DEPARTMENT OF MATHEMATICS

A SEARCH FOR NEW FINITE ELEMENTS TO BE USED IN CONJUNCTION WITH FREE-LAGRANGIAN METHODS

A. Priestley

Numerical Analysis Report 10/90

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Institute of Computational Fluid Dynamics
University of Reading
Department of Mathematics
P.O. Box 220
Reading
RG6 2AX

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Abstract

Lagrangian codes have seemed to rely on either bilinear elements on quadrilaterals or linears on triangles. Both these elements have their advantages and disadvantages. In this paper we shall search for higher order elements and consider whether it succumbs to a spurious mode - the problem with bilinears - or to mesh locking - a problem with the linears. Having found a promising element we then suggest another property that these finite elements ought to have but in fact our new element doesn't.

1. INTRODUCTION

Bilinear elements, using centroid quadrature, have a problem in that a spurious mode exists that allows the nodes to move in a prescribed way that produces no restoring forces, ie there is no resistance to this movement. This phenomenon is called hour-glassing because of the way this velocity pattern distorts the elements. It must be controlled if the spurious mode is not to swamp the solution. However, if the spurious mode is controlled too vigorously then the elements may lock.

Later in this section we will review the problem of hour-glassing for the bilinear element and will look at how it may be controlled.

In Section 2 we shall then look to see what spurious modes are present for other elements and using different quadratures. (In practice centroid quadrature is the one most likely to be used). We shall select the six-noded quadratic triangle with centroid or vertex quadrature as being the most promising candidate.

In Section 3 the problem of mesh locking will be looked at. We tentatively suggest why linear triangular elements tend to lock, why bilinears do if the spurious mode is damped too much and why the quadratic elements won't.

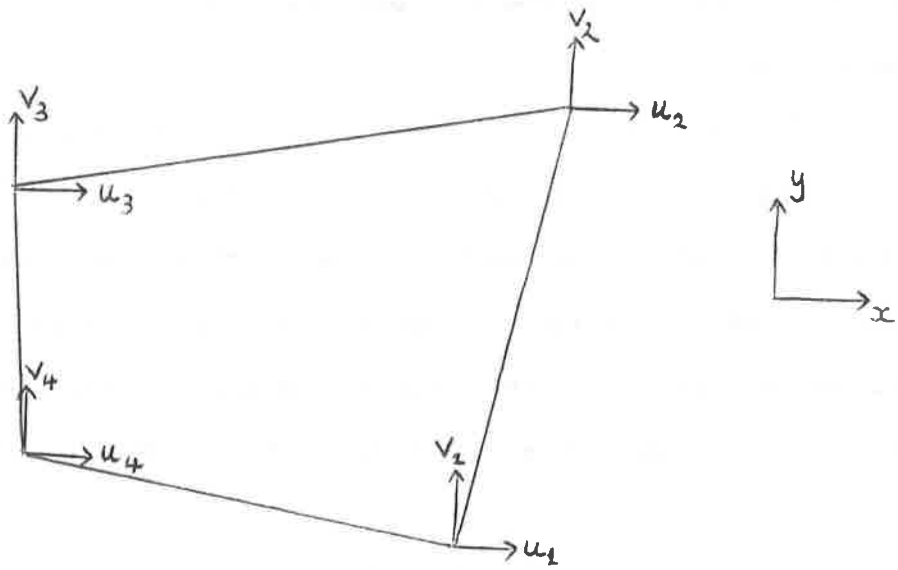
Finally, in Section 4, we discuss the reasons why such a promising element failed to performed in practice and what extra tests an element should be subjected to before being allowed near a Lagrangian code.

Margolin & Pyun (1987) considered the problem of hourglassing on bilinear elements. A spurious or hourglassing mode is defined to be a velocity field that deforms the element whilst producing no restoring forces. In this context no restoring forces results from the element integral (or an approximation thereto) of the divergence of the velocity field is zero on the element, ie

$$\int_E \underline{\nabla} \cdot \underline{v} = 0$$

⇒ No restoring forces on element E.

They consider a quadrilateral with u & v velocity components given as below:



They then represent the 8 velocity components in the cell by an 8-dimensional vector

$$\underline{v} = (u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4).$$

Margolin & Pyun identify six of the eight degrees of freedom with six physical modes of motion and with six mathematical objects. The six physical modes are:

one pattern of horizontal translation
one pattern of vertical translation
one pattern of rotation
one pattern of horizontal strain
one pattern of vertical strain
one pattern of shear strain

The six mathematical objects are then:

$$\tilde{u},\ \tilde{v},\ \frac{\partial u}{\partial x},\ \frac{\partial u}{\partial y},\ \frac{\partial v}{\partial x},\ \frac{\partial v}{\partial y}$$

The idea is then to produce a basis for the eight-dimensional space, $\underline{L}_1$, $\underline{L}_2$, ..., $\underline{L}_8$ where the first six vectors correspond to the six mathematical quantities. L_7 and L_8 are then found by orthogonalisation and must be spurious.

Assuming the basis vectors to be orthonormal, ie $\underline{L}_j \cdot \underline{L}_j = 1\ \forall_j$ we can then damp the spurious mode as follows:

$$\underline{v} = \underline{v} - \alpha \sum_{j=7}^{8} (\underline{v} \cdot \underline{L}_j)\, \underline{L}_j,$$

where α is to be chosen. A range of

$$0.01 < \alpha < 0.05$$

is recommended by Margolin & Pyun. $\alpha = ¼$ would mean the total removal of the spurious mode.

In this paper we will proceed slightly differently in that we will deliberately seek out the non-restoring modes. This is explained by using the bilinear element as an example.

Using centroid quadrature the integrals of the two spatial gradients we are interested in are given by

$$\frac{\partial u}{\partial x} = ((u_1 - u_3)(y_2 - y_4) + (u_2 - u_4)(y_3 - y_1))/2A$$

$$\frac{\partial \underline{v}}{\partial y} = ((v_1 - v_3)(x_4 - x_2) + (v_2 - v_4)(x_1 - x_3))/2A$$

where A = area of the quadrilateral.

By observation we see that the only way for these to be zero is if

$$u_1 = u_3 \quad \text{and} \quad u_2 = u_4$$

and similarly in v.

There are two independent ways of accomplishing this, firstly

$u_1 = u_3 = 1 = u_2 = u_4$

and $u_1 = u_3 = 1$ and $u_2 = u_4 = -1$.

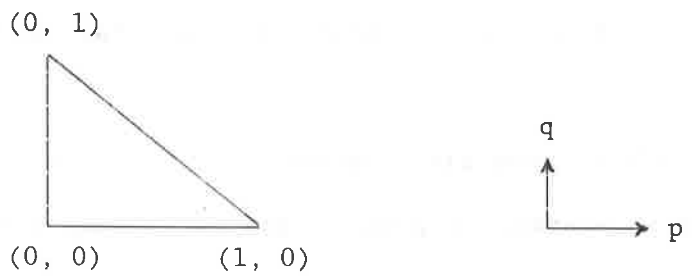
Again similarly in v. The first produces no restoring force but does not deform the element and is allowable. The second is not and corresponds to $\underline{L}_7$ of Margolin & Pyun. $\underline{L}_8$ is just the v version of this, ie

$$\underline{L}_7 = \tfrac{1}{2}(1, -1, 1, -1, 0, 0, 0, 0)$$

$$\underline{L}_8 = \tfrac{1}{2}(0, 0, 0, 0, 1, -1, 1, -1).$$

2. SPURIOUS MODES WITH OTHER ELEMENTS AND OTHER QUADRATURES

We will always transform our (irregular) triangle in (x, y) space on to the standard triangle in (p, q) space which looks like



We then need to calculate a transformation from (p, q) space to (x, y) space giving x = x (p, q) and y = y (p, q).

Having done this we can define the matrix

$$\begin{pmatrix} \frac{\partial x}{\partial p} & \frac{\partial x}{\partial q} \\ \frac{\partial y}{\partial p} & \frac{\partial y}{\partial q} \end{pmatrix}$$

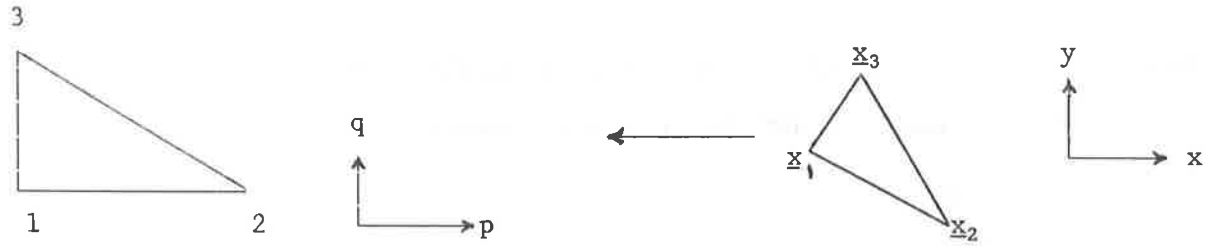
and inverting this gives a matrix which will equal

$$\begin{pmatrix} \frac{\partial p}{\partial x} & \frac{\partial p}{\partial y} \\ \frac{\partial q}{\partial x} & \frac{\partial q}{\partial y} \end{pmatrix}.$$

With this matrix we can then calculate the derivatives we are interested in, namely

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial x}$$

Linear Triangle



The transformation is given by

$$t = (1 - p - q)\, t_1 + pt_2 + qt_3$$

where t can be x, y or even u.

Clearly the derivatives are constants and so the method of integration is irrelevant, we will arrive at

$$\frac{\partial u}{\partial x} = \frac{1}{2A} \left\{ (u_1 - u_3)(y_2 - y_3) - (u_2 - u_3)(y_1 - y_3) \right\}.$$

We are using here, and will continue to use, the very sloppy notation of letting $^{\partial u}/_{\partial x}$ actually equal $\int {}^{\partial u}/_{dx}$.

It is easy to see that for this to be zero we need

$$u_1 = u_3 \text{ and } u_2 = u_3$$

$$\Rightarrow \quad u_1 = u_2 = u_3.$$

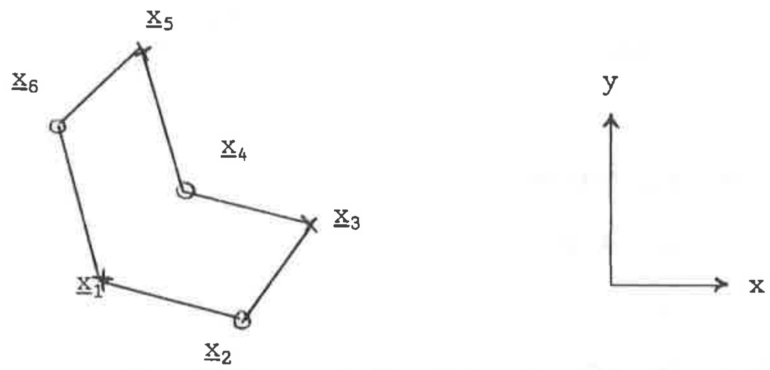
This is a quite allowable mode and so there are no spurious modes. The argument for $^{\partial v}/_{\partial y}$ is trivially the same.

Bilinear Elements and Vertex Quadrature

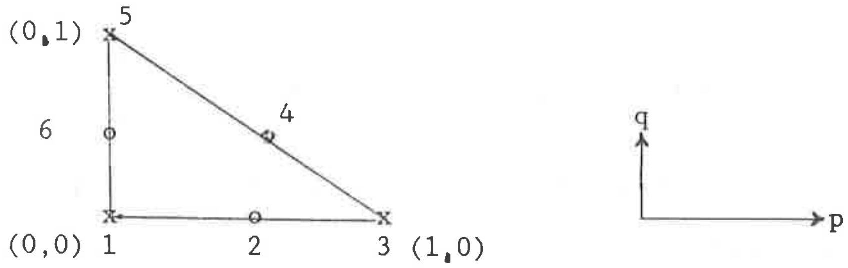
Having seen two simple examples we will not bore the reader with another as there is tedium in excess to come. We just state that bilinear elements when used in conjunction with vertex quadrature do not support any spurious modes.

Six-Noded Quadratic Triangle and Centroid Quadrature

The six-noded quadratic triangle element



is transferred on to the standard triangle in (p, q) space



by the transformation

$$t = (2p + 2q - 1)(p + q - 1)\, t_1 - 4p(p + q - 1)\, t_2 + p(2p - 1)\, t_3 + 4pq t_4 + q(2q - 1)\, t_5 - 4q(p + q - 1)\, t_6,$$

where t can again represent x, y or u.

Putting

$$D = (y_4 - y_6)(4x_5 - 16x_2) + (y_5 - y_1)(4x_6 - x_3)$$
$$+ (y_4 - y_2)(16x_6 - 4x_3) + (y_1 - y_3)(4x_2 - x_5)$$
$$+ (y_5 - y_3)(x_1 - 4x_4) + (y_6 - y_2)(4x_1 - 16x_4)$$

we obtain the following formula for the centroid integral of u_x.

$$\frac{\partial u}{\partial x} = \Big\{ (u_1 - u_3)(y_5 - 4y_2) + (u_1 - u_5)(4y_6 - y_3)$$
$$+ (u_5 - u_3)(4y_4 - y_1) + (u_6 - u_4)(4y_5 - 16y_2)$$
$$+ (u_2 - u_6)(4y_1 - 16y_4) + (u_4 - u_2)(4y_3 - 16y_6) \Big\} /D$$

For this to be zero we require, for example

$$u_1 = u_3 = u_5$$

$$u_2 = u_4 = u_6$$

and similarly

$$v_1 = v_3 = v_5$$

$$v_2 = v_4 = v_6$$

One way of achieving this just leads to uniform translation of the element and so is non-deforming. In the velocity space

$$(u_1, u_2, u_3, u_4, u_5, u_6, v_1, v_2, v_3, v_4, v_5, v_6)$$

two spurious modes are then

(1, -1, 1, -1, 1, -1, 0, 0, 0, 0, 0, 0)

(0, 0, 0, 0, 0, 0, 1, -1, 1, -1, 1, -1)

The two other spurious modes, four if we count the corresponding v-velocity vector, are given by:

(-5, ¼, 4, -1¼, 1, 1)

(2, -3/2, 4, ½, -6, -1)

Six-noded Quadratic Triangles and Vertex Quadrature

Let

$$D = 6(-3x_1y_3 - 16x_6y_2 + 4x_6y_3 + 16x_2y_6 + 4x_5y_2 - 4x_2y_5$$
$$- 3x_5y_1 - 4x_3y_6 + 3x_3y_1 + x_3y_5 - x_5y_3 + 3x_1y_5$$
$$+ 12x_1y_2 - 12x_1, y_6 - 12x_2y_1 + 12x_6y_1)$$

$$E = 6(16x_4y_6 + 4x_1y_4 - 16x_6y_4 - 4x_4y_1 - x_1y_3 + 4x_6y_3$$
$$+ 3x_5y_1 - 4_x3y_6 + x_3y_1 + 3x_3y_5 - 3x_5y_3 - 3x_1y_5$$
$$+ 12x_6y_5 + 12x_5y_4 - 12x_4y_5 - 12x_5y_6)$$

$$F = 6(4x_1y_4 - 4x_4y_1 - 3x_1y_3 - 4x_5y_2 + 4x_2y_5 + x_5y_1 + 3x_3y_1$$
$$- 3x_3y_5 + 16x_4y_2 + 3x_5y_3 = 16x_2y_4 - x_1y_5 + 12x_2y_3$$
$$- 12x_4y_3 - 12x_3y_2 + 12x_3y_4)$$

then

$$\frac{\partial u}{\partial x} = \Big\{ y_2\ (12u_1 + 4u_5 - 16u_6) + y_3\ (-3u_1 - u_5 + 4u_6)$$
$$+ y_5\ (-4u_2 + 3u_1 + u_3) + y_6\ (16u_2 - 12u_1 - 4u_3) \Big\} /D$$

$$+ \Big\{ y_3\ (-u_1 - 3u_5 + 4u_6) + y_4\ (4u_1 - 16u_6 + 12u_5)$$
$$+ y_5\ (-3u_1 + 3u_3 + 12u_6 - 12u_4) + y_6\ (-4u_3 - 12u_5 + 16u_4) \Big\} /E$$

$$+ \Big\{ y_2\ (-4u_5 - 12u_3 + 16u_4) + y_3\ (-3u_1 + 3u_5 - 12u_4 + 12u_2)$$
$$+ y_4\ (4u_1 - 16u_2 + 12u_3) + y_5\ (-4u_2 - u_1 - 3u_3) \Big\} /F$$

(Just in case you are worried about the absence of x_1's and y_1's in the numerator this is because, without loss of generality, they have been set to zero).

From this formula it is then quite straightforward to deduce that the only way the derivatives can be set to zero independently of $\{x_1, x_2, x_3, x_4, x_5, x_6, y_1, y_2, y_3, y_4, y_5, y_6\}$ is if $u_1 = u_2 = u_3 = u_4 = u_5 = u_6$. This is then just horizontal movement and is quite allowable.

Six-noded Quadratic Triangle and the Mid-Edge Rule

Again with $x_1 = y_1 = 0$ let

$$D = 6(-2x_6y_3 + x_5y_3 + 2x_3y_6 - x_3y_5 + 2x_2y_3 - 2x_4y_3 - 2x_3y_2 + 2x_3y_4)$$

$$E = 6(-2x_5y_2 + x_5y_3 + 2x_2y_5 - x_3y_5 - 2x_6y_5 - 2x_5y_4 + 2x_4y_5 + 2x_5y_6)$$

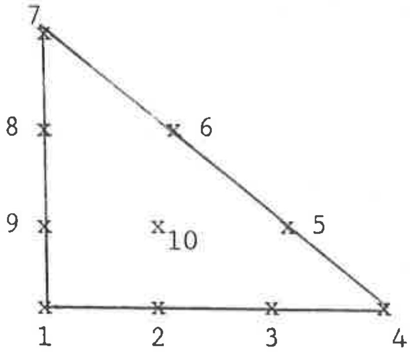
$$\begin{aligned} F = 6(&2x_6y_3 + 2x_5y_2 - x_5y_3 - 2x_2y_5 - 2x_3y_6 + x_3y_5 \\ &+ x_2y_3 - 2x_4y_3 - 2x_3y_2 + 2x_3y_4 - 2x_6y_5 - 2x_5y_4 \\ &+ 2x_4y_5 + 2x_5y_6). \end{aligned}$$

We can then write $\frac{\partial u}{\partial x}$ as

$$\begin{aligned} \frac{\partial u}{\partial x} = &\Big\{ y_4 (-2u_1 + 2u_3) + y_5 (u_1 - u_3) + y_6 (-2u_1 + 2u_3) \\ &+ y_3 (u_1 + u_5 - 2u_6 - 2u_4 + 2u_2) + y_2 (2u_1 + 2u_3) \Big\} /D \\ &+ \Big\{ 2y_2 (u_1 - u_5) + y_3 (u_5 - u_1) + 2y_4 (u_1 - u_5) \\ &+ y_5 (-u_1 - u_3 + 2u_2 + 2u_4 - 2u_6) + y_6 (2u_5 - 2u_1) \Big\} /E \\ &+ \Big\{ 2y_2 (u_5 - u_3) + y_3 (-u_1 - u_5 + 2u_6 - 2u_4 + 2u_2) + 2y_4 (u_3 - u_5) \\ &+ y_5 (u_1 + u_3 - 2u_2 + 2u_4 - 2u_6) + 2y_6 (u_5 - u_3) \Big\} /F \end{aligned}$$

It is now quite clear that to make this derivative zero we must have $u_1 = u_3 = u_5$ and a little deduction then reveals that we must have $u_1 = u_2 = u_3 = u_4 = u_5 = u_6$ as we would hope.

<u>Ten-noded Cubic Triangle with Centroid Quadrature</u>



The transformation of the general 10-noded cubic triangle on to the standard triangle is defined by:

$$
\begin{aligned}
t = & -\tfrac{1}{2}(3p + 3q - 2)(3p + 3q - 1)(p + q - 1)\,t_1 \\
& + \frac{9}{2}p\,(3p + 3q - 2)(p + q - 1)\,t_2 \\
& - \frac{9}{2}p\,(3p - 1)(p + q - 1)\,t_3 + \tfrac{1}{2}p\,(3p - 1)(3p - 2)\,t_4 \\
& + \frac{9}{2}\,pq\,(3p - 1)\,t_5 + \frac{9}{2}\,pq\,(3q - 1)\,t_6 \\
& + \tfrac{1}{2}q\,(3q - 1)(3q - 2)\,t_7 - \frac{9}{2}q\,(3q - 1)(p + q - 1)\,t_8 \\
& + \frac{9}{2}q\,(3p + 3q - 2)(p + q - 1)\,t_9 - 27pq\,(p + q - 1)\,t_{10}
\end{aligned}
$$

Define

$$
\begin{aligned}
D = 2\,(& - x_7y_1 - 3x_7y_3 + 3x_7y_2 + x_7y_4 - 3x_7y_5 + 3x_7y_9 \\
& - 3x_1y_6 + 9x_2y_8 - 9x_3y_6 + 3x_4y_8 - 3x_4y_9 - x_4y_7 \\
& - 3x_5y_1 - 9x_5y_6 + 9x_5y_2 - 9x_5y_8 + 9x_5y_9 + 3x_5y_7 \\
& + 9x_9y_6 + 9x_9y_8 - 3x_9y_7 + 3x_1y_3 - x_1y_4 + 3x_1y_5 \\
& + 3x_6y_1 + 9x_6y_3 - 9x_6y_2 - 3x_6y_4 + 9x_6y_5 - 9x_6y_9 \\
& - 9x_2y_3 + 3x_2y_4 - 9x_2y_5 + 3x_8y_1 + 9x_8y_3 - 9x_8y_2 \\
& - 3x_8y_4 + 9x_8y_5 - 9x_8y_9 - 9x_9y_3 + 3x_9y_4 - 9x_9y_5 \\
& - 3x_1y_8 - 3x_3y_1 + 9x_3y_2 + 9x_2y_6 + 3x_3y_7 + 9x_3y_9 \\
& - 9x_3y_8 + x_1x_7 - 3x_2y_7 + x_4y_1 + 3x_4y_6 - 3x_4y_2).
\end{aligned}
$$

Then

$$\frac{\partial u}{\partial x} = \Big\{ (3y_2 - y_1 + 3y_9)\ (-u_7 - 3u_3 - 3u_5 + 3u_6 + u_4 + 3u_8)$$

$$+ y_4 - 3y_3 - 3y_5 + 3y_6 - y_7 + 3y_9)(- u_7 + 3u_6 + u_1 + 3u_8 - 3u_9 - 3u_2)\Big\}/D$$

Unfortunately there are many spurious ways of making this zero, eg:

$u_1 = u_4 = u_7$ $\qquad$ $u_1 = u_4 = u_7$

$u_5 = u_8 = \begin{matrix} u_9 \\ u_2 \end{matrix}$ $\qquad$ $u_5 = u_6 = \begin{matrix} u_9 \\ u_2 \end{matrix}$

$u_6 = u_3 = \begin{matrix} u_2 \\ u_9 \end{matrix}$ $\qquad$ $u_8 = u_3 = \begin{matrix} u_2 \\ u_9 \end{matrix}$

There are so many spurious modes it doesn't seem worthwhile listing any more.

Another problem is the lack of any appearance of u_{10} and indeed x_{10} and y_{10}. This means we have a mode

$$u_1 = \dots = u_9 = 0$$

$$u_{10} = 1$$

That is not a deforming mode as such but would certainly cause problems if (x_{10}, y_{10}) tried to leave the triangle.

Ten-noded Cubic Triangle with Vertex Quadrature

Define

$$D = 6(99x_3y_1 + 162x_8y_2 + 198x_1y_2 - 22x_4y_1 + 18x_8y_4 - 324x_9y_2$$
$$- 198x_2y_1 + 99x_1y_8 + 36x_4y_9 - 18x_4y_8 - 198x_1y_9 + 36x_2y_7$$
$$+ 22x_1y_4 - 99x_1y_3 - 162x_3y_9 + 324x_2y_9 - 22x_1y_7$$
$$- 162x_2y_8 + 4x_4y_7 - 18x_3y_7 + 81x_3y_8 + 198x_9y_1 - 36x_9y_4$$
$$+ 162x_9y_3 + 22x_7y_1 - 36x_7y_2 - 4x_7y_4 + 18x_7y_3 - 99x_8y_1$$
$$- 81x_8y_3)$$

$$E = 6(36x_3y_1 + 162x_8y_2 + 18x_1y_2 - 22x_4y_1 + 198x_4y_4$$
$$- 81x_9y_2 - 18x_2y_1 + 36x_1y_8 + 99x_4y_9 - 198x_4y_8$$
$$- 18x_1y_9 + 99x_2y_7 + 22x_1y_4 - 36x_1y_3 - 162x_3y_9$$
$$+ 81x_2y_9 - 22x_1y_7 - 162x_2y_8 + 121x_4y_7 - 198x_3y_7$$
$$+ 324x_3y_8 + 18x_9y_1 - 99x_9y_4 + 102x_9y_3 + 22x_7y_1$$
$$- 99x_7y_2 - 121x_7y_4 + 198x_7y_3 - 36x_8y_1 - 324x_8y_3)$$

$$F = - 6(36x_3y_1 + 162x_8y_2 + 18x_1y_2 + 4x_4y_1 - 81x_9y_2 - 18x_2y_1$$
$$+ 36x_1y_8 - 18x_1y_9 - 4x_1y_4 - 36x_1y_3 - 162x_3y_9 + 81x_2y_9$$
$$+ 4x_1y_7 - 162x_2y_8 - 4x_4y_7 - 324x_3y_6 + 162x_3y_5$$
$$- 36x_3y_4 + 324x_3y_8 + 18x_9y_1 + 162x_9y_3 - 4x_7y_1$$
$$+ 4x_7y_4 - 36x_8y_1 - 324x_8y_3 + 162x_6y_8 - 18x_6y_4$$
$$+ 18x_2y_4 - 81x_6y_9 - 324x_5y_8 - 18x_7y_5 - 18x_9y_7$$
$$+ 36x_5y_4 + 162x_5y_9 - 36x_7y_8 + 36x_7y_6 + 18x_7y_9$$
$$+ 162x_2y_6 - 81x_2y_5 - 162x_8y_6 + 324x_8y_5 + 36x_8y_7$$
$$+ 324x_6y_3 - 243x_6y_5 - 36x_6y_7 - 162x_6y_2 + 54x_6y_1$$
$$- 162x_5y_3 + 243x_5y_6 + 18x_5y_7 + 81x_5y_2 - 54x_5y_1$$
$$+ 36x_4y_3 + 18x_4y_6 - 36x_4y_5 - 18x_4y_2 + 81x_9y_6$$
$$- 162x_9y_5 - 54x_1y_6 + 54x_1y_5)$$

We can now write

$$\frac{\partial u}{\partial x} = \Big\{ y_1 (22u_7 - 99u_8 + 198u_9 + 99u_3 - 198u_2 - 22u_4)$$
$$+ y_2 (162u_8 - 36u_7 - 324u_9 + 198u_1) + y_3(- 81u_8 + 18u_7 + 162u_9 - 99u_1)$$
$$+ y_4 (18u_8 - 4u_7 - 36u_9 + 22u_1) + y_7 (- 18u_3 - 22u_1 + 4u_4 + 36u_2)$$
$$+ y_8 (- 18u_4 + 81u_3 + 99u_1 - 162u_2) + y_9 (- 162u_3 - 198u_1 + 36u_4$$
$$+ 324u_2)\Big\} /D$$

$$+ \Big\{ y_1 (22u_7 - 4u_4 - 36u_6 + 18u_5) + y_2 (162u_8 - 99u_7 - 81u_9 + 18u_1)$$
$$+ y_3 (- 324u_8 + 198u_7 + 162u_9 - 36u_1) + y_4 (198u_8 - 121u_7 - 99u_9$$
$$+ 22u_1)$$

$$+ y_7 \, (- \, 22u_1 + 22u_4 - 198u_8 + 198u_6 - 99u_5 + 99u_9)$$

$$+ y_8 \, (- \, 36u_4 + 36u_1 + 324u_8 - 324u_6 + 162u_5 - 162u_9)$$

$$+ y_9 \, (- \, 18u_1 + 18u_4 - 162u_8 + 162u_6 - 81u_5 + 81u_9) \Big\} /E$$

$$+ \Big\{ y_1 \, (4u_7 - 22u_4 - 18u_6 + 36u_5) + y_2 \, (- \, 162u_3 + 81u_2 + 99u_4 - 18u_1)$$

$$+ y_3 \, (- \, 198u_4 + 324u_3 - 162u_2 + 36u_1) + y_4 \, (- \, 4u_7 + 4u_1 - 18u_2 + 36u_3$$

$$+ 18u_6 - 36u_5)$$

$$+ y_5 \, (198u_4 - 54u_1 - 486u_3 + 243u_2 - 81u_6 + 18u_7 + 162u_5)$$

$$+ y_6 \, (- \, 99u_4 + 54u_1 + 486u_3 - 243u_2 + 162u_6 - 36u_7 - 324u_5) + y_7$$

$$(- \, 36u_3 - 4u_1 + 22u_4 + 18u_2)$$

$$+ y_8 \, (- \, 324u_3 - 36u_1 + 162u_2 - 162u_6 + 36u_7 + 324u_5)$$

$$+ y_9 \, (162u_3 + 18u_1 - 81u_2 - 18u_7 + 81u_6 - 162u_5) \Big\} /F$$

Firstly we notice that u_{10} does not appear in this derivative and therefore possibly renders the scheme useless.

The derivative can be made zero in the following

	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9	u_{10}
1	1	1	1	1	1	1	1	1	1	-
2	27/13	27/13	27/13	27/13	1	-1	-27/13	-1	1	-
3	$\frac{449}{260}$	$\frac{449}{260}$	$\frac{449}{260}$	$\frac{449}{260}$	1	$\frac{-7}{20}$	$\frac{-14}{13}$	$\frac{-7}{20}$	1	-
4	0	0	0	0	0	0	0	0	0	1

The first is the only physical mode.

Ten-noded Cubic Triangle and the Mid-Edge Rule

Define

$$D = 6\,(-\,18x_8y_4 - 486x_8y_2 + 18x_8y_1 + 9x_9y_4 + 243x_9y_2 - 243x_9y_3$$
$$- 9x_9y_1 + 8x_7y_4 + 216x_7y_2 - 216x_7y_3 - 8x_7y_1 - 36x_2y_4$$
$$+ 486x_8y_3 - 18x_6y_4 - 486x_6y_2 + 486x_6y_3 - 1458x_{10}y_3$$
$$- 9x_3y_4 + 243x_5y_2 + 18x_6y_1 + 54x_{10}y_4 + 1458x_{10}y_2$$
$$- 54x_{10}y_1 + 18x_4y_8 + 36x_4y_2 + 18x_4y_6 + 9x_4y_3 + 9x_5y_4$$
$$- 243x_5y_3 - 9x_5y_1 + x_1y_4 - 9x_4y_9 - 8x_4y_7 - 54x_4y_{10}$$
$$- 9x_4y_5 - x_4y_1 + 486x_2y_8 - 243x_2y_9 - 486x_2y_6 - 216x_2y_7$$
$$- 1458x_2y_{10} + 1215x_2y_3 - 243x_2y_5 + 9x_2y_1 + 243x_3y_9$$
$$+ 216x_3y_7 - 1215x_3y_2 - 486x_3y_6 + 1458x_3y_{10} + 36x_3y_1$$
$$- 486x_3y_8 - 18x_1y_8 + 9x_1y_9 + 54x_1y_{10} - 36x_1y_3 + 9x_{1y}5$$
$$+ 243x_3y_5 + 8x_1y_7 - 9x_1y_2 - 18x_1y_6)$$

$$E = -\,6\,(1215x_8y_9 - 216x_8y_4 - 243x_8y_2 - 36x_8y_1 + 216x_9y_4$$
$$- 1215x_9y_8 + 9x_8y_7 + 36x_9y_7 - 243x_6y_9 + 243x_6y_8$$
$$- 1458x_{10}y_9 + 1458x_{10}y_8 - 9x_6y_7 + 243x_9y_2 - 54x_{10}y_7$$
$$- 486x_9y_3 + 18x_5y_7 + 486x_5y_9 - 486x_5y_8 - 9x_9y_1$$
$$+ 8x_7y_4 + 9x_7y_2 - 18x_7y_3 + x_7y_1 + 486x_8y_5 - 1458x_8y_{10}$$
$$- 243x_8y_6 - 18x_7y_5 + 54x_7y_{10} + 9x_7y_6 - 36x_7y_9$$
$$- 9x_7y_8 - 486x_9y_5 + 486x_8y_3 + 9x_6y_1 + 54x_{10}y_1$$
$$+ 216x_4y_8 - 18x_5y_1 - 8x_1y_4 - 216x_4y_9 - 8x_4y_7 + 8x_4y_1$$
$$+ 243x_2y_8 - 243x_2y_9 + 1458x_9y_{10} + 243x_9y_6 - 9x_2y_7$$
$$+ 9x_2y_1 + 486x_3y_9 + 18x_3y_7 - 18x_3y_1 - 486x_3y_8$$
$$+ 36x_1y_8 + 9x_1y_9 - 54x_1y_{10} + 18x_1y_3 + 18x_1y_5 - x_1y_7$$
$$- 9x_1y_2 - 9x_1y_6)$$

$$F = -6\,(9x_8y_4 - 18x_9y_4 - 9x_8y_7 + 18x_9y_7 + 486x_6y_9 - 243x_6y_8$$
$$+ 9x_0y_7 - 54x_{10}y_7 + 36x_5y_7 - 486x_5y_9 + 243x_5y_8$$
$$+ 1215x_6y_5 - 1458x_0y_{10} - 1458x_{10}y_5 + 1458x_{10}y_6 + x_7x_4$$
$$- 1215x_5y_6 + 1458x_5y_{10} - 18x_7y_2 + 9x_7y_3 + 8x_7y_1$$
$$- 18x_2y_4 - 243x_8y_5 + 243x_8y_6 - 36x_7y_5 + 54x_7y_{10}$$
$$- 9x_7y_6 - 18x_7y_9 + 9x_7y_8 + 486x_9y_5 - 36x_6y_4$$
$$+ 486x_6y_2 - 243x_6y_3 + 9x_3y_4 - 486x_5y_2 - 216x_0y_1$$
$$+ 54x_{10}y_4 - 9x_4y_8 + 18x_4y_2 + 36x_4y_6 - 9x_4y_3 - 9x_5y_4$$
$$+ 243x_5y_3 + 216x_5y_1 + 8x_1y_4 + 18x_4y_9 - x_4y_7 - 54x_4y_{10}$$
$$+ 9x_4y_5 - 8x_4y_1 - 486x_9y_6 - 486x_2y_6 + 18x_2y_7 + 486x_2y_5$$
$$- 9x_3y_7 + 243x_3y_6 - 216x_1y_5 - 243x_3y_5 - 8x_1y_7 + 216x_1y_6)$$

We can now write the derivative as

$$\frac{\partial u}{\partial x} = \Big\{ y_4\,(-18u_8 + u_1 + 9u_5 - 9u_3 + 54u_{10} - 18u_6 + 36u_2 + 8u_7 + 9u_9)$$
$$+ (y_5 + 6y_{10} - 2y_6 + \frac{8}{9}\,y_7 + y_9 - 2y_8)\,(9u_1 + 243u_3 - 243u_2 - 9u_4)$$
$$+ y_3\,(-36u_1 + 1215u_2 + 9u_4 - 243u_5 - 1458u_{10} + 486u_6 - 216u_7$$
$$- 243u_9 + 486u_8)$$
$$+ y_2\,(-9u_1 - 1215u_3 + 36u_4 + 243u_5 + 1458u_{10} - 486u_6 + 216u_7$$
$$+ 243u_9 - 486u_8)$$
$$+ y_1\,(36u_3 + 9u_2 - u_4 - 9u_5 - 54u_{10} + 18u_6 - 8u_7 - 9u_9 - 18u_8)\Big\}\,/D$$

$$+ \Big\{ y_1\,(18u_3 - 9u_2 - 8u_4 + 18u_5 - 54u_{10} - 9u_6 - u_7 + 9u_9 + 36u_8)$$
$$+ (y_2 - 2y_3 + \frac{8}{9}\,y_4 - 2y_5 + y_6 + 6y_{10})\,(9u_1 - 9u_7 - 243u_9 + 243u_8)$$
$$+ y_7\,(54u_{10} + u_1 - 18u_3 + 9u_2 + 8u_4 - 9u_8 + 9u_6 - 18u_5 - 36u_9)$$
$$+ y_8\,(-36u_1 + 486u_3 - 243u_2 - 216u_4 + 486u_5 + 9u_7 - 1458u_{10}$$
$$- 243u_6 + 1215u_9)$$
$$+ y_9\,(-486u_5 - 9u_1 - 486u_3 + 243u_2 + 216u_4 + 36u_7 + 1458u_{10}$$
$$- 1215U_8 + 243U_6)\Big\}\,/E$$

$$+ \Big\{ (\tfrac{8}{9}y_1 - 2y_2 + y_3 + y_8 - 2y_9 + 6y_{10})(9u_4 - 243u_5 - 9u_7 + 243u_6)$$

$$+ y_4 (- 9u_8 - 8u_1 + 9u_5 - 9u_3 - 54u_{10} + 36u_6 + 18u_2 - u_7 + 18u_9)$$

$$+ y_5 (243u_8 + 216u_1 + 243u_3 - 486u_2 - 9u_4 - 486u_9 - 1215u_6 + 36u_7 + 1458u_{10})$$

$$+ y_6 (- 243u_8 - 216u_1 - 243u_3 + 486u_2 - 36u_4 + 486u_9 + 9u_7 - 1458u_{10} + 1215u_5)$$

$$+ y_7 (54u_{10} + 8u_1 + 9u_3 - 18u_2 + u_4 + 9u_8 - 9u_6 - 36u_5 - 18u_9)\Big\} /F$$

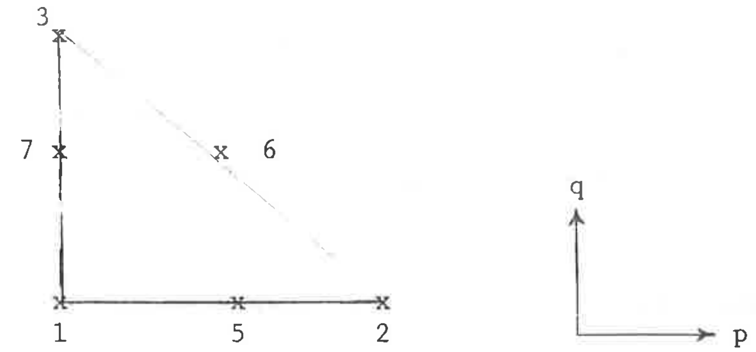
Firstly we notice that u_{10} does take an active part in this expression and so cannot just float around. The derivative can be zero when:-

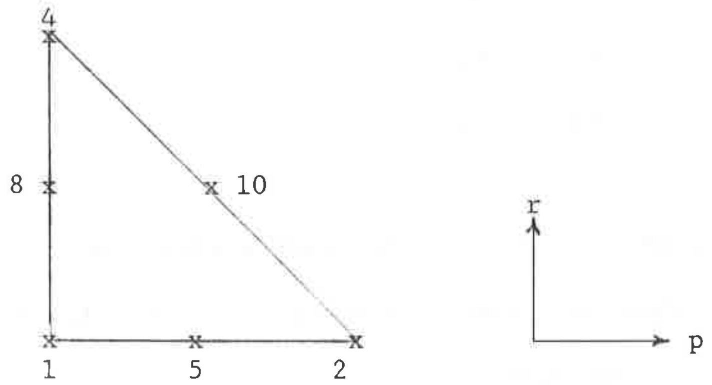
	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u9	u_{10}
1	1	1	1	1	1	1	1	1	1	1
2	9	9	9	9	-1	1	63	1	-1	-1
3	15	-7	$-\frac{37}{5}$	$\frac{21}{5}$	$\frac{13}{5}$	1	-39	-1	1	-1
4	-1	1	$\frac{3}{5}$	$-\frac{59}{5}$	$\frac{3}{5}$	1	-1	-1	-1	1

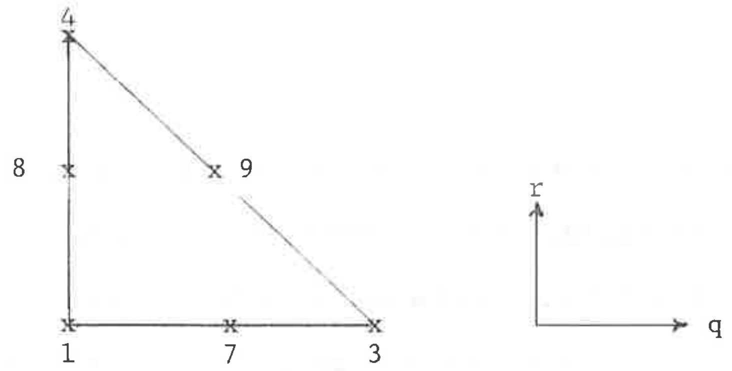
Again just the first is physical.

Ten-noded Quadratic Tetrahedron

Just for fun consider the 10-noded quadratic tetrahedron.







The transformation is given by:-

$$t = (2p + 2q + 2r - 1)(p + q + r - 1)t_1 + p(2p - 1)t_2 + q(2q - 1)t_3 + r(2r - 1)t_4 - 4p(p + q + r - 1)t_5 + 4pqt_6 - 4q(p + q + r - 1)t_7 - 4r(p + q + r - 1)t_8 + 4qrt_9 + 4pr\,t_{10}.$$

The algebra for this problem is even more tedious than that we have gone through before so just the results will be quoted.

Centroid:- The conditions for a zero derivative are

$$u_5 = u_9$$
$$u_6 = u_8$$
$$u_7 = u_{10}$$

There are 10 variables, 3 constraints plus the one physical mode leaves 6 spurious modes. Including the y-velocity and z-velocities as well means that 18 out of 30 modes are spurious.

Vertex and Mid-face Rule:- Both these quadratures just have the one zero derivative mode $u_1 = u_2 = \ldots = u_{10}$.

<u>SUMMARY</u>

Given the disgraceful behaviour of the cubic element using centroid quadrature and it's not much better behaviour with vertex and mid-edge quadrature we are left with the 6-noded quadratic triangle. Since the vertex integrated version of this element, although perhaps better, is more expensive our first choice will be the centroid integrated 6-noded quadratic triangle.

We will now look at this element in a little more detail.

We shall now give a geometric interpretation of the velocity space for the centroid integrated 6-noded quadratic. This is equivalent to some of the work done by Flanagan & Belytechko (1981) for the quadrilateral bilinears.

Firstly we note that we shall only look at the u-velocities, the v-velocities being entirely similar.

The vector space is spanned by the following vectors:-

A = (1,1,1,1,1,1,)

B = (1,-1,1,-1,1,-1,)

C = (-5,¼,4,-1¼,1,1)

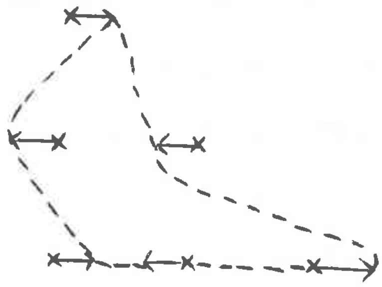
D = (2,-3/2,4,½,-6,1)

E = (1,8,1,-4,-2,-4)

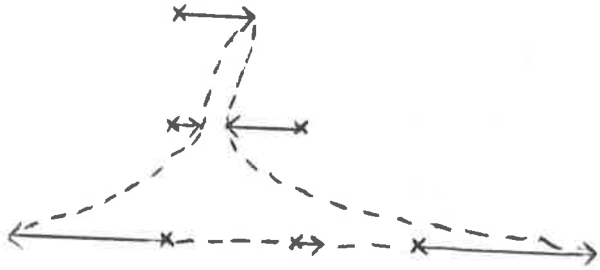
F = (1,0,-1,-4,0,4)

A,B,C & D, A quite justifiably, produce no restoring forces. E and F produce restoring forces. We will not bother to draw A. We will also take liberties with the scaling in some of the others.

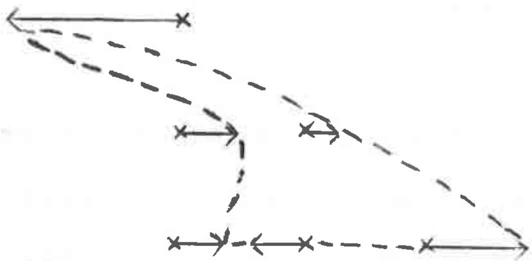
B



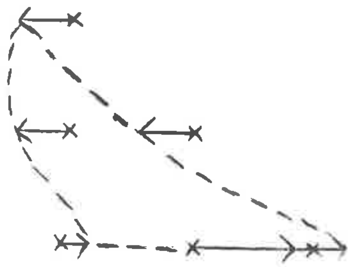
C



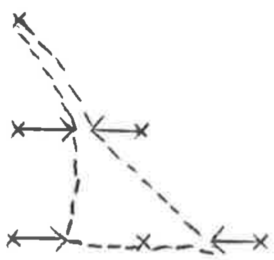
D



E



F



We can now write the original velocity components in terms of the new velocity basis functions.

$$u_1 = \frac{A}{6} + \frac{B}{6} - \frac{40C}{357} + \frac{4D}{119} + \frac{E}{102} + \frac{F}{34}$$

$$u_2 = \frac{A}{6} - \frac{B}{6} + \frac{2C}{357} - \frac{3D}{119} + \frac{4E}{51}$$

$$u_3 = \frac{A}{6} + \frac{B}{6} + \frac{32C}{357} + \frac{8D}{119} + \frac{E}{102} - \frac{F}{34}$$

$$u_4 = \frac{A}{6} - \frac{B}{6} - \frac{10C}{357} + \frac{D}{119} - \frac{2E}{51} - \frac{2F}{17}$$

$$u_5 = \frac{A}{6} + \frac{B}{6} + \frac{8C}{357} - \frac{12D}{119} - \frac{E}{51}$$

$$u_6 = \frac{A}{6} - \frac{B}{6} + \frac{8C}{357} + \frac{2D}{119} - \frac{2E}{51} + \frac{2F}{17}$$

We can also express the element basis functions in terms of A,B etc.

$$\phi_i = \frac{1}{6}A_i + \frac{1}{6}(-8q - 8p + 8p^2 + 8pq + 8q^2 + 1)\,B_i$$

$$+ \frac{8}{357}(18q - 3p^2 - 30pq + 12p - 12q^2 - 5)\,C_i$$

$$+ \frac{4}{119}(2q + 9p^2 + 6pq - 8p - 6q^2 + 1)\,D_i$$

$$+ \frac{1}{102}(-17q - 28p^2 - 28pq + 28p + 14q2 + 1)\,E_i$$

$$+ \frac{1}{34}(13q - 28pq - 2p - 14q^2 + 1)\,F_i$$

3. MESH LOCKING

From talking to people who know far more about Free-Lagrangian methods than myself, see acknowledgements, I have gained the impression that meshes composed of linear triangles are regarded as too "stiff". Using stiffness in this context with finite elements can lead to ambiguity so we will rephrase the statement to meshes composed of linear triangles have a tendency to lock up in that the nodes refuse to move. Meshes composed of bilinear quadrilaterals don't in normal use. Both types of elements have 6 wanted degrees of freedom. The quadrilateral also has 2 unwanted degrees of freedom which we then do our utmost to remove. If this removal is done 'properly' the mesh remains unlocked, if we are over-zealous in our removal the mesh may lock.

Certain things are clear:- locking has nothing to do with the degrees of freedom of the nodes as this is the same for both types of element. It would therefore seem logical to look at the degrees of freedom of the elements. Also, perhaps rather obviously, there is little point looking at one element in isolation. A patch test has therefore been devised. The name has been chosen to deliberately mislead people who use non-conforming elements.

I freely admit that the following arguments lack a certain something, namely rigour. However, the predictions of the "theory" do seem to fit the facts and if the assumptions are not reasonable they are at least not unreasonable.

One conjecture necessary for this "theory" to work is that we will assume the existence of fractional degrees of freedom. However, having once calculated an element to have a fractional degree of freedom we then assume that only the integer part means anything. Hence if we calculate an element to have 1½ degrees of freedom it is actually no freer than an element with just the 1 degree of freedom. More importantly an element with 3/4 of a degree of freedom is no freer than an element with no degrees of freedom at all.

Another assumption, implicitly assumed here, is that pressure is taken to be represented by piecewise constants. If a higher order representation of pressure is used, see Crowley (1985) for example, there will be an increase in the freedom of the system.

Consider three nodes forming a linear triangular element. Each node has 2 degrees of freedom but this does not mean that the element has 6 degrees of freedom because the nodes also belong to other triangles. Typically a node belongs to 6 triangles and can therefore contribute 2/6 degrees of freedom to each element. Each element has 3 such nodes and hence has 1 degree of freedom.

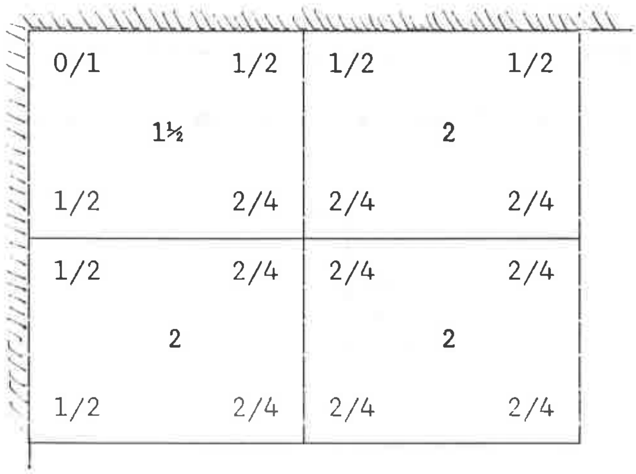
A bilinear quadrilateral node has 2 degrees of freedom shared between 4 quadrilaterals, 2/4 degrees of freedom per element. Each element has 4 such nodes and so the element has 2 degrees of freedom.

This would seem to imply that quadrilaterals are freer than their triangular counterparts. The analysis becomes more interesting when we consider a patch near a boundary.

Numbers in the centres of elements denote the degrees of freedom for that element. The nodal numbers are the degrees of freedom that node contributes to that element which as before is

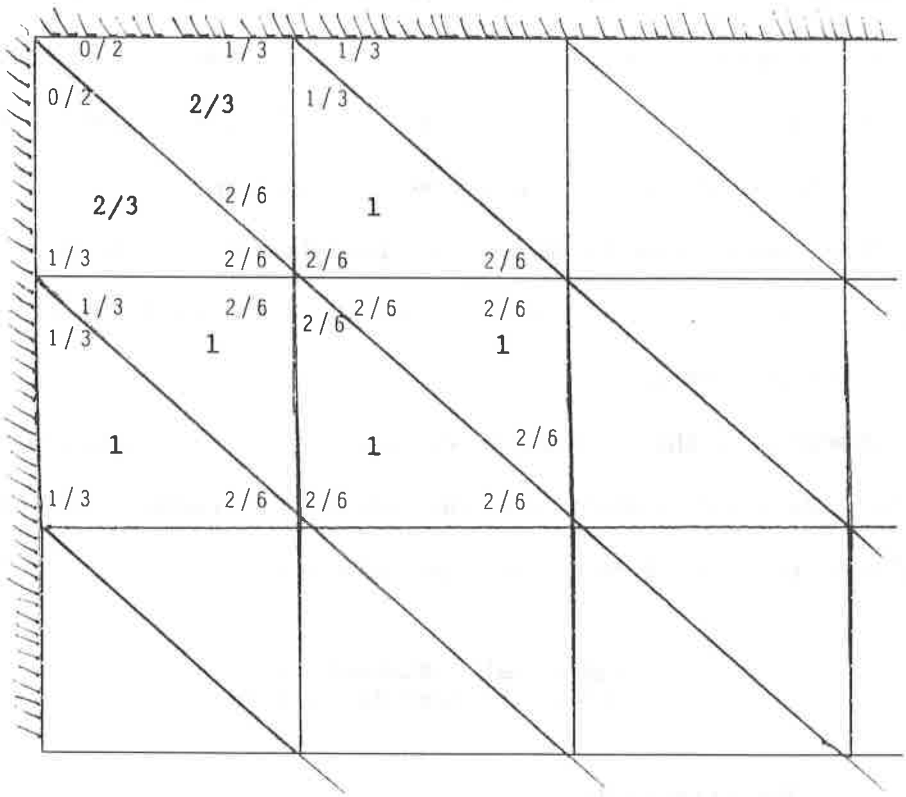
$$\frac{\text{degrees of freedom of node}}{\text{number of elements sharing}}$$

Quadrilaterals

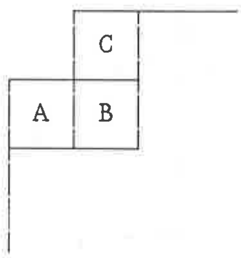


At the boundary we impose a boundary condition on the node reducing its degrees of freedom to 1. However, it only shares this with 2 elements and so the element number remains the same. The corner element, though, loses more degrees of freedom and as a result the degrees of freedom of the element is reduced. However, it is still greater than 1 and so the mesh should still be unlocked.

Triangles



The corner elements have less than 1 degree of freedom. They become stuck. They now act like the boundary, ie we now have a domain



Blocks A and C then become stuck - ad nauseum.

So good so far. Now what about the sticking of the quadrilaterals when the damping of the spurious mode is performed too enthusiastically.

Suppose we wish to damp the two undesired modes by a factor α. If we assume a damping as in Margolin and Pyun (1987) then total damping occurs for $\alpha = \frac{1}{4}$. Therefore we can argue that if we wish to reduce the degrees of freedom of the element by 2 in this total damping situation then a damping by a factor α leads to a reduction of 8α degrees of freedom to the element. (The value of 8 is a negotiable number but will do to illustrate the point.)

This then means that the corner element has $1\frac{1}{2} - 8\alpha$ degrees of freedom. Therefore if $\alpha > \frac{1}{16}$ the bilinear quadrilateral mesh would lock.

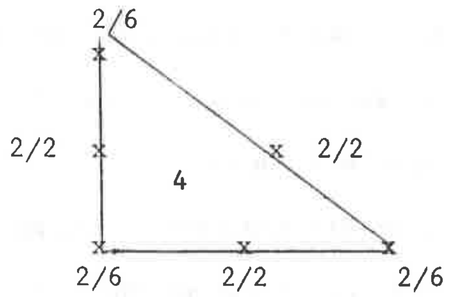
Given that the suggested range is

$$0.01 \leq \alpha \leq 0.05$$

an upper limit of 0.0625 seems as though it might be quite reasonable.

Six-noded Triangle

In an interior triangle



we get 4 degrees of freedom per element. At the corner this just gets reduced to $3\frac{2}{3}$. This would mean that the vertex integrated quadratic would never lock, whilst in the case of the centroid integrated quadratic we may get locking because of the number of spurious modes that need to be damped. If we fully damp the spurious modes on the centroid integrated quadratic triangle it has no more degrees of freedom than the linear triangle had.

Remarks

These element degrees of freedom come entirely from the topology of the grid. Whilst some error may have been made in transferring degrees of freedom from nodes to elements and back again it is relatively unimportant whether we use 4α, 8α or 16α to detract from the degrees of freedom of an element. This does not alter the argument, just the critical value of α.

What is of more concern is the lack of any equation. Perhaps then some trivial equation has implicitly been assumed here and the question of mesh locking been reduced purely to the topology of the mesh. The question is then, in practice will meshes lock harder or more easily than predicted here? Hopefully they would be harder to lock.

4. CONCLUSIONS

It would seem that if a one point evaluation method is required then we should recommend the centroid integrated six-noded quadratic. With very careful damping of the spurious modes it should be possible to keep the mesh unlocked. With the more expensive vertex quadrature this would not be a problem because there are no spurious modes.

There seem to be a number of difficulties in using quadratic triangles.

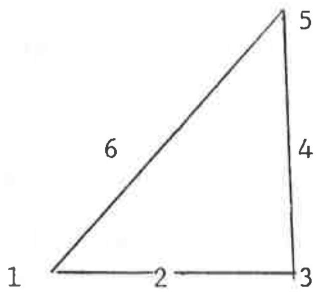
Spurious modes are much more excitable on the triangles than they would appear to be on the bilinears. For the test problem we tried here, that of a piston moving into a gas at rest, which is essentially one-dimensional, the cause of the presence of the spurious mode and the reason why it does not get stimulated in the bilinears is easy to see.

Spurious mode is (1, -1, 1, -1)

Now $u_1 = u_4$ and $u_2 = u_3$) hence

$(u_1, u_2, u_3, u_4) \ . \ (1, -1, 1, -1) = 0.$

This means that there is no spurious mode present if u is constant in y. Another point to note is that the bilinear elements are uniform in y as opposed to



One spurious mode is (1, -1, 1, -1, 1, -1)

$u_2 = u_6$ and $u_3 = u_4 = u_5$ hence

$(u_1, u_2, u_3, u_4, u_5, u_6). (1, -1, 1, -1, 1, -1) = u_1 - 2u_2 + u_3.$

This is not in general zero! The grid, although uniform, is no longer basically on-dimensional itself and this too can lead to problems with the generation of vertical velocities. Going back to the problem of the piston moving into a gas at rest putting $u_3 = u_4 = u_5 = -1$ and the other velocities to zero as initial data we see that this spurious mode is already present accounting for approximately 11% of the initial velocity vector. For the initial data this can be overcome by putting $u_2 = u_6 = -\frac{1}{2}$. Clearly this is only a temporary solution and we will, in general, need to damp the spurious mode much more vigorously than in the bilinear case. It also does nothing for the other spurious modes.

There seems to be a more important problem though. This involves

$$\int_{el} N_j \, dxdy$$

where the N_j ($j = 1,\ldots,6$) are the element basis functions. This arises from the equations for updating the velocities:-

$$\dot{V}_{xj} \int_{el} \rho^e N_j \, dxdy = p^e \int_{el} \frac{\partial N_j}{\partial x} \, dxdy \quad .$$

We have assumed here, without loss of generality that the pressure and density are element constants ρ^e and p^e.

This integral is interpreted as contributing to the nodal mass of node j. The actual nodel mass is then just the sum of these contributions from all elements with node j as a node.

The mass associated with a node ought to be positive. With quadratic N_j this cannot be got. The integrals may well be negative and even on a regular, uniform mesh all nodes that are vertices of triangles will receive a zero mass contribution. From a programming angle this is even worse than the mass being negative as it leads to accelerations that are very large and totally dominated by rounding error.

It would seem then that apart from spurious modes and problems of stiffness an ever more basic requirement of the elements to be used in these codes is that

$$\int_{el} N_j \, dxdy > 0 \quad \forall_j, \ \forall \text{ elements.}$$

We note that linears on triangles and bilinears on quadrilaterals do possess this property! We also note that the vertex integrated version, with no spurious modes, failed in the same way although fractionally later.

What of the future? All classical high order finite elements are going to suffer the same fate because of their polynomial nature. What perhaps may be a possibility is to use a high-order polynomial that is limited to being monotone by restrictions being placed upon its derivatives, see Fritsch and Carlson (1980) for a discussion of monotonic cubic interpolation. Perhaps there is a finite element equivalent.

Supposing it was decided to use the 6-noded quadratic triangle with vertex quadrature. This overcomes the spurious mode and locking problems. Given that we are using quadratic elements it seems a shame to throw away all this information by lumping the mass matrix, especially as how the mass lumping leads to even more problems of its own. The argument against the full mass matrix is the cost of inversion. However, since we are not primarily using quadratics to increase the accuracy we could reduce the number of elements used in the calculation. Secondly the sparse, symmetric highly structured nature of the mass matrix make its inversion by conjugate gradients very efficient. The whole inversion process is also highly parallelizable.

The use of 6-noded quadratic triangular elements with vertex integration and the use of the full mass matrix would seem to be the most promising avenue to explore further.

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