

# Singular Values and the Distance to Instability

D. Kirkland

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## **Abstract**

A numerical algorithm for constructing a state feedback for a controllable stable linear system is presented and tested. Main attention is given to maximizing the distance to instability such that the system remains stable. Two methods are considered, namely robust eigenstructure assignment and singular value assignment. Examples are looked at to illustrate the theoretical results discussed. A comparison between these two methods is considered and conclusions drawn from the numerical results.

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# Contents

List of Figures	iv
Notation	v
<b>1 Introduction</b>	<b>1</b>
<b>2 Problem Formulation</b>	<b>4</b>
2.1 Feedback . . . . .	4
2.2 Controllability . . . . .	7
2.3 Observability . . . . .	8
2.4 Motivation for Pole and Singular Value Assignment . . . .	10
2.5 Basic Matrix Theory . . . . .	12
<b>3 Robust Eigenstructure Assignment</b>	<b>14</b>
3.1 Introduction . . . . .	14
3.2 Background Theory for Linear State Feedback . . . . .	15
3.3 Numerical Algorithm . . . . .	20
3.4 Examples . . . . .	22
<b>4 Singular Value Assignment</b>	<b>25</b>
4.1 Preliminary Theory . . . . .	25

4.2	Numerical Algorithm . . . . .	29
4.3	Examples . . . . .	30
<b>5</b>	<b>Distance to Instability</b>	<b>33</b>
5.1	Bisection Method . . . . .	34
5.2	Examples . . . . .	37
<b>6</b>	<b>Methods for Increasing the Distance to Instability</b>	<b>40</b>
6.1	Numerical Algorithm . . . . .	40
6.2	Singular Value Assignment . . . . .	42
6.3	Eigenstructure Assignment . . . . .	49
	<b>Conclusion</b>	<b>54</b>
	<b>Appendix 1</b>	<b>55</b>
	<b>Bibliography</b>	<b>65</b>

# List of Figures

1.1	An open loop system . . . . .	2
1.2	A closed loop system . . . . .	2
2.1	A feedback control system . . . . .	4
5.1	Example 1 . . . . .	39
5.2	Example 2 . . . . .	39
6.1	Example 3 . . . . .	52
6.2	Example 4 . . . . .	52
6.3	Example 5 . . . . .	53

# Notation

## Symbols

$A \in R^{n \times n}$

$B \in R^{n \times m}$

$C \in R^{p \times n}$

$K \in R^{m \times n}$

$F \in R^{p \times n}$

$X \in R^{n \times n}$

$\mathcal{N}$

$\mathcal{U}$

$A+BK \in R^{n \times n}$

$A+BFC \in R^{n \times n}$

$\Lambda \in R^{n \times n}$

$\mathbf{x} \in R^{n \times 1}$

$\mathbf{u} \in R^{m \times 1}$

$\mathbf{y} \in R^{p \times 1}$

$\mathbf{v} \in R^{m \times 1}$

$\lambda_j^{(*)}$

$\kappa_2(X)$

$\sigma_i^{(*)}$

$\sigma_{min}^{(*)}$

$\sigma_{max}^{(*)}$

$\| * \|$

## Meaning

state matrix

input matrix

output matrix

state feedback matrix

output feedback matrix

matrix of eigenvectors

null space

the set of unstable matrices

state closed loop matrix

output closed loop matrix

diagonal matrix of eigenvalues ( $\lambda_j$ )

state vector

control vector

output vector

input vector

$j$ th eigenvalues of  $*$

condition number of  $X$

$i$ th largest singular value of  $*$

minimum singular value of  $*$

maximum singular value of  $*$

the 2-norm of  $*$

# Chapter 1

## Introduction

The aim of this project is to construct a numerical algorithm for finding a state feedback, for a linear time-invariant control system, which will increase the distance to instability and keep the eigenvalues stable.

Before we consider this work in more detail, it is necessary to give some basic definitions.

A control system may be defined as an arrangement of physical components connected or related in such a manner as to command or regulate itself or another system. Control systems influence every way of modern life. Automatic washers and dryers, microwave ovens, aircraft control, chemical plants, even economic modelling are a few examples of where control systems are used.

In general, control systems can be thought of being either *open loop* or *closed loop*. An open loop system is one in which the control action is independent of the output. An idealised example is the heating system of a room, with a radiator which is fed with water of constant temperature. Such a system of a room is not perfect, since any change in the status of the room will effect the temperature of



the room. For example, changes in the outside temperature or the opening and closing of doors. A closed loop system is one in which the control action depends on the output in some way. The use of thermostats in order to control the heating system of a room or a house is a well known example of a closed loop system. Both of these systems are given in the following figures.

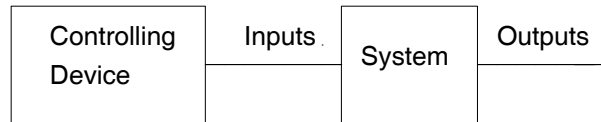


Figure 1.1: An open loop system

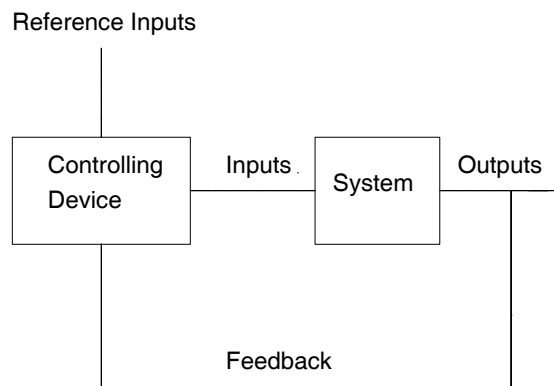


Figure 1.2: A closed loop system

In practical control problems, analysis starts with the formulation of a mathematical model of the physical system under investigation. This is done in chapter 2, and there the problem is formulated and the conditions given for the system to be controllable and observable.

In chapter 3 we look at the method of eigenstructure assignment. This method is used most commonly and is known to produce good results. In this method we are given a set of eigenvalues that we wish to assign and a feedback is sought that will assign these eigenvalues to our system. A numerical algorithm is given for finding a state feedback. Some examples are also given of finding a state feedback. It is known that if we assign the eigenvalues robustly we get a good distance to instability.

In chapter 4 we consider the method of singular value assignment. Here we have some fixed singular values and we wish to assign the remaining singular values. A numerical algorithm is given with some examples. In this chapter we look at finding a state feedback, since algorithms are available that assign singular values in this case. The area of finding an output feedback is not discussed as it is a recent area of research. It is not known what happens to the eigenvalues if we try to increase our distance to instability.

In chapter 5 we look at the distance to instability. A definition is given as well as some theory. We see that the distance to instability depends on the minimum singular value and so the method of singular value assignment would be a good way of increasing this singular value.

In chapter 6 the numerical algorithm for increasing the distance to instability is discussed. We look at some examples where we use this algorithm. We then use robust eigenstructure assignment to find a state feedback and compare whether this method give a better distance to instability than the algorithm discussed.

# Chapter 2

## Problem Formulation

In this chapter we introduce the concept of feedback, the equations that describe it, and the motivation behind pole or singular value assignment. We differentiate between state and output feedback and provide definitions of controllability and observability. We begin with the basic system.

### 2.1 Feedback

The most general form of feedback control system is shown in Figure 2.1:

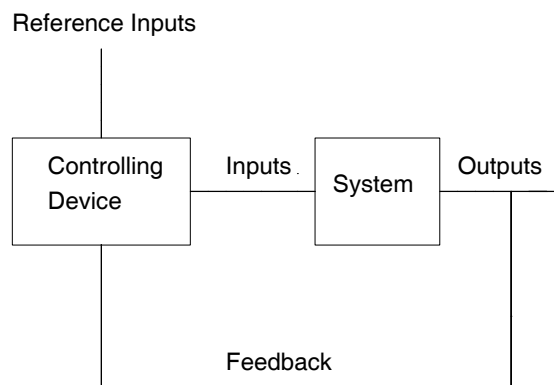


Figure 2.1: A feedback control system

Figure 2.1 shows a simple closed loop feedback control system which is employed in order to achieve or maintain a prescribed behaviour (ie stability). The controller examines the difference between the output of the process and the input and so employs a function to control the system. The equations describing the system in Figure 2.1 are

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}(t) + B\mathbf{u}(t) \quad (2.1)$$

$$\mathbf{y}(t) = C\mathbf{x}(t) \quad (2.2)$$

where,

$\mathbf{x} \in R^{n \times 1}$  is the system state vector,

$\mathbf{u} \in R^{m \times 1}$  is the system input vector,

$\mathbf{y} \in R^{p \times 1}$  is the system output vector,

$A \in R^{n \times n}$  is the state matrix,

$B \in R^{n \times m}$  is the input matrix,

$C \in R^{p \times n}$  is the output matrix.

Both the matrices B and C are assumed to be of full rank. If A, B and C are constant then our system is known to be *time invariant*; otherwise it is *time varying*.

Additionally we have the feedback

$$\mathbf{u} = F\mathbf{y} + B\mathbf{v} \quad (2.3)$$

where,

$F \in R^{m \times p}$  is the constant gain matrix and

$\mathbf{v} \in R^m$  is a reference input.

Then equation (2.1) becomes :

$$\frac{d\mathbf{x}}{dt} = (A + BFC)\mathbf{x} + B\mathbf{v} \quad (2.4)$$

We can see from (2.4) that the state matrix of the closed loop system is now given by  $A + BFC$ . We can see that this change in the state matrix may produce a change in the system behaviour, and that the feedback matrix  $F$  will control the way that the system behaves. The choice of  $F$  is therefore critical if we wish the system to behave in a certain way. If we employ a feedback that only uses the system outputs, then we have an output feedback problem and this has a closed loop matrix given by  $A + BFC$ . In the case of state feedback we assume that  $C$  is the  $n \times n$  identity matrix and assume that we may measure all of the system states. Notationally we make a distinction between the state feedback and output feedback matrices by labelling the state feedback matrix  $K \in R^{m \times n}$ .

In this case, substitution of

$$\mathbf{u} = K\mathbf{x} + \mathbf{v} \quad (2.5)$$

into (2.1) yields the following equation:

$$\frac{d\mathbf{x}}{dt} = (A + BK)\mathbf{x} + B\mathbf{v} \quad (2.6)$$

Now the closed loop matrix has the form  $A + BK$ .

So from equations (2.6) and (2.2) and  $C = I$  we have a state feedback problem and from (2.4) and (2.2) an output feedback problem. The problem of state feedback will be defined more formally in the next chapter, but before we do that we give conditions for controllability and observability and then look at the motivation behind pole and singular value assignment.

## 2.2 Controllability

Here we introduce the concept of controllability, that is, the ability of a given system described by (2.1), to achieve some desired final state by an admissible control.

### **Definition 2.1** [1]

*We say that a system is completely controllable if for any  $t_0$ , any initial state  $\mathbf{x}(t_0) = \mathbf{x}_o$  and any given final state  $\mathbf{x}_f$  there exists a finite time  $t_1 > t_0$  and a control  $\mathbf{u}(t)$ ,  $t_0 < t < t_1$ , such that  $\mathbf{x}(t_1) = \mathbf{x}_f$ .  $\square$*

An equivalent definition is given by:

**Theorem 2.2** *The system (2.1) is completely controllable if and only if the  $n \times nm$  controllability matrix associated with  $(A, B)$ ,*

$$W = [B, AB, A^2B, \dots, A^{n-1}B]$$

*has rank  $n$ .*

For proof see [1].  $\square$

Now we give another alternative definition of controllability :

**Theorem 2.3** *The system is completely controllable if and only if:*

$$\mathbf{s}^t A = \mu \mathbf{s}^t \text{ and } \mathbf{s}^t B = \mathbf{0} \Leftrightarrow \mathbf{s}^t = \mathbf{0}, \quad \forall \mu \in \mathbb{C}$$

Proof [9] :

i) Complete controllability  $\Rightarrow [\mathbf{s}^t A = \mu \mathbf{s}^t \text{ and } \mathbf{s}^t B = \mathbf{0} \Leftrightarrow \mathbf{s}^t = \mathbf{0}.]$

Suppose complete controllability and that there exists an  $\mathbf{s}^t \neq \mathbf{0}$  such that,  $\mathbf{s}^t A = \mu \mathbf{s}^t$  and  $\mathbf{s}^t B = \mathbf{0}$  Then

$$\mathbf{s}^t[\mathbf{B}, \mathbf{A}\mathbf{B}, \dots, \mathbf{A}^{n-1}\mathbf{B}] = [\mathbf{s}^t\mathbf{B}, \mu \mathbf{s}^t\mathbf{B}, \dots, \mu^{n-1}\mathbf{s}^t\mathbf{B}] = \mathbf{0}.$$

This implies that there exists an  $\mathbf{s}^t \neq \mathbf{0}$  such that  $\mathbf{s}^t\mathbf{W} = \mathbf{0}$  which implies that  $\mathbf{W}$  has rank  $< n$

and hence the system is not controllable, by Theorem 2.2 which implies a contradiction.

ii)  $[\mathbf{s}^t \mathbf{A} = \mu \mathbf{s}^t \text{ and } \mathbf{s}^t \mathbf{B} = \mathbf{0} \Leftrightarrow \mathbf{s}^t = \mathbf{0}.] \Rightarrow$  complete controllability.

For proof of this part see [9].  $\square$ .

We have now looked at some conditions which are needed for the existence of a controller  $\mathbf{u}$ . We next look at the conditions needed for the existence of an observer.

## 2.3 Observability

Twinned with the concept of controllability is that of observability, or more precisely the ability to determine the state of a system by measurements of its outputs only. In this section we give equivalent definitions of observability and conditions to be satisfied for the existence of an output feedback matrix.

**Definition 2.4** [1] *A system is observable if for any  $t_0$  and any initial state  $\mathbf{x}(t_0) = \mathbf{x}_0$ , there exists a finite time  $t_1 > t_0$  such that knowledge of  $\mathbf{u}(t)$  and  $\mathbf{y}(t)$  for  $t_0 < t < t_1$  suffices to determine  $\mathbf{x}_0$ .  $\square$ .*

Again we give a equivalent definition which may be used to determine observability.

**Theorem 2.5** *The system given by (2.1)-(2.2) is completely observable if and only if the  $n \times n$  observability matrix associated with  $(\mathbf{A}, \mathbf{C})$ ,*

$$V = \begin{bmatrix} C \\ CA \\ CA^2 \\ \cdot \\ CA^{n-1} \end{bmatrix}$$

has rank  $n$ .

For proof see [1].  $\square$

**Theorem 2.6** *The system (2.1)-(2.2) is completely observable if and only if:*

$$A\mathbf{s} = \lambda \mathbf{s} \text{ and } C\mathbf{s} = 0 \Rightarrow \mathbf{s} = 0, \quad \forall \lambda \in \mathbb{C}$$

For proof of we make use the duality theorem in [1].  $\square$

By comparison of the controllability matrix associated with the pair  $(A^t, C^t)$  and the observability matrix associated with the pair  $(A^t, B^t)$ , we have the following theorem:

**Theorem 2.7** *The system (2.1)-(2.2) is completely controllable if and only if the dual system*

$$\frac{d\mathbf{x}}{dt} = -A^t\mathbf{x}(t) + C^t\mathbf{u}(t) \tag{2.7}$$

$$\mathbf{y}(t) = B^t\mathbf{x}(t) \tag{2.8}$$

*is completely observable and vice versa.*

So we have that controllability and observability are dual concepts.



We have now looked at conditions that are needed for a system to have either a controller or an observer, that is, the system to be completely controllable or completely observable. In the next chapter we define the problem for pole assignment and singular value assignment.

## 2.4 Motivation for Pole and Singular Value Assignment

Now we look at the motivation behind pole assignment and we examine the use of a feedback in a particular way to achieve some property. A general time-continuous system can be described by the differential problem

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0, \quad (2.9)$$

where the matrix  $A$  is of dimension  $n \times n$  and is constant.

Equation (2.9) has the following solution:

$$\mathbf{x}(t) = \exp(At)\mathbf{x}_0 \quad (2.10)$$

If we expand the exponential term and take norms, we have

$$\|\exp(At)\| \leq \sum_{k=1}^q \sum_{j=1}^{\alpha_k} t^{j-1} \exp(\operatorname{Re}(\lambda_k)t) \|Z_{kj}\| \quad (2.11)$$

where,

$\operatorname{Re}(\lambda_k)$  denotes the real part of the eigenvalues of  $A$ ,

$q$  is the number of distinct eigenvalues,

$\alpha_k$  is the order of the largest Jordan block associated with the eigenvalues of  $A$

[1].

$Z_{kj}$  are constant matrices determined entirely by  $A$  and finally  $\|\cdot\|$  is the  $l_2$  norm.

From (2.11),  $\|exp(At)\| \rightarrow 0$  as  $t \rightarrow \infty$ , provided that  $Re(\lambda_k) < 0 \forall k$ , since (2.11) is a finite sum of terms which each tend to zero as  $t \rightarrow \infty$ . Now we need the following definitions [1]:

**Definition 2.8** *We say that the equilibrium state  $\mathbf{x}=\mathbf{0}$  is stable if for any positive scalar  $\epsilon$  there exists a positive scalar  $\delta$  such that  $\|\mathbf{x}(t_0)\|_2 < \delta \Rightarrow \|\mathbf{x}(t)\|_2 \leq \epsilon, \forall t \geq t_0$ .  $\square$ .*

An necessary and sufficient condition for stability can be given as follows:

**Theorem 2.9** *Let the eigenvalues of  $A$  be  $\lambda_k, k = 1, 2, \dots, n$ . Then system (2.9) is stable if and only if  $Re(\lambda_k) \leq 0$  and for any  $\lambda_k$  with  $Re(\lambda_k)=0$ , the eigenvalues are simple, ie the eigenvalues are non defective and have a full set of associated independent eigenvectors.  $\square$*

**Definition 2.10** *We say that the system is asymptotically stable if  $\mathbf{x}=\mathbf{0}$  is stable and if:*

$$\mathbf{x}(t) \rightarrow 0 \text{ as } t \rightarrow \infty.$$

$\square$ .

From (2.10) it follows that  $\|\mathbf{x}(t)\| \leq \|exp(At)\| \|\mathbf{x}_0\|$  and therefore the system (2.9) is asymptotically stable provided that  $Re(\lambda_k) < 0$ .

We see then for stability it is sufficient that the eigenvalues of the system state matrix have negative real parts. We aim to design a feedback that is able to alter the state matrix so that this is the case. Specifically, the method we

plan to use will assign eigenvalues or poles to precise locations and is commonly known as pole placement. So the motivation behind pole assignment is to find a feedback matrix which makes the system stable by moving the eigenvalues to new locations.

We now look at the motivation behind singular value assignment. It is assumed that we have a stable matrix  $A$  in the sense that all the eigenvalues of  $A$  have negative real parts. We consider the set of matrices  $\mathcal{U}$  which have at least one eigenvalue on the imaginary axis and so are unstable. Then the distance from  $A$  to set  $\mathcal{U}$  is defined to be:

$$\beta(A) = \min_{E \in \mathbf{C}^{n \times n}} [\|E\| | A + E \in \mathcal{U}]$$

This is a measure of how 'nearly unstable' is the stable matrix  $A$  (ie the distance to instability). So we need to find a feedback such that we maximise the distance between our closed loop matrix and the set of the set of unstable matrices. In a later chapter we see that the distance to instability is related to singular values and so the motivation behind singular value assignment is that we wish to find a feedback such the distance to instability is as large as possible. More details on how this is done will be discussed in a later chapter. In either case we require that our new system matrix is stable. Before we look at these methods, we need to give some basic matrix theory.

## 2.5 Basic Matrix Theory

In this section we describe two basic decompositions of a matrix. Throughout this dissertation we make extensive use of the singular value decomposition(SVD)

and the QR decomposition of a matrix  $M \in R^{n \times m}$ . In the usual notation the SVD is given by:

$$M=U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^t$$

where  $U$  and  $V$  are  $n \times n$  and  $m \times m$  orthogonal matrices, respectively, and  $\Sigma$  is a  $\text{rank}(M) \times \text{rank}(M)$  diagonal matrix with positive diagonal entries. Also we refer to the orthogonal reduction of  $M$  to diagonal form:

$$U^t M V = \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}$$

as an SVD of  $M$  because we always need it in this form.

The  $\sigma_i$  are the singular values of  $M$  and the vectors  $u_i, v_i$  are the  $i$ th *left singular vector* and the  $i$ th *right singular vector*, respectively. It is easy to see that by comparing columns in the equations  $MV=\Sigma U$  and  $M^tU=\Sigma^tV$  that

$$Mv_i=\sigma_i u_i$$

$$M^t u_i=\sigma_i v_i$$

The next decomposition is the QR decomposition. Again in the usual notation a QR decomposition of the matrix  $M$  is given by:

$$M=QR,$$

where  $Q \in R^{n \times n}$  is orthogonal and  $R \in R^{n \times m}$  is upper triangular.

More information about numerical techniques for computing these decompositions are given in [5].

# Chapter 3

## Robust Eigenstructure

### Assignment

#### 3.1 Introduction

In this chapter we look at a way of assigning eigenvalues and eigenvectors by state feedback in the linear time invariant system described by the equations (2.1)-(2.2). There are two approaches for doing this:

- by Linear State Feedback.
- by Output Feedback.

In the following section we discuss how we assign eigenvalues and eigenvectors by linear state feedback. We could find a feedback by output feedback but this is not discussed here.

## 3.2 Background Theory for Linear State Feedback

The state feedback pole assignment problem in control system design is essentially an inverse eigenvalue problem; that is, we assign eigenvalues and find the system which has these assigned eigenvalues. A desirable property of any system design is that the poles should be insensitive to perturbations in the coefficients matrices of the system equations. There are many way of assigning eigenvalues discussed in earlier papers [4, 11] but in this section we look for ways of obtaining a robust solution that is '*a well conditioned solution.*'

We now consider the completely controllable, time invariant, linear, multi-variate system (2.1)-(2.2). In this section  $C = I$  the  $n \times n$  identity matrix. The behaviour of the system (2.1) is governed by the eigenvalues of the matrix  $A$ . If we have an unstable system, then it is often desirable to make the system stable, and this is done by pole(eigenvalue) assignment. This is achieved by using the state feedback control

$$\mathbf{u} = K\mathbf{x} + \mathbf{v} \quad (3.1)$$

where the matrix  $K$  is called the **feedback** or **gain matrix** and is chosen such that the modified dynamic system :

$$\frac{d\mathbf{x}}{dt} = (A + BK)\mathbf{x}(t) + B\mathbf{v}(t), \quad (3.2)$$

now with an input  $\mathbf{v}$ , has the desired poles. In this case  $K$ , the feedback matrix, is found by assigning linearly independent eigenvectors corresponding to the required eigenvalues, such that the matrix of eigenvectors is as well conditioned as possible. Thus this method is called eigenstructure assignment.

The state feedback pole assignment problem for system (2.1) can be formulated as:

**Problem 3.1** [6]

Given real matrices  $(A, B)$  of orders  $(n \times n, n \times m)$  respectively, and a set of  $n$  complex numbers  $\Delta = (\lambda_1, \lambda_2, \dots, \lambda_n)$ , closed under complex conjugation, find a real  $m \times n$  matrix  $K$  such that the eigenvalues of  $A + BK$  are  $\lambda_j, j = 1, 2, \dots, n$ .

Given the Problem 3.1 can we find a solution to this?. Conditions for a solution to exist are well known and the following theorem is well established.

**Theorem 3.2** [13]

A solution  $K$  to Problem 3.1 exists for every set  $\Delta$  of self conjugate complex numbers if and only if the pair  $(A, B)$  is **completely controllable**, that is, if and only if:

$$\mathbf{s}^t A = \mu \mathbf{s}^t \text{ and } \mathbf{s}^t B = \mathbf{0} \Leftrightarrow \mathbf{s}^t = \mathbf{0}$$

If the pair  $(A, B)$  was not controllable, i.e. there exists  $\mathbf{s}^t \neq \mathbf{0}$  such that  $\mathbf{s}^t A = \mu \mathbf{s}^t$  and  $\mathbf{s}^t B = \mathbf{0}$ , thus  $\mathbf{s}^t (A + BK) = \mu \mathbf{s}^t$  for all  $K$ . Then we can see that  $\mu$  is an eigenvalue of  $A + BK$  for all  $K$  and must belong to any set  $\Delta$  of poles to be assigned. The pole  $\mu$  is said to be *uncontrollable*, and it cannot be modified by any feedback control.

In the case where  $m = n$ , a solution always exists, since  $rank(B) = n$  implies that the left null space of  $B$  contains only the trivial solution and the pair  $(A, B)$  is always *completely controllable*.

If we restrict the choice of the feedback matrix such that the resulting system matrix  $A + BK$  is nondefective, then Problem 3.1 can be reformulated as the

following:

**Problem 3.3 [6]**

Given  $(A, B)$  and  $\Delta$  (as in Problem 3.1), we need to find a real matrix  $K$  such that

$$(A + BK)X = X\Lambda \tag{3.3}$$

for some nonsingular  $X$ , where

$$\Lambda = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n).$$

From equation (3.3) it can be seen that the columns  $\mathbf{x}_j$ ,  $j = 1, 2, \dots, n$  of the matrix  $X$  are the right eigenvectors of  $A+BK$  corresponding to the assigned eigenvalues  $\lambda_j$ . Similarly, the rows  $(\mathbf{y}_j)^t$ ,  $j = 1, 2, \dots, n$  of the matrix  $Y^t = X^{-1}$  are the corresponding left eigenvectors. It has been shown by Wilkinson [12] that the sensitivity of the eigenvalues  $\lambda_j$  to perturbations in the components of  $A, B$ , and  $K$  depends upon the magnitude of the *condition number*  $c_j = 1/s_j$ , where:

$$s_j \equiv \frac{|(\mathbf{y}_j)^t \mathbf{x}_j|}{\|\mathbf{y}_j\|_2 \|\mathbf{x}_j\|_2} \leq 1$$

In the case of multiple eigenvalues, a particular choice of eigenvector is assumed. (For  $\lambda_j$  the sensitivity  $s_j$ , is just the cosine of the angle between the right and left eigenvectors corresponding to  $\lambda_j$ ).

We also observe that a bound on the sensitivity of the eigenvalues is given by Wilkinson [12], and is

$$\max c_j \leq \kappa_2(X) \equiv \|X\|_2 \|X^{-1}\|_2,$$



where  $\kappa_2(X)$  is the condition number of the matrix  $X = [\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n]$

Bearing in mind what we have just discussed, we can now reformulate Problem 3.3 such that we have a *robust* pole assignment problem to solve. It can be stated as follows:

**Problem 3.4** [6] *Given  $(A, B)$  and  $\Delta$  (as in Problem 3.1) we need to find a matrix  $K$  such that:*

$$(A + BK)X = X\Lambda \quad (3.4)$$

*for some nonsingular  $X$ , where*

$$\Lambda = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n),$$

*such that some measure  $\nu$  of the conditioning, or robustness of the eigenproblem is optimized.*

We could take this measure  $\nu$  to be  $\nu_1 = \|\mathbf{c}\|_\infty$  where  $\mathbf{c}^t = [c_1, c_2, \dots, c_n]$  is the vector of the condition numbers (ie. the condition number  $c_j$  defined above) corresponding to the selected matrix  $X$  of eigenvectors. Alternatively, we could take the measure of robustness to be  $\nu_2 = \kappa_2(X)$ , the condition number of the matrix  $X$ . The measure  $\nu_2$  then gives an upper bound on the measure  $\nu_1$ , and both attain their common minimum values simultaneously.

It would be a good idea to ask under what conditions a given nonsingular matrix  $X$  can be assigned to the a system problem. The following theorem can be used to demonstrate the conditions that are needed:

**Theorem 3.5** [6]

*Given  $\Lambda = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$  and  $X$  nonsingular, then there exists a solution to Problem 3.4 if and only if*

$$U_1^t(AX - X\Lambda) = 0, \quad (3.5)$$

where

$$B = [U_0, U_1] \begin{bmatrix} Z \\ 0 \end{bmatrix}, \quad (3.6)$$

with  $U = [U_0, U_1]$  orthogonal and  $Z$  is nonsingular. Then  $K$  is given explicitly by:

$$K = Z^{-1}U_0^T(X\Lambda X^{-1} - A). \quad (3.7)$$

*Proof:* See ref [6]

The assumption that  $B$  is of full rank implies the existence of the decomposition (3.6). From (3.4),  $K$  must satisfy

$$BK = X\Lambda X^{-1} - A \quad (3.8)$$

and pre-multiplication by  $U^t$  then gives the two equations

$$ZK = U_0^t(X\Lambda X^{-1} - A) \quad (3.9)$$

$$0 = U_1^t(X\Lambda X^{-1} - A), \quad (3.10)$$

from which (3.5) and (3.7) follow directly, since  $X$  is invertible from our condition that  $X$  is nonsingular.  $\square$

We observe that the decomposition of  $B$  in (3.6) is in fact a QR decomposition in which  $Z$  is an upper triangular matrix. Alternatively we could take the decomposition to be the Singular Value Decomposition in which we have  $Z = \Sigma V^t$ , where  $\Sigma = \text{diag}(\sigma_1, \sigma_2, \dots, \sigma_n)$  is a positive matrix and  $V$  is orthogonal.

Now we have looked at some of the theory behind the assigning of eigenvalues and eigenvectors, we present an algorithm that will do this.

### 3.3 Numerical Algorithm

We now consider the practical implementation of the theory discussed in the previous section for the linear state feedback design. The following algorithm can be found in [6]. The procedure consists of three basic steps :

**•Step 1:**

*Compute the decomposition of matrix B by either using SVD or QR, to find the matrices  $U_0, U_1$  and  $Z$ , and construct the orthonormal bases, comprised of the columns of the matrices  $S_j, \hat{S}_j$ , for the null space  $S_j = \mathcal{N}[U_1^t(A - \lambda_j I)]$  and its complement  $\hat{S}_j$  for  $\lambda_j \in \Delta, j = 1, 2, \dots, n$ .*

Standard library software is available to compute the decomposition of B using either SVD or QR. We see that QR is less expensive to compute than SVD but doesn't give as much information as SVD does about the system.

We consider two methods to find the orthonormal bases  $S_j$  and  $\hat{S}_j$ :

Case 1(SVD):

We determine the singular value decomposition of  $U_1^t(A - \lambda_j I)$  in the form:

$$U_1^t(A - \lambda_j I) = Z_j[\Gamma, 0][\hat{S}_j, S_j]^t. \quad (3.11)$$

Then the columns of  $S_j$  and  $\hat{S}_j$  give the required orthonormal bases.

Case 2(QR):

We determine the QR decomposition of  $(U_1^t(A - \lambda_j I))^t$  partitioned as the

following:

$$(U_1^t(A - \lambda_j I))^t = [\hat{S}_j, S_j] \begin{bmatrix} R_j \\ 0 \end{bmatrix}. \quad (3.12)$$

Then  $S_j$ , and  $\hat{S}_j$  are the required matrices.

**•Step 2:**

*Select vectors  $\mathbf{x}_j = S_j \mathbf{w}_j \in S_j$  with  $\|\mathbf{x}_j\|_2=1$  and set  $X=[\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n]$*

There are four basic methods which are used for this step namely Methods 0, 1 and 2/3 which can be found in [6]. Each of the methods aims to minimize a different measure of the conditioning of matrix  $X$ , although two of them use relatively simple measures. Each of these methods are based on an iteration where we have an initial set of eigenvectors  $\mathbf{x}_j \in S_j$  and  $X_j=[\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_{j-1}, \dots, \mathbf{x}_{j+1}, \dots, \mathbf{x}_n]$ . At each sweep of the iteration, the vector  $\mathbf{x}_j$  is replaced by a new vector  $\mathbf{x}_j \in S_j$ , selected to improve the conditioning of  $X$ . A complete sweep has been made when  $j$  has run from 1 to  $n$ . The iteration is then continued with the new matrix  $X$  until it becomes well conditioned in some sense (ie the condition number of  $X$  becomes unchanged for some tolerance). A more detailed description of these methods are given elsewhere [6] and shall not be discussed any further in this dissertation.

**•Step 3:**

*Find the matrix  $M=A+BK$  by solving  $MX=X\Lambda$  and compute  $K$  explicitly from  $K=Z^{-1}U_0^t(X\Lambda X^{-1} - A)$*

The matrix  $M=X\Lambda X^{-1}$  is constructed in Step 3 by solving the following equation  $X^t M^t = (X\Lambda)^t$  for  $M^t$  using direct LU decomposition or Gaussian Elimination methods.

•Step 4:

All of the above steps can be carried out using the system MATLAB [7]. This system uses standard library routines from software packages such as LINPACK and EISPACK.

### 3.4 Examples

In this section we look at some examples which have been collected from the literature [6] for which the numerical procedures in the earlier sections have been used. In two of the examples a linear state feedback control has been used.

Example 1: Chemical Reactor [6]

$$n = 4, m = 2$$

$$A = \begin{bmatrix} 1.380 & -0.0277 & 6.715 & -5.676 \\ -0.5814 & -4.290 & 0 & 0.6750 \\ 1.067 & 4.273 & -6.654 & 5.893 \\ 0.0480 & 4.273 & 1.343 & -2.104 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ 5.679 & 0 \\ 1.136 & -3.146 \\ 1.136 & 0 \end{bmatrix}$$

$$\text{EIG}(A) = (1.991, 6.351 \times 10^{-2}, -5.057, -8.6666)$$

This system can be seen to be unstable (ie.  $Re(\lambda_j) > 0$ ) and a feedback gain matrix is required to stabilize the system. We therefore assign the following eigenvalues  $\Delta = (-0.2, -0.5, -5.0566, -8.6659)$ . If the procedure of Section 3.3 is carried out to find a state linear feedback control of the form  $\mathbf{u} = \mathbf{K}\mathbf{x}$ , we get the following feedback gain matrix (using Method 2/3 in Step 2 [6]):

$$K = \begin{bmatrix} 0.10277 & -0.63333 & -0.11872 & 0.14632 \\ 0.83615 & 0.52704 & -0.25775 & 0.54269 \end{bmatrix}$$

The conditioning of the results are given in the following table [6]:

(a) sol. after two sweeps					(b) sol. at convergence				
Method	$\ \mathbf{c}\ _\infty$	$\kappa_2(X)$	$\ \mathbf{c}\ _2$	$\ K\ _2$	$\ \mathbf{c}\ _\infty$	$\kappa_2(X)$	$\ \mathbf{c}\ _2$	$\ K\ _2$	Sweeps
0	1.82	3.43	3.28	1.47	-	-	-	-	-
1	1.79	3.38	3.27	1.44	1.76	3.32	3.23	1.40	106
2/3	2.36	4.56	3.71	1.16	2.37	4.54	3.68	1.17	6

Table 3.1 Conditioning

The last column in the table is the number of sweeps needed for convergence. From the table the magnitude of the gain matrix using Method 2/3 is  $\|K\|_2=1.17$  and the condition number of the matrix of eigenvectors is  $\kappa_2(X)=4.54$ . The matrix which has these assigned eigenvalues is:

$$A+BK = \begin{bmatrix} 1.38 & -0.20770 & 6.715 & -5.6760 \\ 0.0022062 & -7.8867 & -0.67420 & 1.5059 \\ -1.4468 & 1.8955 & -5.9780 & 4.3519 \\ 0.16474 & 3.5535 & 1.2081 & -1.9378 \end{bmatrix}.$$

The condition number of  $X$  is not too large so we conclude that we have found a well-conditioned solution. If Method 0, is used the best result is obtained after one sweep; if Method one is used, then we have convergence within 106 sweeps compared with 6 when Method 2/3 is used. Although Method 1 gives a better condition number for  $X$  which is 3.32, we use a lot of sweeps to achieve this. The

maximum condition number  $\|\mathbf{c}\|_\infty$  using method 2/3 is increased slightly as is the magnitude of  $\|K\|_2$  of the gains.

We now go on and look at a different example which comes from the area of aircraft control. We wish to move the eigenvalues such that they are all real.

**Example 2:** Aircraft control [6]

$$n = 4, m = 3$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0.00014 & -2.04 & -1.95 & 0.013 \\ -0.00025 & 1 & -1.32 & -0.024 \\ -0.56 & 0 & 0.36 & -0.28 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 & 0 \\ -5.33 & 0.0065 & -0.27 \\ -0.16 & -0.012 & -0.25 \\ 0 & 0.11 & 0.086 \end{bmatrix}$$

$$\text{EIG}(A) = (-3.12 \times 10^{-2}, -2.46 \times 10^{-1}, -1.68 \pm 1.35i)$$

This time we assign the eigenvalues  $\Delta = (-1, -2, -3, -4)$ , and so we want all the eigenvalues to be real. Again, if Method 2/3 is used, then we get a state feedback matrix which has  $\|K\| = 28.255$  after two sweeps and has converged at this point. With the other methods we get the same sort of results with the condition number of  $K$  ranging from 25-30. The errors introduced are due to rounding error. More details about this example can be found in [6].

In the examples we have illustrated the method of eigenstructure assignment. In the next chapter we look at the method of singular value assignment. The theory is discussed and then the numerical algorithm is stated to achieve this. In all of the examples either the system given is unstable or we just wish to move the eigenvalues to obtain different system behaviour.

# Chapter 4

## Singular Value Assignment

In this chapter we again consider the time invariant continuous dynamical system of the form (2.1)-(2.2) with state feedback (3.1). The closed loop system takes the form (3.2).

The closed loop matrix  $A + BK$  gives us the response of the system and therefore we have to choose  $K$  to obtain the required behaviour. In this chapter we are interested in assigning singular values which give the system certain properties (i.e. to make the matrix  $A + BK$  as well-conditioned as possible or, equivalently, to make the distance to instability as large as possible). The method presented is a numerically stable method. To obtain the feedback matrix we apply a method which employs a number of orthogonal matrix decompositions.

### 4.1 Preliminary Theory

Again our system has to be completely controllable. The following theorem gives us the basic tool and provides a 'canonical form' for our system, which can be obtained in a numerically stable way. The theorem is a modification of the theory



presented in [2].

**Theorem 4.1** *Let  $A \in R^{n \times n}$ ,  $B \in R^{n \times m}$  and let  $\text{rank}(B)=m \leq n$ . Then there exists orthogonal matrices  $Q, U, V$  such that:*

$$QAU = \begin{bmatrix} \Sigma_1 & 0 & 0 \\ A_{21} & A_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad QBV = \begin{bmatrix} 0 \\ \Sigma_B \\ 0 \end{bmatrix}$$

where  $\Sigma_1, \Sigma_B$  are  $l \times l$  and  $m \times m$  diagonal matrices respectively with positive diagonal entries and  $A_{22}$  is a matrix with full column rank. The partitioning in  $QAU$  and  $QBV$  is conformable.

*Proof.* Let

$$\hat{P}BV = \begin{bmatrix} \Sigma_B \\ 0 \end{bmatrix}$$

be an SVD of the matrix  $B$ . Now let

$$P = \begin{bmatrix} 0 & I_{n-m} \\ I_m & 0 \end{bmatrix} \hat{P}$$

Then we obtain

$$PBV = \begin{bmatrix} 0 \\ \Sigma_B \end{bmatrix}, \quad PA = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$$

with a compatible partitioning. Let

$$WA_1Z_1 = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix}$$

be an SVD of  $A_1$ , where  $\Sigma_1$  is an  $l \times l$  diagonal matrix with positive entries. Then

$$\begin{bmatrix} W & 0 \\ 0 & I_m \end{bmatrix} \text{PA}Z_1 = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \\ A_{21} & \hat{A}_{22} \end{bmatrix}$$

where  $[A_{21}, \hat{A}_{22}]$  is a compatible partitioning of  $A_2Z_1$ . Let  $Z_2$  be an orthogonal matrix which does a 'column compression'

$$\hat{A}_{22}Z_2 = [A_{22}, 0]$$

on  $\hat{A}_{22}$ , such that  $A_{22}$  has full column rank. This matrix could, for example, be derived from an QR decomposition of  $\hat{A}_{22}^t$ .

Then from the above matrices we get the desired transformation as :

$$\begin{bmatrix} I_l & 0 & 0 \\ 0 & 0 & I_m \\ 0 & I_{n-m-l} & 0 \end{bmatrix} \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \\ A_{21} & \hat{A}_{22} \end{bmatrix} \begin{bmatrix} I_l & 0 \\ 0 & Z_2 \end{bmatrix} = \begin{bmatrix} \Sigma_1 & 0 & 0 \\ A_{21} & A_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} I_l & 0 & 0 \\ 0 & 0 & I_m \\ 0 & I_{n-m-l} & 0 \end{bmatrix} \begin{bmatrix} W & 0 \\ 0 & I_m \end{bmatrix} \text{PB}V = \begin{bmatrix} 0 \\ \Sigma_B \\ 0 \end{bmatrix} \quad \square.$$

**Theorem 4.2** [8] *If the system pair  $(A, B)$  is completely controllable, then we have the decompositions:*

$$QAU = \begin{bmatrix} \Sigma_1 & 0 \\ A_{21} & [A_{22}, 0] \end{bmatrix} \quad QBV = \begin{bmatrix} 0 \\ \Sigma_B \end{bmatrix}$$

*Proof*

We know from the definitions given in Section 2.2 that we have:

controllability if and only if  $rk[A - \lambda I, B] = n \quad \forall \lambda$ .

In particular when  $\lambda=0$ , for the system to be completely controllable we need

$$rk[A, B] = rk\left[Q[A, B] \begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix}\right] = rk[QAU, QBV] = n. \text{ Therefore we require}$$

$$rk \left[ \begin{array}{ccc|c} \Sigma_1 & 0 & 0 & 0 \\ A_{21} & A_{22} & 0 & \Sigma_B \\ 0 & 0 & 0 & 0 \end{array} \right] = n,$$

which implies that we can only have a controllable system, if we haven't got the last row of zero's, and hence we obtain our two decompositions.  $\square$ .

Now  $[A_{22}, 0]$  is a square matrix of dimension  $n-l \times n-l$  if the system is completely controllable, and therefore:

$$Q(A + BK)U = QAU + QBVV^tKU = \begin{bmatrix} \Sigma_1 & 0 \\ A_{21} + \Sigma_B \hat{K}_1 & [A_{22}, 0] + \Sigma_B \hat{K}_2 \end{bmatrix},$$

and we can assign a set of desired singular values by choosing  $\hat{K} = [\hat{K}_1, \hat{K}_2] = V^t K U$  appropriately. In particular, we obtain:

$$Q(A+BK)U = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix},$$

where  $\Sigma_1$  contains  $n-m$  fixed singular values and  $\Sigma_2$  contains  $m$  assigned singular values, by taking

$$\hat{K}_1 = -\Sigma_B^{-1} A_{21}$$

$$\hat{K}_2 = \Sigma_B^{-1} (\Sigma_2 - [A_{22}, 0]).$$

We can then recover  $K$  from  $V\hat{K}U^t$ . So we have now found a  $K$  that will assign the required singular values to our system. In all the numerical work described here, orthogonal transformations are used to achieve the required  $K$ .

## 4.2 Numerical Algorithm

In this section a numerical algorithm is presented where we compute the decompositions of  $A$  and  $B$  and then find the feedback  $K$  which gives the system the required singular values.

**•Step 1:**

Find orthogonal matrices  $\hat{P}$  and  $V$  such that,  $\hat{P}BV = \begin{bmatrix} \Sigma_B \\ 0 \end{bmatrix}$ , using the singular value decomposition of  $B$ .

**•Step 2:**

Let  $P = \begin{bmatrix} 0 & I_{n-m} \\ I_m & 0 \end{bmatrix}$  and partition  $P\hat{P}A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$ , compatibly with  $P\hat{P}BV = \begin{bmatrix} 0 \\ \Sigma_B \end{bmatrix}$ .

**•Step 3:**

Find orthogonal matrices  $W$  and  $Z_1$  such that

$$WA_1Z_1 = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix}, \Sigma_1 = \text{diag}(\sigma_1, \dots, \sigma_l),$$

where  $l$  is the rank of  $A_1$ , by the singular value decomposition of  $A_1$ . Here the singular values are ordered such that  $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_l > 0$ .

**•Step 4:**

Partition  $A_2Z_1 = [A_{21}, \hat{A}_{22}]$  compatibly with  $WA_1Z_1$  and find an orthogonal matrix

$Z_2$  such that  $\hat{A}_{22}Z_2=[A_{22},0]$  where  $A_{22}$  is of full rank. This is achieved by the Q-R decomposition of  $A_{22}^t$ .

•**Step 5:**

Then let

$$Q = \begin{bmatrix} I_l & 0 & 0 \\ 0 & 0 & I_m \\ 0 & I_{n-l-m} & 0 \end{bmatrix} \begin{bmatrix} W & 0 \\ 0 & I_m \end{bmatrix} P\hat{P}, \quad U = Z_1 \begin{bmatrix} I_l & 0 \\ 0 & Z_2 \end{bmatrix}$$

•**Step 6:**

Now we have to choose our assigned singular values. We choose them to be such that  $\Sigma_2 = \text{diag}(\sigma_{l+1}, \dots, \sigma_n)$  where  $\sigma_l(\Sigma_1) \leq \sigma_j(\Sigma_2) \leq \sigma_1(\Sigma_1), j = l+1, \dots, n$

•**Step 7:**

We now find the feedback matrix  $K$  such that  $A+BK$  has these assigned singular values. Let  $\hat{K} = [\hat{K}_1, \hat{K}_2]$  where:

$$\begin{aligned} \hat{K}_1 &= -\Sigma_B^{-1}A_{21}, \\ \hat{K}_2 &= \Sigma_B^{-1}(\Sigma_2 - [A_{22}, 0]) \end{aligned}$$

and set  $K = V\hat{K}U^t$ .

## 4.3 Examples

### Example 1

In this section we consider the following numerical example:

$$A = \begin{bmatrix} 4 & 2 & 1.2 \\ 2 & 1.2 & 0.8 \\ 1.2 & 0.8 & 0.5663 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\sigma_{max}(A)=5.462, \sigma_{min}(A)=0.003108, \kappa_2(A)=1757.$$

The matrix A can be seen to be fairly ill conditioned. We therefore want to design a K that will modify this system such that the matrix becomes well conditioned. When the numerical algorithm of Section 4.2 is applied, we get the state feedback matrix:

$$K = \begin{bmatrix} -2.5996 & -3.2883 & -2.7685 \\ -0.8547 & -2.5265 & 1.1601 \end{bmatrix}$$

and

$$A+BK = \begin{bmatrix} 1.4004 & -1.2884 & -1.5685 \\ 2 & 1.2 & 0.8 \\ 0.34530 & -1.7265 & 1.7265 \end{bmatrix}.$$

Now the closed loop matrix  $A + BK$  has singular values  $\sigma_1 = \sigma_2 = \sigma_3 = 2.466$  and the matrix is well conditioned with condition number one.

### Example 2

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0.00014 & -0.04 & -1.95 & 0.013 \\ -0.00025 & 1 & -1.32 & -0.024 \\ -0.56 & 0 & 0.36 & -0.28 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 & 0 \\ -5.33 & 0.0065 & -0.27 \\ -0.16 & -0.012 & -0.25 \\ 0 & 0.11 & 0.086 \end{bmatrix}$$

$$\sigma_{max}(A)=2.463, \sigma_{min}(A)=0.018, \kappa_2(A)=136.789.$$

Again when the numerical procedure is applied we get the state feedback matrix :

$$K = \begin{bmatrix} 0.1578 & -0.2298 & -0.2843 & 0.2045 \\ 1.0596 & -3.3687 & 0.8932 & 11.8373 \\ -3.001 & 4.3088 & -5.2886 & -3.5991 \end{bmatrix}$$

and

$$A+BK = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -0.0239 & 0 & 0.9993 & -0.0283 \\ 0.7121 & 0 & 0.0369 & 0.7010 \\ -0.7015 & 0 & 0.0034 & 0.7125 \end{bmatrix}.$$

The singular values of this closed loop matrix  $A + BK$  are then  $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 = 1$ . Again this matrix is well conditioned with condition number equal to one.

We now look at the distance to instability and how singular value assignment can be used to widen the distance to instability.

# Chapter 5

## Distance to Instability

If we consider a matrix  $A$  that is stable in the sense that all its eigenvalues lie in the open left half plane, then the distance to instability is a measure of 'how stable' matrix  $A$  is. In this chapter we describe a bisection method which enables us to find this distance.

Suppose that  $A \in \mathbb{C}^{n \times n}$  has no eigenvalues on the imaginary axis. Let  $\mathcal{U} \subset \mathbb{C}^{n \times n}$  be the set of matrices with at least one eigenvalue on the imaginary axis. The distance from  $A$  to  $\mathcal{U}$  is defined to be:

$$\beta(A) = \min_{E \in \mathbb{C}^{n \times n}} [\|E\| | A + E \in \mathcal{U}]$$

**Theorem 5.1** [3]

$$\beta(A) = \min_{\omega \in \mathbb{R}} (\sigma_{\min}(A - i\omega I))$$

*Proof* See [3] for reference.

If matrix  $A$  is stable, let  $B$  be the closest unstable matrix to  $A$  (i.e.  $B$  is unstable and minimizes  $\|A - C\|$  over all unstable  $C$ .) Then  $B$  has an eigenvalue on the imaginary axis with the same imaginary part as some of the eigenvalues



of  $A$ , then one may conclude that,

$$\|A - B\| = \min_{\omega \in \mathbb{R}} (\sigma_{\min}(A - i\omega I)).$$

where  $\sigma_{\min}(A - i\omega I)$  is the smallest singular value of  $A - \omega I$  (i.e. the distance from  $A$  to  $B$  an unstable matrix).  $\square$ .

So for any real  $\omega$ , an upper bound on  $\beta(A)$  is

$$\beta(A) \leq \sigma_{\min}(A - i\omega I)$$

In the next section we describe a bisection method which will enable us to find this distance.

## 5.1 Bisection Method

If we are given  $\sigma \geq 0$  and  $A \in \mathbb{R}^{n \times n}$ , then we may define the  $2n \times 2n$  matrix  $H=H(\sigma)$  by:

$$H=H(\sigma) = \begin{bmatrix} A & -\sigma I_n \\ \sigma I_n & -A^H \end{bmatrix},$$

where  $I_n$  denotes the  $n$  by  $n$  identity matrix and  $A^H$  represents the complex transpose.

The following theorem shows how the eigenvalues of  $H(\sigma)$  distinguish the cases  $\sigma \geq \beta(A)$  from  $\sigma < \beta(A)$ .

**Theorem 5.2**  *$H(\sigma)$  has an eigenvalue whose real part is zero if and only if  $\sigma \geq \beta(A)$ .*

*Proof* Can be found in [3]  $\square$ .

Suppose that  $\alpha$  is a lower bound and  $\gamma$  is an upper bound on  $\beta(A)$ . The bounds can be improved by choosing a number  $\sigma$  that lies between  $\alpha$  and  $\gamma$  and checking to see if  $H(\sigma)$  has any eigenvalue with a zero real part. The following algorithm gives an estimate of the distance to instability,  $\beta(A)$  to within a factor of ten. Also this algorithm uses the naive upper bound  $\beta(A) \leq 1/2\|A + A^H\|$  found in [3].

*Bisection Algorithm.*

**•Step 1:**

*Input:*  $A \in \mathbb{C}^{m \times n}$  and a tolerance  $\tau > 0$

**•Step 2:**

*Finding  $\alpha$  and  $\gamma$ :*

$$\alpha = 0, \gamma = 1/2\|(A + A^H)\|$$

**WHILE**  $\gamma > 10\text{MAX}(\tau, \alpha)$

$$\sigma = \sqrt{\gamma \text{MAX}(\tau, \alpha)}$$

**IF**  $H(\sigma)$  has an eigenvalue with zero real part **THEN**  $\gamma = \sigma$  **ELSE**  $\alpha = \sigma$

**•Step 3**

*Output:*  $\alpha \in \mathbb{R}$  and  $\gamma \in \mathbb{R}$  such that either  $\gamma \setminus 10 \leq \alpha \leq \beta(A) \leq \gamma$  or  $0 = \alpha \leq \beta(A) \leq \gamma \leq 10\tau$ .

With the choice of  $\tau = 1/2(10^{-8}\|A + A^H\|)$ , then at most we require three bisection steps.

Now we require to know the value of  $\omega$  which gives the smallest singular value, as it is this that we are trying to maximise. There are two way of doing this: either by plotting  $\omega$  against  $\sigma_{\min}(A - i\omega I)$  for some range of  $\omega$  or by simply modifying the bisection algorithm.

**•Step 1:**

*Input:  $A \in \mathbb{C}^{m \times n}$ ,  $\zeta$  and a tolerance  $\tau > 0$ .*

**•Step 2:**

*Finding  $\alpha$  and  $\gamma$ :*

$$\alpha = 0, \gamma = 1/2 \| (A + A^H) \|$$

**WHILE**  $\gamma > \zeta \mathbf{MAX}(\tau, \alpha)$

$$\sigma = \sqrt{\gamma \mathbf{MAX}(\tau, \alpha)}$$

**IF**  $H(\sigma)$  has an eigenvalue with zero real part **THEN**  $\gamma = \sigma$  **ELSE**  $\alpha = \sigma$

**•Step 3:**

*Finding  $\omega$ :*

*Take the singular value  $\hat{\sigma} = \gamma$*

*Calculate the eigenvalues of  $H(\hat{\sigma})$  and find the eigenvalues  $\lambda = \pm \omega i$  which have real part which is zero.*

*Calculate  $\sigma_{\min}(A - i\omega I)$  for each  $\omega$  and take  $\omega$  for which  $\sigma_{\min}(A - i\omega I) = \hat{\sigma}$ .*

**•Step 4:**

*Output:  $\alpha \in \mathbb{R}$  and  $\gamma \in \mathbb{R}$  such that either  $\gamma \setminus \zeta \leq \alpha \leq \beta(A) \leq \gamma$  or  $0 = \alpha \leq \beta(A) \leq \gamma \leq \zeta \tau$ , and  $\omega$  and  $\hat{\sigma}$*

In the modified algorithm we again take the tolerance  $\tau$  to be as before, but this time  $\zeta$  is taken to be less than 10 as we want the error on  $\beta(A)$  to be quite small. Then our estimate of the minimum singular value will be as accurate as possible, and our estimate of  $\omega$  will be close to the real value of  $\omega$ .

## 5.2 Examples

In the examples to follow we use the modified bisection algorithm of Section 5.1 to calculate the distance to instability and then find the corresponding value of  $\omega$ .

### Example 1[10]

$$A = \begin{bmatrix} * & 4 & -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & -10 & 4 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & * & 4 & -1 & -1 & -1 & -1 \\ 0 & 0 & -1 & * & 4 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & * & 4 & -1 & -1 \\ 0 & 0 & 0 & 0 & -4 & * & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & * & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & -6 & * \end{bmatrix}$$

Note in this example  $* = -1 \times 10^{-5}$

The eigenvalues are  $-10^{-5}$ ,  $-10$ ,  $-10^{-5} \pm 2i$ ,  $-10^{-5} \pm 4i$ , and  $-10^{-5} \pm 6i$  and can be seen to be distinct.

From the modified algorithm we get a value of  $0.29738124 \times 10^{-5}$  for  $\beta(A)$  with  $\omega = \pm 3.99$  and  $\zeta = 1.00001$ . This is verified by plotting and can be seen in Figure 5.1.

**Example 2**[10]

$$A = \begin{bmatrix} -0.01 & 5 & -1 & -1 \\ -5 & -0.01 & 5 & -1 \\ 0 & 0 & -0.01 & 5 \\ 0 & 0 & -5 & -0.01 \end{bmatrix}$$

Here we have the following eigenvalues  $-0.01 \pm 5i$  which can be seen to be double defective.

From the modified algorithm we get  $\omega = \pm 4.9995$  and a value of  $0.3170150 \times 10^{-4}$  for  $\beta(A)$  with  $\zeta = 1.00000001$ . Again this is verified by plotting and can be seen in Figure 5.2.

The graphs of the minimum singular values against  $\omega$  shown in the Figures 5.1 and 5.2 were found by the program plot1.m in Appendix 1. The programs were written using the package MATLAB.

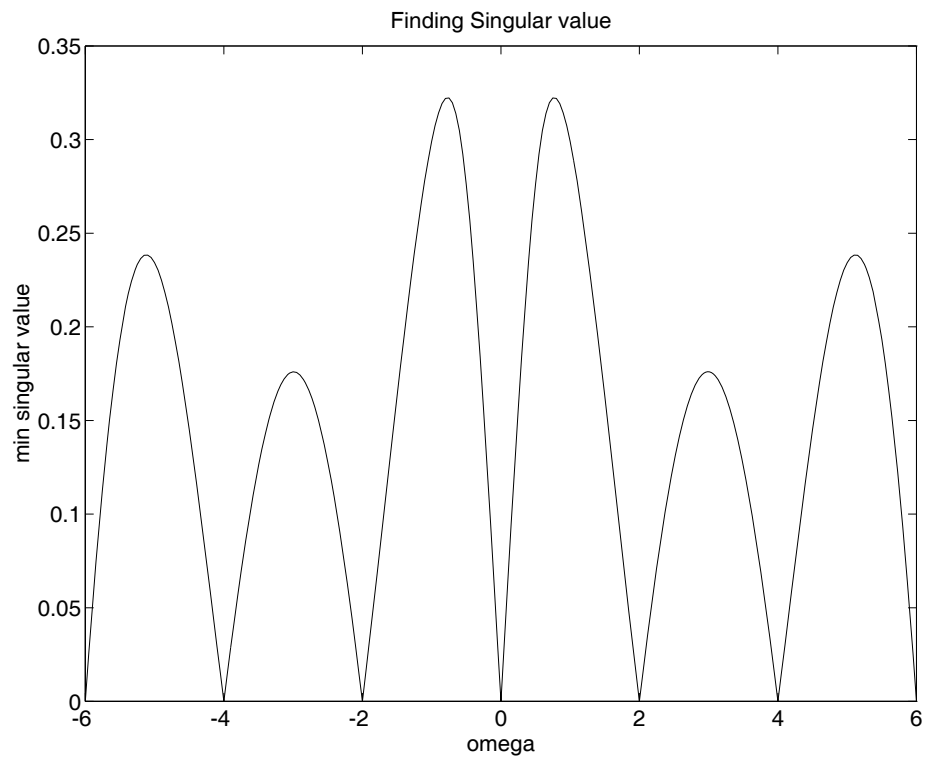


Figure 5.1: Example 1

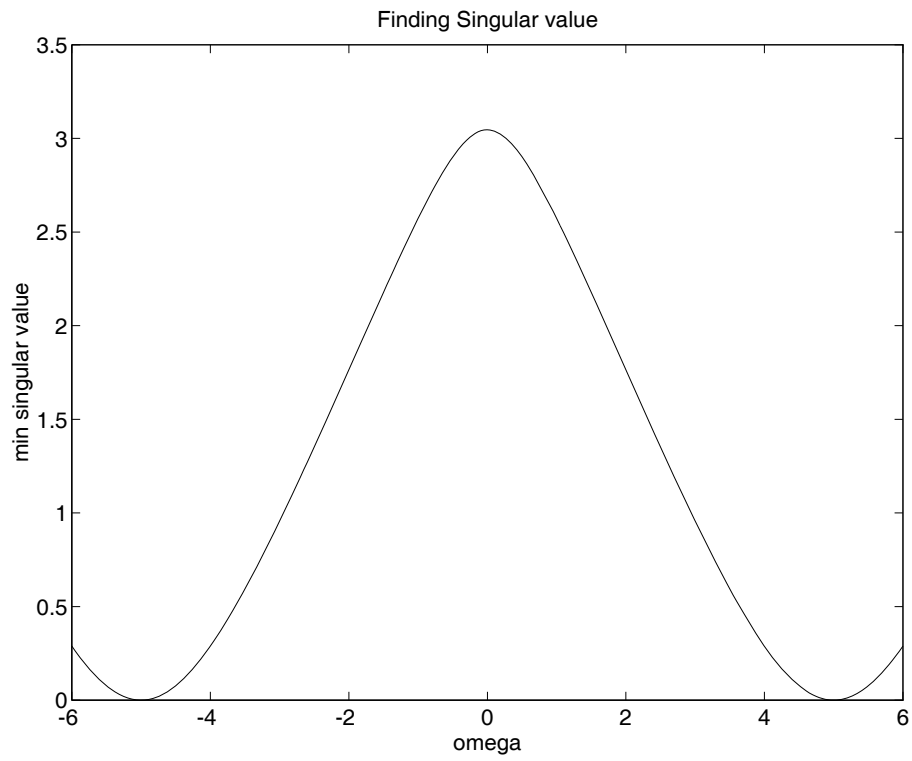


Figure 5.2: Example 2

# Chapter 6

## Methods for Increasing the Distance to Instability

In this chapter we look at some numerical examples where we have used two different ways of achieving a state feedback namely singular value assignment and robust eigenstructure assignment. In either case we are interested in increasing the distance to instability. We state the numerical algorithm for increasing the distance to instability by singular value assignment. Then we consider some numerical examples where we have used this algorithm. We then look at some examples where we have used the method of eigenstructure assignment to increase the distance to instability and then make a comparison of the two methods.

### 6.1 Numerical Algorithm

The numerical algorithm for constructing a state feedback which increases the distance to instability via singular assignment consists of two steps :

**•Step 1**

*Input:  $A, B, \zeta, \gamma, \tau, \alpha, H(\sigma)$ .*

**•Step 2**

In this step we calculate the minimum singular values and the corresponding  $\omega$ .

**•Step 2.1:**

*First set  $\sigma = \sqrt{\gamma \text{MAX}(\tau, \alpha)}$  and calculate the eigenvalues  $\lambda_j$  of  $H(\sigma)$ . If any of  $\lambda_j$  have  $\text{Re}(\lambda_j) > 0$ , then set  $\gamma = \sigma$ , else  $\alpha = \sigma$ .*

*Repeat this until  $\gamma < \zeta \text{MAX}(\tau, \alpha)$ .*

**•Step 2.2:**

*Take  $\hat{\sigma} = \gamma$  and calculate the eigenvalues of  $H(\hat{\sigma})$ . Store  $\omega_j = \text{Im}(\lambda_j)$  of any eigenvalues which have  $\text{Re}(\lambda_j) = 0$ .*

**•Step 2.3:**

*Calculate  $A - i\omega_j I$ , and find  $\sigma_{\min}(A - i\omega_j I)$ . If  $\sigma_{\min}(A - i\omega_j I) = \hat{\sigma}$  then  $\omega_j = \omega$  else repeat until such  $\omega_j$  is found.*

**•Step 3**

In this step we aim to find the state feedback  $K$  which increases the distance to instability.

**•Step 3.1**

*Calculate  $\hat{A} = A - i\omega I$  with  $\omega$  found by the previous step. Find orthogonal matrices  $Q, U$  and  $V$  such that:*

$$Q\hat{A}U = \begin{bmatrix} \Sigma_1 & 0 \\ A_{21} & [A_{22}, 0] \end{bmatrix}, \quad QBV = \begin{bmatrix} 0 \\ \Sigma_B \end{bmatrix},$$

*where  $\Sigma_1 \in R^{l \times l}, [A_{21}, 0] \in R^{n-l \times n-l}$ .*



•**Step 3.2**

Choose singular values such that:

$$\sigma_l(\Sigma_1) \leq \sigma_i(\Sigma_2) \leq \sigma_1(\Sigma_1), i = l + 1, \dots, n,$$

where  $\Sigma_2 = \text{diag}(\sigma_{l+1}, \dots, \sigma_n)$ .

•**Step 3.3**

Calculate  $K$  from  $K = V\hat{K}U^t$  where  $\hat{K} = [\hat{K}_1, \hat{K}_2]$  and,

$$\hat{K}_1 = -\Sigma_B^{-1}A_{21}$$

$$\hat{K}_2 = \Sigma_B^{-1}(\Sigma_2 - [A_{22}, 0])$$

•**Step 3.4**

Repeat step one with  $A + BK$  where  $K$  is the state feedback just found, to find  $\hat{\sigma}_1$  and  $\omega_1$ .

**Step 4**

Output:  $K, \hat{\sigma}_1, \omega_1, \omega, \hat{\sigma}, A + BK$ .

## 6.2 Singular Value Assignment

The following examples were computed using the MATLAB package. The programs are given in Appendix 1. In each example we apply the numerical algorithm described in Section 6.1.

**Example 1**[6]

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0.00014 & -2.04 & -1.95 & 0.013 \\ -0.00025 & 1 & -1.32 & -0.024 \\ -0.56 & 0 & 0.36 & -0.28 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 \\ -5.33 & 0.0065 & -0.27 \\ -0.16 & -0.012 & -0.25 \\ 0 & 0.11 & 0.086 \end{bmatrix}.$$

When the algorithm of Section 6.1 is applied we get the state feedback:

$$K = \begin{bmatrix} 0.15786 & -0.6184 & -0.2843 & 0.2045 \\ 1.0597 & -3.5707 & 0.8932 & 11.837 \\ -3.0017 & 4.5672 & -5.2887 & -3.59913 \end{bmatrix}$$

where the singular values of  $A+BK$  are  $\sigma_i = 1, i = 1, 2, 3, 4$ .

The results are as follows:

*	$\beta(*)$	$\omega$	$\lambda_j(*)$
$A$	0.010912	0	-0.031, -0.2473, $-1.6809 \pm 1.3504i$
$A + BK$	0.53813	$\pm 0.85$	$-0.535 \pm 0.8443i, 0.9106 \pm 0.4312i$

Table 6.1 Example 1

As we can see from the Table 6.1 we have managed to increase our distance to instability, but the eigenvalues have moved from eigenvalues which were stable to eigenvalues that are unstable. We observe that  $\|K\|=13.8512$ . Let see if we get the same sort of results with another example.

**Example 2**[6]

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -10.940 & -6.4894 & 1.5838 & 0.023645 \\ -1.5163 & 0.16176 & -0.51425 & 0.042692 \\ -0.44748 & -0.087530 & 0.20686 & -2.9964 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 \\ -0.172 & 0.0000745 \\ -0.0238 & -0.0000778 \\ 0 & 0.00369 \end{bmatrix}.$$

When the algorithm is applied we get the state feedback matrix:

$$K = 10^2 \times \begin{bmatrix} -0.5785 & -0.3689 & 0.085 & 0.001585 \\ 1.20939 & 0.29728 & -0.6476 & 9.47286 \end{bmatrix}$$

where the singular values of  $A+BK$  are  $\sigma_1 = 1.54989, \sigma_2 = 1, \sigma_3 = 0.5$  and  $\sigma_4 = 0.466129$ .

The results for this example are:

*	$\beta(*)$	$\omega$	$\lambda_j(*)$
A	0.463111	0	-1, -2, -3, -4
A+BK	1.265059	$\pm 0.90$	2.86922, -1.2999 $\pm$ 6.47732i -3.3044

Table 6.2 Example 2

We see from Table 6.2 that we have increased the distance to instability but again the eigenvalues have moved. In this example we find  $\|K\|=9.5767 \times 10^2$ . If all we wanted was to maximise the distance to instability and were not worried about the eigenvalues this, would be fine. Unfortunately we require the distance to instability to be increased and our eigenvalues to remain stable. We now consider  $A + \alpha BK$ , instead of  $A + BK$ , and find the value of  $\alpha$  where the eigenvalues change from a stable set to an unstable set. In the examples to follow we wish to find this  $\alpha$ . In all of the examples  $\alpha \in [0, 1]$ .

**Example 3**[6]

$$A = \begin{bmatrix} -3.6240 & 0.049567 & -0.24564 & -0.013853 \\ 0.33486 & -1.8875 & -0.81251 & -0.28102 \\ -0.19958 & -1.1335 & -2.2039 & -0.45523 \\ 0.13784 & -0.47140 & -0.33229 & -0.40605 \end{bmatrix},$$
$$B = \begin{bmatrix} 0.23122 & 0.88339 \\ 0.30761 & 0.21460 \\ 0.36164 & 0.56642 \\ 0.33217 & 0.50153 \end{bmatrix}.$$

where  $A + BK$  has the singular values  $\sigma_1 = 3.367, \sigma_2 = 2.8, \sigma_3 = 1.5$  and  $\sigma_4 = 1.388172$ .

When the algorithm is applied we get the state feedback matrix:

$$K = \begin{bmatrix} -4.2039 & 2.06077 & 5.0511 & 5.2694 \\ 5.78910 & 0.90811 & 0.31872 & 0.0253 \end{bmatrix}$$

The results for this example are as follows:

$\alpha$	$\beta(A + \alpha BK)$	$\omega$	$eig(A + \alpha BK)$	$cond(\lambda_j)$
0	1.04	0	-1.04	1.01
			-3.62	1.01
			-2.86	1.12
			-4.29	1.02
0.3	1.09	0	-3.68	1.04
			-1.68	1.12
			-2.66	1.06
			-1.13	1.15
0.6	0.23	0	0.23	1.001
			-3.43	1.07
			-2.01	1.59
			-1.32	1.54
0.8	1.26	0	1.55	1
			-3.35	1.07
			$-1.489 \pm 0.424$	1.28
1	1.26	$\pm 0.6$	2.8692	1
			$-1.299 \pm 0.647$	1.05
			-3.30	1.06

Table 6.3 Example 3

The comments on these results will be discussed after we have considered another example.

**Example 4**[6]

$$A = \begin{bmatrix} 1.38 & -0.20770 & 6.715 & -5.676 \\ 0.0022062 & -7.8867 & -0.67420 & 1.5059 \\ -1.4468 & 1.8955 & -5.9780 & 4.3519 \\ 0.16474 & 3.5535 & 1.2081 & -1.9378 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 5.679 & 0 \\ 1.136 & -3.146 \\ 1.136 & 0 \end{bmatrix}.$$

We get the state feedback matrix,

$$K = \begin{bmatrix} 0.6616 & 1.4225 & 0.7077 & 0.7158 \\ 2.1271 & 1.0166 & -2.7255 & 0.93764 \end{bmatrix}.$$

when the algorithm of Section 6.1 is applied. The closed loop matrix  $A + BK$  has the singular values  $\sigma_1 = 9.3212, \sigma_2 = 8.5, \sigma_3 = 7.5$  and  $\sigma_4 = 4.926141$

The results are shown in the following table:

$\alpha$	$\beta(A + \alpha BK)$	$\omega$	$eig(A + \alpha BK)$	$cond(\lambda_j)$
0	0.087	0	-5.06	2.35
			-0.2	2.37
			-8.66	1.08
			-0.5	1.07
0.03	0.1875	0	-4.38	2.9454
			-0.649	2.774
			-0.298	1.387
			-8.54	1.0820
0.08	0.005	0	$-2.31 \pm 0.616i$	8.463
			-8.344	1.072
			0.006	1.206
0.3	1.29	0	$-1.41 \pm 4.11i$	1.43
			-7.62	1.04
			1.50	1.09
1	4.89	$\pm 1$	$1.725 \pm 7.854i$	1.040
			-6.493	1.057
			6.890	1.034

Table 6.4 Example 4

From the Tables and Figures 6.1-6.2, we see that there seems to be a general trend. As we increase  $\alpha$ , the distance to instability reaches an optimum where the distance to instability has increased as well as the eigenvalues staying stable. In Example 3 the optimum value is 0.3 and the eigenvalues become unstable at 0.6. In Example 4 we get an optimum value of 0.03 and the eigenvalues become

unstable at 0.08. In all of the cases we notice that when the eigenvalues go from a real pair to a complex pair the condition number increases and then decreases. When our eigenvalues become unstable then our distance to instability decreases to zero and increases soon afterwards. If a different set of assigned singular values were assigned, as in the next example, we observed the same behaviour as in the previous examples.

### **Example 5**

In this example we take the system matrices  $A$  and  $B$  to be same as the ones in Example 4 but this time our closed loop matrix  $A + BK$  has singular values  $\sigma_1 = 9.3212, \sigma_2 = 7.5, \sigma_3 = 5$  and  $\sigma_4 = 4.926141$ . Again from Figure 6.3 we observe the same behaviour; that is, we reach an optimum value where the distance to instability is increased and the eigenvalues remain stable. The optimum value occurred at  $\alpha = 0.042$  and the eigenvalues became unstable at  $\alpha = 0.15$ .

From the examples in this section we observed that if we assign singular values to increase the distance to instability then we can't guarantee that the eigenvalues to remain stable. In all cases except Example 1 we could assign 2 singular values and these were  $\sigma_2$  and  $\sigma_3$  but in the case of Example 1 they were  $\sigma_i, i = 2, 3, 4$ . We go on now and consider the distance to instability when we assign eigenvalues using robust eigenstructure assignment.

## **6.3 Eigenstructure Assignment**

The distance to instability was observed for some examples where we assign the eigenvalues using the robust eigenstructure technique of Chapter 3.



### Example 6

The matrices A and B of Example 1 in Section 6.2 are used. The following eigenvalues were assigned  $\Delta = (-1, -2, -3, -4)$ . When the algorithm of Chapter 3 is used we get the state feedback matrix using method 2/3:

$$K = \begin{bmatrix} 0.79689 & 0.35594 & -0.54029 & -0.089527 \\ 5.9292 & -3.1747 & -6.0894 & -26.640 \\ -0.78300 & 3.9039 & 3.335 & 1.1934 \end{bmatrix}$$

The magnitude of the state feedback matrix is  $\|K\|=28.255$ . We know from Example 1 that the distance to instability of A is 0.010912 and that when we apply the algorithm in Section 6.1 the distance to instability is increased.

The distance to instability of the closed loop matrix found by the method of robust eigenstructure assignment is 0.620017 with  $\omega = 0$ . The optimum value for Example 1 was 0.20064 with  $\omega = 0$  and occurs at  $\alpha = 0.0015$ . By comparison we have managed to increase the distance to instability and see that the method of robust eigenstructure assignment gives a higher distance to instability than the method of singular value assignment. The norm of K using the method of robust eigenstructure assignment is higher than the norm of K when singular value assignment is used. We now go on and look at another example.

### Example 7

This example uses the system matrices A and B from Example 1 in Section 3.4 and the eigenvalues to be assigned are  $\Delta = (-0.2, -0.5, -5.0566, -8.6659)$ . When the algorithm of Chapter 3 is used we get the state feedback matrix using method 2/3:

$$K = \begin{bmatrix} 0.10277 & -0.63333 & -0.11872 & 0.14632 \\ 0.83615 & 0.52704 & -0.25775 & 0.54269 \end{bmatrix}$$

The magnitude of the state feedback matrix is  $\|K\|=1.17$ .

The distance to the instability of the closed loop matrix found by the method of robust eigenstructure assignment is 0.087 with  $\omega = 0$ . The optimum value for this example when the numerical algorithm of Section 6.1 is used, was found to be 2.75135 with  $\omega = 0$  and occurs at  $\alpha = 0.6$ . Unfortunately the matrix  $A+BK$  is unstable and so this distance is the distance to the imaginary axis. The matrix  $A$  is also unstable and has a distance of 0.06 from the imaginary axis, so we have managed to increase this distance but the eigenvalues remain unstable. If the method of robust eigenstructure assignment is used we have increased this distance as well as making our eigenvalues stable. In this case the norm of  $K$  obtained using the algorithm in Section 6.1 is higher than using the eigenstructure assignment method.

From the examples that we have observed we deduce that if our system is already stable then the method of robust eigenstructure assignment gives a higher distance to instability than what we get with the algorithm that assigns singular values to increase the distance to instability. If the system is unstable then the method of robust eigenstructure assignment give the best results as we can make the eigenvalues stable and increase the distance to instability. We see that if we use the method of robust eigenstructure assignment we can guarantee our eigenvalues are stable and that the distance in increased.

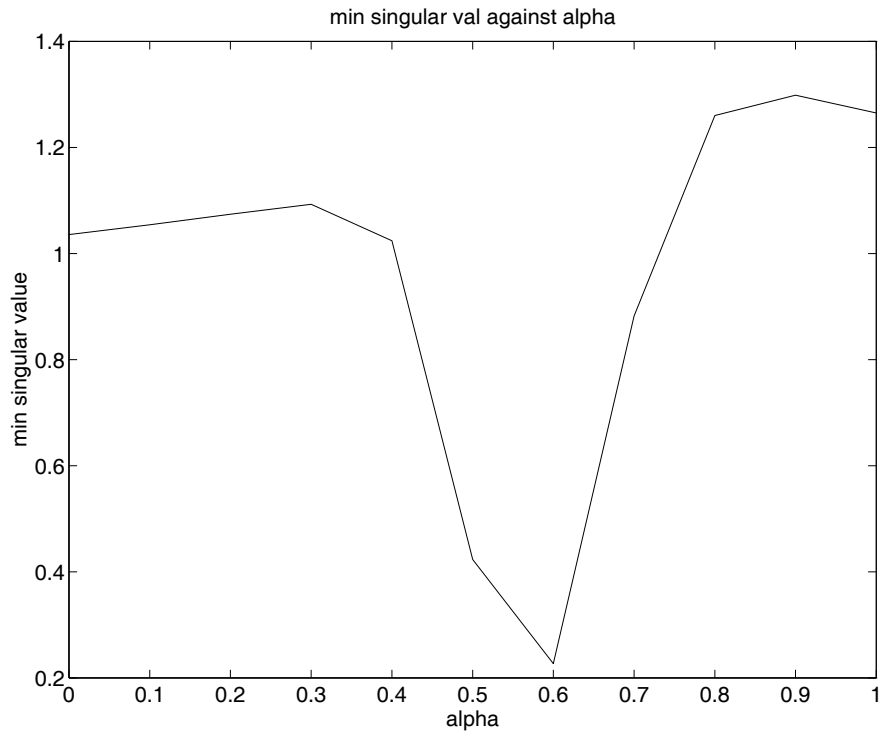


Figure 6.1: Example 3

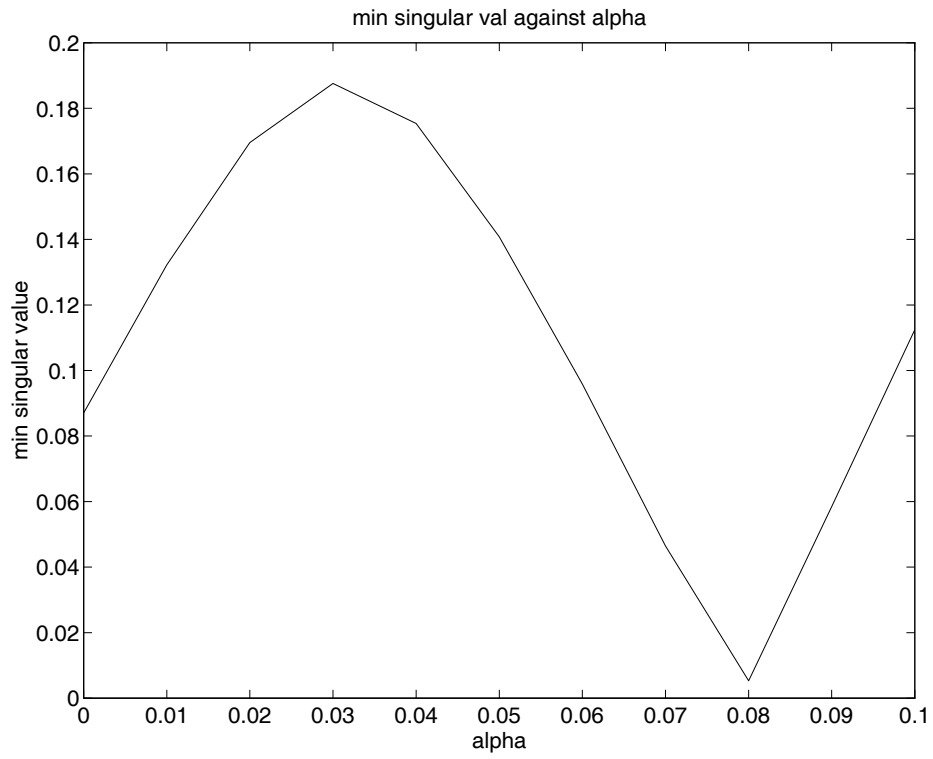


Figure 6.2: Example 4

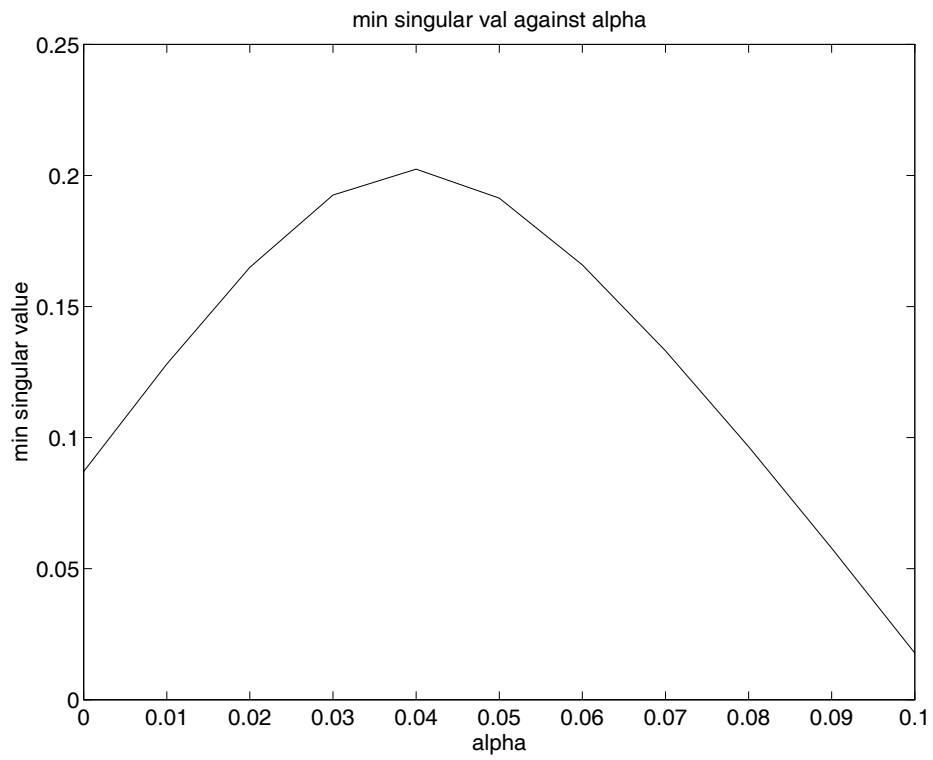


Figure 6.3: Example 5

# Conclusions

In this dissertation we have described an algorithm that aims to construct a state feedback to maximize the distance to instability. The method is based on a numerically stable approach. This method however turns out not to be a good one if we want to increase the distance to instability as well as keeping the eigenvalues stable. We see from the numerical results that in fact we have an optimum value where the distance to instability is increased as well as the eigenvalues staying stable, although this value tends to be lower than the distance observed by the algorithm. Unfortunately there is no time in this dissertation to construct an algorithm that will find the optimum state feedback and so opens up a new area of research. If on the other hand we obtain this feedback by robust eigenstructure assignment then we can guarantee that we have a stable set of eigenvalues and have increased the distance to instability.

# Appendix 1

## Programs in Matlab Notation

Throughout this dissertation the following programs were used to generate the results. The programs were written using the MATLAB package [7].

- **Plot1.m**
- **Eigplot.m**
- **Byers1.m**
- **Sing1.m**

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