2M\rho^{-n}$. This leads to

$$
\|u-u_p\|^2_{L_2[-1,1]} = \int_{-1}^{1}\left(\sum_{n=p+1}^{\infty} a_n T_n(t)\right)^2 dt
$$

$$
\leq \int_{-1}^{1}\left(\sum_{n=p+1}^{\infty} |a_n|\,|T_n(t)|\right)^2 dt \leq \int_{-1}^{1}\left(\sum_{n=p+1}^{\infty} |a_n|\right)^2 dt
$$

$$
= 2\left(\sum_{n=p+1}^{\infty} |a_n|\right)^2 \leq 2\left(\sum_{n=p+1}^{\infty} 2M\rho^{-n}\right)^2 = 8M^2\left(\frac{\rho^{-p}}{\rho-1}\right)^2
$$

Therefore

$$
\|u-u_p\|_{L_2[-1,1]} \leq M2\sqrt{2}\frac{\rho^{-p}}{\rho-1}
$$

□

This is a standard bound from which current approximation theories are developed. However it can be improved by noting that the proof effectively approximates $T_n(x)$ by 1 $\forall n$. Given it is an oscillatory function (and increasingly oscillatory with increasing n), this can be improved upon by taking the complete product of the infinite sum and then integrating the Chebyshev products exactly.

$$
\|u-u_p\|^2_{L_2[-1,1]} = \left\|\sum_{n=p+1}^{\infty} a_n T_n(t)\right\|^2_{L_2[-1,1]}
$$

$$= \int_{-1}^{1} \left(\sum_{n=p+1}^{\infty} a_n T_n(t) \right) \left(\sum_{n=p+1}^{\infty} \overline{a_n} T_n(t) \right) dt$$

$$= \sum_{n=p+1}^{\infty} |a_n|^2 \int_{-1}^{1} T_n^2(t) dt \; + \; 2 \sum_{n=p+1}^{\infty} \sum_{m=n+1}^{\infty} \mathrm{Re}(a_n \overline{a_m}) \int_{-1}^{1} T_n(t) T_m(t) dt$$

Chebyshev polynomials are orthogonal with respect to the norm

$$(f, g) = \int_{-1}^{1} \frac{f(t)\bar{g}(t)}{\sqrt{1-t^2}} dt$$

but not the $L_2[-1,1]$ norm. In this case, with an appropriate coordinate change,

$$\int_{-1}^{1} T_n(t) T_m(t) dt \; = \int_0^{\pi} \cos(mx) \cos(nx) \sin(x) dx$$

$$= \begin{cases} 1 - \frac{1}{4n^2-1} & \text{if } m = n \\ 0 & \text{if } m+n \text{ is odd} \\ -\left[\frac{1}{(m-n)^2-1} + \frac{1}{(m+n)^2-1} \right] & \text{if } m+n \text{ is even} \end{cases}$$

Thus

$$\sum_{n=p+1}^{\infty} |a_n|^2 \int_{-1}^{1} T_n^2(t) dt \; + \; 2 \sum_{n=p+1}^{\infty} \sum_{m=n+1}^{\infty} \mathrm{Re}(a_n \overline{a_m}) \int_{-1}^{1} T_n(t) T_m(t) dt$$

$$= \sum_{n=p+1}^{\infty} |a_n|^2 \left(1 - \frac{1}{4n^2-1} \right) - 2 \sum_{n=p+1}^{\infty} \sum_{k=1}^{\infty} \mathrm{Re}(a_n \overline{a_{n+2k}}) \left[\frac{1}{4k^2-1} + \frac{1}{4(n+k)^2-1} \right]$$

$$\leq \sum_{n=p+1}^{\infty} |a_n|^2 \left(1 - \frac{1}{4n^2-1} \right) + \sum_{n=p+1}^{\infty} \sum_{k=1}^{\infty} 2 \left| \mathrm{Re}(a_n \overline{a_{n+2k}}) \right| \left[\frac{1}{4k^2-1} + \frac{1}{4(n+k)^2-1} \right]$$

which is an equality for non-trivial cases. These sums are not possible to evaluate exactly, even given the bounds on $|a_n|$. But by noting $1 - \frac{1}{4n^2-1} \leq 1$, $2\left|\mathrm{Re}(a_n \overline{a_{n+2k}})\right| \leq 8M^2 \rho^{-2n-2k}$, and that each denominator can be bounded above by the first denominator, we find

$$\sum_{n=p+1}^{\infty} \sum_{k=1}^{\infty} 2 \left| \mathrm{Re}(a_n \overline{a_{n+2k}}) \right| \left[\frac{1}{4k^2-1} + \frac{1}{4(n+k)^2-1} \right]$$

$$\leq \sum_{n=p+1}^{\infty} 8M^2 \left[\frac{1}{3} + \frac{1}{4(n+1)^2-1} \right] \rho^{-2n} \sum_{k=1}^{\infty} \rho^{-2k}$$

$$
\begin{aligned}
&= \frac{8M^2}{\rho^2-1} \sum_{n=p+1}^{\infty} \left[\frac{1}{3} + \frac{1}{4(n+1)^2-1}\right] \rho^{-2n} \\
&\leq \frac{8M^2}{\rho^2-1} \left[\frac{1}{3} + \frac{1}{4(p+2)^2-1}\right] \sum_{n=p+1}^{\infty} \rho^{-2n} \\
&\leq \frac{8M^2\rho^{-2p}}{(\rho^2-1)^2} \left[\frac{1}{3} + \frac{1}{4(p+2)^2-1}\right]
\end{aligned}
$$

and so

$$
\begin{aligned}
\|u-u_p\|_{L_2[-1,1]} &\leq \sqrt{\frac{4M^2\rho^{-2p}}{\rho^2-1} + \frac{8M^2\rho^{-2p}}{(\rho^2-1)^2}\left[\frac{1}{3} + \frac{1}{4(p+2)^2-1}\right]} \\
&= \frac{2M\rho^{-p}}{\rho-1}\left[\frac{\sqrt{\rho^2-\frac{1}{3}+\frac{2}{4(p+2)^2-1}}}{\rho+1}\right]
\end{aligned}
$$

This is better than the previous bound by a factor of

$$
K_{\rho,p} := \frac{\sqrt{\rho^2-\frac{1}{3}+\frac{2}{4(p+2)^2-1}}}{(\rho+1)\sqrt{2}}
$$

which, as can be shown by taking limits of ρ and p, satisfies

$$
\frac{1}{\sqrt{12}} < K_{\rho,p} < \frac{1}{\sqrt{2}}.
$$

Specifically, $K_{\rho,p}$ is an increasing function in ρ and a decreasing function in p, which are in the ranges $(1,\infty)$ and $[0,\infty)$ respectively. The case $\rho=\infty$ corresponds to a function analytic everywhere in the complex plane with an approximation error of zero for any p. This is because the definition of $u(z)$ was that it was a bounded analytic function in E_ρ, and the only bounded analytic function in the entire complex plane is the constant function, which would be approximated identically. This then fits with all but the first Chebyshev coefficients being necessarily zero.

2.2 hp-approximation, a Case Study

Definition 2.5 *A geometric mesh of parameter σ and size n on $[0, \beta]$ has mesh points $x_0 = 0, x_i = \beta\sigma^{n-i}$ for $i = (1, \dots, n)$, where $\beta > 0, \sigma \in (0, 1), n \in \mathbb{N}$*

Definition 2.6 *Considering a mesh $[0 = x_0, x_1, \dots, x_n = 1]$, $\Pi_{p,n}u$ denotes the best approximation to u in the L_2 norm on $[0, 1]$ from the space $S_{p,n} := \left\{v : [x_1, 1] \to \mathbb{R} : v|_{(x_i,x_{i+1})} \in P_p, i = 0, \dots, n-1\right\}$*

Consider the case when $u(z)$ is analytic and bounded (with bound M) for $\mathrm{Re}(z) > 0$, and is approximated on a general mesh on $[x_1, 1]$ by $\Pi_{p,n}u$ and by zero on $[0, x_1]$ for reasons which shall soon become apparent. Clearly

$$\begin{aligned}
\|u - \Pi_{p,n}u\|^2_{L2[0,1]} &= \int_0^1 |u(t) - \Pi_{p,n}|^2 \, dt \\
\leq \int_0^{x_1} |u(t)|^2 \, dt &+ \sum_{i=1}^{n-1} \int_{x_i}^{x_{i+1}} |u(t) - \Pi_{p,n}u(t)|^2 \, dt \\
&\leq M^2 x_1 + \sum_{i=1}^{n-1} \int_{x_i}^{x_{i+1}} |u(t) - \Pi_{p,n}u(t)|^2 \, dt
\end{aligned}$$

The results of the previous section can be applied to each mesh cell (save for the first) by applying an affine transformation, mapping the cell $[x_i, x_{i+1}]$ to $[-1, 1]$. Because it is the leftmost point of analyticity, we use the origin as the leftmost point of the ellipse, thereby determining ρ_i for each mapped cell and thus the accuracy of the approximation. It is clear that the function

$$f(x) = \frac{2x - x_{i+1} - x_i}{x_{i+1} - x_i}$$

is the unique affine transformation that maps the mesh cell $[x_i, x_{i+1}]$ to $[-1, 1]$. Note that the first cell can not follow this method as the ellipse is collapsed onto $[-1, 1]$, resulting in $\rho_0 = 1$ and infinite error. Hence we approximate by zero.

For all other mesh cells, the origin is mapped to the co-ordinate

$$-\frac{x_{i+1} + x_i}{x_{i+1} - x_i} = -\frac{1}{2}\left(\rho_i + \rho_i^{-1}\right)$$

which results in a quadratic equation with largest root (i.e. the root necessarily larger than 1)

$$\rho_i = \frac{\sqrt{x_{i+1}} + \sqrt{x_i}}{\sqrt{x_{i+1}} - \sqrt{x_i}}$$

and so the L_2 error on the transformed approximation is bounded by

$$M2\sqrt{2}\frac{\rho_i^{-p}}{\rho_i - 1} = M\sqrt{2}\left(\frac{\sqrt{x_{i+1}} - \sqrt{x_i}}{\sqrt{x_{i+1}} + \sqrt{x_i}}\right)^p \frac{\sqrt{x_{i+1}} - \sqrt{x_i}}{\sqrt{x_i}}$$

A simple argument to do with areas shows the square of the L_2 error on $[-1, 1]$ is a linear multiple of the square of the L_2 error on the original mesh. The appropriate scale factor to apply is $\frac{x_{i+1}-x_i}{2}$, meaning the total error bound becomes

$$M^2 x_1 + \sum_{i=1}^{n-1} \int_{x_i}^{x_{i+1}} |u(t) - \Pi_{p,n} u(t)|^2 \, dt$$

$$\leq M^2 \left(x_1 \; + \; \sum_{i=1}^{n-1} \left(\frac{\sqrt{x_{i+1}} - \sqrt{x_i}}{\sqrt{x_{i+1}} + \sqrt{x_i}}\right)^{2p} \frac{\left(\sqrt{x_{i+1}} - \sqrt{x_i}\right)^2}{x_i} (x_{i+1} - x_i) \right) \quad (2.1)$$

This applies for any mesh on $[0, 1]$, and so in theory it would be possible to differentiate w.r.t. the x_i and thus calculate the optimum mesh of size n on which we approximate with piecewise polynomials of degree $\leq p$. But as is clear from the expression above, the resulting system is nonlinear and would be very difficult to solve (though an iterative process might be possible, noting that $x_n = 1$ and the expressions for the derivatives are identical for $i = 2, 3, \ldots n-1$). Instead we shall apply a geometric mesh of parameter σ, which makes the sum far easier to evaluate largely due to the simplification:

$$\rho_i = \frac{1 - \sqrt{\sigma}}{1 + \sqrt{\sigma}} \; \forall i$$

The overall error bound becomes

$$\leq M^2 \left(\sigma^{n-1} \; + \; \sum_{i=1}^{n-1} \left(\frac{1 - \sqrt{\sigma}}{1 + \sqrt{\sigma}}\right)^{2p} \sigma^{n-i-1} \left(1 - \sqrt{\sigma}\right)^2 \left(\sigma^{-1} - 1\right) \right)$$

$$= M^2 \left(\sigma^{n-1} \; + \; \left(\frac{1 - \sqrt{\sigma}}{1 + \sqrt{\sigma}}\right)^{2p} \left(1 - \sqrt{\sigma}\right)^2 \left(\sigma^{-1} - 1\right) \sigma^{n-1} \sum_{i=1}^{n-1} \sigma^{-i} \right)$$

$$= M^2 \left(\sigma^{n-1} \; + \; \left(\frac{1 - \sqrt{\sigma}}{1 + \sqrt{\sigma}}\right)^{2p} \left(1 - \sqrt{\sigma}\right)^2 (1 - \sigma) \, \sigma^{n-2} \frac{\sigma^{-1} \left(\sigma^{1-n} - 1\right)}{\sigma^{-1} - 1} \right)$$

$$= M^2 \left(\sigma^{n-1} \ + \ \left(\frac{1-\sqrt{\sigma}}{1+\sqrt{\sigma}} \right)^{2p} \frac{\left(1-\sqrt{\sigma}\right)^2 \left(1-\sigma^{n-1}\right)}{\sigma} \right)$$

Thus we get the final result

$$\|u - \Pi_{p,n} u\|_{L_2[0,1]} \leq M \left(\sigma^{n-1} \ + \ \left(\frac{1-\sqrt{\sigma}}{1+\sqrt{\sigma}} \right)^{2p} \frac{\left(1-\sqrt{\sigma}\right)^2 \left(1-\sigma^{n-1}\right)}{\sigma} \right)^{\frac{1}{2}}$$

A key point to note is the conditions on $u(z)$. Initially it was stated that u should be analytic and bounded for $\mathrm{Re}(z) > 0$, but the mathematics used for this bound merely require u to be analytic in a set of ellipses centred on the midpoints of the mesh cells. A simple geometrical argument shows that the last ellipse encompasses all previous ellipses, meaning for this bound to hold we only need u analytic in

$$\{z \in \mathbb{C} | \ \ |z - \sigma| + |z - 1| < 1 + \sigma\}$$

Now when it comes to approximating $u(z)$ on $[0, 1]$, there are two exponentially decaying terms, one in p and one in n. It is sensible to have these decay at roughly the same rate, so that increasing either variable will have the same increase in error. It can be shown that this is the optimal situation, because if one variable is significantly larger than the other, increasing the lesser one makes the biggest decrease in error. Consider the number of degrees of freedom $N = n(p + 1)$ as fixed and minimise the decaying terms w.r.t. σ. The relevant equation for balancing the terms is

$$\sigma^n \approx \left(\frac{1-\sqrt{\sigma}}{1+\sqrt{\sigma}} \right)^{\frac{2N}{n}}$$

Equating both sides and taking logarithms reveals

$$n = \sqrt{\frac{2N}{\log \sigma} \log \left(\frac{1-\sqrt{\sigma}}{1+\sqrt{\sigma}} \right)}$$

and the decaying term becomes

$$\sigma^{\sqrt{\frac{2N}{\log \sigma} \log \left(\frac{1-\sqrt{\sigma}}{1+\sqrt{\sigma}} \right)}}$$

It would seem sensible to minimise this term with respect to σ and thus maximise the decay rate. Assuming this function has a stationary value in

$(0,1)$, the function resulting from taking the logarithm would have a stationary value at the same point. Thus

$$\begin{aligned}
&\log \sigma^{\sqrt{\frac{2N}{\log \sigma} \log\left(\frac{1-\sqrt{\sigma}}{1+\sqrt{\sigma}}\right)}} \\
&= \sqrt{2N \log\left(\frac{1-\sqrt{\sigma}}{1+\sqrt{\sigma}}\right) \log \sigma}
\end{aligned}$$

Squaring this function will not change the stationary value of σ, and will make the result much easier to differentiate. Though it is obvious at this point that the constant $2N$ can be removed. Noting that

$$\frac{d}{d\sigma} \log\left(1 \pm \sqrt{\sigma}\right) = \frac{\pm 1}{2\sqrt{\sigma}\,(1 \pm \sqrt{\sigma})}$$

we find

$$\begin{aligned}
&\frac{d}{d\sigma} \log\left(\frac{1-\sqrt{\sigma}}{1+\sqrt{\sigma}}\right) \log \sigma \\
&= \frac{1}{\sigma} \log\left(\frac{1-\sqrt{\sigma}}{1+\sqrt{\sigma}}\right) \;+\; \log \sigma \left(\frac{-1}{2\sqrt{\sigma}\,(1-\sqrt{\sigma})} - \frac{1}{2\sqrt{\sigma}\,(1+\sqrt{\sigma})}\right) \\
&= \frac{1}{\sigma} \log\left(\frac{1-\sqrt{\sigma}}{1+\sqrt{\sigma}}\right) \;-\; \frac{\log \sigma}{\sqrt{\sigma}\,(1-\sigma)}
\end{aligned}$$

The minimum value is therefore reached when

$$(1-\sigma) \log\left(\frac{1-\sqrt{\sigma}}{1+\sqrt{\sigma}}\right) = \sqrt{\sigma} \log \sigma$$

which is at precisely $\sigma = \sigma_{opt} = \left(\sqrt{2}-1\right)^2 \approx 0.17$ which matches the result in [3]. Finally, with a value of σ we can go back to the previous result for the value of n to see how it should grow with N:

$$n = \sqrt{\frac{2N}{\log \sigma_{opt}} \log\left(\frac{1-\sqrt{\sigma_{opt}}}{1+\sqrt{\sigma_{opt}}}\right)} = \sqrt{N}$$

which is a surprisingly compact result, as is the optimum value of σ. More generally, there is a constant C_σ such that the optimal choice of n and p is

$$n = C_\sigma N^{\frac{1}{2}} \quad (p+1) = \frac{1}{C_\sigma} N^{\frac{1}{2}}$$

or equivalently

$$n = C_\sigma^2 (p+1)$$

where

$$C_\sigma := \sqrt{\frac{2}{\log \sigma} \log \left(\frac{1 - \sqrt{\sigma}}{1 + \sqrt{\sigma}} \right)}$$

Note C_σ is a strictly increasing function of σ over $(0, 1)$. Note further that I have excluded the factor $K_{\rho,p}$ from the working out simply because it is a constant multiplied onto the second term in the expression for $\|u - \Pi_{p,n} u\|_{L_2[0,1]}$ and does not affect the results of the decay rates or the optimum value of σ.

2.3 hp-approximations of Smooth Functions

We now draw our attention to the case when u is analytic for $\mathrm{Re}(z) > 0$ and is approximated over $[0, \beta]$ by a geometric mesh as in the previous section, and for some $\alpha \in (0, \frac{1}{2})$, $M > 0$, $u(z)$ satisfies

$$|u(z)| \leq \begin{cases} M\,|z|^{-\alpha}, & |z| \leq 1 \\ M\,|z|^{-\frac{1}{2}}, & |z| \geq 1 \end{cases}$$

or, more generally

$$|u(z)| \leq F\left(|z|\right)$$

where F is decreasing on $(0, \infty)$. The error in approximating u by zero on the first cell is

$$\|u\|_2 \leq \int_0^{x_1} M^2 x^{-2\alpha} dx = M^2 \left[\frac{x^{1-2\alpha}}{-2\alpha}\right]_0^{x_1} = M^2 \frac{x_1^{1-2\alpha}}{1-2\alpha} < \infty$$

For the other mesh cells, an important issue arises. If we approximate with Chebyshev polynomials as before, the ellipse centred on each cell can not extend to the origin, because then $|u(z)|$ is not bounded in the ellipse. So we need to choose a point somewhere between the origin and the mesh cell to mark as the leftmost point of the ellipse.

Consider a mesh cell $[x_a, x_b]$ where the leftmost point is A. It can be shown that the appropriate ellipse parameter ρ satisfies

$$\rho^2 + \left(\frac{4A - 2x_a - 2x_b}{x_b - x_a}\right)\rho + 1 = 0$$

$$\rho = \frac{\left(\sqrt{x_b - A} + \sqrt{x_a - A}\right)^2}{x_b - x_a}$$

Results from section 2.2 show that the L_2 error in approximating $u(z)$ on a given mesh cell is bounded by

$$\frac{\rho^{-p}}{\rho - 1} M' 2\sqrt{x_b - x_a}$$

where M' is the maximum modulus of $u(z)$ in the ellipse, so in this case $M' = F(A)$. So to minimise the bound, we need to minimise

$$\frac{\rho^{-p}}{\rho - 1} F(A) \tag{2.2}$$

with respect to A. Key points to note from this expression are

- As $A \to 0$, $F(A) = A^{-\alpha} \to \infty$
- As $A \to x_a$, $\rho \to 1$ and so $\frac{1}{\rho - 1} \to \infty$

Therefore there exists a minimum value between 0 and x_a, which will occur at a stationary point. In fact there is only one stationary point in $(0, x_a)$, and the function is meaningless outside it. Differentiating w.r.t. A results in the following equation to be solved

$$\frac{\rho'}{\rho}\left(1 + p + \frac{1}{\rho - 1}\right) = \frac{F'(A)}{F(A)} \tag{2.3}$$

where we have assumed that F is smooth. Particular results are

$$\frac{d\rho}{dA} = -\frac{\rho(x_b - x_a)}{\sqrt{(x_a - A)(x_b - A)}}$$

and, if $F(A) = MA^{-\alpha}$,

$$\frac{F'(A)}{F(A)} = \frac{-\alpha}{A}$$

However manipulating (2.3) to remove the square roots and get an equation for A is very difficult and results in a cubic equation with quite complicated coefficients. It is much faster to differentiate (2.2) with respect to ρ which will allow us to find the value of ρ at which the error is minimised. Firstly, we must find A in terms of ρ:

$$\begin{aligned}
\rho &= \frac{\left(\sqrt{x_b - A} + \sqrt{x_a - A}\right)^2}{x_b - x_a} \\
&= \frac{x_b + x_a - 2A + 2\sqrt{A^2 - (x_a + x_b)A + x_a x_b}}{x_b - x_a} \\
&= \frac{v - 2A + 2\sqrt{A^2 - Av + \frac{1}{4}(v^2 - w^2)}}{w} \\
\rho w - v + 2A &= 2\sqrt{A^2 - Av + \frac{1}{4}(v^2 - w^2)}
\end{aligned}$$

where $v = x_b + x_a$, $w = x_b - x_a$. Squaring and cancelling terms reveals

$$\begin{aligned}
4\rho w A &= 2\rho w v - w^2 - \rho^2 w^2 \\
A &= \frac{v}{2} - \frac{w}{4}(\rho + \rho^{-1})
\end{aligned}$$

and so (2.2) becomes

$$\frac{\rho^{-p}}{\rho - 1}\left(\frac{v}{2} - \frac{w}{4}(\rho + \rho^{-1})\right)^{-\alpha} \tag{2.4}$$

Note that for any three functions f, g, h the chain rule dictates that

$$(fgh)' = fgh\left(\frac{f'}{f} + \frac{g'}{g} + \frac{h'}{h}\right)$$

and thus the product fgh is only stationary when the term in the brackets is zero. For (2.4) this results in

$$\frac{\alpha(\rho - \rho^{-1})}{\rho^2 - 2\frac{v}{w}\rho + 1} + \frac{p}{\rho} + \frac{1}{\rho - 1} = 0 \tag{2.5}$$

This clearly results in a cubic equation for ρ for which a simple expression for the solution appears impossible, and even if it were simple then the expression for A would likely be unpleasant. But there is one interesting characteristic - the points of the mesh, x_a and x_b, only occur as ratios of their sum and difference. This means that if a geometric mesh is used, the equation for ρ is identical for all mesh points:

$$\frac{v}{w} = \frac{x_b + x_a}{x_b - x_a} = \frac{\beta\sigma^{n-i-1} + \beta\sigma^{n-i}}{\beta\sigma^{n-i-1} - \beta\sigma^{n-i}} = \frac{\sigma^{n-i-1}}{\sigma^{n-i-1}}\frac{1+\sigma}{1-\sigma} = \frac{1+\sigma}{1-\sigma}\ \forall i$$

This means if one could numerically find the root of the equation, then the value of ρ would be known for all i meaning the error calculation is simplified in the same way as in section 2.2. Furthermore,

$$A = \frac{v}{2} - \frac{w}{4}(\rho + \rho^{-1}) = \beta\sigma^{n-i-1}\left(\frac{1+\sigma}{2} - \frac{1-\sigma}{4}(\rho + \rho^{-1})\right)$$

meaning the minimum point A has a constant ratio to each of x_a and x_b. We define θ such that $A = x_a\theta = \beta\sigma^{n-i}\theta$ (or equivalently $\theta = \sigma^b$ for some $b \in \mathbb{R}^+$). This gives

$$\begin{aligned}\rho(A) &= \frac{(\sqrt{\beta\sigma^{n-i-1} - \beta\sigma^{n-i}\theta} + \sqrt{\beta\sigma^{n-i} - \beta\sigma^{n-i}\theta})^2}{\beta\sigma^{n-i-1} - \beta\sigma^{n-i}} \\ &= \frac{(\sqrt{1 - \sigma\theta} + \sqrt{\sigma - \sigma\theta})^2}{1 - \sigma}\end{aligned}$$

So back on the original problem, if we assume we know the value of ρ (or equivalently θ), and define $A_i = \beta\sigma^{n-i}\theta$, results from the previous chapter show us that the square of the L_2 error in the best approximation is bounded above by

$$\int_0^{\beta\sigma^{n-1}} F(x)^2 dx + \frac{4\rho^{-2p}}{(\rho-1)^2} \sum_{i=1}^{n-1} F(A_i)^2 \beta\sigma^{n-i-1}(1-\sigma)$$

$$\int_0^{\beta\sigma^{n-1}} F(x)^2 dx + \frac{4\rho^{-2p}(1-\sigma)}{(\rho-1)^2} \beta\sigma^{n-1} \sum_{i=1}^{n-1} F(A_i)^2 \sigma^{-i}$$

The function $F(A_i)$ has been left in because for the specific problem of this chapter the exponent for (2.2) changes at the point $A = 1$. The means the root of (2.5) is different for $A < 1$ and $A > 1$ meaning to find the minimum bound we need to solve two cubic equations. We continue with the approximation using the first root, and recall that the F we consider is strictly decreasing, so we only need to find out when $A_i = 1$ to determine what power it is raised to for each summand.

$$A_i = 1 \Leftrightarrow \beta\sigma^{n-i+b} = 1 \Leftrightarrow \sigma^i = \beta\sigma^{b+n}$$

$$i = \frac{\log\beta}{\log\sigma} + b + n$$

which is not necessarily an integer, so we define I as the largest integer for which $A_i \leq 1$ for all $i \leq I$. To be precise,

$$I = \left\lfloor \frac{\log\beta}{\log\sigma} + b + n \right\rfloor = \left\lfloor \frac{\log\beta}{\log\sigma} + b \right\rfloor + n$$

Combining all these results we find

$$\|u - \Pi_{p,n}u\|_2^2 \leq \frac{M^2\beta^{1-2\alpha}\sigma^{(n-1)(1-2\alpha)}}{1-2\alpha} + \frac{4M^2\beta\rho^{-2p}(1-\sigma)\sigma^{n-1}}{(\rho-1)^2} \times$$

$$\left(\sum_{i=1}^{I} \beta^{-2\alpha}\sigma^{-2\alpha(n+b)}\sigma^{-i(1-2\alpha)} + \sum_{i=I+1}^{n-1} \beta^{-1}\sigma^{-(n+b)}\sigma^i\sigma^{-i} \right)$$

where the sums evaluate to

$$\beta^{-2\alpha}\sigma^{-2\alpha n}\theta^{-2\alpha} \frac{\sigma^{2\alpha-1}\left(\sigma^{(2\alpha-1)I} - 1\right)}{\sigma^{2\alpha-1} - 1} + \beta^{-1}\sigma^{-n}\theta^{-1}(n - I - 2)$$

giving the overall bound as

$$\frac{M^2\beta^{1-2\alpha}\sigma^{(n-1)(1-2\alpha)}}{1-2\alpha} + \frac{4M^2\beta^{1-2\alpha}\rho^{-2p}(\sigma^{-1}-1)\sigma^{(n-1)(1-2\alpha)}\theta^{-2\alpha}}{(\rho-1)^2}\frac{\sigma^{(2\alpha-1)I}-1}{\sigma^{2\alpha-1}-1}$$

$$+ \frac{4M^2\rho^{-2p}(\sigma^{-1}-1)\theta^{-1}(n-I-2)}{(\rho-1)^2}$$

$$= \frac{M^2\beta^{1-2\alpha}\sigma^{(n-1)(1-2\alpha)}}{1-2\alpha} + \frac{4M^2\beta^{1-2\alpha}\rho^{-2p}(\sigma^{-1}-1)\sigma^{(n-I-1)(1-2\alpha)}\theta^{-2\alpha}}{(\rho-1)^2}\frac{1-\sigma^{(1-2\alpha)I}}{\sigma^{2\alpha-1}-1}$$

$$+ \frac{4M^2\rho^{-2p}(\sigma^{-1}-1)\theta^{-1}(n-I-2)}{(\rho-1)^2}$$

Note the expressions $(n-I-2)$ and $\sigma^{(n-I-1)(1-2\alpha)}$. By the definition of I above, $(n-I) = \left\lfloor \frac{\log\beta}{\log\sigma} + b \right\rfloor$ is always constant with respect to n. This means that though there are σ terms present in the second term, that term remains constant with changing n. So again there are two different decay rates for the error terms, and it would make sense to balance them in the same way as before. Setting $N = n(p+1)$ again,

$$\sigma^{n(1-2\alpha)} \approx \rho^{-2(p+1)}$$

$$n(1-2\alpha)\log\sigma = 2\frac{N}{n}\log\rho^{-1}$$

$$n = \sqrt{\frac{2N}{1-2\alpha}\frac{\log\rho^{-1}}{\log\sigma}}$$

A difficulty with this is that ρ is unknown, as it is the solution of (2.5) and so is a function of α, p and σ. Equivalently we have the expression

$$\rho = \frac{(\sqrt{1-\sigma\theta}+\sqrt{\sigma-\sigma\theta})^2}{1-\sigma}$$

where θ is a function of the same variables as ρ. We know $\theta \in (0,1)$ which corresponds to

$$\rho \in \left(1, \frac{1-\sqrt{\sigma}}{1+\sqrt{\sigma}}\right)$$

but little else of even approximations to the best value of ρ. However if we assume we have chosen a value of θ (meaning ρ and A_i are 'known'), the theory predicts that the resulting bound will be non-optimal, so minimising it does not violate the bound. So the ideal decaying term is bounded by

$$\sigma^{\sqrt{\frac{2N}{1-2\alpha}\frac{-\log\rho}{\log\sigma}}}$$

and minimising it over σ is equivalent to minimising

$$\log\rho\log\sigma$$

which we can differentiate to attempt to find the minimum value. This results in having to solve the following equation

$$\sigma\log\sigma\left(\frac{1}{1-\sigma}+\frac{\theta}{\sqrt{1-\sigma\theta}\sqrt{\sigma-\sigma\theta}}-\frac{1}{\sqrt{\sigma-\sigma\theta}(\sqrt{1-\sigma\theta}-\sqrt{\sigma-\sigma\theta})}\right)$$
$$=2\log(\sqrt{1-\sigma\theta}-\sqrt{\sigma-\sigma\theta})-\log(1-\sigma)$$

Unfortunately this appears to be impossible algebraically, so like the cubic root, the optimum value of σ is only obtainable numerically.

3 Part II: High Frequency Boundary Element Methods for Scattering by Convex Polygons

3.1 Scattering by Convex Polygons and the Analytic Solution

In the case of a sound-soft smooth shape, when an incident plane wave hits the object there is primarily a reflection and a small amount of diffraction which decays exponentially. However the sharp corners of a polygon cause a strong diffraction, such that the theory predicts u to be unbounded at each corner. In fact it is shown in [steve and simon paper] that the polynomials on each edge satisfy

$$k^{-n}\left|v_{\pm}^{(n)}(s)\right| \leq \begin{cases} C_n(ks)^{-\alpha_{\pm}-n}, & ks \leq 1 \\ C_n(ks)^{-\frac{1}{2}-n}, & ks \geq 1 \end{cases}$$

where

$$\alpha_{\pm} := 1 - \frac{\pi}{a_{\pm}}$$

and $a_{\pm}$ is the external angle at the corner at which $v_{\pm}$ is highly peaked. Because the polygon is convex, this gives $\alpha_{\pm} \in (0, \frac{1}{2})$. Thus the work done in the previous section will give a suitable bound on the error between $v_{\pm}$ and the approximation polynomials. Recalling that on each edge

$$\frac{1}{k}\frac{\partial u}{\partial n}(s) = \Psi(s) + \frac{\mathrm{i}}{2}\left[\mathrm{e}^{iks}v_{+}(s) + \mathrm{e}^{-iks}v_{-}(s)\right]$$

and that we know Ψ precisely, if we denote $\phi = \frac{1}{k}\frac{\partial u}{\partial n}$ and ϕ' as its approximation, as well as $v'_{\pm}$ the approximation to $v_{\pm}$, we can see

$$\begin{aligned}
\|\phi - \phi'\| &= \left\|\Psi + \frac{\mathrm{i}}{2}\left[\mathrm{e}^{iks}v_{+} + \mathrm{e}^{-iks}v_{-}\right] - \Psi - \frac{\mathrm{i}}{2}\left[\mathrm{e}^{iks}v'_{+} + \mathrm{e}^{-iks}v'_{-}\right]\right\| \\
&= \left\|\frac{\mathrm{i}}{2}\left[\mathrm{e}^{iks}[v_{+} - v'_{+}] + \mathrm{e}^{-iks}[v_{-} - v'_{-}]\right]\right\| \\
&\leq \frac{1}{2}\left\|v_{+} - v'_{+}\right\| + \frac{1}{2}\left\|v_{-} - v'_{-}\right\|
\end{aligned}$$

and so we have a bound on the approximation to the normal derivative of u.

For an overall bound for the whole shape we would need an accurate solution for the problem in the previous section for each value of α. However it would suffice to use the maximum value of α determined by the minimum angle in the polygon, and then multiply the bound by the number of sides. In fact, under that assumption we get

$$\|\phi - \phi'\| \leq B \left\| v_+ - v'_+ \right\|$$

where B is the number of sides of the polygon.

3.2 A High Frequency hp-version Galerkin Method

The hp-version Galerkin Method (convpolyhp.m) composed by Dr. S. Langdon works to solve the integral equation

$$(I+K)\frac{\partial u}{\partial n} = f \text{ on } \partial D$$

by removing the leading order behaviour (i.e. the reflected wave) and trying to approximate the remaining function $\phi(s)$ by its composition

$$\phi(s) = \mathrm{e}^{iks}v_+(s) + \mathrm{e}^{-iks}v_-(s), \quad s \in [0, L]$$

where $v_+(s)$ and $v_-(s)$ are polynomials in s. The program (implemented in Matlab 7) solves the integral equation for the coefficients of the basis functions of Legendre polynomials of degree $\leq p$ on a geometric mesh of size n and parameter $\sigma = 0.15$. The program computes the products of all the basis functions using numerical quadrature rules and solves the resulting matrix system.

The polygon in question is a right-angled triangle with acute angles of $\frac{\pi}{4}$ and straight sides of length 2π, and with an incident wave travelling perpendicular to the hypotenuse of the triangle. This situation is highly symmetric, and has the minimum number of sides required to fit the scattering problem. This choice minimises the number of degrees of freedom ($2n(p+1)$ per side) and thus minimises the number of matrix entries, enabling comparison of higher frequencies with theoretical results without requiring unfeasibly long running times.

Definition 3.1 *(p,n) denotes the approximation space of piecewise polynomials of degree $\leq p$ on a geometric mesh of size n, where the mesh parameter σ is fixed*

When n is increased, only the first mesh cell is split and all others remained the same. As $P_p \subset P_{p+1}$ for any p, we have the statement on subspaces

$$(p, n) \subseteq (p+a, n+b) \quad \forall a, b \in \mathrm{N}_0$$

However in the theory we have sought the minimum error for a fixed number of degrees of freedom, $N = n(p+1)$. For a given N, the number of possible approximation spaces with exactly N degrees of freedom is equal to the number of divisors of N. For example, when $N = 6$ the possible spaces are

$$(1,6),(2,3),(3,2),(6,1)$$

But to compare errors in the solution, we must realise that as n and p increase, the numerical solution tends to the analytic solution. So to make an estimate of the error we need an approximation space with large n and p such that all other approximation spaces are contained in it. The smallest space which contains all possible spaces with N degrees of freedom is the space (N, N), which has N^2 degrees of freedom and so is a much more accurate result.

For the programming part of this project we will run the code for various subspaces of $(8, 8)$, checking all possible spaces with 1-6, and 8 degrees of freedom, as well as selected ones with 12, 16, 20 and 24 degrees of freedom that are still subspaces. $(8, 8)$ is as far as Matlab can reasonably run programs for the wavenumbers chosen, without requiring weeks of total running time. Furthermore, as $(8, 8)$ is the approximation space used for a comparison, the subspaces can not get too 'close' to this space because the approximate error in the subspace $(8, 8)$ is zero.

Recalling from section 2.2 the optimal ratio between n and p for a given σ was

$$n = C_\sigma^2(p + 1)$$

where

$$C_\sigma := \sqrt{\frac{2}{\log\sigma}\log\left(\frac{1-\sqrt{\sigma}}{1+\sqrt{\sigma}}\right)}$$

In the code, the value of σ is taken to be 0.15 rather than 0.17 because some overrefinement is recommended (see [4]). This gives the value of C_σ as approximately 0.9282, i.e. the theory suggests that for a minimised error we need

$$n \approx 0.862(p + 1)$$

Given that n and $p+1$ are integers, this ratio can only be roughly met up until $n = p = 6$. For large n the ratio can be met much more closely, and this predicts that the computed results should show an ideal 'direction' in which the error decreases fastest. Note that the theory predicts an exponential convergence with respect to the degrees of freedom.

3.3 Numerical Results

The program was ran for wavenumbers 1-6, 8, 11 and 16. The increasing computing time meant that showing the changes as k grows geometrically was only plausible for a geometric progression of approximately $\sqrt{2}$. Thus the sequence 2, 3, 4, 6, 8, 11, 16 should show some level of example of how k affects convergence.

For each wavenumber, the 'exact' solution taken was the one in the space $(8, 8)$. All possible degrees of freedom less than or equal to 8 are included, as well as ones up to 24 degrees of freedom. The latter ones will not be fully represented, but the theory predicts that the best choice for a fixed number of degrees of freedom is to have n and p close. Thus we should have the most accurate approximation for degrees of freedom in the set $\{9, 10, 12, 14, 15, 16, 18, 20, 24\}$ without having to compare to the best approximation in the space $(24, 24)$.

Another limiting factor is that these are relative errors compared to the solution in the space $(8, 8)$. The solutions in all subspaces should converge to that solution, and so the relative errors would tend to zero and in fact be identically zero when $(8, 8)$ is compared to itself. To avoid this unneccessary confusion, 24 degrees of freedom was chosen to be the highest value represented because it should not be close enough to $(8, 8)$ for the relative error to be affected, and it is the last value in the range $[1, 64]$ that has 4 different factorisations using numbers in the range $[1, 8]$.

Relative errors for $k = 1$

	n	$\rightarrow$						
p	1.07	1.09	1.12	1.12	1.12	1.15	1.72	1.17
$\downarrow$	1.11	1.07	1.14	1.19	1.72	1.79	1.83	1.87
	1.41	1.71	1.86	1.94	2.00	2.04	2.08	2.10
	1.66	1.97	2.09	2.20	2.18	2.18		
	1.83	2.16	2.16	2.01				
	1.74	1.98	2.03	2.29				
	1.73	1.97	2.03					
	1.73	1.97	2.03					

Relative errors for $k = 2$

	n	$\longrightarrow$						
p	0.96	0.88	0.79	0.74	0.63	0.60	0.54	0.49
$\downarrow$	0.84	0.70	0.61	0.54	0.49	0.43	0.39	0.35
	0.67	0.49	0.38	0.31	0.26	0.22	0.19	0.17
	0.47	0.26	0.18	0.16	0.18	0.19		
	0.32	0.22	0.18	0.08				
	0.33	0.13	0.09	0.09				
	0.34	0.13	0.09					
	0.34	0.13	0.09					

Relative errors for $k = 3$

	n	$\longrightarrow$						
p	0.97	0.91	0.82	0.76	0.70	0.62	0.59	0.51
$\downarrow$	0.87	0.72	0.61	0.53	0.48	0.42	0.38	0.33
	0.68	0.49	0.38	0.30	0.26	0.23	0.22	0.22
	0.47	0.26	0.22	0.24	0.27	0.29		
	0.34	0.30	0.25	0.15				
	0.34	0.19	0.16	0.15				
	0.34	0.19	0.16					
	0.34	0.19	0.16					

Relative errors for $k = 4$

	n	$\longrightarrow$						
p	0.99	0.97	0.89	0.81	0.77	0.72	0.68	0.65
$\downarrow$	0.91	0.79	0.70	0.63	0.58	0.54	0.51	0.47
	0.75	0.60	0.51	0.46	0.42	0.40	0.38	0.38
	0.58	0.42	0.38	0.37	0.38	0.38		
	0.46	0.39	0.36	0.33				
	0.47	0.35	0.33	0.33				
	0.47	0.35	0.33					
	0.47	0.35	0.33					

Relative errors for $k = 5$

	$n \rightarrow$							
p	0.98	0.95	0.88	0.84	0.77	0.73	0.68	0.64
$\downarrow$	0.91	0.80	0.69	0.62	0.55	0.50	0.46	0.43
	0.74	0.57	0.47	0.39	0.33	0.29	0.26	0.24
	0.54	0.33	0.25	0.22	0.23	0.24		
	0.37	0.24	0.21	0.18				
	0.38	0.20	0.17	0.17				
	0.39	0.20	0.17					
	0.39	0.20	0.17					

Relative errors for $k = 6$

	$n \rightarrow$							
p	1.00	0.97	0.94	0.90	0.88	0.85	0.83	0.80
$\downarrow$	0.95	0.89	0.84	0.80	0.77	0.75	0.73	0.71
	0.86	0.78	0.73	0.70	0.68	0.67	0.66	0.65
	0.76	0.68	0.65	0.65	0.65	0.65		
	0.69	0.65	0.65	0.64				
	0.69	0.64	0.64	0.64				
	0.69	0.64	0.64					
	0.69	0.64	0.64					

Relative errors for $k = 8$

	$n \rightarrow$							
p	0.99	0.96	0.91	0.86	0.81	0.76	0.72	0.67
$\downarrow$	0.94	0.82	0.73	0.65	0.58	0.52	0.46	0.42
	0.77	0.59	0.46	0.38	0.31	0.27	0.23	0.21
	0.54	0.31	0.21	0.20	0.21	0.22		
	0.35	0.23	0.19	0.12				
	0.35	0.14	0.10	0.09				
	0.36	0.14	0.10					
	0.36	0.14	0.10					

Relative errors for $k = 11$

	n	$\longrightarrow$						
p	1.00	0.96	0.94	0.88	0.85	0.78	0.76	0.70
$\downarrow$	0.95	0.85	0.75	0.67	0.60	0.54	0.48	0.44
	0.79	0.61	0.48	0.39	0.32	0.26	0.23	0.20
	0.55	0.31	0.21	0.19	0.21	0.22		
	0.35	0.24	0.19	0.09				
	0.34	0.12	0.06	0.05				
	0.35	0.12	0.06					
	0.35	0.12	0.06					

Relative errors for $k = 16$

	n	$\longrightarrow$						
p	1.00	0.97	0.95	0.90	0.88	0.82	0.80	0.74
$\downarrow$	0.97	0.88	0.79	0.71	0.64	0.58	0.52	0.48
	0.82	0.64	0.51	0.41	0.33	0.28	0.24	0.21
	0.57	0.33	0.22	0.20	0.22	0.23		
	0.35	0.25	0.20	0.08				
	0.34	0.11	0.05	0.02				
	0.35	0.11	0.05					
	0.35	0.11	0.05					

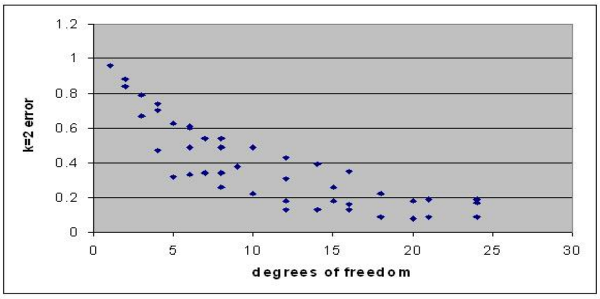


Figure 3.1: Scatter plot demonstrating exponential convergence of error with degrees of freedom

3.4 Conclusions

Firstly, the results for $k = 1$ appear to be anomalous, as the error appears to increase with the degrees of freedom rather than decrease. It is likely a side-effect of choosing such a low parameter for the convpolyhp.m program, which was intended for high frequencies. As a result there will be no further comment on these results.

The graphs of degrees of freedom versus error indeed show an exponential convergence. Finding an equation for the graphs was not done because they were merely meant to confirm what has been theorised many times before. In fact the worst error values for each degree of freedom show the same convergence, though they have left out the cases when n or p are equal to 1, and the surface plots indicate that those have the highest relative error.

The tables of numbers reveal a curious result. It appears the optimum ratio of p to n, that is the fastest decay slope on the surface plots, is not $n < (p+1)$ after all. It appears to be the other way round (i.e. values below the diagonal are less than those above the diagonal), and the best results for the degrees of freedom versus error plots show the best ratio is somewhere around 0.7.

Perhaps the reason for this is that the results of section 2.2 do not apply to the functions $v_{\pm}(s)$, because the results assume that the best approximation on the first mesh cell is zero. Clearly that is not the case for such a singular function, meaning the first term in the error bound is incorrect. Given that a large amount of algebra resulted from that choice of approximation, it is unfortunate that it is not useful in this case (although the results will be very strong for a function which is best approximated by zero near the origin).

Nevertheless the theory suggested an avenue to investigate, and though the results were incorrect the preferred results were found i.e. exponential convergence with respect to degrees of freedom and an ideal ratio of n to p.

Another odd artifact of the results is that some wavenumbers have significantly lower errors for given values compared to others, particularly for $k = 6$ and $k = 4$. This is likely to do with a physical complication due to the relation of the wavelength to the obstacle which makes the solution converge more slowly, e.g. the polynomials $v_{\pm}$ may be significantly more oscillatory than for other wavenumbers.

3.5 Recommendations for further work

In the work here there were a number of points where certain algebraic problems were posed but remained unsolved, and if these were solved in future then the methods involved here could be optimised:

1. In section 2.1 there were the two sums (up to taking out a factor of ρ)

$$\sum_{n=p+1}^{\infty} \rho^{-2n} \left(1 - \frac{1}{4n^2 - 1}\right)$$

$$\sum_{n=p+1}^{\infty} \rho^{-2n} \frac{1}{4n^2 - 1}$$

   the first of which approximated the bracketed term by 1, the second approximating all denominators by the first denominator. If these sums could be more accurately bounded above then the bound on each mesh cell would be tightened. This essentially decreases the range of values of $K_{\rho,p}$.

2. In section 2.2 there was equation (2.1) which described the bound for a general mesh for $u(z)$ bounded and analytic for $\mathrm{Re}(z) > 0$. For any mesh, there are a set of ellipses resulting from the mesh points which produce the given bound, meaning that u only needs to be bounded in the union of these ellipses. The simplest bound for all possible unions of ellipses is the domain

$$\{z \in \mathbb{C} | \ |z - 1| < 1\}$$

   i.e. we only require u to be analytic in the above set. This set is also the limiting ellipse around the last mesh cell as $\sigma \to 1$.

   The result could be improved if one could differentiate (2.1) with respect to $x_i, i = 1, \ldots, n-1$) and solve the resulting non-linear system to get the optimal mesh spacings for approximating u on a mesh of size n with piecewise polynomials of degree $\leq p$.

   The theory suggests that the mesh must get more refined close to the origin, so a geometric mesh is a good choice (aside from making the algebra considerably easier). There is also the possibility of a polynomial mesh, where the points satisfy

$$x_i = \left(\frac{i}{n}\right)^q, \; q > 0$$

and a graded (i.e. non-uniform) mesh. Indeed in [1] a composite mesh was used for a Galerkin approximation by piecewise polynomials, and it was shown that for a given level of accuracy the number of mesh points need only grow logarithmically with respect to k.

But as was mentioned in the conclusion, the fact that this setup approximates by zero on the first cell means that minimising the result over the x_i is not the best bound in general, and certainly not for the polygon scattering problem. The program convpolyhp.m approximates the first mesh cell by a polynomial degree p, thus the points x_i will only be optimal for approximating a function u which is best approximated by zero close to the origin.

Improving the first expression in the error bound will also enable a better error analysis with respect to the parameters, meaning the peculiar result of the ideal ratio of n and p being about 0.7 could be shown algebraically to be near-optimal.

3. In section 2.3 there are numerous mentions of a cubic equation which needs to be solved for the cases $\alpha \in (0, \frac{1}{2})$ and $\alpha = \frac{1}{2}$. Taking limits of the parameters of the equation (p, α, σ) to limit the range of values of ρ and θ would significantly improve the error approximation.

   Similarly, using the roots for the different values of α and altering the two summations accordingly would enable a better expression for the overall bound.

4. A particular result from plotting graphs in Excel, approximating the gradient and observing where the derivative crosses the x-axis showed that the value of σ at which

   $$\log \rho \log \sigma$$

   is minimised is not heavily dependent on the unknown parameter θ. It appears as $\theta \to 0$, the minimum value is at $\sigma = \sigma_{opt} = (\sqrt{(2)} - 1)^2$ as in section 2.2. Meanwhile as $\theta \to 1$, in the limit the minimum value is somewhere near to $\sigma = 0.2032$. It is important to note that this root only exists in the limit, and not at when $\theta = 1$ because that results in the singular case $\rho = 1$. Algebraic work has failed to discover this limiting root, but it suggests that for the approximation of $v_\pm$ we should take $\sigma \in (\sigma_{opt}, 0.2032)$ rather than $\sigma = 0.15$ as in the convpolyhp.m program and as recommended in [4].

4 References

1. A Galerkin Boundary Element Method for High Frequency Scattering by Convex Polygons [S.N. Chandler-Wilde and S. Langdon]

2. Boundary Element Methods for Acoustics [S.N. Chandler-Wilde and S. Langdon]

3. Notes on hp-FEM [J.M. Melenk]

4. hp-FEM Lecture Course [J.M. Melenk]