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DIRECT COMPUTATION OF STOCHASTIC FLOW IN RESERVOIRS WITH UNCERTAIN PARAMETERS

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Direct Computation of Stochastic Flow in Reservoirs with Uncertain Parameters[1]

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Abstract

A direct method is presented for determining the uncertainty in reservoir pressure, flow, and net present value (NPV) using the time-dependent, one phase, two or three dimensional reservoir flow equations. The uncertainty in the solution is modelled as a probability distribution function and is computed from given statistical data for input parameters such as permeability.

The method involves a perturbation expansion about a mean of the parameters. Coupled equations for second order approximations to the mean at each point and to the field covariance of the pressure are developed and solved numerically. The procedure is then used to find the statistics of the flow and the risked value of the field, defined by the Net Present Value (NPV), for a given development scenario.

This method involves only one (albeit complicated) solution of the equations and contrasts with the more usual Monte-Carlo approach where many such solutions are required. The procedure can be applied easily to other physical systems modelled by partial differential equations with uncertain data.

Key words : Flow in porous media. stochastic modelling, numerical solution of uncertain systems

AMS subject classification : 65M, 65U, 76M. 76S, 86

1 Introduction

Difficulty in the mathematical and numerical modelling of flow through porous media in underground reservoirs often arises because a precise knowledge of data is not available. Specifically, reservoir data may only be known within

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certain limits of accuracy, or it may only be possible to specify certain statistical properties of the data. This may be due to inaccuracy in measuring equipment or to inaccessibility and a high level of heterogeneity in the reservoir materials.

The usual approach to problems of this kind is to use Monte-Carlo methods. However, in some cases the number of realisations that need to be generated may be prohibitively large, and for this reason we have aimed to develop a more direct method for assessing the uncertainty in the solution. The procedure described here uses an approach similar to that of [8] [9] and is an extension of techniques that we have previously developed for the stochastic steady-state reservoir flow problem and for a transient mass-balance model with uncertain parameters [1] [2] [3]. Preliminary results of this work have been published in [4] [5]. The method that we present can be applied easily to other physical systems governed by partial differential equations with stochastic data.

We restrict our study to a fairly straightforward two-dimensional model equation (with the implicit assumption that the results obtained may be generalised to the three-dimensional case). The model is obtained by combining Darcy's law for flow in a porous medium [6] with the equation for single-phase flow in a fluid with a constant compressibility to give

$$\gamma\frac{\partial p}{\partial t} - \nabla(k\nabla p) = f(\mathbf{r},t), \tag{1}$$

where γ is the compressibility, p the pressure, k the permeability, and $f(\mathbf{r},t)$ is some forcing function. The risked value of the field is assessed using the net present value (NPV), defined by

$$NPV = \int_0^\infty \|\mathbf{Q}(t)\| e^{-\delta t} dt, \tag{2}$$

where $\mathbf{Q}(t)$ is the flow at the relevant production well and δ is some discounting factor.

In the mathematical modelling of the field for a deterministic case, the flow term $\mathbf{Q}(\mathbf{t})$ may easily be obtained if values for the pressure are known or the field flow equations have been solved at each time-step. For the simple model used here, the flow can be obtained directly from the formula

$$\mathbf{Q}(t) = -k\nabla(p), \tag{3}$$

where k is the permeability and p the pressure.

In the first part of this paper, we deal with cases where uncertainties in the permeabilities cause corresponding uncertainties in the solutions for the pressure. In the second part we investigate how these uncertainties propagate into uncertainties in the flow and, more importantly, in the NPV.

We make the assumption that the statistical behaviour of the permeability field may be characterised by its mean value, $\langle k \rangle$, and the permeability auto-correlation function (P.A.F.), written as a function of two spatial positions, $\mathbf{r}_1$ and $\mathbf{r}_2$. The P.A.F is defined explicitly as

$$\rho(\mathbf{r}_1, \mathbf{r}_2) = \frac{\langle (k(\mathbf{r}_1) - k_0(\mathbf{r}_1))(k(\mathbf{r}_2) - k_0(\mathbf{r}_2)) \rangle}{\sigma_k(\mathbf{r}_1)\sigma_k(\mathbf{r}_2)}, \tag{4}$$

and can be thought of as a measure of how strongly the statistical properties at points $\mathbf{r}_1$ and $\mathbf{r}_2$ are related. For practical purposes, the distribution is assumed to be of a lognormal form.

2 Hierarchical Equations

We begin by developing a set of hierarchical equations for a general admissible realisation. By developing these systems of equations as far as possible, before taking mean values on either side, we can obtain equations that allow us to solve for the statistical properties of the numerical solution for the pressure.

2.1 Standard Form

For a permeability distribution function that is symmetric about the mean value, a simple linear perturbation about the mean can be considered. We therefore treat the two-dimensional permeability field for a single realisation as a perturbation about some pre-defined mean value field and write

$$k = k_0 + \alpha k_1. \tag{5}$$

We assume that $k_0 = \langle k \rangle$ is a deterministic mean, knowledge of which is available.

Equation (1) can then be written

$$\gamma \frac{\partial p}{\partial t} - \nabla((k_0 + \alpha k_1)\nabla(p)) = f_0(\mathbf{r}, t) + \alpha f_1(\mathbf{r}, t), \tag{6}$$

where p is the pressure solution for a specific realisation.

As in much work by Dagan [9] and Dupuy and Schwydler [8], we assume that the pressure solution can be expressed in a series form

$$p = \sum_{m=0}^{N} \alpha^m p_m + R_{N+1}, \tag{7}$$

where R_{N+1} is the residue due to truncating the series for N^{th} order accuracy. Substituting equation (7) into (6) gives

$$\gamma\frac{\partial}{\partial t}(\sum_{m=0}^{N}\alpha^m p_m + R_{N+1}$$

$$- \quad \nabla((k_0+\alpha k_1)\nabla\left(\sum_{m=0}^{N}\alpha^m p_m + R_{N+1}\right) = f_0(\mathbf{r},t)+\alpha f_1(\mathbf{r},t). \qquad (8)$$

If we define p_0 to be the solution of the mean value problem (also known as the deterministic problem)

$$\gamma\frac{\partial p_0}{\partial t} - \nabla k_0 \nabla p_0 = f_0, \qquad (9)$$

then, by equating successive powers of α, equation (8) can be split into the set of $N+1$ hierarchical equations

$$\gamma\frac{\partial p_0}{\partial t} - \nabla k_0 \nabla p_0 = f_0, \qquad (10)$$

$$\gamma\frac{\partial p_1}{\partial t} - \nabla k_0 \nabla p_1 - \nabla k_1 \nabla p_0 = f_1, \qquad (11)$$

$$\gamma\frac{\partial p_2}{\partial t} - \nabla k_0 \nabla p_2 - \nabla k_1 \nabla p_1 = 0, \qquad (12)$$

$$\vdots$$

$$\gamma\frac{\partial p_m}{\partial t} - \nabla k_0 \nabla p_m - \nabla k_1 \nabla p_{m-1} = 0, \qquad (13)$$

$$\vdots$$

$$\gamma\frac{\partial p_N}{\partial t} - \nabla k_0 \nabla p_N - \nabla k_1 \nabla p_{N-1} = 0, \qquad (14)$$

$$\gamma\frac{\partial R_{N+1}}{\partial t} - \nabla(k_0+\alpha k_1)\nabla R_{N+1} - \alpha^{N+1}\nabla k_1 \nabla p_N = 0. \qquad (15)$$

This represents a set of coupled partial differential equations for each admissible realisation. By truncating this series at the N^{th} term, we impose a level of accuracy on the possible solutions. In a statistical sense, we are not able to solve the $N+1^{st}$ equation (15), and so the equations are of N^{th} order accuracy.

It may, of course, be possible to obtain bounds on the size of the residue terms over all admissible realisations. In [2] and [3] an analysis of the residual error in the expansion for the pressure is given for the steady-state problem and bounds on the residuals are derived in terms of bounds on the range of possible values for the permeability. This effectively gives a measure of the accuracy of the hierarchical approximations in the limit as the system tends to steady-state.

2.2 Lognormal Distribution

If a lognormal distribution function is assumed for the permeability, the expansion must be formed about the geometric mean [9]. This is equivalent to a linear expansion about the log of the permeability of form

$$ln(k) = z = z_0 + \beta z_1,$$

where $z_0 = \langle z \rangle$. Hence

$$\begin{aligned} k &= e^{z_0} + \beta z_1 e^{z_0} + \frac{\beta^2 z_1^2}{2} e^{z_0} + \cdots \\ &= \kappa_g + \beta\kappa_1 + \beta^2\kappa_2 + \cdots = \kappa_g + \sum_{j=1}^{\infty} \beta^j \kappa_j, \end{aligned} \tag{16}$$

where κ_g is the geometric mean.

If we perform the same procedure as in Section 2.3, assuming that the pressure has the form

$$p = \sum_{m=0}^{N} \beta^m p_m + S_{N+1} \tag{17}$$

and substituting for pressure and permeability into equation (1), we obtain

$$\gamma\frac{\partial}{\partial t}(\sum_{m=0}^{N} \beta^m p_m + S_{N+1}) - \nabla(\kappa_g + \sum_{j=1}^{\infty} \beta^j \kappa_j)\nabla(\sum_{m=0}^{N} \beta^m p_m + S_{N+1}) = f(\mathbf{r}, t). \tag{18}$$

Writing

$$f(\mathbf{r}, t) = f_0(\mathbf{r}, t) + \beta f_1(\mathbf{r}, t)$$

and equating powers of β then gives the system of hierarchical equations

$$\gamma\frac{\partial p_0}{\partial t} - \nabla\kappa_g\nabla p_0 = f_0, \tag{19}$$

$$\gamma\frac{\partial p_1}{\partial t} - \nabla\kappa_g\nabla p_1 - \nabla\kappa_1\nabla p_0 = f_1, \tag{20}$$

$$\gamma\frac{\partial p_2}{\partial t} - \nabla\kappa_g\nabla p_2 - \nabla\kappa_1\nabla p_1 - \nabla\kappa_2\nabla p_0 = 0, \tag{21}$$

$$\vdots$$

$$\gamma\frac{\partial p_j}{\partial t} - \nabla\kappa_g\nabla p_j - \sum_{m=0}^{j-1} \nabla\kappa_{j-m}\nabla p_m = 0, \tag{22}$$

$$\vdots$$

$$\gamma\frac{\partial p_N}{\partial t} - \nabla\kappa_g\nabla p_N - \sum_{m=0}^{N-1} \nabla\kappa_{N-m}\nabla p_m = 0, \tag{23}$$

$$\gamma\frac{\partial S_{N+1}}{\partial t} - \nabla\kappa_g\nabla S_{N+1} - \nabla(\sum_{j=1}^{\infty} \beta^j\kappa_j)\nabla S_{N+1} - \sum_{j=N+1}^{\infty}\sum_{m=0}^{N} \beta^j \nabla\kappa_{j-m}\nabla p_m = 0. \tag{24}$$

3 Statistical Properties of Analytical Equations

To progress further, we must now consider the statistical properties of the solutions to all of the hierarchical equations.

3.1 Standard Form

For the purposes of this research, we restrict our consideration to second order approximations for symmetric, or standard-form, permeability distribution functions.

Taking mean values on both sides of equations (10)–(12) and assuming k_1 is a perturbation about the absolute mean, so that $\langle k_1 \rangle = 0$, we obtain

$$\gamma \frac{\partial p_0}{\partial t} - \nabla k_0 \nabla p_0 = f_0, \tag{25}$$

$$\gamma \frac{\partial \langle p_1 \rangle}{\partial t} - \nabla k_0 \nabla \langle p_1 \rangle = \langle f_1 \rangle, \tag{26}$$

$$\gamma \frac{\partial \langle p_2 \rangle}{\partial t} - \nabla k_0 \nabla \langle p_2 \rangle - \nabla \langle k_1 \nabla p_1 \rangle = 0, \tag{27}$$

and

$$\gamma \frac{\partial \langle R_3 \rangle}{\partial t} - \nabla (k_0 \nabla \langle R_3 \rangle) + \nabla \langle \alpha k_1 \nabla R_3 \rangle - \alpha^3 \nabla \langle k_1 \nabla p_2 \rangle = 0. \tag{28}$$

As they stand, these equations are not solvable, even just up to second order, due to the presence of the cross-correlation term $\nabla \langle k_1 \nabla p_1 \rangle$. In order to obtain a solution, a method for evaluating the correlation function $\langle k_1 \nabla p_1 \rangle$ is needed.

We consider multiplying k_1 into the grad of equation (11) to give an extra partial differential equation. The result of this is to introduce higher order cross-correlation terms, such as $\langle k_1 \nabla^2 k_1 \nabla p_0 \rangle$, into the equations. The evaluation of these terms involves subsequently higher and higher order cross-correlation terms. This process, of course, is only feasible if a closure can be imposed on the system of equations under consideration. As they stand, this is not possible.

3.2 Lognormal Distribution

Applying the same procedure as in Section 3.1 to the set of equations (19)–(21) for the lognormal permeability distribution function gives the similar, but modified, equations

$$\gamma \frac{\partial p_0}{\partial t} - \nabla \kappa_g \nabla p_0 = f_0, \tag{29}$$

$$\gamma \frac{\partial \langle p_1 \rangle}{\partial t} - \nabla \kappa_g \nabla \langle p_1 \rangle = \langle f_1 \rangle, \tag{30}$$

$$\gamma \frac{\partial \langle p_2 \rangle}{\partial t} - \nabla \kappa_g \nabla \langle p_2 \rangle - \nabla \langle \kappa_1 \nabla p_1 \rangle - \nabla \langle \kappa_2 \rangle \nabla p_0 = 0. \tag{31}$$

The difference here is the presence of the third term, $\nabla \langle \kappa_2 \rangle \nabla p_0$, in equation (31). This term just links in the first equation in the series with an extra moment of the distribution, $\langle \kappa_2 \rangle$ which is a known property of the distribution.

However, the basic problem remains the same: the presence of $\langle \kappa_1 \nabla p_1 \rangle$, which must be solved for simultaneously in order to obtain closure of the equations, as for example in [11] and [9].

3.3 Variance

A second order approximation to the covariance can be found in a similar way to [10] by considering

$$\gamma \frac{\partial}{\partial t}(p_1(\mathbf{r_1}, t) p_1(\mathbf{r_2}, t)) = p_1(\mathbf{r_1}, t) \gamma \frac{\partial p_1(\mathbf{r_2}, t)}{\partial t} + p_1(\mathbf{r_2}, t) \gamma \frac{\partial p_1(\mathbf{r_1}, t)}{\partial t} \tag{32}$$

and substituting for $\gamma \frac{\partial p_1}{\partial t}$, etc. from (11) to obtain

$$\begin{aligned} &\gamma \frac{\partial}{\partial t}(p_1(\mathbf{r_1}, t) p_1(\mathbf{r_2}, t)) \\ &- \nabla_2 k_0(\mathbf{r_2}) \nabla_2 p_1(\mathbf{r_1}, t) p_1(\mathbf{r_2}, t) - \nabla_2 k_1(\mathbf{r_2}) p_1(\mathbf{r_1}, t) \nabla_2 p_0(\mathbf{r_2}, t) \\ &- \nabla_1 k_0(\mathbf{r_1}) \nabla_1 p_1(\mathbf{r_2}, t) p_1(\mathbf{r_1}, t) - \nabla_1 k_1(\mathbf{r_1}) p_1(\mathbf{r_2}, t) \nabla_1 p_0(\mathbf{r_1}, t) = 0, \end{aligned} \tag{33}$$

where ∇_1 and ∇_2 denote the grad with respect to $\mathbf{r_1}$ and $\mathbf{r_2}$, respectively. Taking the mean value on either side of this equation results in an equation for the behaviour of the covariance of the pressure given by

$$\begin{aligned} &\gamma \frac{\partial}{\partial t}(\langle p_1(\mathbf{r_1}, t) p_1(\mathbf{r_2}, t) \rangle) \\ &- \nabla_2 k_0(\mathbf{r_2}, t) \nabla_2 \langle p_1(\mathbf{r_1}, t) p_1(\mathbf{r_2}, t) \rangle - \nabla_2 \langle k_1(\mathbf{r_2}) p_1(\mathbf{r_1}, t) \rangle \nabla_2 p_0(\mathbf{r_2}, t) \\ &- \nabla_1 k_0(\mathbf{r_1}) \nabla_1 \langle p_1(\mathbf{r_2}, t) p_1(\mathbf{r_1}, t) \rangle - \nabla_1 \langle k_1(\mathbf{r_1}) p_1(\mathbf{r_2}, t) \rangle \nabla_1 p_0(\mathbf{r_1}, t) = 0. \end{aligned} \tag{34}$$

If the covariance at time t between pressure values at two points $\mathbf{r_1}$, and $\mathbf{r_2}$ is denoted by $C(\mathbf{r_1}, \mathbf{r_2}, t)$, then these equations are

$$\begin{aligned} &\gamma \frac{\partial}{\partial t}(C(\mathbf{r_1}, \mathbf{r_2}, t)) \\ &- \nabla_2 k_0(\mathbf{r_2}, t) \nabla_2 C(\mathbf{r_1}, \mathbf{r_2}, t) - \nabla_2 \langle k_1(\mathbf{r_2}) p_1(\mathbf{r_1}, t) \rangle \nabla_2 p_0(\mathbf{r_2}, t) \\ &- \nabla_1 k_0(\mathbf{r_1}) \nabla_1 C(\mathbf{r_2}, \mathbf{r_1}, t) - \nabla_1 \langle k_1(\mathbf{r_1}) p_1(\mathbf{r_2}, t) \rangle \nabla_1 p_0(\mathbf{r_1}, t) = 0. \end{aligned} \tag{35}$$

In the case of a lognormal distribution. the covariance takes the same form, with κ_0, κ_1 replacing k_0, k_1, respectively.

Evaluation of the terms in this expression is again rendered impossible if no method of solving for the cross-correlation term is available.

We conclude that developing a method for determining the lowest moments of the distribution function of the solution to (1), in this case, second order accurate approximations to mean and variance, requires some method for obtaining the cross-correlation terms $\langle k_1 \nabla p_1 \rangle$ for values of spatial separation and time. Finding a solvable equation for these terms is problematic, but we now establish that we can obtain closure if we consider the discretised equations.

4 Discretisation

We now show that the problem of providing a solution for $\langle k_1 \nabla p_1 \rangle$, or $\langle \kappa_1 \nabla p_1 \rangle$, may be overcome by consideration of the discretised versions of the hierarchical equations derived in Sections 2.1 and 2.2.

4.1 Standard Form

We consider a discretisation of the equations (10)–(12) with a simple explicit time scheme and a general (unspecified) spatial difference scheme of the form

$$\frac{\gamma p_{0\ ij}^{n+1} - \gamma p_{0\ ij}^{n}}{\Delta t} - \nabla_h (k_{ij}^0 \nabla_h p_{0\ ij}^n) = f_{0\ ij}^n, \tag{36}$$

$$\frac{\gamma p_{1\ ij}^{n+1} - \gamma p_{1\ ij}^{n}}{\Delta t} - \nabla_h (k_{ij}^0 \nabla_h p_{1\ ij}^n) - \nabla_h (k_{ij}^1 \nabla_h p_{0\ ij}^n) = f_{1\ ij}^n, \tag{37}$$

and

$$\frac{\gamma p_{2\ ij}^{n+1} - \gamma p_{2\ ij}^{n}}{\Delta t} - \nabla_h (k_{ij}^0 \nabla_h p_{2\ ij}^n) - \nabla_h (k_{ij}^1 \nabla_h p_{1\ ij}^n) = 0. \tag{38}$$

where the (i, j) indices refer to spatial points $(i\Delta x, j\Delta y)$ in Cartesian co-ordinates. and $p_{m\ ij}^n$ refers to the numerical solution for $p_m(\mathbf{r}, n\Delta t)$. where $\mathbf{r}$ is also in Cartesian co-ordinates.

We now denote a general value of the perturbation k_1 at a discrete point $(i\Delta x, j\Delta y)$ by k_{ij}^1, and consider the value $k_{i'j'}^1$ at a second point. $(i'\Delta x, j'\Delta y)$. Multiplying this into equation (37) and taking mean values throughout gives

$$\frac{\gamma p_{0\ ij}^{n+1} - \gamma p_{0\ ij}^{n}}{\Delta t} - \nabla_h (k_{ij}^0 \nabla_h p_{0\ ij}^n) = f_{0\ ij}^n, \tag{39}$$

$$\frac{\gamma \langle k_{i'j'}^1 p_{1\ ij}^{n+1} \rangle - \gamma \langle k_{i'j'}^1 p_{1\ ij}^{n} \rangle}{\Delta t} - \langle k_{i'j'}^1 \nabla_h (k_{ij}^0 \nabla_h p_{1\ ij}^n) \rangle - \langle k_{i'j'}^1 \nabla_h (k_{ij}^1 \nabla_h p_{0\ ij}^n) \rangle = \langle k_{i'j'}^1 f_{1\ ij}^n \rangle, \tag{40}$$

$$\frac{\gamma\langle p_2{}^{n+1}_{ij}\rangle - \gamma\langle p_2{}^n_{ij}\rangle}{\Delta t} - \nabla_h(k^0_{ij}\nabla_h\langle p_2{}^n_{ij}\rangle) - \langle\nabla_h(k^1_{ij}\nabla_h p_1{}^n_{ij})\rangle = 0. \tag{41}$$

This is now a complete set of coupled (numerical) equations that have an explicit solution. When these equations are solved simultaneously, the cross-correlation function is found from equation (40) and then substituted into equation (41). In this form, the cross-correlation is a function of two (discretised) spatial points. The discretised autocorrelation function of the permeability field occurs in the $\langle k^1_{i'j'}\nabla_h(k^1_{ij}\nabla_h p_0{}^n_{ij})\rangle$ terms. These are basically just linear combinations of the autocorrelation parameters, with coefficients specifically dependent on the selected spatial discretisation scheme. The boundary conditions are incorporated into the right hand side terms of the equations.

4.2 Lognormal Form

Performing the expansion for a lognormal distribution function about the geometric mean results in an extra term in the second order equation, as seen in equation (21). In discretised form, with the obvious notation, the set of coupled numerical equations becomes

$$\frac{\gamma p_0{}^{n+1}_{ij} - \gamma p_0{}^n_{ij}}{\Delta t} - \nabla_h(\kappa^g_{ij}\nabla_h p_0{}^n_{ij}) = f_0{}^n_{ij}, \tag{42}$$

$$\frac{\gamma\langle\kappa^1_{i'j'}p_1{}^{n+1}_{ij}\rangle - \gamma\langle\kappa^1_{i'j'}p_1{}^n_{ij}\rangle}{\Delta t} - \langle\kappa^1_{i'j'}\nabla_h(\kappa^g_{ij}\nabla_h p_1{}^n_{ij})\rangle - \langle\kappa^1_{i'j'}\nabla_h(\kappa^1_{ij}\nabla_h p_0{}^n_{ij})\rangle = \langle\kappa^1_{i'j'}f_1{}^n_{ij}\rangle, \tag{43}$$

$$\frac{\gamma\langle p_2{}^{n+1}_{ij}\rangle - \gamma\langle p_2{}^n_{ij}\rangle}{\Delta t} - \nabla_h(\kappa^g_{ij}\nabla_h\langle p_2{}^n_{ij}\rangle) - \langle\nabla_h(\kappa^1_{ij}\nabla_h p_1{}^n_{ij})\rangle - \nabla_h\langle\kappa^2_{ij}\rangle\nabla_h p_0{}^n_{ij} = 0. \tag{44}$$

4.3 Variance Equations

The same discretisation performed on the covariance equations (35) (which have the same form in the linear and lognormal distributions, but with k and κ interchanged) results in the following equations

$$\begin{aligned} &\frac{\gamma C^{n+1}_{i'j'ij} - \gamma C^n_{i'j'ij}}{\Delta t} \\ &- \nabla_h k^0_{ij}\nabla_h C^n_{i'j'ij} - \nabla_h\langle k^1 p_1\rangle^n_{i'j'ij}\nabla_h p_0{}^n_{ij} \\ &- \nabla_h k^0_{i'j'}\nabla_h C^n_{iji'j'} - \nabla_h\langle k^1 p_1\rangle^n_{iji'j'}\nabla_h p_0{}^n_{i'j'} = 0. \end{aligned} \tag{45}$$

The quantity of particular interest is the variance of the pressure distribution, an important characterisation of the complete distribution function. In discretised form, the variance for time level $n\Delta t$, at spatial position $(i\Delta x, j\Delta y)$

is the value of C^n_{ijij}. Unfortunately, in the process of solving for this value, the correlation values for distinct points, $C^n_{i'j'ij}$ must also be computed and stored at each time-level. These values can be considered as a bonus to the required information, having an academic rather than a practical interest. An indication of the correlation length of the solution variable is, however, now directly available through this technique.

4.4 Summary

The discretization of the hierarchical equations (10)–(12) and (19)–(21) gives us a set of coupled numerical equations for the first two moments characterising the probability distribution function of the pressure solution. These are equations (39)–(41) and (45) for the standard form and (42)–(45) for the log-normal form. They can be solved at each successive time-level to follow their progression in time. This results in an approximation to the time development of the distribution function.

5 Application to Pressure Equations

We now apply this technique to a specific example of a discretisation for a lognormal distribution of the permeability.

We consider a simple explicit five-point difference scheme, where the value of the permeability at points halfway between adjacent gridpoints (i, j) and $(i \pm 1, j)$ or $(i, j \pm 1)$ is always approximated by an average of the two values at the grid-points. Equation (42) in this case becomes,

$$\begin{aligned}
&\frac{\gamma p_{0\ ij}^{n+1} - \gamma p_{0\ ij}^{n}}{\Delta t} \\
&+ \frac{(\kappa^g_{i+1j} + \kappa^g_{ij})}{2\Delta x^2} p_{0\ i+1j} + \frac{(\kappa^g_{i-1j} + \kappa^g_{ij})}{2\Delta x^2} p_{0\ i-1j} \\
&+ \frac{(\kappa^g_{ij+1} + \kappa^g_{ij})}{2\Delta y^2} p_{0\ ij+1} + \frac{(\kappa^g_{ij-1} + \kappa^g_{ij})}{2\Delta y^2} p_{0\ ij-1} \\
&- \left\{ \frac{(\kappa^g_{i+1j} + \kappa^g_{i-1j} + 2\kappa^g_{ij})}{2\Delta x^2} + \frac{(\kappa^g_{ij+1} + \kappa^g_{ij-1} + 2\kappa^g_{ij})}{2\Delta y^2} \right\} p_{0\ ij} = f_{0\ ij}. \quad (46)
\end{aligned}$$

The equations (43) – (45) are discretized similarly. Provided that the pressure is specified as a deterministic function of time at one point in the region or on its boundary, it can be shown that the numerical scheme is *stable* if the condition

$$\frac{4\Delta t \kappa_g}{\gamma h^2} < 1 \quad (47)$$

holds, where $h = \Delta x = \Delta y$. In practice the pressure at a well site is controlled and, therefore, the assumption that the pressure is specified deterministically at some point is a natural constraint on the system.

If the pressure is not specified as a deterministic function at some point in the region or on its boundary, then the approximation (46) to the equation for the deterministic solution p_0 is stable under the condition (47), but the numerical scheme for the complete hierarchical equations is unconditionally *unstable* and errors are expected to propagate with a polynomial growth rate.

5.1 Results

In this section we present some examples illustrating the results obtained by this method for the full statistical problem.

In each case we consider a single Fourier mode as the initial condition for the pressure, with no flow conditions around the boundary and zero forcing function. The region under investigation is a square of unit length. The parameter γ is assumed to be deterministic with unit value. The initial values for the mean and variance of the pressure are taken to be zero throughout the region (equivalent to a deterministic initial condition). The pressure at the centre of the region is assumed to be deterministic and is held fixed at a value of zero for all time; the higher moments are, thus, also zero at this point for all time.

All lengths and times are normalised. It is assumed here that one unit of length corresponds to one kilometre. If one unit of time is taken to represent ten years, then one pressure unit corresponds to 450 pounds per square inch.

Using a single Fourier mode as the initial condition means that in the case of a homogeneous geometric mean value, κ_g, for the permeability, the solution to the deterministic equation (9) may be expressed as the Fourier mode

$$p_0(x, y, t) = e^{-\pi^2 \frac{\kappa_g}{\gamma} t} cos(\pi x) \tag{48}$$

with an exponentially decaying amplitude. It is fairly trivial to show by substitution that (48) is a solution to the model equation satisfying the zero boundary conditions. We choose this test function as it is a straightforward solution whose deterministic behaviour is well-known.

In the experiments presented here, the values for the geometric mean of the permeability and for the variance of the log of the permeability, $z \equiv ln(k)$, are taken to be constants, $\kappa_g \equiv e^{\langle z \rangle} = 0.2$ and $\sigma_z = 0.1$, respectively. The P.A.F. of the log of the permeability is given by

$$\rho(x, y, x', y') = e^{-(\frac{\pi}{2}\frac{(x-x')}{\lambda_x})^2} e^{-(\frac{\pi}{2}\frac{(y-y')}{\lambda_y})^2}. \tag{49}$$

Both the isotropic case, where $\lambda_x = \lambda_y$, and the anisotropic case, $\lambda_x \neq \lambda_y$, are considered. Solutions are computed with $h = \frac{1}{18}$ and $\Delta t = \frac{1}{500}$. Other experiments have been tried for different means and different sizes of variance and for different computational time and spatial steps.

In Figures 1 and 2, we show the evolution of the deterministic pressure solution, firstly at time $t = 0.1$ and then at the final time value $t = 1.0$. Figures 3 and 4 then show the correction for the mean at the two time values and Figures 5 and 6 demonstrate the values of the variance at the same time points.

The next set of four figures shows the case where the correlation lengths are anisotropic. Figures 7 and 8 show plots at the final time interval where the correlation length is short in the x-direction, and long in the y-direction, with $\lambda_x = 0.1$ and $\lambda_y = 1.0$. The plots are for the mean correction to the deterministic solution and for the variance, respectively, after time interval $t = 1.0$. Figures 9 and 10 show plots for the same values at $t = 1.0$, but with anisotropic correlation lengths reversed, so that $\lambda_x = 1.0$ and $\lambda_y = 0.1$.

5.2 Discussion

The deterministic solution behaves as expected, decaying exponentially whilst retaining the basic shape of the (one-dimensional) mode. The numerical amplitude at time $t = 1.0$ is 0.140 compared to the analytic value of $e^{-\pi^2 \times 0.2} = 0.139$.

We can see in Figures 3–6 how the statistical moments grow from very low values, close to zero at the initial time to more significant values at the final time. This is to be expected as the initial conditions are assumed to be deterministic and the statistical moments are zero at $t = 0$.

The variance is seen to reach a maximum at around $t = 0.5$, thereafter gradually decreasing, with the maximum concentrating in the corners as it decays.

In comparison with experiments using a higher mean value, we observe a slower decay rate; for example, when $\kappa_g = 0.1$, the numerical decay rate is halved. The general shape assumed by the variance and second order approximations after one time unit are the same. The numerical value of the variance is, however, higher due to a greater relative spread in admissible realisations.

In the case of strong correlation in the y-direction, and much less correlation in the x-direction, we find that the statistical properties throughout the region are more homogeneous in themselves than in the case where the strong correlation is in the x-direction, and there are much higher variances concentrated in the corners. In the case where we considered small isotropic correlation lengths in both directions we observed a similar concentration of variance in the corners, with numerical values of one order of magnitude lower, which is the sort of behaviour we would expect if the statistical properties are weakly correlated.

6 Treatment of the Fluid Flow and NPV

6.1 Fluid Flow

The equation for flow in a porous medium can be obtained from the pressure in the fluid using Darcy's law, which is given in simplest form by

$$\mathbf{Q} = -k\nabla p. \tag{50}$$

In the case of a lognormal probability distribution we may substitute the perturbation expansion (16) for the permeability into the equation (50). Assuming, as previously, that the pressure may be approximated by a truncated series of form (17), we find

$$\mathbf{Q} \simeq -\left(\kappa_g + \beta\kappa_1 + \beta^2\kappa_2\right)\nabla\left(p_0 + \beta p_1 + \beta^2 p_2\right), \tag{51}$$

where all terms up to and including second order have been retained.

If we now take mean values on either side, then, since $\langle p_1\rangle = 0$, we obtain a vector expression for the mean value of the flow given by

$$\langle\mathbf{Q}\rangle \simeq -\kappa_g\nabla p_0 - \beta^2\left(\langle\kappa_1\nabla p_1\rangle + \langle\kappa_2\rangle\nabla p_0 + \kappa_g\nabla\langle p_2\rangle\right). \tag{52}$$

The covariance of the flow may be written

$$Cov_q \simeq \langle\kappa_1\kappa_1\rangle\left(\nabla p_0\right)^2 + 2\kappa_g\nabla p_0\cdot\langle\kappa_1\nabla p_1\rangle + (\kappa_g)^2\langle(\nabla p_1)\cdot(\nabla p_1)\rangle. \tag{53}$$

Using the computational results obtained by the methods described in the previous sections, we can now compute the first two statistical moments for the flow. These only require statistical information for the pressure which is already available. Both these terms can then be used to calculate the mean of the net present value and its statistical moments up to second order.

It is fairly straightforward to approximate equation (50) with a central difference approximation so that the flow at the point $(i\Delta x, j\Delta y)$ can be written

$$\mathbf{Q_{ij}} = -k_{ij}\nabla_h p_{ij}. \tag{54}$$

The equation for the mean value of the flow then takes the form

$$\langle\mathbf{Q_{ij}}\rangle \simeq -\kappa^g_{ij}\nabla_h p^0_{ij} - \beta^2\left(\langle\kappa^1_{ij}\nabla_h p^1_{ij}\rangle + \langle\kappa^2_{ij}\rangle\nabla_h p^0_{ij} + \kappa^g_{ij}\nabla\langle p^2_{ij}\rangle\right), \tag{55}$$

and the equivalent covariance term is

$$Cov_{q_{ij}} \simeq \langle\kappa^1_{ij}\kappa^1_{ij}\rangle\left(\nabla_h p^0_{ij}\right)^2 + 2\kappa^g_{ij}\nabla_h p^0_{ij}\cdot\langle\kappa^1_{ij}\nabla_h p^1_{ij}\rangle + (\kappa^g_{ij})^2\langle(\nabla_h p^1_{ij})\cdot(\nabla_h p^1_{ij})\rangle. \tag{56}$$

These discretised forms for the statistical moments of the flow are used to calculate numerical approximations to the NPV.

In the case of a standard probability distribution function for the permeability, similar results can be derived.

6.2 Net Present Value

To assess the Net Present Value of the systems we are considering, we must treat the NPV as a time-dependent variable; that is, we define

$$NPV(t) = \int_0^t \|\mathbf{Q}\| e^{-\delta s} ds, \tag{57}$$

where $\mathbf{Q}$ is the flow at a specified position, and let $t \to \infty$. Here $\| \cdot \|$ denotes the L_2 vector norm. The mean value of the NPV can then be shown to be

$$\langle NPV \rangle \simeq \int_0^t \|\langle \mathbf{Q_{ij}} \rangle\| e^{-\delta s} ds, \tag{58}$$

to second order accuracy, and an approximation to the second moment may be written as

$$\langle NPV_2 \rangle \simeq \int_0^t \langle (Q_{ij} - \langle Q_{ij} \rangle)^2 \rangle e^{-\delta s} ds \equiv \int_0^t Cov_{q_{ij}} e^{-\delta s} ds. \tag{59}$$

We are chiefly interested in how the mean value of the NPV compares with the deterministic solution, obtained by operating the numerical process on the mean value of the permeability field to give

$$N\check{P}V = \int_0^t \|\tilde{\mathbf{Q}}\| e^{-\delta s} ds, \tag{60}$$

where

$$\tilde{\mathbf{Q}} = -\kappa_{ij}^g \nabla_h p_{ij}^0. \tag{61}$$

6.3 Results

We now give examples of risked values of a field that have been computed by the methods described here for finding the low order moments of the probablilty distribution function of the NPV. We take the same data as in Section 5.1 for the test problem. The discount factor is taken to be $\delta = 1.0$. Integrals are computed using the trapeziodal quadrature rule with time step $\Delta t = \frac{1}{100}$.

As before, we consider a single Fourier mode as the initial pressure condition in the reservoir, with no flow conditions around the boundary and zero forcing function. The region under investigation is a square of unit length, and all lengths and times are normalised. Using the single Fourier mode as the initial condition means that, in the case of a homogeneous geometric mean value for the permeability, the deterministic solution to equation (1) is given by equation (48).

We observe the values for the NPV over the time interval $[0, 2]$ determined by the flow at the centre of the region. At this point the pressure p is deterministic and is held constant for all times t. These conditions correspond to those that hold at a well site. Figure 11 shows the various mean values for the NPV with different permeability variances, compared with the deterministic solution. The homogeneous geometric mean value of the permeability is $kmean \equiv \kappa_g = 0.2$. In Figure 12 the corresponding relative variances are shown for the NPV for the same permeability variances.

In Figure 13, we show the equivalent plots in the case of a smaller permeability mean. Here, $kmean \equiv \kappa_g = 0.1$. In Figure 14, we show the plots of the mean of the NPV for a larger mean permeability field with $kmean \equiv \kappa_g = 0.4$.

In Figure 11 we can see that the mean values for the NPVs corresponding to the smaller values of the permeability field seem to converge to a similar order of magnitude, but to a significantly different value from the deterministic solution ($var(k) \equiv \sigma_z = 0.0$). The value for the case where the covariance of the permeability field is large with respect to its mean seems not to show convergence over the specified time period.

This effect is repeated in Figures 13 and 14, with significant convergence being shown in Figure 14, where the mean is always larger than the deterministic value of the NPV.

7 Conclusions

In this paper we establish a new method for computing the statistical moments of the probabilty distribution of a reservoir pressure field directly from statistical data describing the stochastic properties of the reservoir, such as permeability and porosity. We show also that the probability distributions of the flow and the net present value (NPV) of the field can be assessed from these results. The advantage of this method is that it requires only one solution of the field equations, in contrast with the more usual Monte-Carlo procedure where many such solutions are required.

The proposed method uses a perturbation expansion about the mean of the input parameters to derive coupled equations for the moments of the stochastic variables. The key to the success of this approach lies in finding a closure of the equations for the first N moments of the probablility distributions. We show here that this can be achieved using numerical approximations to the pressure, flow and NPV for a specific realization of the field.

The feasibility of this approach is demonstrated for a simple example of one phase flow in a two dimensional reservoir where the permeability field is charaterised by its mean value and auto-correlation function and is assumed to be of lognormal form. Simple explicit finite difference schemes are used to approximate the pressure and flow equations. Second order approximations to the mean and variance of the pressure field are calculated and the risked value of the field is estimated for various statistical descriptions of the permeability field. The results indicate that the estimated mean of the NPV varies significantly with the variance of the permeability field.

These results demonstrate that the direct approach described here can be used effectively to assess the potential of reservoirs with uncertain data. Further studies are needed to improve the efficiency and range of applicability of the process. The limitations imposed by the stability conditions can easily be removed by applying implicit difference schemes to obtain the numerical approximations. Efficiency could be improved by reducing the computation of the cross-correlation terms only to those making significant contributions to the moments.

The approach presented here can be extended to uncertain nonlinear multi-phase flow problems. In these cases the method is expected to be particularly competitive, because the equations for the higher moments are linear and can be solved rapidly and efficiently, in contrast to Monte-Carlo methods, which require repeated solution of the full nonlinear models. The procedure can also be applied to other physical systems modelled by partial differential equations with uncertain data.

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Figure 1: Deterministic Solution for Pressure at t=0.1

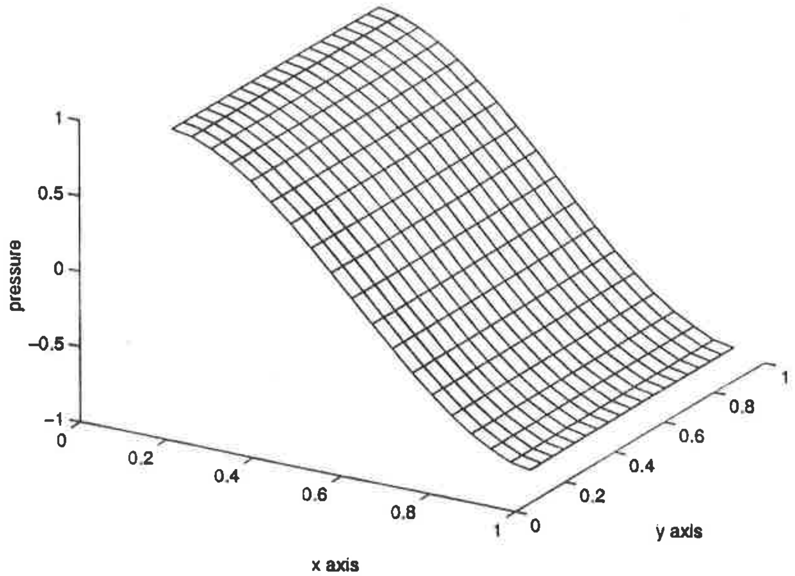

Figure 2: Deterministic Solution for Pressure at t=1.0

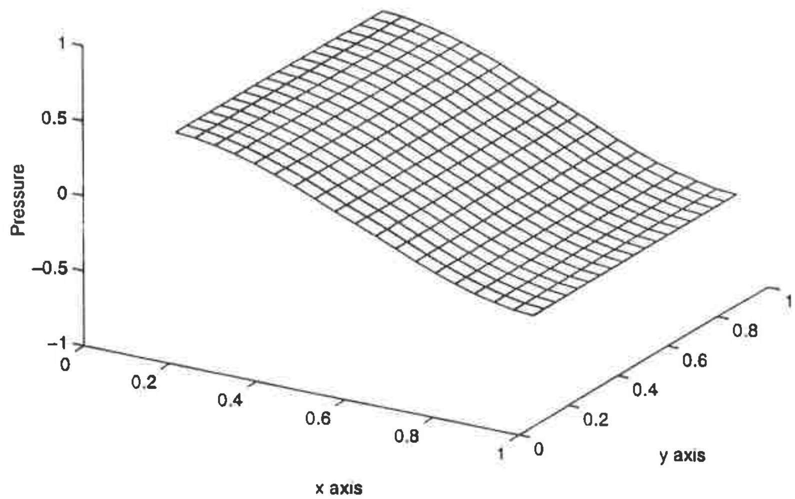

Figure 3: Mean Correction to the Deterministic Pressure at t=0.1

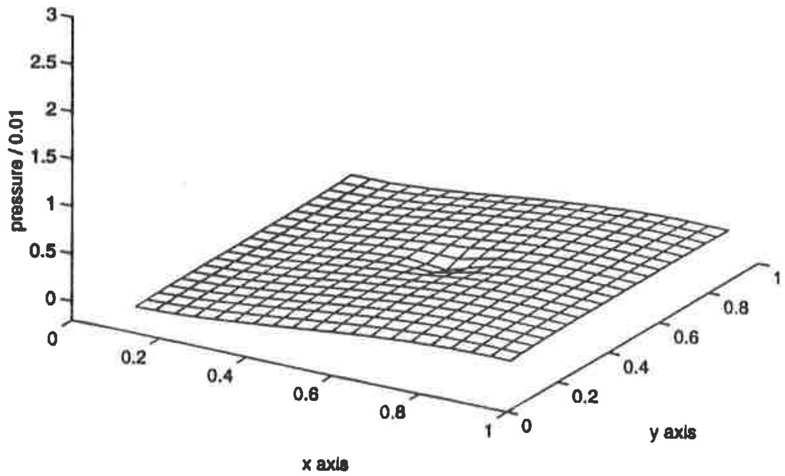

Figure 4: Mean Correction to the Deterministic Pressure at t=1.0

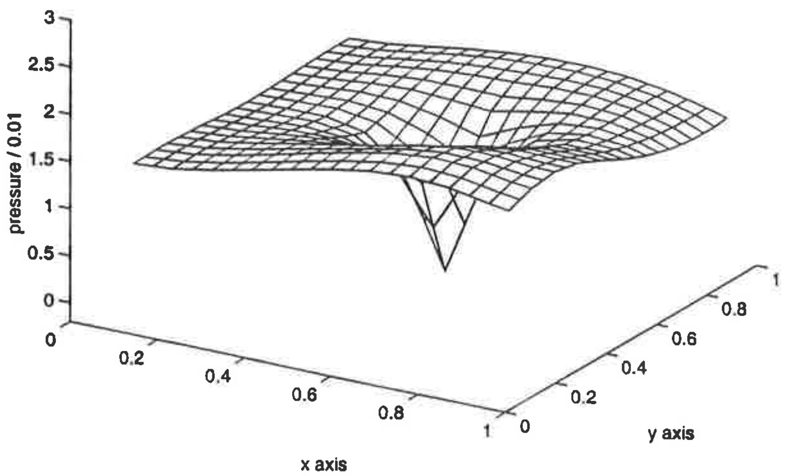

Figure 5: Pressure Variance t=0.1

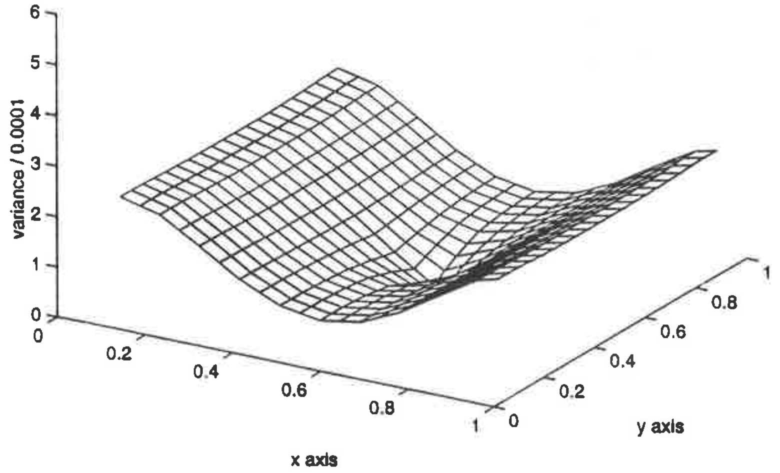

Figure 6: Pressure Variance at t=1.0

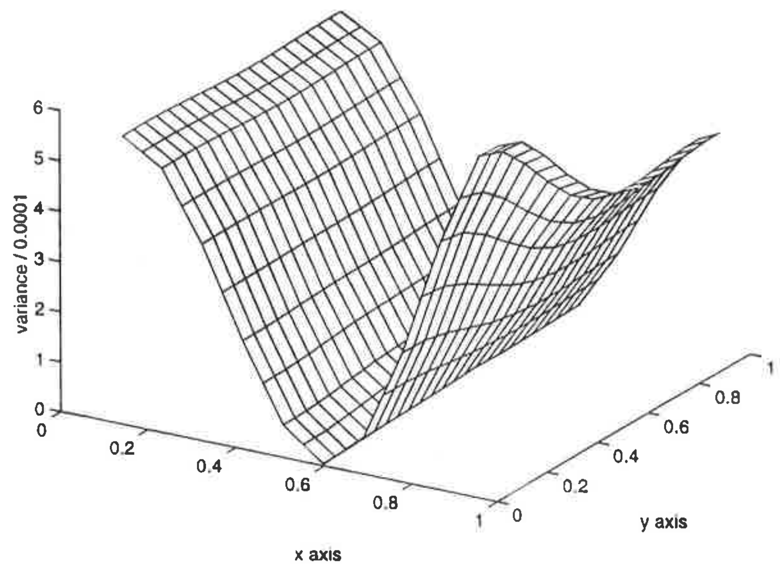

Figure 7: Mean Correction for Anisotropic Correlation Lengths at t=1.0

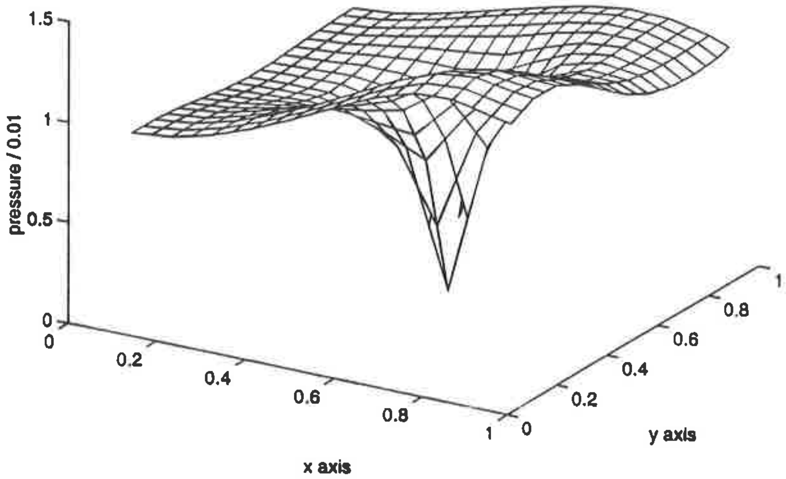

Figure 8: Pressure Variance for Anisotropic Correlation Lengths at t=1.0

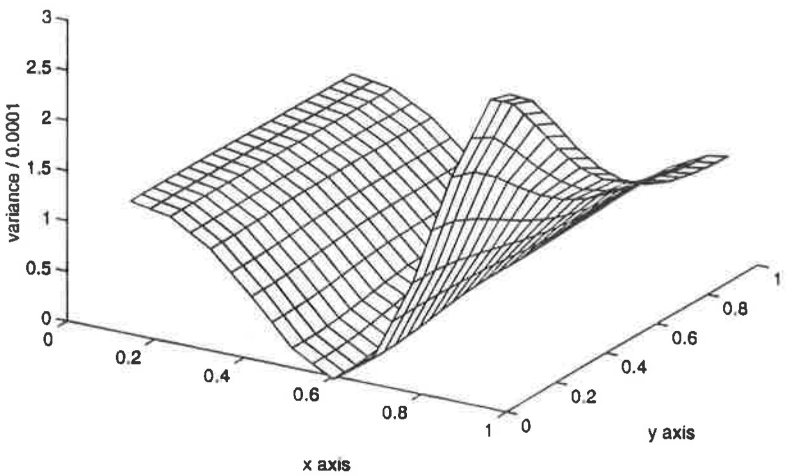

Figure 9: Mean Correction for Anisotropic Correlation Lengths at t=1.0

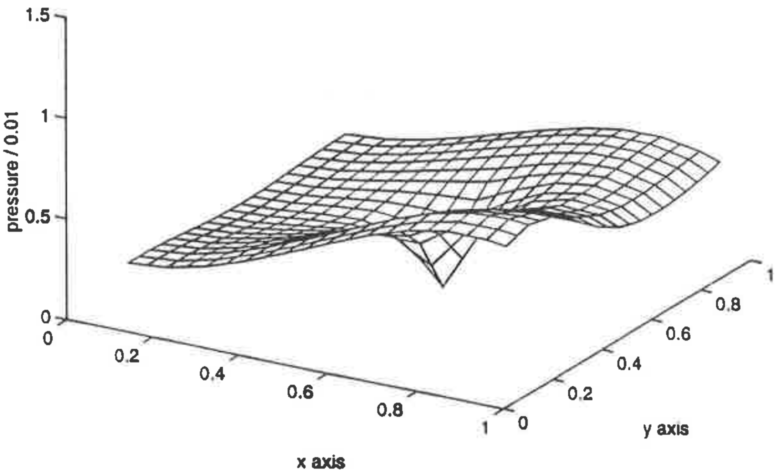

Figure 10: Pressure Variance for Anisotropic Correlation Lengths at t=1.0

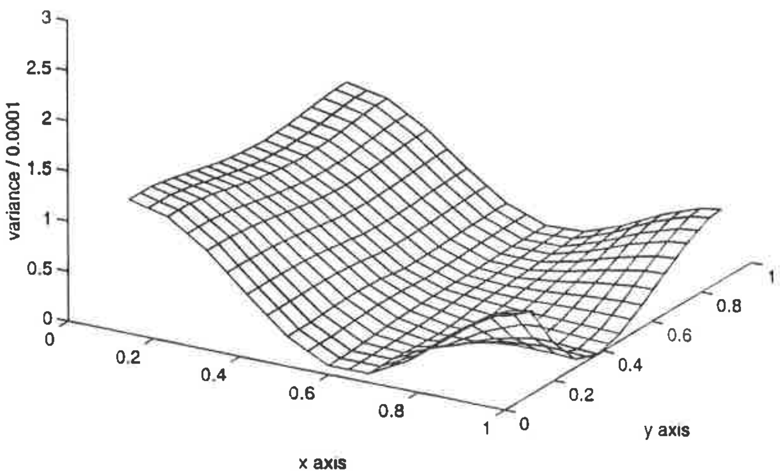

Figure 11: Evolution of Means of NPV for various σ_z^2

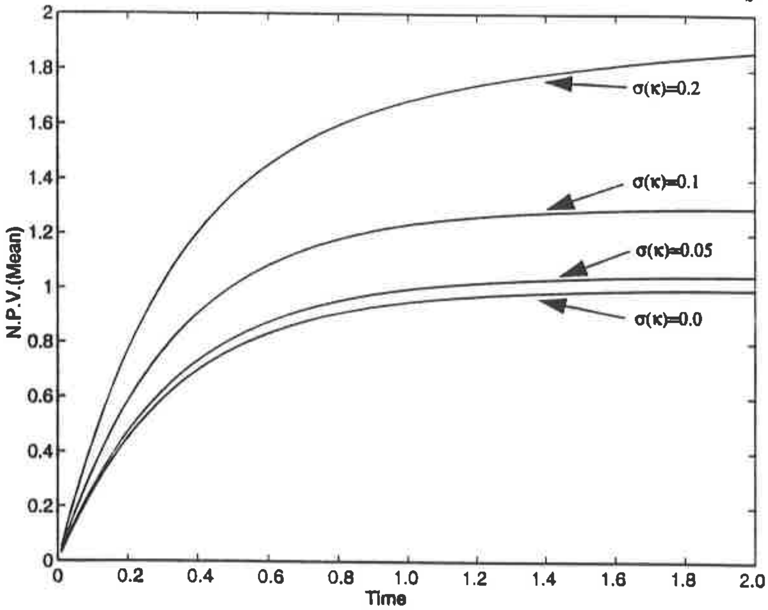

Figure 12: Approximation of Variance of NPVs for various σ_z^2

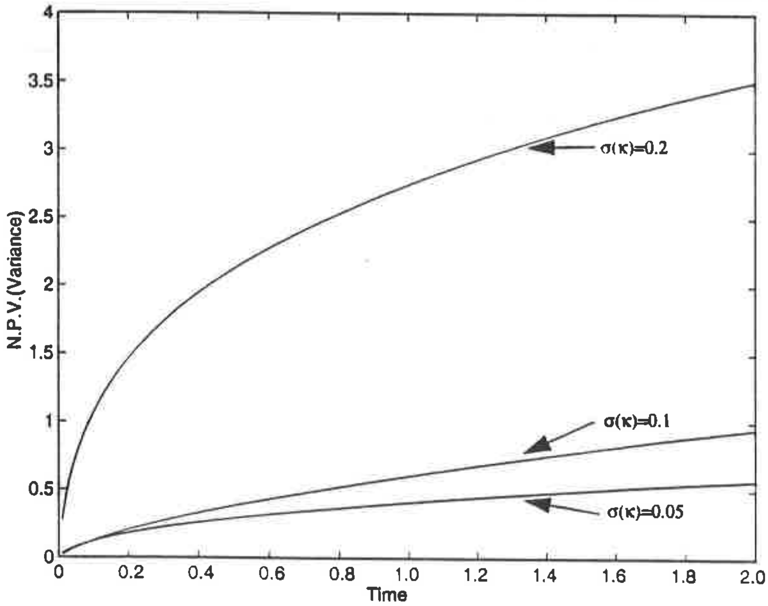

Figure 13: Means of NPV for small κ_g for various σ_z^2

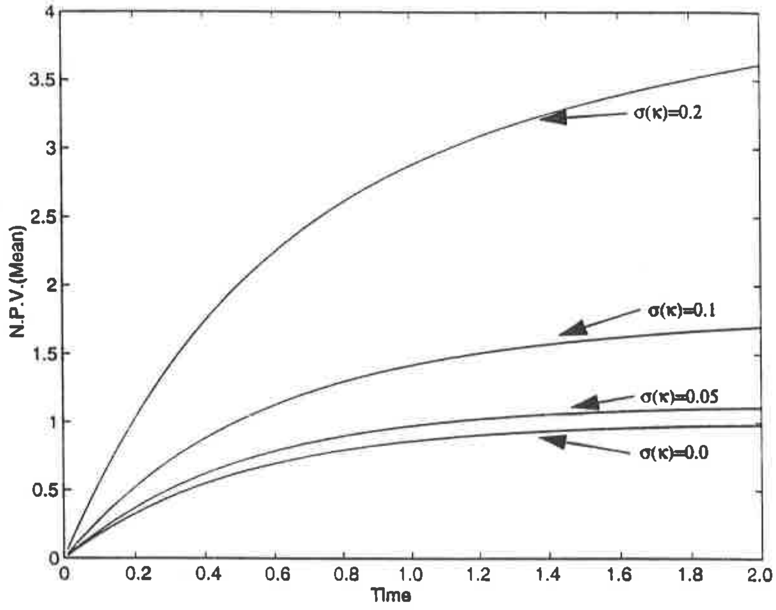

Figure 14: Means of NPV for large κ_g for various σ_z^2