SOME OBSERVATIONS ON THE MOVING FINITE ELEMENT METHOD AND ITS IMPLEMENTATION

PETER K. SWEBY

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Abstract

Some observations are made on the nature of the moving finite element equations, namely their invertibility for co-incident nodes, and conditions for equivalence of local and global methods when using Mueller inner products. Techniques are then described for two-dimensional recovery and regularisation of nodal movement.

Key words: Moving Finite Elements

1. Introduction

The moving finite element (MFE) method, originally devised by Miller and Miller [6], and being still in its infancy has prompted much work in its development and implementation, see e.g. [10], [11], [3], as well as associated work, e.g. [2]. In this report we describe some aspects of MFE and its implementation not described elsewhere in the literature.

In the rest of this section we describe briefly the Moving Finite Element method in one and two dimensions, both local and global versions. An indepth account is not given, the reader being referred to more detailed works for this, the purpose of this section being to set the framework and notation for the remainder of the report.

In section 2 we demonstrate the non-singularity of the MFE matrix for coincident nodes, a result not appreciated in early work. In section 3 conditions are derived for the equivalence of the local and global versions of MFE when using Mueller inner products to deal with diffusion terms, while in section 4 a form of recovery to deal with such terms in two dimensions is described. Finally in section 5 a technique of regularisation (see also [1]) is described which imposes a constraint on nodal movement in order to prevent element folding.

1.1 Moving Finite Elements in one-dimension

Consider the scalar evolutionary equation

$$u_t = L(u) \tag{1.1}$$

where L is a spatial differential operator. We approximate the solution v by piecewise linear finite elements, i.e.

$$v(x,t) = \sum_{0}^{N+1} a_j \alpha_j(x, \underset{\sim}{s}(t)) \tag{1.2}$$

where a_j are the nodal amplitudes and α_j are piecewise linear basis functions (see Figure 1.1), depending on the nodal positions s_j which vary with time. $\underset{\sim}{s}(t)$ is the vector $(s_0, \ldots s_j, \ldots s_{N+1})$ of these nodal positions.

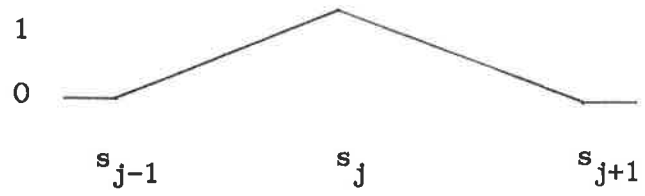

Figure 1.1 α_j

Partial differentiation of (1.2) with respect to time gives

$$v_t = \sum_{1}^{N} \{\dot{a}_j \alpha_j(x, \underset{\sim}{s}(t)) + \dot{s}_j \beta_j(x, \underset{\sim}{a}(t), \underset{\sim}{s}(t))\} \tag{1.3}$$

where

$$\beta_j = - v_x \alpha_j \tag{1.4}$$

(see [5] for full details), which have a form shown in Figure 1.2.

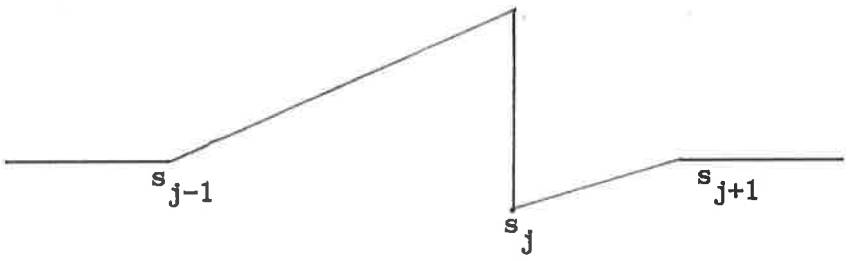

Figure 1.2 β_j

The 2N unknowns $\dot{a}_j$ and $\dot{s}_j$ are obtained by minimising the L_2 norm of the residual, namely

$$\| v_t - L(v) \|_2, \tag{1.5}$$

with respect to these parameters, giving the set of 2N equations

$$\left.\begin{aligned} \langle v_t - L(v), \alpha_j \rangle &= 0 \\ \langle v_t - L(v), \beta_j \rangle &= 0 \end{aligned}\right\} \quad j = 1, \ldots, N \tag{1.6}$$

where $\langle .,. \rangle$ is the L_2 inner product. Substitution of (1.3) into

(1.6) yields the MFE equations

$$A(\underset{\sim}{y})\dot{\underset{\sim}{y}} = \underset{\sim}{g}(\underset{\sim}{y}) \tag{1.7}$$

where

$$\underset{\sim}{y} = (a_1, s_1; a_2, s_2; \ldots; a_N, s_N)^T \tag{1.8}$$

$$\underset{\sim}{g} = (\langle \alpha_1, L(v) \rangle, \langle \beta_1, L(v) \rangle; \ldots\ldots \langle \alpha_N, L(v) \rangle, \langle \beta_N, L(v) \rangle)^T \tag{1.9}$$

and $A(\underset{\sim}{y})$ is the MFE matrix, which is square, symmetric and 2×2 block tridiagonal, the ijth block given by

$$A_{ij} = \begin{bmatrix} \langle \alpha_i, \alpha_j \rangle & \langle \alpha_i, \beta_j \rangle \\ \langle \beta_i, \alpha_j \rangle & \langle \beta_i, \beta_j \rangle \end{bmatrix}. \tag{1.10}$$

The system (1.7) is solved for $\dot{\underset{\sim}{y}}$ using, for example, pre-conditioned conjugate gradients and the solution updated using Euler time-stepping on a_j and s_j,

$$\begin{aligned} a_j(t+\Delta t) &= a_j(t) + \Delta t \dot{a}_j(t) \\ s_j(t+\Delta t) &= s_j(t) + \Delta t \dot{s}_j(t) \ . \end{aligned} \tag{1.11}$$

Various time-stepping strategies can be used to select Δt. For example, choosing Δt to be half the time to when the first pair of nodes would coincide. A more sophisticated strategy also prevents the arc length $(\sqrt{\Delta a^2 + \Delta s^2})$ from passing through an extrema during the time-step.

1.2 The local method

MFE as described so far involves the global solution of a set of 2N coupled equations for $\dot{a}_j$ and $\dot{s}_j$. This system however will decouple to produce a two-step locally based method.

Firstly we define two linear basis functions $\phi^{(1)}_{j-\frac{1}{2}}$ and $\phi^{(2)}_{j-\frac{1}{2}}$ as shown in Figure 1.3. It is easily seen that

$$\alpha_j = \phi^{(2)}_{j-\frac{1}{2}} + \phi^{(1)}_{j+\frac{1}{2}} \tag{1.12}$$

and

$$\beta_j = m_{j-\frac{1}{2}}\phi^{(2)}_{j-\frac{1}{2}} + m_{j+\frac{1}{2}}\phi^{(1)}_{j+\frac{1}{2}} \tag{1.13}$$

where

$$m_{j-\frac{1}{2}} = \frac{a_j - a_{j-1}}{s_j - s_{j-1}} \tag{1.14}$$

is the gradient of the piecewise linear segment of the solution in element $j-\frac{1}{2}$.

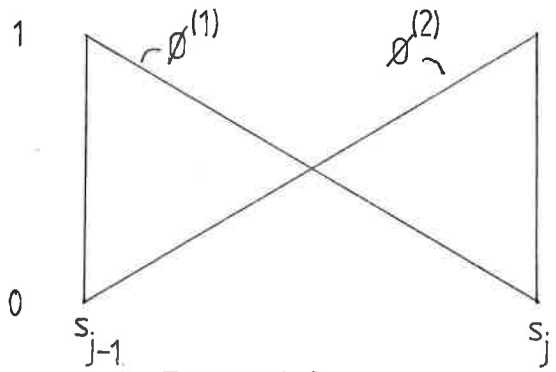

Figure 1.3

The inner product $\langle \alpha_j, L(v) \rangle$ can be decomposed into

$$\langle \alpha_j L(v) \rangle = \langle \phi^{(2)}_{j-\frac{1}{2}}, L(v) \rangle + \langle \phi^{(1)}_{j+\frac{1}{2}}, L(v) \rangle$$

and similarly (1.15)

$$\langle \beta_j, L(v) \rangle = m_{j-\frac{1}{2}} \langle \phi^{(2)}_{j-\frac{1}{2}}, L(v) \rangle + m_{j+\frac{1}{2}} \langle \phi^{(1)}_{j+\frac{1}{2}}, L(v) \rangle \ .$$

The inner products $\langle \alpha_i, \alpha_j \rangle$, $\langle \beta_i, \alpha_j \rangle$ etc. also decompose in a similar fashion allowing us to rewrite the system (1.7) as

$$M^T C M \dot{\underset{\sim}{\chi}} = M^T \underset{\sim}{b} \tag{1.16}$$

where the matrix M is a 2×2 block diagonal matrix, whose block entries are

$$M_j = \begin{bmatrix} 1 & -m_{j-\frac{1}{2}} \\ & \\ 1 & -m_{j+\frac{1}{2}} \end{bmatrix} , \tag{1.17}$$

the matrix C is also 2×2 block diagonal, but with its blocks, staggered with respect to those of M , of form

$$C_{j-\frac{1}{2}} = \frac{\Delta s_{j-\frac{1}{2}}}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \tag{1.18}$$

and $\underset{\sim}{b}$ is the vector of inner products of $L(v)$ with the elementwise basis functions ϕ

$$\begin{aligned} \underset{\sim}{b} &= (\underset{\sim}{b}^T_{\frac{1}{2}}; \underset{\sim}{b}^T_{\frac{3}{2}}; \quad)^T \\ &= (\langle \phi^{(1)}_{\frac{1}{2}}, L(v) \rangle . \langle \phi^{(2)}_{\frac{1}{2}}, L(v) \rangle ; \ldots .)^T . \end{aligned} \tag{1.19}$$

Since M is square and invertible (assuming $m_{j-\frac{1}{2}} \neq m_{j+\frac{1}{2}}$, in which case we have the singularity of parallelism [10]) we can write (1.16) as

$$C M \dot{\underset{\sim}{\chi}} = \underset{\sim}{b} \tag{1.20}$$

which may be solved in two local stages:

1. In each element solve

$$C_{j-\frac{1}{2}} \underset{\sim}{w}_{j-\frac{1}{2}} = \underset{\sim}{b}_{j-\frac{1}{2}} \tag{1.21}$$

for

$$\underset{\sim}{w}_{j-\frac{1}{2}} = \begin{bmatrix} w^{(1)}_{j-\frac{1}{2}} \\ w^{(2)}_{j-\frac{1}{2}} \end{bmatrix} . \tag{1.22}$$

2. At each node solve

$$M_j \dot{\underset{\sim}{y}}_j = \underset{\sim}{w}_j = \begin{bmatrix} w^{(2)}_{j-\frac{1}{2}} \\ w^{(1)}_{j+\frac{1}{2}} \end{bmatrix} . \tag{1.23}$$

At each stage only a 2×2 inversion is necessary.

The local and global methods are equivalent (but see section 3) in the one-dimensional case.

1.3 Two-dimensions - the global method

In two-dimensions the nodal positions now have two coordinates, i.e. $\underset{\sim}{s}_j = (x_j, y_j)^T$ and (1.3) becomes

$$v_t = \sum_1^N \{\dot{a}_j \alpha_j(x, \underset{\sim}{s}(t)) + \dot{x}_j \beta_j(x, \underset{\sim}{a}(t), \underset{\sim}{s}(t)) + \dot{y}_j \gamma_j(x, \underset{\sim}{a}(t), \underset{\sim}{s}(t))\} \tag{1.24}$$

where the extra function

$$\gamma_j = - v_y \alpha_j \tag{1.25}$$

now appears and the elements are now triangular.

The MFE matrix A is now 3×3 block tridiagonal, its entries being

$$A_{ij} = \begin{bmatrix} \langle \alpha_i, \alpha_j \rangle & \langle \alpha_i, \beta_j \rangle & \langle \alpha_i, \gamma_j \rangle \\ \langle \beta_i, \alpha_j \rangle & \langle \beta_i, \beta_j \rangle & \langle \beta_i, \gamma_j \rangle \\ \langle \gamma_i, \alpha_j \rangle & \langle \gamma_i, \beta_j \rangle & \langle \gamma_i, \gamma_j \rangle \end{bmatrix} \tag{1.26}$$

and the vectors $\dot{\underline{\chi}}$ and $\underline{g}$ are now composed of triplets. The method then proceeds analogously with the one-dimensional form described above.

1.4 Two-dimensions - the local methods

Just as in the one-dimensional case the MFE equations may be written in the form of (1.16) where now

$$C_k = \{\langle \phi_k^{(i)}, \phi_k^{(j)} \rangle\} = \frac{A_k}{12} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \tag{1.27}$$

A_k being the area of element k and $\phi_k^{(i)}$ being the element basis function of element k which is unity at node i. The matrix M_j is given by

$$M_j = \begin{bmatrix} 1 & -m_k & -n_k \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} \tag{1.28}$$

where the j subscript refers to node j and k spans over surrounding elements. The piecewise constant y gradient of element k is denoted here by n_k.

Since M_j is rectangular we cannot multiply through by its inverse, and the dimensions do not permit its pseudo inverse to be exact. Instead we solve the following local element and nodal problems.

1. For each element k solve

$$C_k \underset{\sim}{w}_k = \underset{\sim}{b}_k \tag{1.29}$$

2. For each node j solve

$$M_j^T D_j M_j \dot{\underset{\sim}{y}}_j = M_j^T D_j \underset{\sim}{w}_j \tag{1.30}$$

where D_j is the diagonal of C_j. (See (5.2) for the explicit form of these equations).

Here, due to the non-invertibility of M_j, the local method is not equivalent to the global method.

2. Non-singularity of coincident nodes

Early work (e.g. [10]) assumed the singularity of the MFE matrix when two nodes coincided in physical space. This misconception arose from the decomposition of the matrix into

$$A = M^T C M \tag{2.1}$$

where the matrices M and C are block diagonal, namely

$$M = \begin{bmatrix} & & & \\ & M_i & & \\ & & \ddots & \\ & & & \end{bmatrix} \qquad C = \begin{bmatrix} C_{i-\frac{1}{2}} & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \end{bmatrix} \tag{2.2}$$

where

$$M_i = \begin{bmatrix} 1 & -m_{i-\frac{1}{2}} \\ 1 & -m_{i+\frac{1}{2}} \end{bmatrix} \qquad C_{i-\frac{1}{2}} = \frac{\Delta s_{i-\frac{1}{2}}}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} . \tag{2.3}$$

(Note the staggering of the blocks of M compared with C , thus giving the block tridiagonal A).

When two nodes, s_{i-1} and s_i , merge, then $\Delta s_{i-\frac{1}{2}}$ becomes zero and C becomes singular. Clearly if M remains finite then A is also singular, however this is not the case. To best see this, let us decompose C further into

$$C = E^T D E \tag{2.4}$$

where E is diagonal and D block diagonal, with

$$C_{i-\frac{1}{2}} = E^{T}_{i-\frac{1}{2}} D_{i-\frac{1}{2}} E_{i-\frac{1}{2}} \tag{2.5}$$

where

$$E_{i-\frac{1}{2}} = \begin{bmatrix} \Delta s^{\frac{1}{2}}_{i-\frac{1}{2}} & 0 \\ 0 & \Delta s^{\frac{1}{2}}_{i-\frac{1}{2}} \end{bmatrix} \qquad D_{i-\frac{1}{2}} = \frac{1}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} . \tag{2.6}$$

The potential singularity has now been removed from the block diagonal matrix, into the matrix E. We now recombine,

$$A = M^{T}E^{T}D\,E\,M = N^{T}D\,N \tag{2.7}$$

the potential singularity being taken into N,

$$N = E\,M = \begin{bmatrix} & & & \\ & & & \\ & & N_i & \\ & & & \end{bmatrix} \tag{2.8}$$

where

$$N_i = \begin{bmatrix} \Delta s^{\frac{1}{2}}_{i-\frac{1}{2}} & -m_{i-\frac{1}{2}}\Delta s^{\frac{1}{2}}_{i-\frac{1}{2}} \\ \Delta s^{\frac{1}{2}}_{i+\frac{1}{2}} & -m_{i+\frac{1}{2}}\Delta s^{\frac{1}{2}}_{i+\frac{1}{2}} \end{bmatrix} = \begin{bmatrix} \Delta s^{\frac{1}{2}}_{i-\frac{1}{2}} & -\Delta u_{i-\frac{1}{2}} & \Delta s^{-\frac{1}{2}}_{i-\frac{1}{2}} \\ \Delta s^{\frac{1}{2}}_{i+\frac{1}{2}} & -\Delta u_{i+\frac{1}{2}} & \Delta s^{-\frac{1}{2}}_{i+\frac{1}{2}} \end{bmatrix} \tag{2.9}$$

since

$$m_{i+\frac{1}{2}} = \frac{\Delta u_{i+\frac{1}{2}}}{\Delta s_{i+\frac{1}{2}}} .$$

Note the effect of the staggering of the blocks of M and E.

The matrix D is now always non-singular and finite, so we consider the matrix N. In particular since it is block diagonal we consider one of its blocks,

$$N_i = \begin{bmatrix} \Delta s_{i-\frac{1}{2}}^{\frac{1}{2}} & -\Delta u_{i-\frac{1}{2}} \Delta s_{i-\frac{1}{2}}^{-\frac{1}{2}} \\ \Delta s_{i+\frac{1}{2}}^{\frac{1}{2}} & -\Delta u_{i+\frac{1}{2}} \Delta s_{i+\frac{1}{2}}^{-\frac{1}{2}} \end{bmatrix} .$$

Its determinant is given by

$$\det(N_i) = \left[\frac{\Delta s_{i+\frac{1}{2}}}{\Delta s_{i-\frac{1}{2}}} \right]^{\frac{1}{2}} \left\{ \Delta u_{i-\frac{1}{2}} - \frac{\Delta s_{i-\frac{1}{2}}}{\Delta s_{i+\frac{1}{2}}} \Delta u_{i+\frac{1}{2}} \right\} \qquad (2.10a)$$

or equivalently by

$$\det(N_i) = \left[\frac{\Delta s_{i-\frac{1}{2}}}{\Delta s_{i+\frac{1}{2}}} \right]^{\frac{1}{2}} \left\{ \frac{\Delta s_{i+\frac{1}{2}}}{\Delta s_{i-\frac{1}{2}}} \Delta u_{i-\frac{1}{2}} - \Delta u_{i+\frac{1}{2}} \right\} \qquad (2.10b)$$

and its inverse is given by

$$N_i^{-1} = \frac{1}{\det(N_i)} \begin{bmatrix} -\Delta u_{i+\frac{1}{2}} \Delta s_{i+\frac{1}{2}}^{-\frac{1}{2}} & \Delta u_{i-\frac{1}{2}} \Delta s_{i-\frac{1}{2}}^{-\frac{1}{2}} \\ -\Delta s_{i+\frac{1}{2}}^{\frac{1}{2}} & \Delta s_{i-\frac{1}{2}}^{\frac{1}{2}} \end{bmatrix} . \qquad (2.11)$$

We now consider the cases where either $\Delta s_{i-\frac{1}{2}}$ or $\Delta s_{i+\frac{1}{2}}$ tend to zero, i.e. either $s_i \to s_{i-1}$ or $s_i \to s_{i+1}$, (but not both, which would be classified as the true singularity of parallelism).

Consider first the case $\Delta s_{i-\frac{1}{2}} \to 0$, choosing expression (2.10a) for the determinant, we have

$$N_i^{-1} = \left\{ \Delta u_{i-\frac{1}{2}} - \frac{\Delta s_{i-\frac{1}{2}}}{\Delta s_{i+\frac{1}{2}}} \Delta u_{i+\frac{1}{2}} \right\}^{-1} \begin{bmatrix} -\Delta u_{i+\frac{1}{2}} \Delta s_{i-\frac{1}{2}}^{\frac{1}{2}} \Delta s_{i+\frac{1}{2}}^{-1} & \Delta u_{i-\frac{1}{2}} \Delta s_{i+\frac{1}{2}}^{-\frac{1}{2}} \\ -\Delta s_{i-\frac{1}{2}}^{\frac{1}{2}} & \Delta s_{i+\frac{1}{2}}^{-\frac{1}{2}} \Delta s_{i-\frac{1}{2}} \end{bmatrix} \quad (2.12)$$

which as $\Delta s_{i-\frac{1}{2}} \to 0 \quad (\Delta s_{i+\frac{1}{2}} \neq 0)$

$$N_i^{-1} \to \Delta u_{i-\frac{1}{2}}^{-1} \begin{bmatrix} 0 & \Delta u_{i-\frac{1}{2}} \Delta s_{i+\frac{1}{2}}^{-\frac{1}{2}} \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \Delta s_{i+\frac{1}{2}}^{-\frac{1}{2}} \\ 0 & 0 \end{bmatrix} \quad (2.13)$$

If, alternatively, $\Delta s_{i+\frac{1}{2}} \to 0$ we choose (2.10b) for the determinant, and obtain

$$N_i^{-1} = \left\{ \frac{\Delta s_{i+\frac{1}{2}}}{\Delta s_{i-\frac{1}{2}}} \Delta u_{i-\frac{1}{2}} - \Delta u_{i+\frac{1}{2}} \right\}^{-1} \begin{bmatrix} -\Delta u_{i+\frac{1}{2}} \Delta s_{i-\frac{1}{2}}^{-\frac{1}{2}} & \Delta u_{i-\frac{1}{2}} \Delta s_{i+\frac{1}{2}}^{\frac{1}{2}} \Delta s_{i-\frac{1}{2}}^{-1} \\ -\Delta s_{i+\frac{1}{2}} \Delta s_{i-\frac{1}{2}}^{-\frac{1}{2}} & \Delta s_{i+\frac{1}{2}}^{\frac{1}{2}} \end{bmatrix}$$

$$\to \Delta u_{i+\frac{1}{2}}^{-1} \begin{bmatrix} -\Delta u_{i+\frac{1}{2}} \Delta s_{i-\frac{1}{2}}^{-\frac{1}{2}} & 0 \\ 0 & 0 \end{bmatrix} \text{ as } \Delta s_{i+\frac{1}{2}} \to 0 \ , \ \Delta s_{i-\frac{1}{2}} \neq 0$$

$$= \begin{bmatrix} -\Delta s_{i-\frac{1}{2}}^{-\frac{1}{2}} & 0 \\ 0 & 0 \end{bmatrix} \quad (2.14)$$

Note that in each case N_i^{-1} exists but N_i does not, in the sense that N_i^{-1} is not invertible. This should be interpreted that, although the minimisation equations cannot be explicitly written down for coincident

nodes, an inverse exists, and so, provided the right hand side, $\underset{\sim}{g}$, or more precisely $A^{-1}\underset{\sim}{g}$ exists and is finite, the solution does not have an analytic singularity for coincident nodes and the solution can be continued through such a situation. In such circumstances, however, care should be taken to use an inner product which integrates along arc length, such as the gradient-weighted MFE of Miller [8] , so that the norm of minimisation remains appropriate for overturned manifolds.

3. A condition for the equivalence of local and global MFE in one-dimension when using Mueller inner products

When using Mueller inner products [9] for second derivatives in the right-hand side of diffusion problems, the integration by parts produces terms involving u_x evaluated at the nodes. In the global method these terms cancel as the contributions from adjacent elements are assembled and therefore are not explicitly calculated. However, in the local method, since the assembly matrix M is not applied to the element inner products, this cancellation does not occur. The question then arises as to what value to take for u_x at nodes, since due to the piecewise linear nature of the solution it is underfined there.

For the local and global methods to be equivalent, the requirement is that these additional terms should cancel when assembled using the matrix M, we therefore compare

$$\underset{\sim}{g} = \begin{Bmatrix} \langle \alpha_i, (D(u)u_x)_x \rangle \\ \langle \beta_i, (D(u)u_x)_x \rangle \end{Bmatrix} \tag{3.1}$$

the r.h.s. of the global system, with

$$M^T \underset{\sim}{b} = \begin{bmatrix} 1 & 1 \\ -m_{i-\frac{1}{2}} & -m_{i+\frac{1}{2}} \end{bmatrix} \begin{bmatrix} \langle \phi^{(2)}_{i-\frac{1}{2}}, (D(u)u_x)_x \rangle \\ \langle \phi^{(1)}_{i+\frac{1}{2}}, (D(u)u_x)_x \rangle \end{bmatrix} \tag{3.2}$$

the assembled element inner products from the local methods.

Simple integration by parts gives

$$\langle \phi^{(2)}_{i-\frac{1}{2}}, (D(u)u_x)_x \rangle = D(u_i)u_x^{(L)}\Big|_i - \frac{m_{i-\frac{1}{2}}}{\Delta s_{i-\frac{1}{2}}} \int_{s_{i-1}}^{s_i} D(u)dx \qquad (3.3a)$$

$$\langle \phi^{(1)}_{i+\frac{1}{2}}, (D(u)u_x)_x \rangle = D(u_i)u_x^{(R)}\Big|_i + \frac{m_{i+\frac{1}{2}}}{\Delta s_{i+\frac{1}{2}}} \int_{s_i}^{s_{i+1}} D(u)dx \qquad (3.3b)$$

where $u_x^{(L)}\Big|_i$ and $u_x^{(R)}\Big|_i$ are the values of u_x at node i approached from the left and right respectively. On the other hand we have

$$\langle \alpha_i, (D(u)u_x)_x \rangle = -\frac{m_{i-\frac{1}{2}}}{\Delta s_{i-\frac{1}{2}}} \int_{s_{i-1}}^{s_i} D(u)dx + \frac{m_{i+\frac{1}{2}}}{\Delta s_{i+\frac{1}{2}}} \int_{s_i}^{s_{i+1}} D(u)dx \qquad (3.4)$$

which, comparing with $\langle (\phi^{(2)}_{i-\frac{1}{2}} + \phi^{(1)}_{i+\frac{1}{2}}), (D(u)u_x)_x \rangle$ gives

$$u_x^{(L)}\Big|_i = u_x^{(R)}\Big|_i = u_x\Big|_i \qquad (3.5)$$

i.e. we must use a consistent value for u_x at the nodes, and not, for example $m_{i-\frac{1}{2}}$ in (3.3a) and $m_{i+\frac{1}{2}}$ in (3.3b) as we might be tempted. This is only natural if we require the terms to cancel.

If we now compare the β equation

$$\langle \beta_i, (D(u)u_x)_x \rangle = \int_{s_{i-1}}^{s_{i+1}} D(u)u_x^2(\alpha_i)_x dx - \tfrac{1}{2}\int_{s_{i-1}}^{s_{i+1}} u_x^2(D(u)\alpha_i)_x dx$$

(using $\beta_i = -u_x\alpha_i$ and integration by parts)

$$= \frac{m^2_{i-\frac{1}{2}}}{\Delta s_{i-\frac{1}{2}}} \int_{s_{i-1}}^{s_i} D(u)dx - \frac{m^2_{i+\frac{1}{2}}}{\Delta s_{i+\frac{1}{2}}} \int_{s_i}^{s_{i+1}} D(u)dx$$

$$+ \tfrac{1}{2}\left\{m^2_{i+\frac{1}{2}} - m^2_{i-\frac{1}{2}}\right\} D(u_i) \qquad (3.6)$$

with

$$\langle (-m_{i-\frac{1}{2}}\phi^{(2)}_{i-\frac{1}{2}} - m_{i+\frac{1}{2}}\phi^{(1)}_{i-\frac{1}{2}}), (D(u)u_x)_x \rangle$$

gives

$$(m_{i+\frac{1}{2}} - m_{i-\frac{1}{2}})u_x\Big|_i = \tfrac{1}{2}(m^2_{i+\frac{1}{2}} - m^2_{i-\frac{1}{2}})$$

i.e.

$$u_x\Big|_i = \tfrac{1}{2}(m_{i+\frac{1}{2}} + m_{i-\frac{1}{2}}) \qquad (3.7)$$

So for equivalence of the local and global methods in 1-D when using Mueller inner products, we must take the value of u_x at nodes to be the average of the slopes on either side. Of course, we need not stipulate that the two methods be equivalent - we could, for instance, choose a weighted average instead, thus generating an alternative method.

In two-dimensions, although the global and local methods cannot be equivalent, due to the different spaces in which (α,β,γ) and $(\phi^{(1)},\phi^{(2)},\phi^{(3)})$ lie, we can adopt the same strategy of taking u_x, u_y to be the average of adjacent slopes at the nodes.

4. Recovery in two-dimensions

An alternative to using Mueller inner products for second derivative terms in the right-hand side of (1.1) is to use recovery [10]. This entails fitting a piecewise polynomial to the first derivative and then differentiating. Typically a quadratic would be fitted to the first derivative giving a piecewise linear representation of the second derivative. This process is simple and straightforward in one-dimension, best results being obtained using weighted averages of gradients at some knots [3]: however in two-dimensions the expressions derived by Johnson [4] become complex.

In this section we show how, by using area coordinates, the expressions for quadratic recovery in two-dimensions are much simplified. We also show how, by inclusion of a cubic term, the quality of the recovery is improved.

Consider a patch of elements as in Figure 4.1.

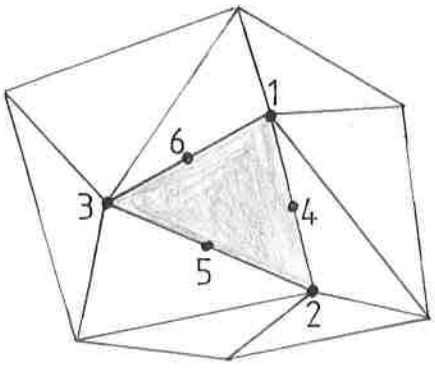

Figure 4.1

Fit a quadratic to points 1-6 for u_x and for u_y, such that

$$w_x^{(i)} = w_x(\underset{\sim}{x}_i) = \begin{cases} \dfrac{\sum_{j=1}^{s_i} \rho_j m_j}{\sum_{j=1}^{s_i} \rho_j} & i = 1,2,3 \\ \dfrac{\rho_E m_E + \rho_{N_i} m_{N_i}}{P_E + \rho_{N_i}} & i = 4,5,6 \end{cases} \qquad (4.1)$$

where

ρ_j = weighting function for element j,

m_j = x gradient in element j,

E = indicates element of interest,

N_i = neighbouring element on side containing i,

s_i = number of surround elements

and similarly for $w_y(\underset{\sim}{x}_i)$, m being replaced by n the y gradient. That is, the gradients interpolated at the knots 1-6 are given by a weighted average determined by the ρ_j.

We now construct the interpolant to the first derivatives.

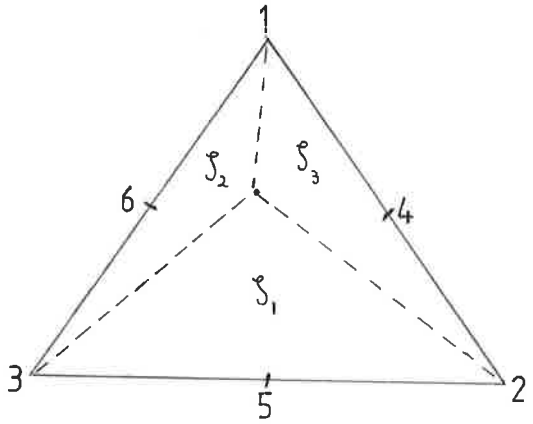

Figure 4.2

Use area coordinates ζ_i : $\zeta_i(\underset{\sim}{x}_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$ $i,j = 1,2,3$

with $\zeta_1 + \zeta_2 + \zeta_3 = 1$.

Then points have area coordinates:

$$\begin{aligned} 1 &: (1,0,0) \\ 2 &: (0,1,0) \\ 3 &: (0,0,1) \\ 4 &: (½,½,0) \\ 5 &: (0,½,½) \\ 6 &: (½,0,½) \end{aligned} \qquad (4.2)$$

as in Figure 4.2.

Hence

$$\begin{aligned} w_z(\zeta_1,\zeta_2,\zeta_3) &= 2\zeta_1(\zeta_1-½)w_z^{(1)} + 2\zeta_2(\zeta_2-½)w_z^{(2)} + 2\zeta_3(\zeta_3-½)w_2^{(3)} \\ &+ 4\zeta_1\zeta_2 w_z^{(4)} + 4\zeta_2\zeta_3 w_z^{(5)} + 4\zeta_1\zeta_3 w_z^{(6)} \qquad (4.3) \\ & \qquad z = x \text{ or } y \end{aligned}$$

and therefore

$$\begin{aligned} w_{zz}(\zeta_1,\zeta_2,\zeta_3) &= (4\zeta_1-1)w_z^{(1)}\zeta_1 + (4\zeta_2-1)w_z^{(2)}\zeta_{2z} + (4\zeta_3-1)w_z^{(3)}\zeta_3 \\ &+ 4(\zeta_1\zeta_{2z} + \zeta_{1z}\zeta_2)w_z^{(4)} + 4(\zeta_2\zeta_{3z} + \zeta_{2z}\zeta_3)w_z^{(5)} \\ &+ 4(\zeta_1\zeta_{3z} + \zeta_{1z}\zeta_3)w_z^{(6)} \qquad (4.4) \\ & \qquad z = x \text{ or } y . \end{aligned}$$

We need now expressions for ζ_{ix} and ζ_{iy}. In terms of x and y the area coordinates are given by

$$\zeta_i = a_i(x - x_j) + b_i(y - y_j) \qquad (4.5a)$$

with

$$\begin{aligned} a_i(x_k - x_j) + b_i(y_k - y_j) &= 0 \\ a_i(x_i - x_j) + b_i(y_i - y_j) &= 1 \end{aligned} \qquad (4.5b)$$

where

$$(i,j,k) = \text{perm } (1,2,3) \qquad (\underline{\text{ordered}} \text{ permutation})$$

$$\text{therefore} \qquad a_i = \frac{(y_j - y_k)}{(x_k - x_j)(y_i - y_j) - (y_k - y_j)(x_i - x_j)} = \frac{(y_j - y_k)}{2A} \qquad (4.6)$$

and

$$b_i = \frac{(x_k - x_j)}{2A}$$

where A = area of element = ½ × element jacobian.

Hence

$$\zeta_{ix} = a_i = \frac{(y_j - y_k)}{2A} \; ; \qquad (4.7)$$

$$\zeta_{iy} = b_i = \frac{(x_k - x_j)}{2A} \; .$$

The question now arises as to the weights ρ_j used in the averages. $\rho_j \equiv 1$ is a straightforward average, whilst $\rho_j = A_j$, the area of element j, is a possible analogue of the element weighted recovery so successful in one-dimension. Another possible analogue of this one-dimensional recovery is to weight according to transversals, see Figure 4.3, which appears more robust in practice.

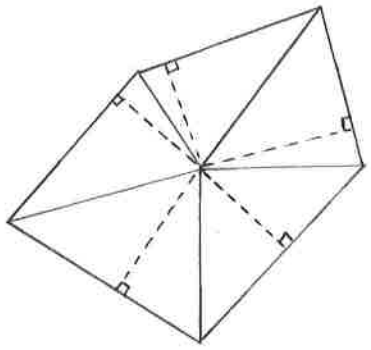

Figure 4.3

Both of these possible analogue differ in one important feature from the one-dimensional recovery, and that is that the gradient of the

element 'being recovered' is not used singly at any of the nodes, whereas in the 1-D recovery, the situation is as in Figure 4.4.

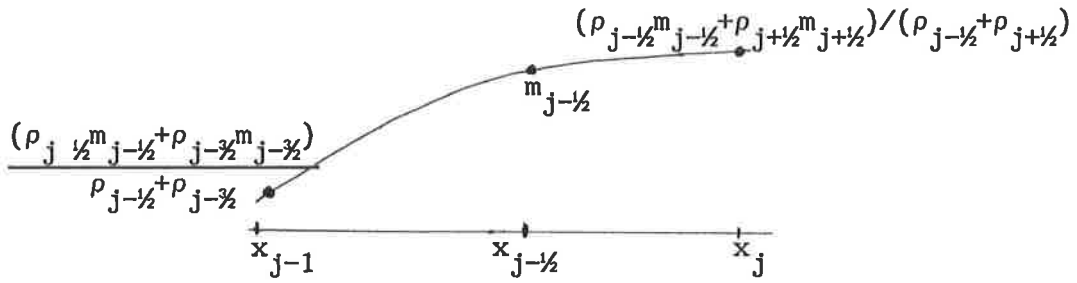

Figure 4.4

We can strengthen the analogy by introducing a seventh knot at the centroid of the element as in Figure 4.5.

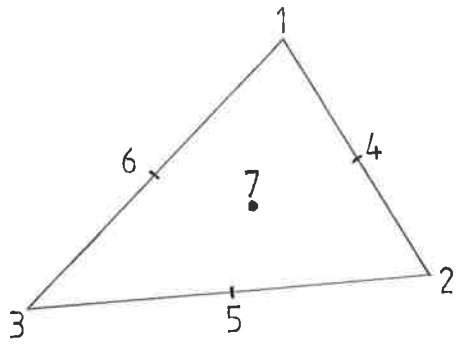

Figure 4.5

This point has the area coordinates $\zeta_1 = \zeta_2 = \zeta_3 = 1/3$, and so (4.3) has added an extra term

$$\zeta_1 \zeta_2 \zeta_3 w_z^{(7)} \tag{4.8}$$

where, to interpolate m_i (or n_i) at this point, we take

$$w_z^{(7)} = m_i + \frac{1}{9}\left[w_x^{(1)} + w_x^{(2)} + w_x^{(3)} - 4(w_x^{(4)} + w_x^{(5)} + w_x^{(6)})\right] \tag{4.9}$$

and (4.4) gains the extra term

$$(\zeta_{1z}\zeta_2\zeta_3 + \zeta_1\zeta_{2z}\zeta_3 + \zeta_1\zeta_2\zeta_{3z})w_z^{(7)} \quad . \tag{4.10}$$

The effect of this is to add a cubic term to the interpolating quadratic, although the interpolant is not a full cubic. Numerical results indicate that this form of recovery is far less prone to spurious over/under shoots than that given by (4.3),(4.4).

5. Regularisation

In one-dimension the problem of node merging can usually be easily overcome using a time-stepping strategy. However in two-dimensions the equaivalent problem of element folding is not so readily overcome. This is possibly due to the extra degree of freedom in nodal movement.

An alternative ploy is to use penalty functions [7] or regularisation [1], both of which effectively constrain nodal movement. Baines [1] gives a general overview of regularisation strategy, but here we describe an ad hoc regularisation which works well in practice for the two-dimensional local method.

Consider the second stage of the local method in two-dimensions, where we are solving

$$M_i^T D_i M_i \dot{\underset{\sim}{y}}_i = M_i^T D_i \underset{\sim}{w}_i \tag{5.1}$$

at node i, i.e.

$$\begin{bmatrix} \sum A_j & -\sum A_j m_j & -\sum A_j n_j \\ -\sum A_j m_j & \sum A_j m_j^2 & \sum A_j m_j n_j \\ -\sum A_j n_j & \sum A_j m_j n_j & \sum A_j n_j^2 \end{bmatrix} \begin{bmatrix} \dot{a}_i \\ \dot{x}_i \\ \dot{y}_i \end{bmatrix} = \begin{bmatrix} \sum A_j w_j^i \\ -\sum A_j m_j w_j^i \\ -\sum A_j n_j w_j^i \end{bmatrix} \tag{5.2}$$

where summation is over elements j surrounding node i. It is easily seen that if we were to overwrite the centre term $\sum A_j m_j^2$ of the matrix $M_i^T D_i M_i$ with 'infinity' this would produce a zero $\dot{x}_i$ on solution, similar treatment of the $\sum A_j n_j^2$ entry would produce the same effect on $\dot{y}_i$. We can therefore control the nodal movement by modification of these two diagonal terms of the matrix $M_i^T D_i M_i$, whilst

the first equation of the system ensures that the nodal amplitude is correctly adjusted. We achieve this by multiplying the second two diagonal terms by limiters, ϕ_i^x and ϕ_i^y giving the new system

$$\begin{bmatrix} \sum A_j & -\sum A_j m_j & -\sum A_j n_j \\ -\sum A_j m_j & \phi_i^x \sum A_j m_j^2 & \sum A_j m_j n_j \\ -\sum A_j n_j & \sum A_j m_j n_j & \phi_i^y \sum A_j n_j^2 \end{bmatrix} \begin{bmatrix} \dot{a}_i \\ \dot{x}_i \\ \dot{y}_i \end{bmatrix} = \begin{bmatrix} \sum A_j w_j^i \\ -\sum A_j m_j w_j^i \\ -\sum A_j n_j w_j^i \end{bmatrix} \qquad (5.3)$$

The choice of these limiters now give great latitude to control of the nodes. Choosing them to be different will change the direction θ of movement, modifying it by an angle $\delta\theta$ such that

$$\theta + \delta\theta = \tan^{-1}\left[\frac{\phi_i^x}{\phi_i^y} \tan\theta \right] \qquad (5.4)$$

including even a reversal of direction by taking them both negative. The larger the values of the limiters the smaller the resulting velocities.

The crudest possible regularisation is to use the same limiter on both terms, $\phi_i^x = \phi_i^y = \phi_i$, and to choose it in such a way that if one of the surrounding elements is much smaller than the rest the limiter is large, slowing the node down to prevent element folding. Two such regularisations have been used successfully in practice. The first takes

$$\phi_i = \frac{A_{ave}}{A_{min}} \qquad (5.5)$$

where the minimum and average of the areas are taken over all the

surrounding elements, the second takes

$$\phi_i = \frac{T_{ave}}{T_{min}} \tag{5.6}$$

where T is the transversal of the element j passing through node i. (See Figure 4.3). Just as in the case of the recovery (see section 4) the regularisation using transversals works better in practice than regularisation using areas.

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