MATHEMATICS DEPARTMENT

ROE - TYPE SCHEMES FOR SUPER-CRITICAL FLOWS IN RIVERS.

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Numerical Analysis Report 13/89

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ROE - TYPE SCHEMES FOR SUPER-CRITICAL FLOWS IN RIVERS.

by

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Abstract

In this report we consider the flow in an open channel with a region of geometry-induced super-critical flow. A shock capturing scheme is used to resolve the resulting discontinuous solutions.

1. Introduction

Although discontinuities are not normally associated with the flows in rivers they can, and do, occur.

In this report we will consider the flow in an open channel with rectangular cross-section. The channel width and the bed-slope will be such as to induce a sub-critical → super-critical → sub-critical flow regime.

A shock-capturing scheme is used and although it is an inefficient method for smooth flows some examples of non-discontinuous flows are included to show that the method also copes with these types of flows. However, the shock-capturing scheme comes into its own as the bed-slope is increased and the flow becomes super-critical. Discontinuities are formed and many of the classical methods fail at this point, or at best contaminate the solution with oscillations. We shall show that the shock-capturing scheme takes all this in its stride.

In Section 2 the equations will be introduced and the method of solution will be described in some detail.

In Section 3 the model problem will be described and the results given.

Finally some remarks are made about work that will be done in the future to (hopefully) improve the results yet further.

2. The St. Venant Equations and Roe's Scheme

The St. Venant equations for rough-turbulent flow in an open channel are:-

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = 0 \qquad (1a)$$

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\left[\frac{Q^2}{A}\right] + gA\left[\frac{\partial h}{\partial x} + \frac{Q|Q|}{K^2}\right] = 0 \ . \qquad (1b)$$

Where

A = cross-sectional area = breadth x depth = B x d

(only rectangular channels are considered).

Q = massflow

u = velocity

g = gravity

h = height = d + z where z is the height of the river bed.

$\frac{Q|Q|}{K^2}$ is the friction term chosen to give the fully rough-turbulent form in this case.

$K = \frac{A}{M}\,(\text{hydraulic radius})^{2/3}$

where the hydraulic radius = A/wetted perimeter. M is Manning's constant for which we will take a value of 0.03.

The scheme we shall use for the solution of these equations is an approximate Riemann solver due to Roe (1981).

To get the equations in a form to which Roe's scheme can be applied we rewrite (1) as

$$A_t + Q_x = 0 \tag{2a}$$

$$Q_t + \left[\frac{Q^2}{A} + \frac{gA^2}{2B}\right]_x = gA\left[\frac{AB_x}{2B^2} - \frac{Q^2}{K^2} - \beta\right] \tag{2b}$$

where we have replaced h by $A/B + z$ and $\beta = z_x$. An extra $\frac{1}{2}g\ A^2\ B_x/B^2$ is added to both sides of equation (2b) for reasons we will give later.

The equations are now in the form

$$\underline{q}_t + \underline{F}_x = \underline{b}$$

where

$$\underline{q} = \begin{bmatrix} A \\ Q \end{bmatrix} \tag{3a}$$

$$\underline{F} = \begin{bmatrix} Q \\ Q^2/_A + \frac{1}{2}g\ A^2/_B \end{bmatrix} \tag{3b}$$

$$\underline{b} = \begin{bmatrix} 0 \\ gA(AB_{x}/B^2 - \frac{Q^2}{K^2} - \beta) \end{bmatrix} . \tag{3c}$$

Following Roe (1981) we define an intermediate, or parameter, vector

$$\underline{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} A^{\frac{1}{2}} \\ uA^{\frac{1}{2}} \end{bmatrix}$$

and now express the vectors $\underline{q}$ and $\underline{F}$ in terms of the parameter vector to get

$$\underline{q} = \begin{bmatrix} w_1^2 \\ w_1 \; w_2 \end{bmatrix}, \quad \underline{F} = \begin{bmatrix} w_1 \; w_2 \\ w_2^2 + \tfrac{1}{2}\tfrac{g}{B}\, w_1^4 \end{bmatrix}.$$

Using the standard notation of $\Delta\underline{x} = \underline{x}_R - \underline{x}_L$ and $\bar{\underline{x}} = \tfrac{1}{2}(\underline{x}_R + \underline{x}_L)$ we proceed to calculate matrices $B(\underline{w})$ and $C(\underline{w})$ such that

$$\Delta\underline{q} = B(\underline{w})\ \Delta\underline{w}$$

$$\Delta\underline{F} = C(\underline{w})\ \Delta\underline{w}\ .$$

This leads to

$$B = \begin{bmatrix} 2\bar{w}_1 & 0 \\ \bar{w}_2 & \bar{w}_1 \end{bmatrix}$$

and

$$C = \begin{bmatrix} \bar{w}_2 & \bar{w}_1 \\ \dfrac{2g}{B^{*2}}\ \bar{B}\ \overline{w_1^2}\ \bar{w}_1 & 2\bar{w}_2 \end{bmatrix},$$

where $B^* = \sqrt{B_L\ B_R}$.

At least this is what we have tried to do, but not all the terms fit into this form which is why we added the term $\tfrac{1}{2}g\ A^2\ B_x/B^2$ earlier to cancel the additional term out (see equation 2b).

We proceed to find λ such that

$$\det(\lambda B - C) = 0 \; ,$$

giving

$$\lambda_1 = \frac{\bar{w}_2}{\bar{w}_1} - \sqrt{g\bar{B}\;\overline{w_1^2}/B^{*2}}$$

$$\lambda_2 = \frac{\bar{w}_2}{\bar{w}_1} + \sqrt{g\bar{B}\;\overline{w_1^2}/B^{*2}} \; .$$

Reassuringly, if we ignore the averaging, these are then $u \pm \sqrt{gd}$ as we would expect.

These eigenvalues give two eigenvectors which, after multiplication by B, are given by:-

$$\underline{e}_1 = \begin{bmatrix} \bar{w}_1 \\ \bar{w}_2 - \sqrt{g\bar{B}\;\overline{w_1^2}\;\bar{w}_1^2/B^{*2}} \end{bmatrix} , \quad \underline{e}_2 = \begin{bmatrix} \bar{w}_1 \\ \bar{w}_2 + \sqrt{g\bar{B}\;\overline{w_1^2}\;\bar{w}_1^2/B^{*2}} \end{bmatrix} .$$

Two α's are now found such that $\sum_i \alpha_i \underline{e}_i = \Delta \underline{q}$ (and by construction $\sum_i \alpha_i \lambda_i \underline{e}_i = \Delta \underline{F}$) , i.e.

$$\alpha_1 = \Delta w_1 - \frac{\bar{w}_1 \Delta w_2 - \bar{w}_2 \Delta w_1}{2\sqrt{gB\overline{w_1^2}\;\bar{w}_1^2/B^{*2}}}$$

$$\alpha_2 = \Delta w_1 + \frac{\bar{w}_1 \Delta w_2 - \bar{w}_2 \Delta w_1}{2\sqrt{gB\overline{w_1^2}\;\bar{w}_1^2/B^{*2}}}$$

If we define $\phi_{j,i+½}$ to be the signal from the j^{th} eigenvalue at the jump at $i + ½$, i.e.

$$\phi_{j,i+½} = -\frac{\Delta t}{\Delta x} \lambda_j \alpha_j \underline{e}_j$$

then the first order upwind algorithm is defined by

Algorithm U1

$$\text{if} \quad \begin{Bmatrix} \lambda_j & > & 0 \\ \lambda_j & < & 0 \end{Bmatrix} \quad \text{then add} \quad \phi_{j,i+½} \quad \text{to} \quad \begin{Bmatrix} \underline{q}_{i+1} \\ \underline{q}_i \end{Bmatrix} .$$

If we in addition transfer an amount $a_{j,i+½}$ against the direction of the flow we can achieve second order accuracy in smooth regions by choosing

$$a_j = ½(1 - |v_j|)$$

where v_j is the CFL number of j^{th} wave.

Defining a transfer function, Baines (1983), by

$$B(a_{j,i+½}\, \phi_{j,i+½}\, ,\, a_{j,i+½-\sigma_j}) = B(b_1, b_2) ,$$

say, where σ_j =sign (λ_j), we can arrive at various second order schemes. Linear functions of b_1 & b_2 tend to give classical second order schemes with all their faults for discontinuous solutions. Taking non-linear functions, however, enables us to arrive at oscillation-free second order schemes (see Sweby (1984, 1985) for a fuller discussion).

We note, for further reference, that if we wished to project the right hand side of the equation, i.e. (3c), in the following fashion

$$\begin{bmatrix} 0 \\ gA\left(\frac{AB_x}{B^2} - \frac{Q^2}{K^2} - \beta\right) \end{bmatrix} = \beta_1 \underset{\sim}{e}_1 + \beta_2 \underset{\sim}{e}_2$$

then β_1 and β_2 would be

$$\beta_1 = \frac{-\sqrt{g}\,(AB_x/B^{*2} - Q^2/K^2 - \beta)B^*}{\sqrt{\bar{B}}} \tag{4a}$$

and

$$\beta_2 = -\beta_1 \,. \tag{4b}$$

3. The Problem and the Results

We consider only flow in a rectangular channel. It is desired that the flow be sub-critical → super-critical → sub-critical which means that one boundary condition needs to be applied to each end of the channel. At the left-hand end we fix the massflow, Q, and at the right-hand end we fix the depth, d, by extrapolation from the interior values.

The parameters available to us in choosing the channel are its breadth and slope. The channel is 10,000 metres long and has a smooth constriction that goes from a breadth of 10 metres → 5 metres → 10 metres, see Figure 1.

The bed-slope was taken to be a constant value except between 4500 and 5500 metres where twice this value was taken. See Figure 2 for a typical cross-section.

Depth, mass flow and Froude number have been plotted out for the following slopes:

$\frac{1}{10,000}$	Figures 3-5
$\frac{1}{10,000}$	Figures 6-8
$\frac{1}{500}$	Figures 9-11
$\frac{1}{500}$	Figures 12-14
$\frac{1}{50}$	Figures 15-17.

These figures show that Roe's scheme copes with both the smooth flows and the discontinuous flows equally well. We should just remark that because the flow in Figures 15-17 is super-critical throughout, the boundary conditions are not correct for this problem.

4. Conclusion

It has been shown that Roe's scheme can cope with these flows. However, further improvements could be made to the treatment of the source terms. Here it has been found necessary to evaluate them pointwise, in line with the experience of Sweby (1988) but in contrast to the experience of Priestley (1987) and Glaister (1987). Using a second order scheme exaggerated problems with the source terms. This obviously needs to be looked at if we do not want to keep changing our codes every time the source terms change.

5. References

1. Baines, M.J., 1983: Numerical Algorithms for the Non-linear Scalar Wave Equation. University of Reading Numerical Analysis Report 1/83.

2. Glaister, P., 1987: Second Order Difference Schemes for Hyperbolic Conservation Laws with Source Terms. University of Reading Numerical Analysis Report 6/87.

3. Priestley, A., 1987: The Use of a Characteristic Based Scheme for the 2-D Shallow Water Equations. Techniques for Horizontal Discretization in Numerical Weather Prediction Models. ECMWF Workshop Proceedings November 1987.

4. Roe, P.L., 1981: Approximate Riemann Solvers, Parameter Vectors, and Difference Schemes. Journal of Computational Physics, 43.

5. Sweby, P.K., 1984: High Resolution Schemes using Flux Limiters for Hyperbolic Conservation Laws. SIAM J. of Numerical Analysis, Vol. 21, No. 5.

6. Sweby, P.K., 1985: High Resolution TVD Schemes Using Flux Limiters. Lectures in Applied Mathematics, Vol. 22. (Springer).

7. Sweby, P.K., 1988: "TVD" Schemes for Inhomogeneous Conservation Laws. Nonlinear Hyperbolic Equations - Theory, Computational Methods, and Applications. Notes on Numerical Fluid Mechanics Vol. 24 (eds. Josef Ballmann and Rolf Jeltsch). Vieweg.

AP:sd/89

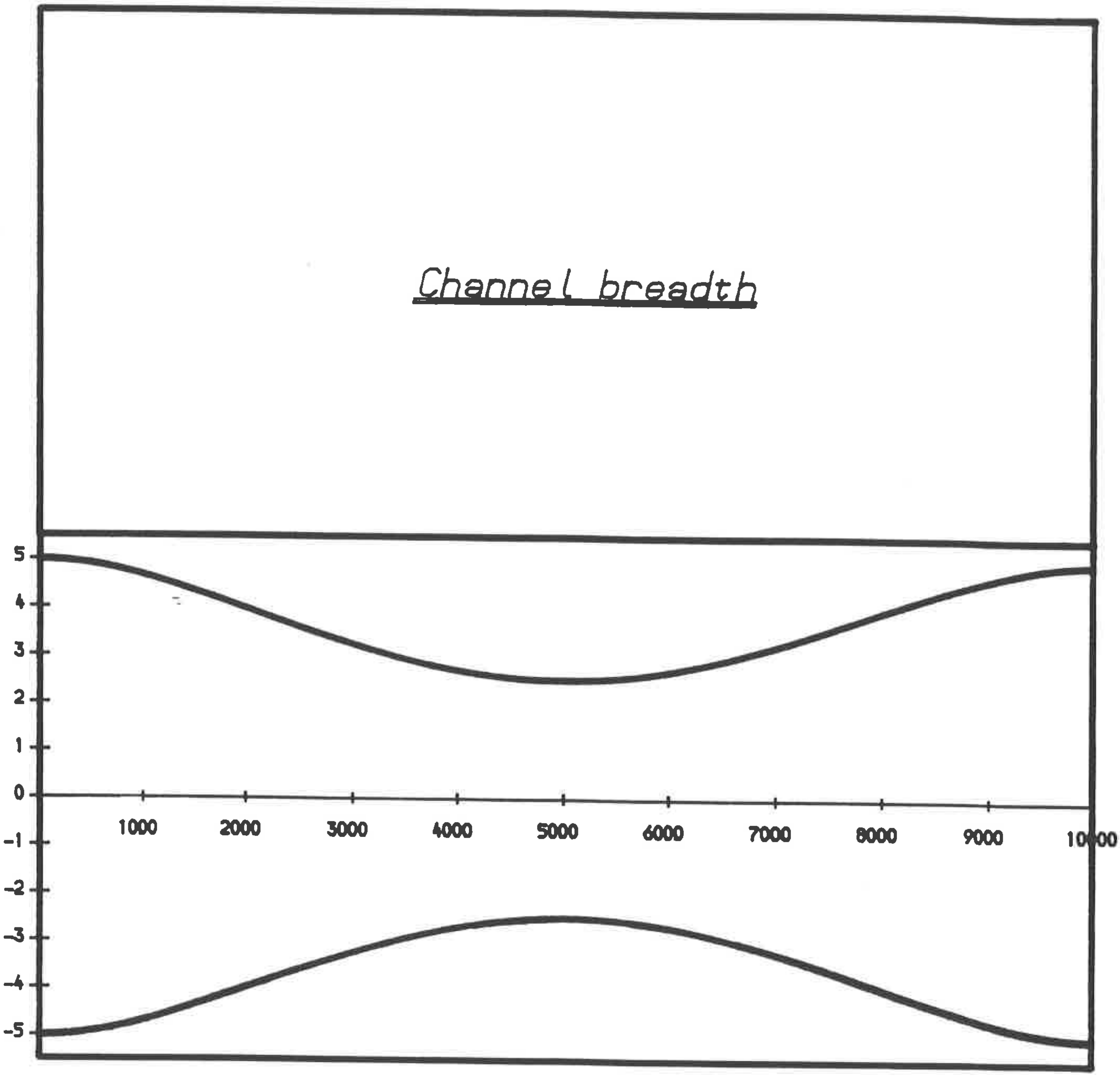

Channel breadth
5
4
3
2
1
0
-1
-2
-3
-4
-5
1000
2000
3000
4000
5000
6000
7000
8000
9000
10000

Figure 1

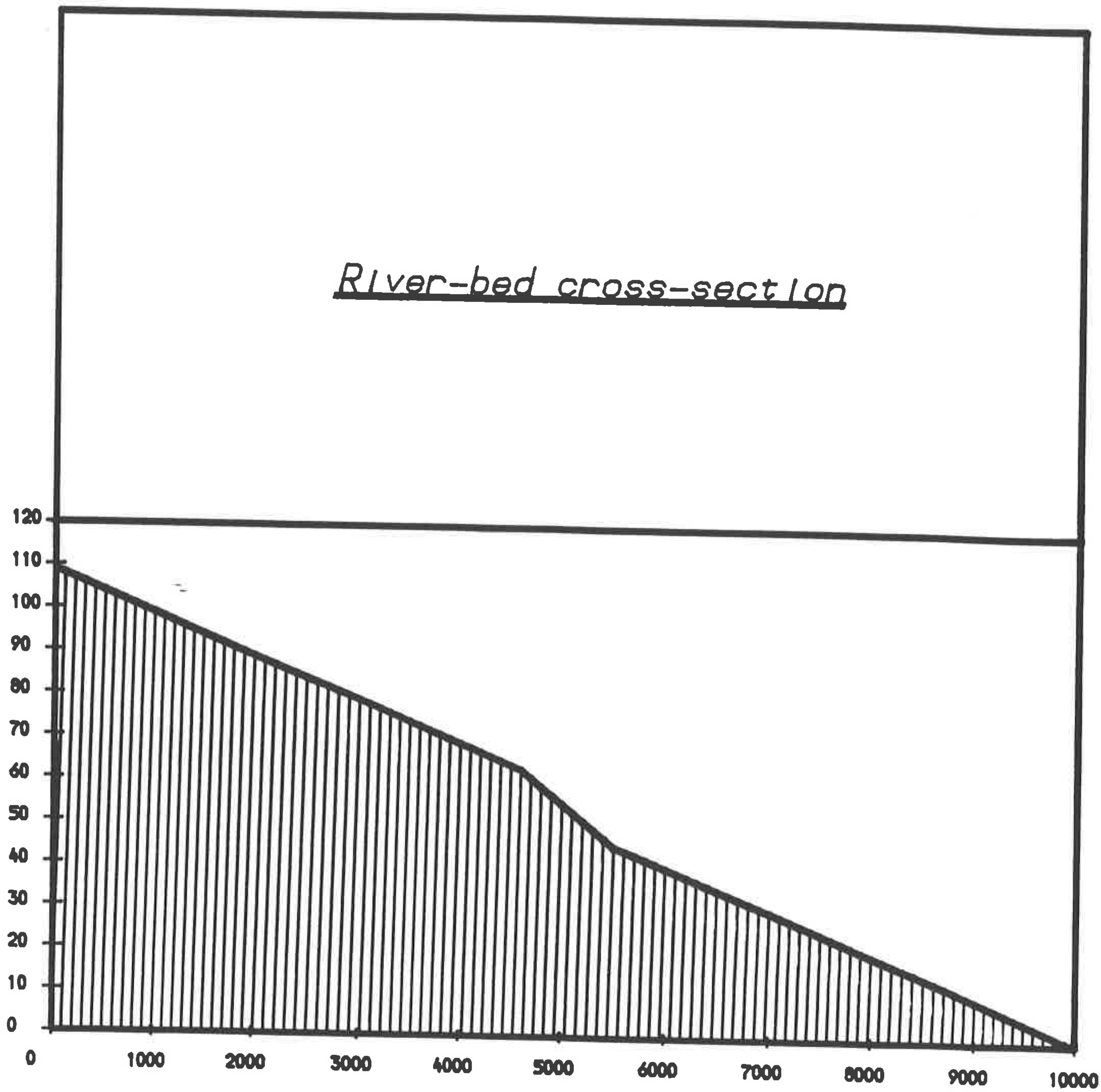

River-bed cross-section
120
110
100
90
80
70
60
50
40
30
20
10
0
0
1000
2000
3000
4000
5000
6000
7000
8000
9000
10000

Figure 2

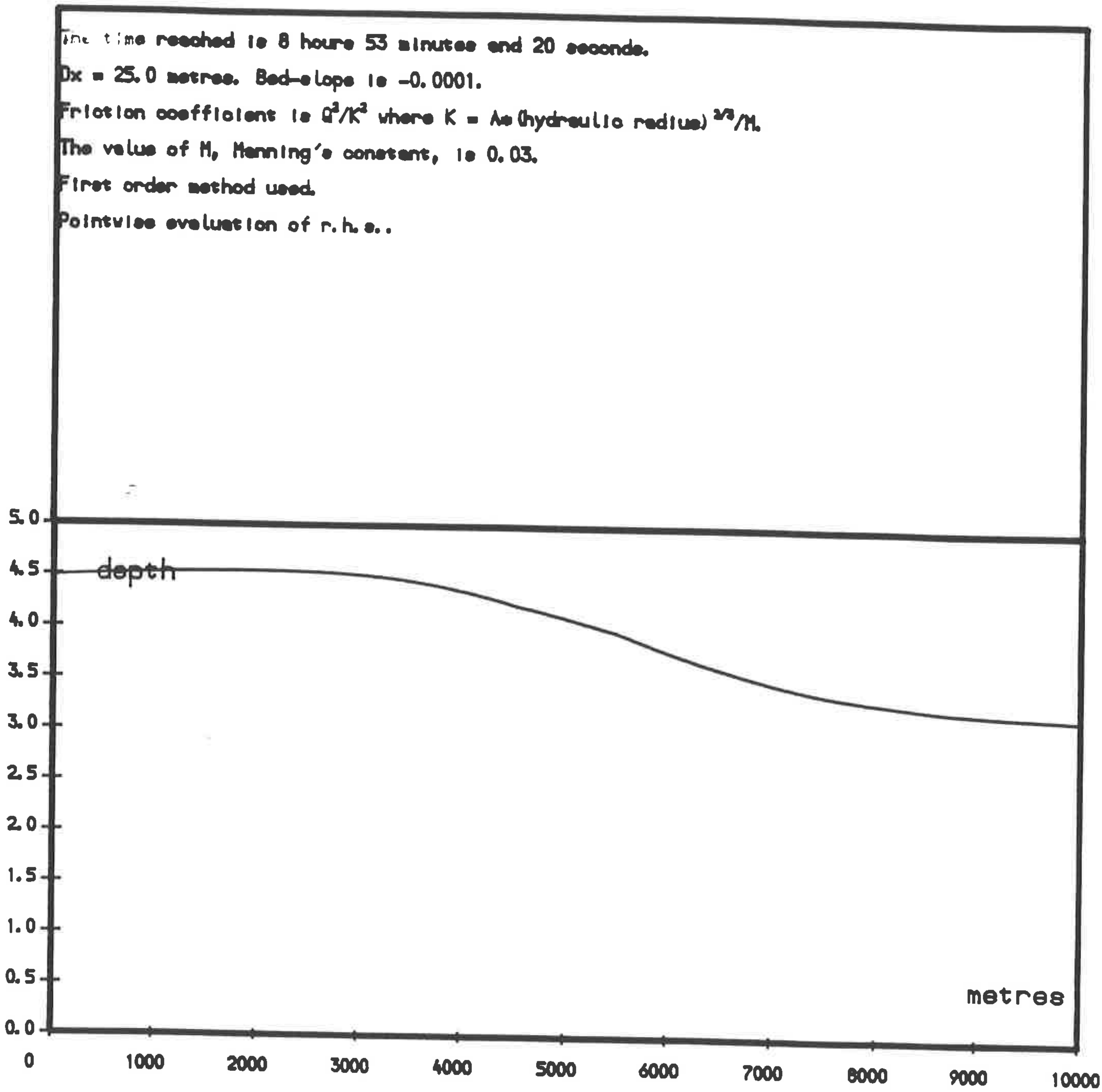
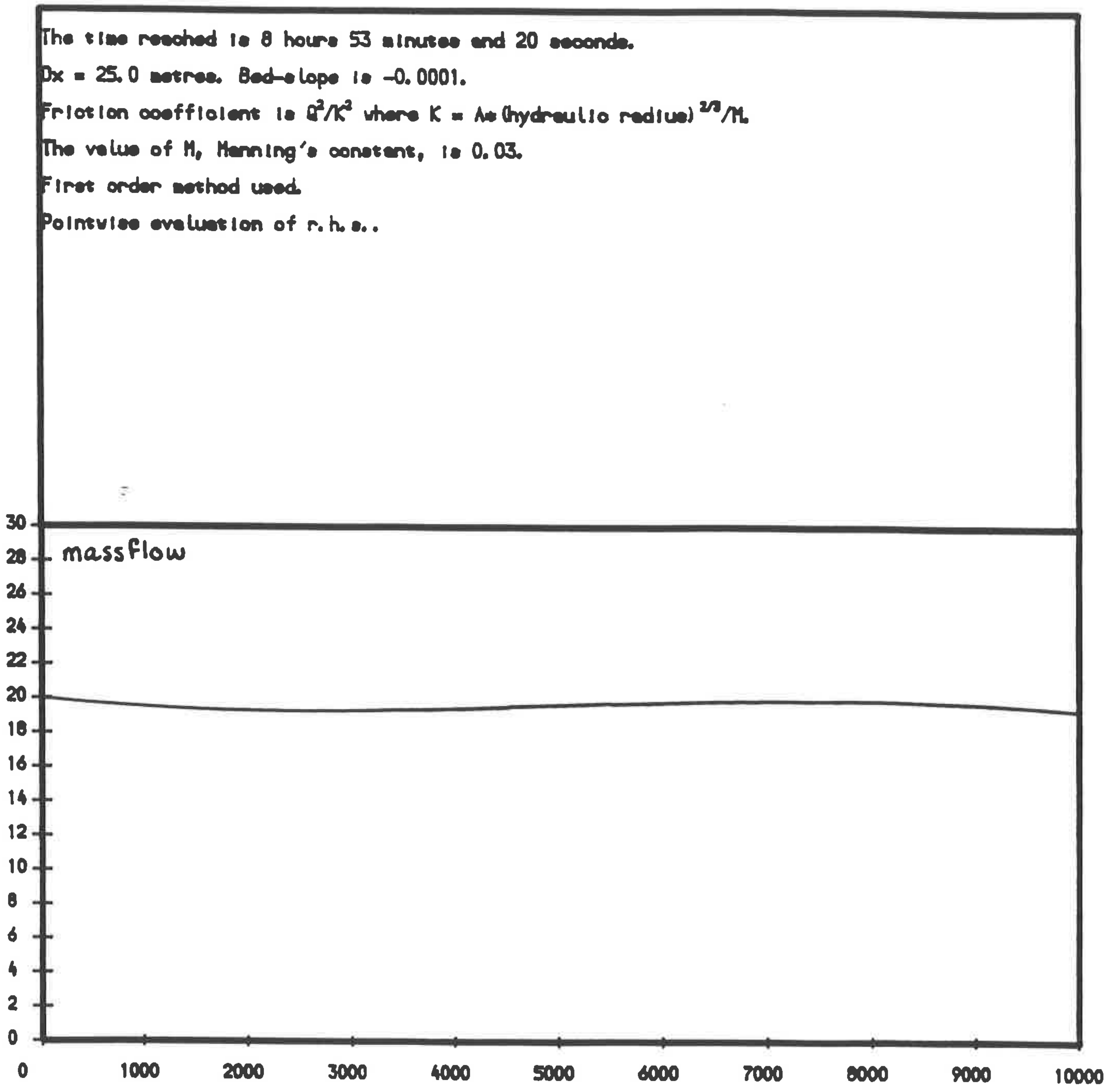

The time reached is 8 hours 53 minutes and 20 seconds.
Dx = 25.0 metres. Bed-slope is -0.0001.
Friction coefficient is Q²/K² where K = A*(hydraulic radius)^2/3/M.
The value of M, Manning's constant, is 0.03.
First order method used.
Pointwise evaluation of r.h.s..
depth
metres
5.0
4.5
4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
0
1000
2000
3000
4000
5000
6000
7000
8000
9000
10000

Figure 3



The time reached is 8 hours 53 minutes and 20 seconds.
Dx = 25.0 metres. Bed-slope is -0.0001.
Friction coefficient is Q²/K² where K = A*(hydraulic radius)^2/3/M.
The value of M, Manning's constant, is 0.03.
First order method used.
Pointwise evaluation of r.h.s..
massflow
30
28
26
24
22
20
18
16
14
12
10
8
6
4
2
0
0
1000
2000
3000
4000
5000
6000
7000
8000
9000
10000

Figure 4

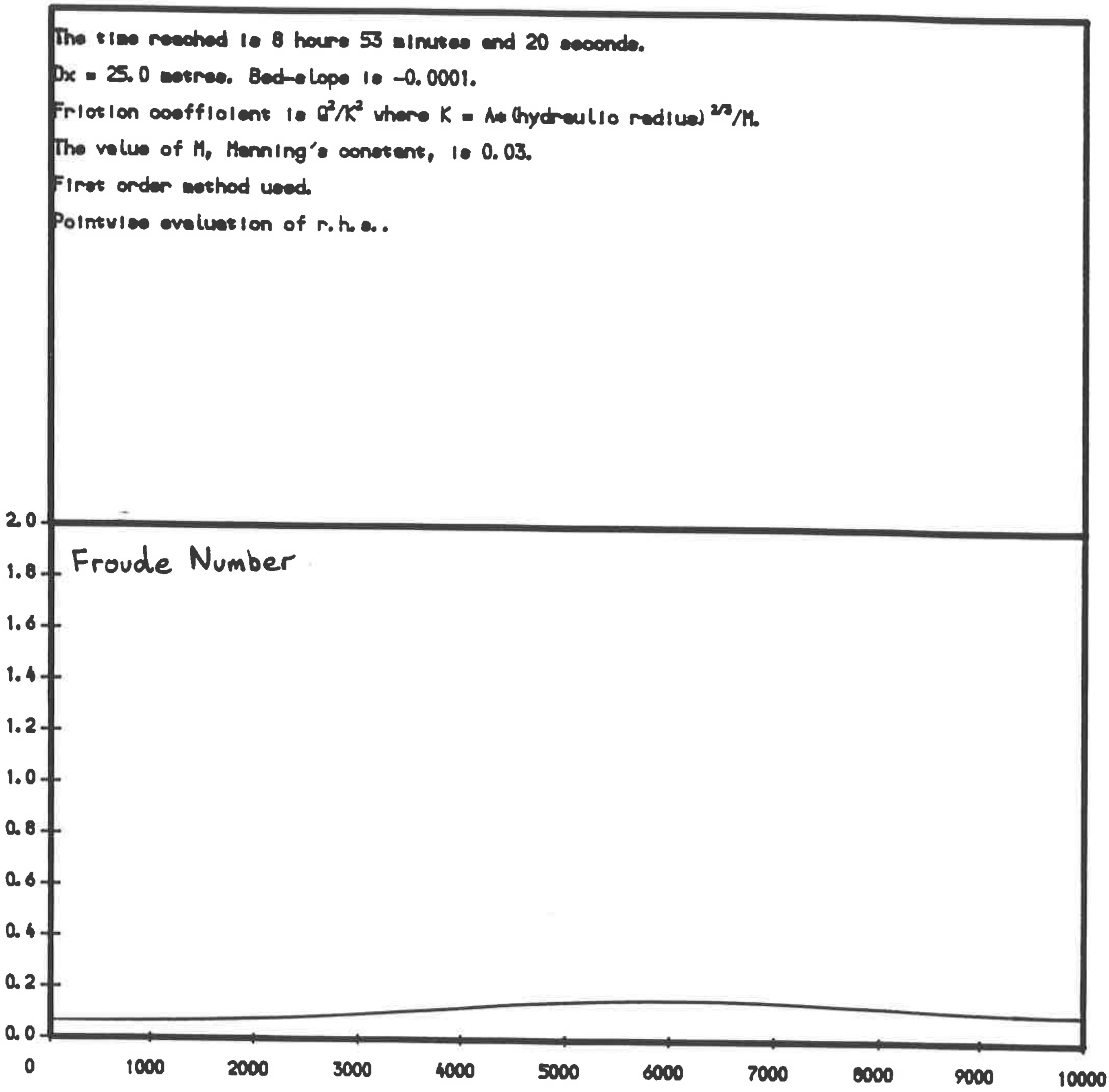

The time reached is 8 hours 53 minutes and 20 seconds.
Dx = 25.0 metres. Bed-slope is -0.0001.
Friction coefficient is Q^2/K^2 where K = A*(hydraulic radius)$^{2/3}$/M.
The value of M, Manning's constant, is 0.03.
First order method used.
Pointwise evaluation of r.h.s..
Froude Number
2.0
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
0
1000
2000
3000
4000
5000
6000
7000
8000
9000
10000

Figure 5

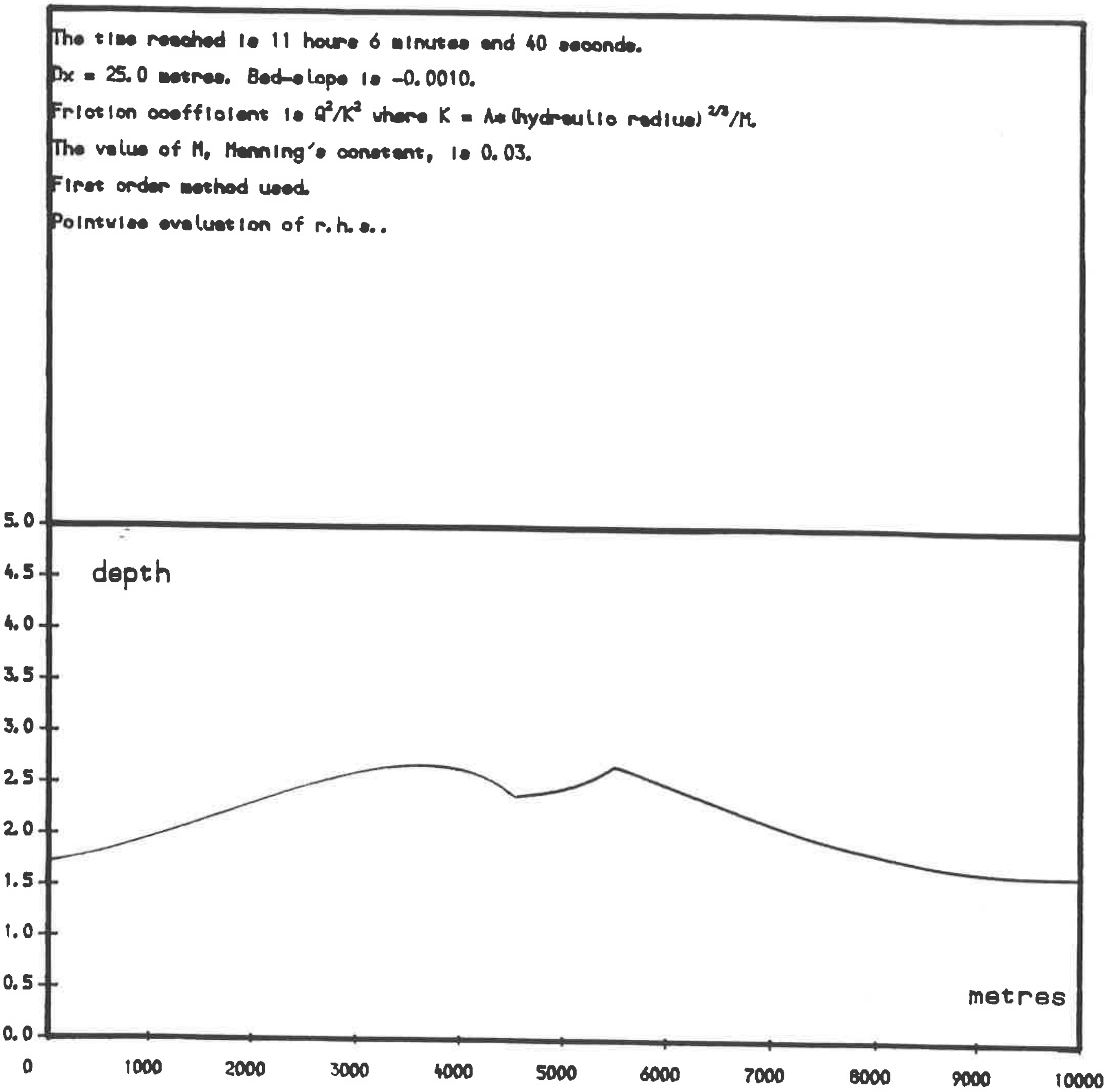

The time reached is 11 hours 6 minutes and 40 seconds.
Dx = 25.0 metres. Bed-slope is -0.0010.
Friction coefficient is Q²/K² where K = A*(hydraulic radius)^2/3/M.
The value of M, Manning's constant, is 0.03.
First order method used.
Pointwise evaluation of r.h.s..
depth
5.0
4.5
4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
metres
0
1000
2000
3000
4000
5000
6000
7000
8000
9000
10000

Figure 6

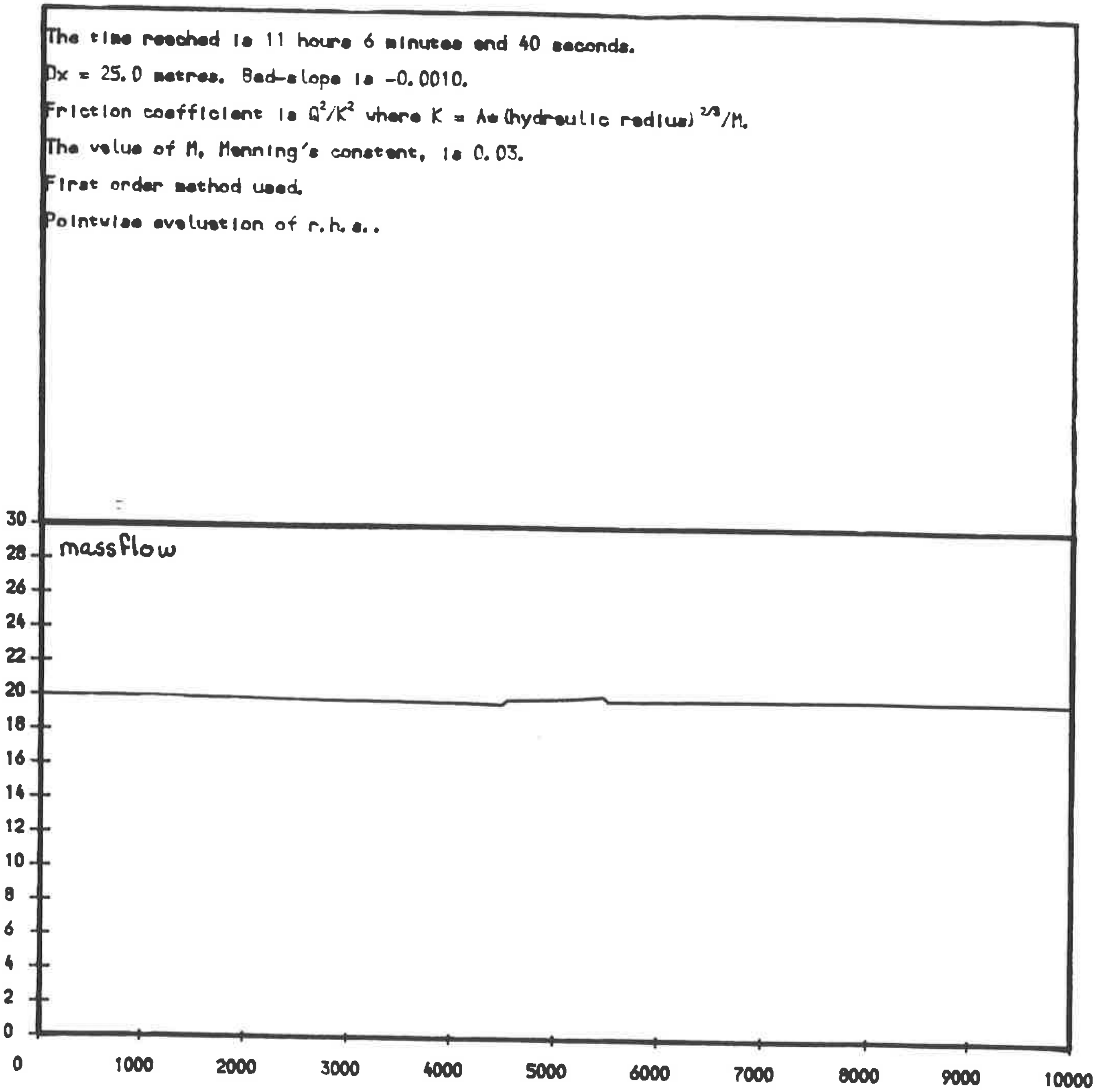

The time reached is 11 hours 6 minutes and 40 seconds.
Dx = 25.0 metres. Bed-slope is -0.0010.
Friction coefficient is Q²/K² where K = A×(hydraulic radius)^2/3/M.
The value of M, Manning's constant, is 0.03.
First order method used.
Pointwise evaluation of r.h.s..
massflow
30
28
26
24
22
20
18
16
14
12
10
8
6
4
2
0
0
1000
2000
3000
4000
5000
6000
7000
8000
9000
10000

Figure 7

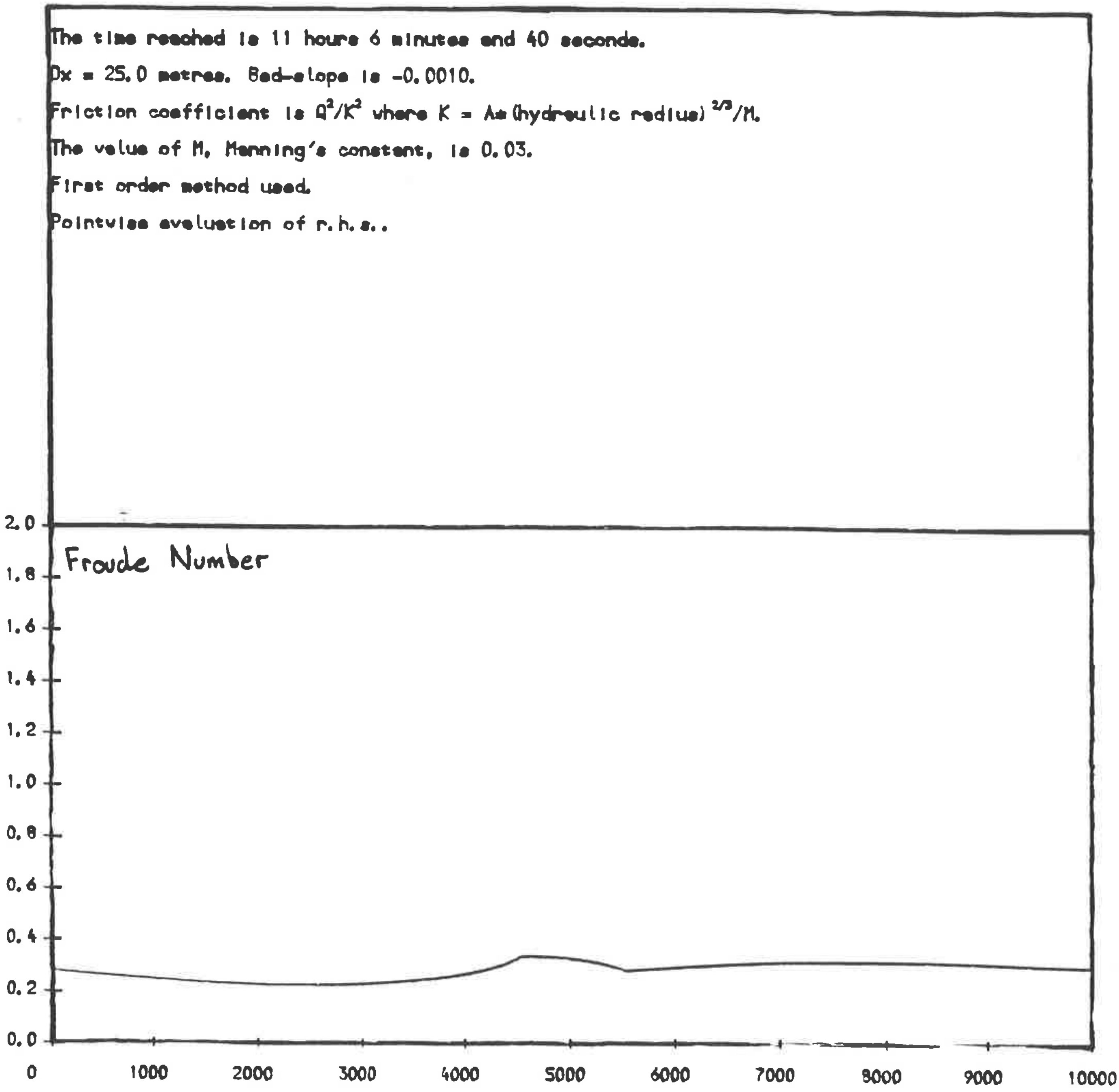

The time reached is 11 hours 6 minutes and 40 seconds.
Dx = 25.0 metres. Bed-slope is -0.0010.
Friction coefficient is Q²/K² where K = A*(hydraulic radius)^2/3/M.
The value of M, Manning's constant, is 0.03.
First order method used.
Pointwise evaluation of r.h.s..
Froude Number
2.0
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
0
1000
2000
3000
4000
5000
6000
7000
8000
9000
10000

Figure 8

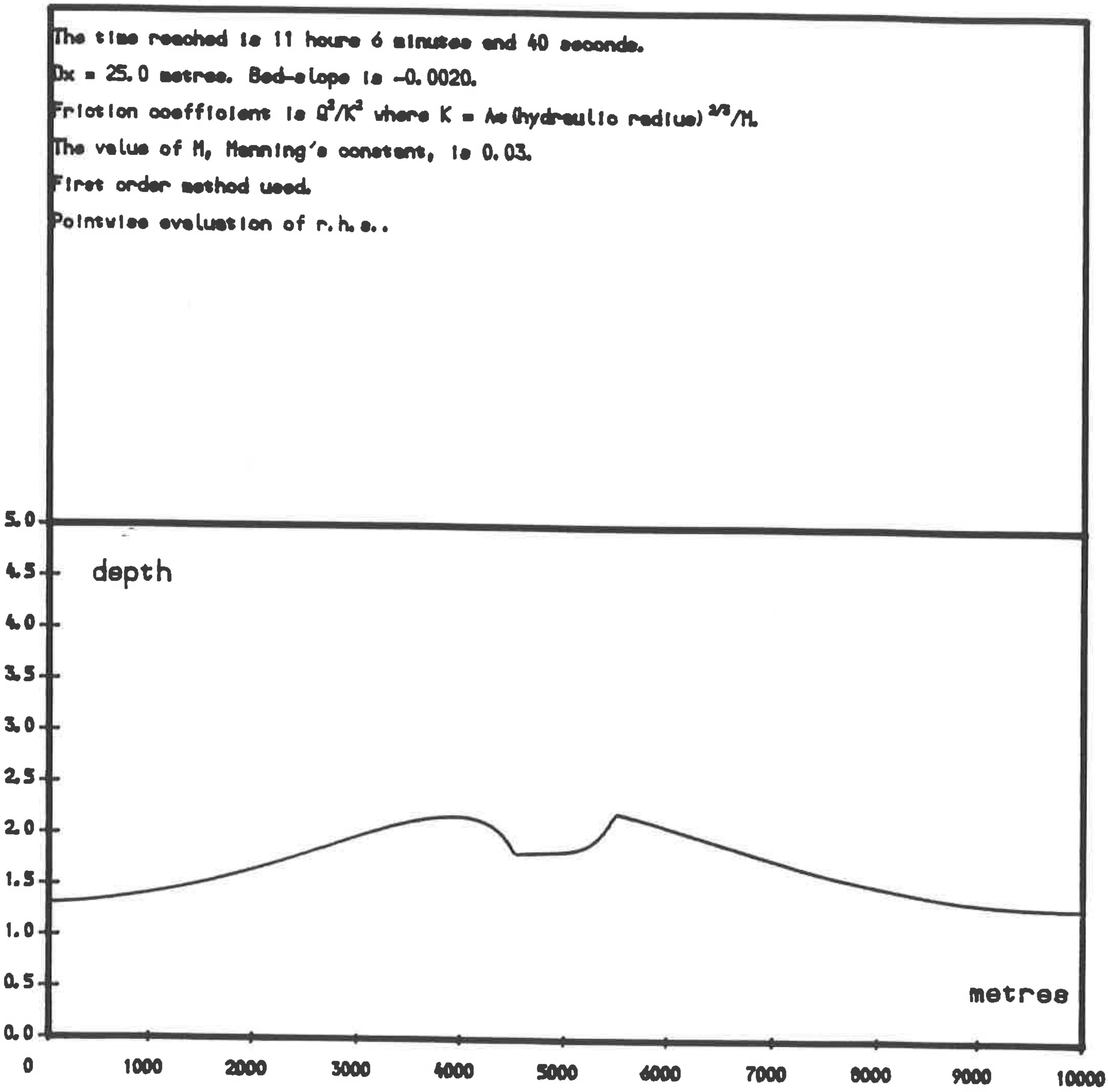
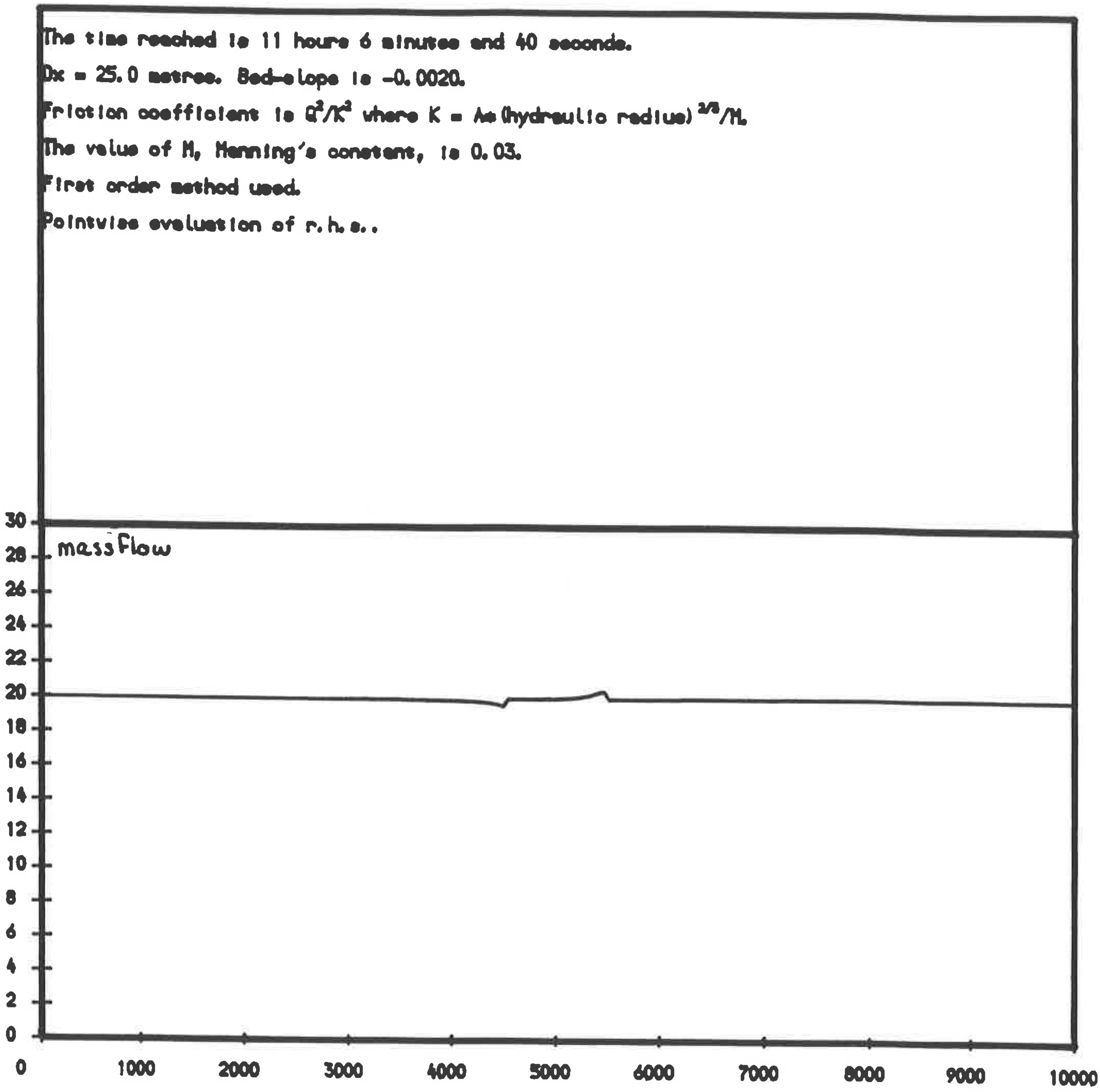

The time reached is 11 hours 6 minutes and 40 seconds.
Dx = 25.0 metres. Bed-slope is -0.0020.
Friction coefficient is Q²/K² where K = A*(hydraulic radius)^2/3/M.
The value of M, Manning's constant, is 0.03.
First order method used.
Pointwise evaluation of r.h.s..
depth
5.0
4.5
4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
metres
0
1000
2000
3000
4000
5000
6000
7000
8000
9000
10000

Figure 9


The time reached is 11 hours 6 minutes and 40 seconds.
Dx = 25.0 metres. Bed-slope is -0.0020.
Friction coefficient is Q^2/K^2 where K = A*(hydraulic radius)$^{2/3}$/N.
The value of N, Manning's constant, is 0.03.
First order method used.
Pointwise evaluation of r.h.s..
massflow
30
28
26
24
22
20
18
16
14
12
10
8
6
4
2
0
0
1000
2000
3000
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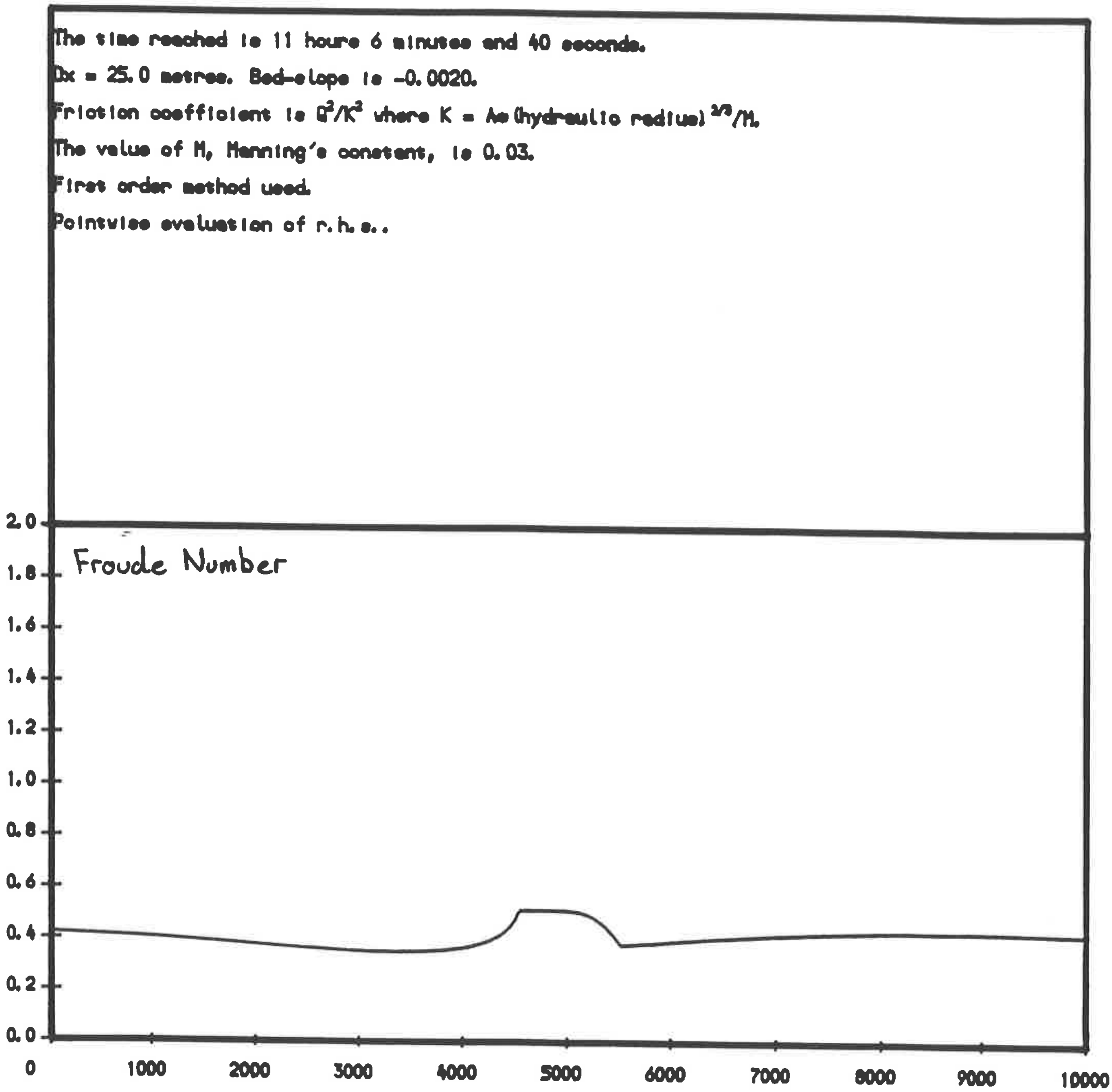
5000
6000
7000
8000
9000
10000

Figure 10


The time reached is 11 hours 6 minutes and 40 seconds.
Dx = 25.0 metres. Bed-slope is -0.0020.
Friction coefficient is Q^2/K^2 where K = A*(hydraulic radius)$^{2/3}$/M.
The value of M, Manning's constant, is 0.03.
First order method used.
Pointwise evaluation of r.h.s..
Froude Number
2.0
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
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1000
2000
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5000
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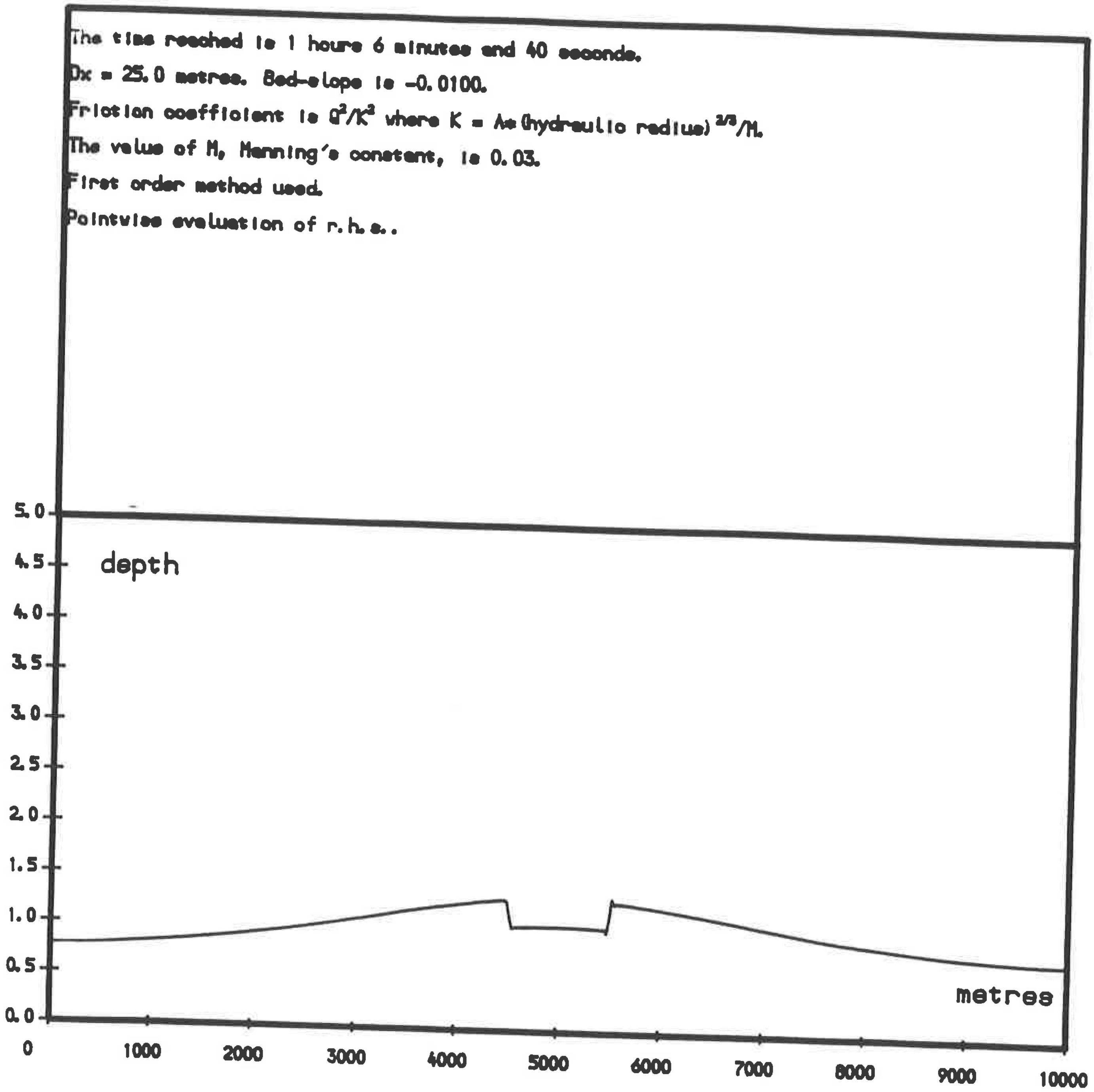
7000
8000
9000
10000

Figure 11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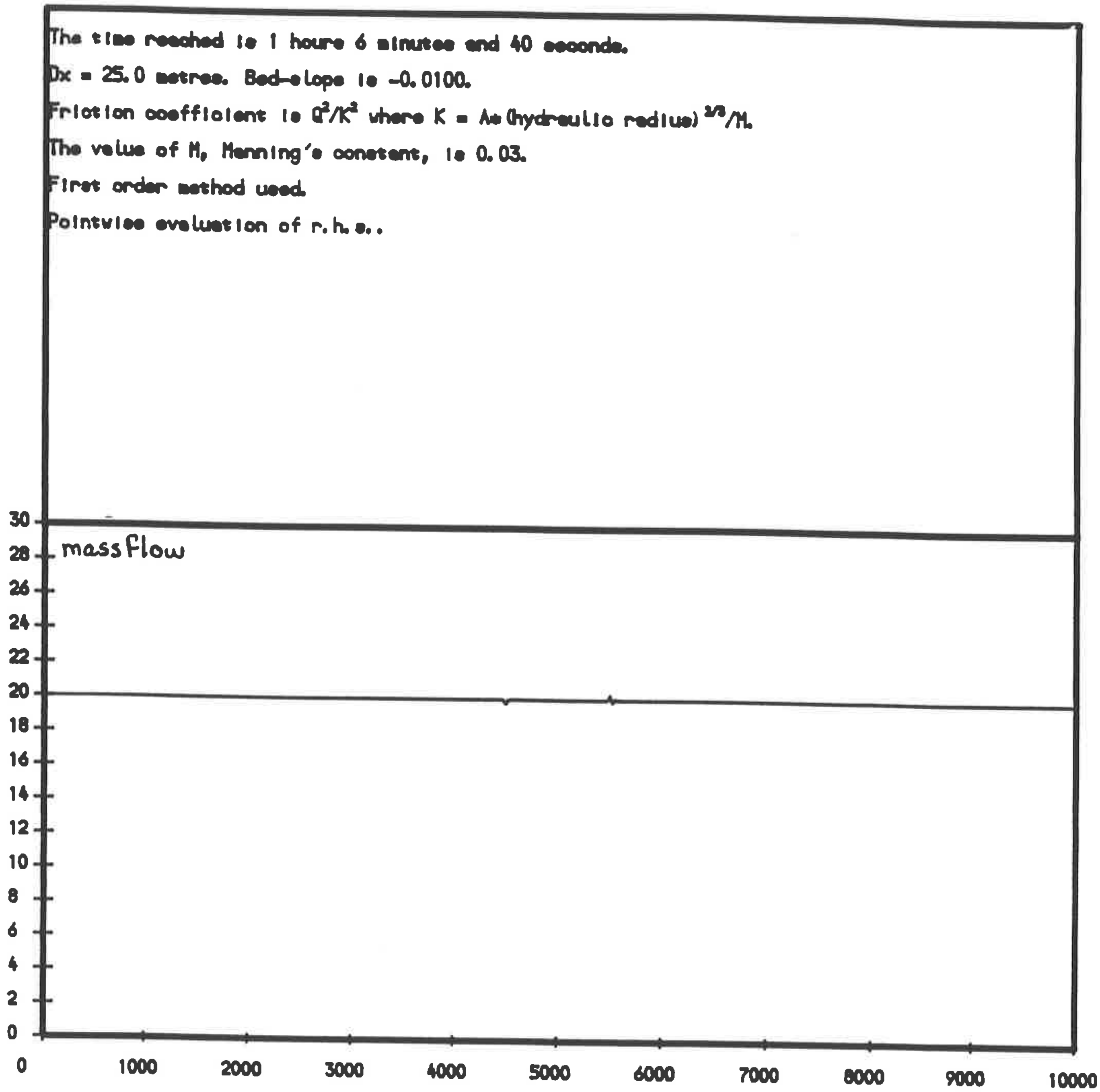
Dx = 25.0 metres. Bed-slope is -0.0100.
Friction coefficient is Q²/K² where K = A*(hydraulic radius) 2/3/M.
The value of M, Manning's constant, is 0.03.
First order method used.
Pointwise evaluation of r.h.s..
depth
metres
5.0
4.5
4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
0
1000
2000
3000
4000
5000
6000
7000
8000
9000
10000

Figure 12


The time reached is 1 hours 6 minutes and 40 seconds.
Dx = 25.0 metres. Bed-slope is -0.0100.
Friction coefficient is Q²/K² where K = A*(hydraulic radius)^2/3/M.
The value of M, Manning's constant, is 0.03.
First order method used.
Pointwise evaluation of r.h.s..
mass flow
30
28
26
24
22
20
18
16
14
12
10
8
6
4
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1000
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5000
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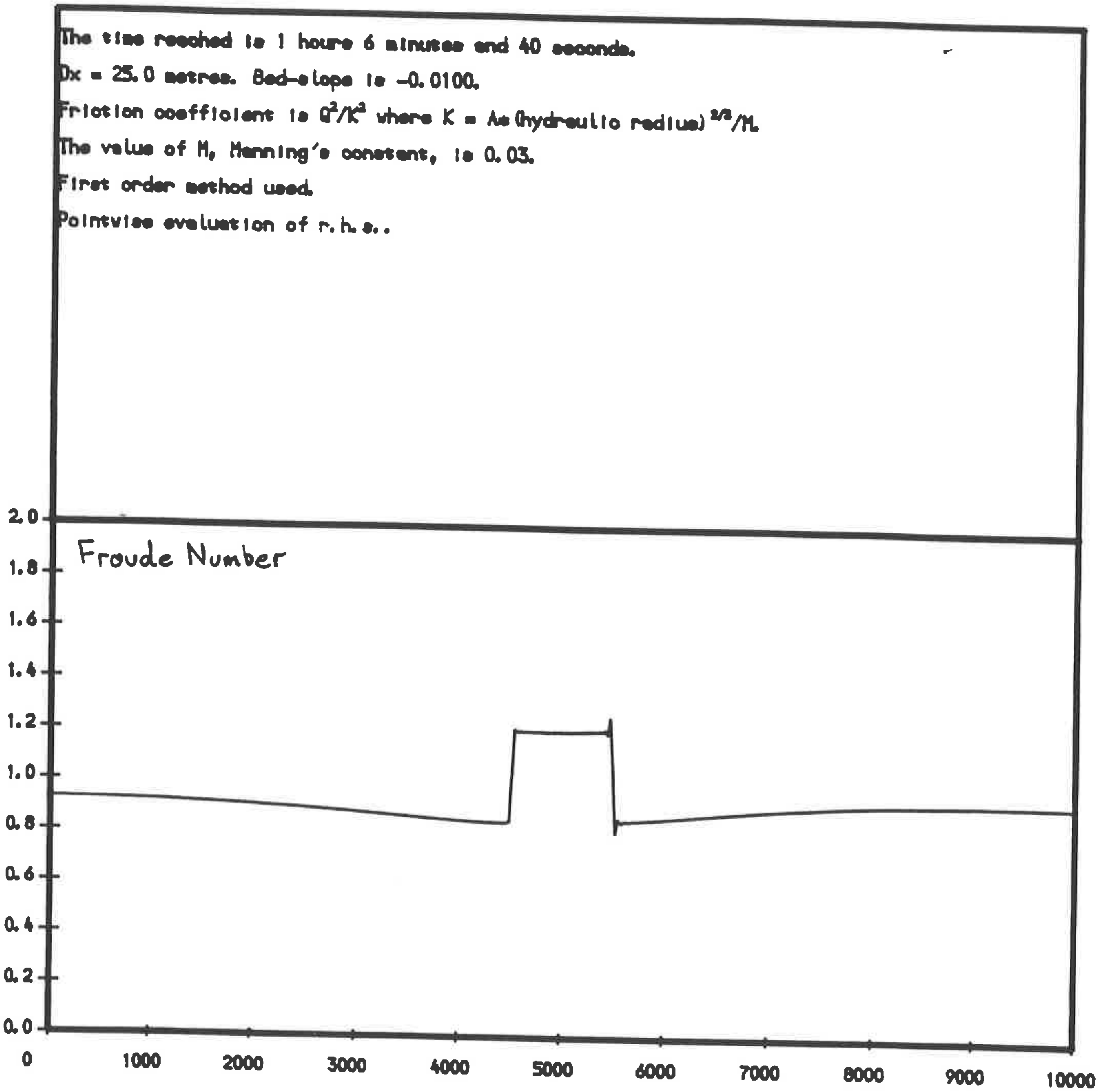
7000
8000
9000
10000

Figure 13



The time reached is 1 hours 6 minutes and 40 seconds.
Dx = 25.0 metres. Bed-slope is -0.0100.
Friction coefficient is Q^2/K^2 where K = A*(hydraulic radius)$^{2/3}$/M.
The value of M, Manning's constant, is 0.03.
First order method used.
Pointwise evaluation of r.h.s..
Froude Number
2.0
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
0
1000
2000
3000
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7000
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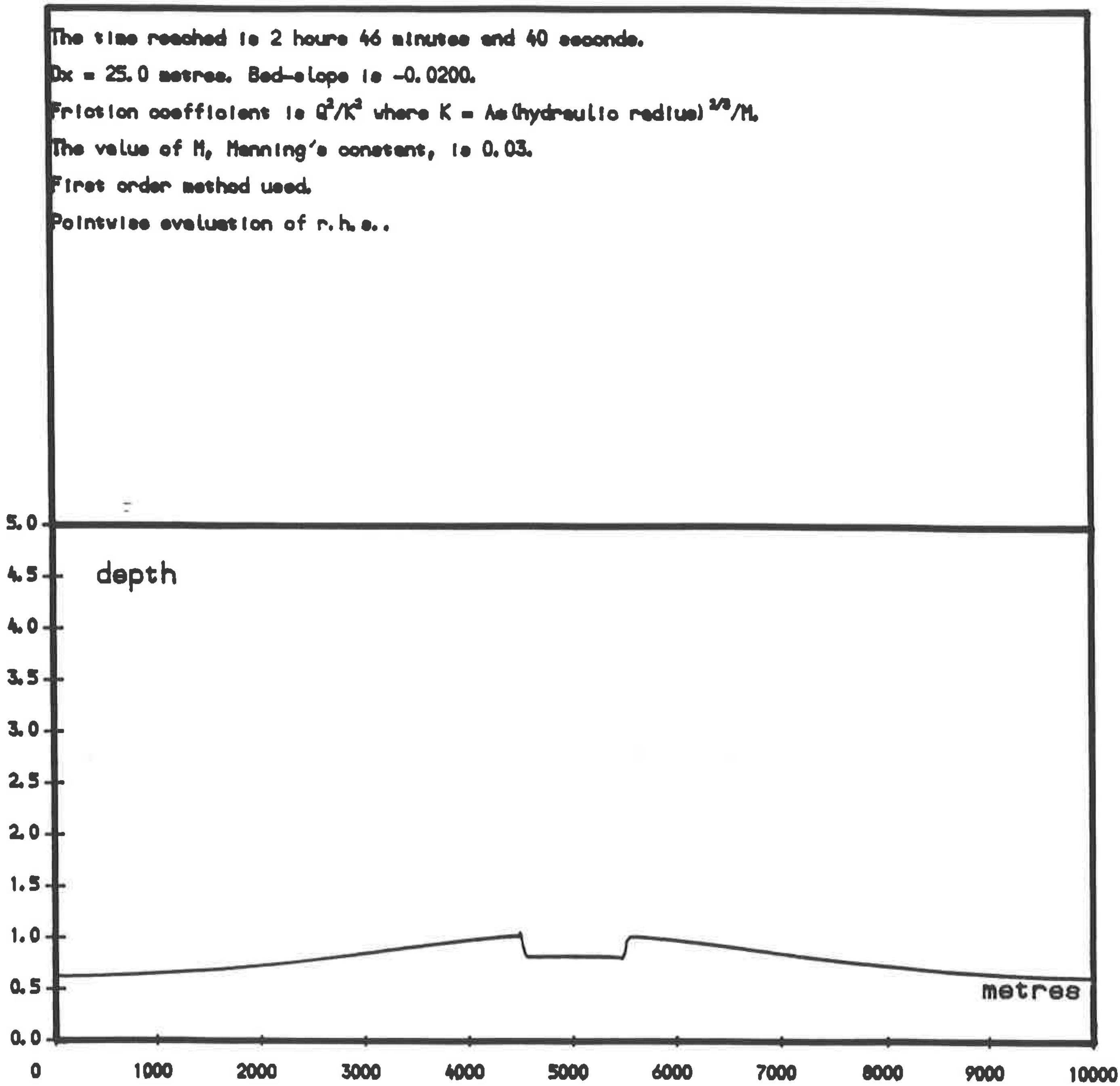
9000
10000

Figure 14



The time reached is 2 hours 46 minutes and 40 seconds.
Dx = 25.0 metres. Bed-slope is -0.0200.
Friction coefficient is Q^2/K^2 where $K = A*(\text{hydraulic radius})^{2/3}/M$.
The value of M, Manning's constant, is 0.03.
First order method used.
Pointwise evaluation of r.h.s..
depth
metres
5.0
4.5
4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
0
1000
2000
3000
4000
5000
6000
7000
8000
9000
10000

Figure 15

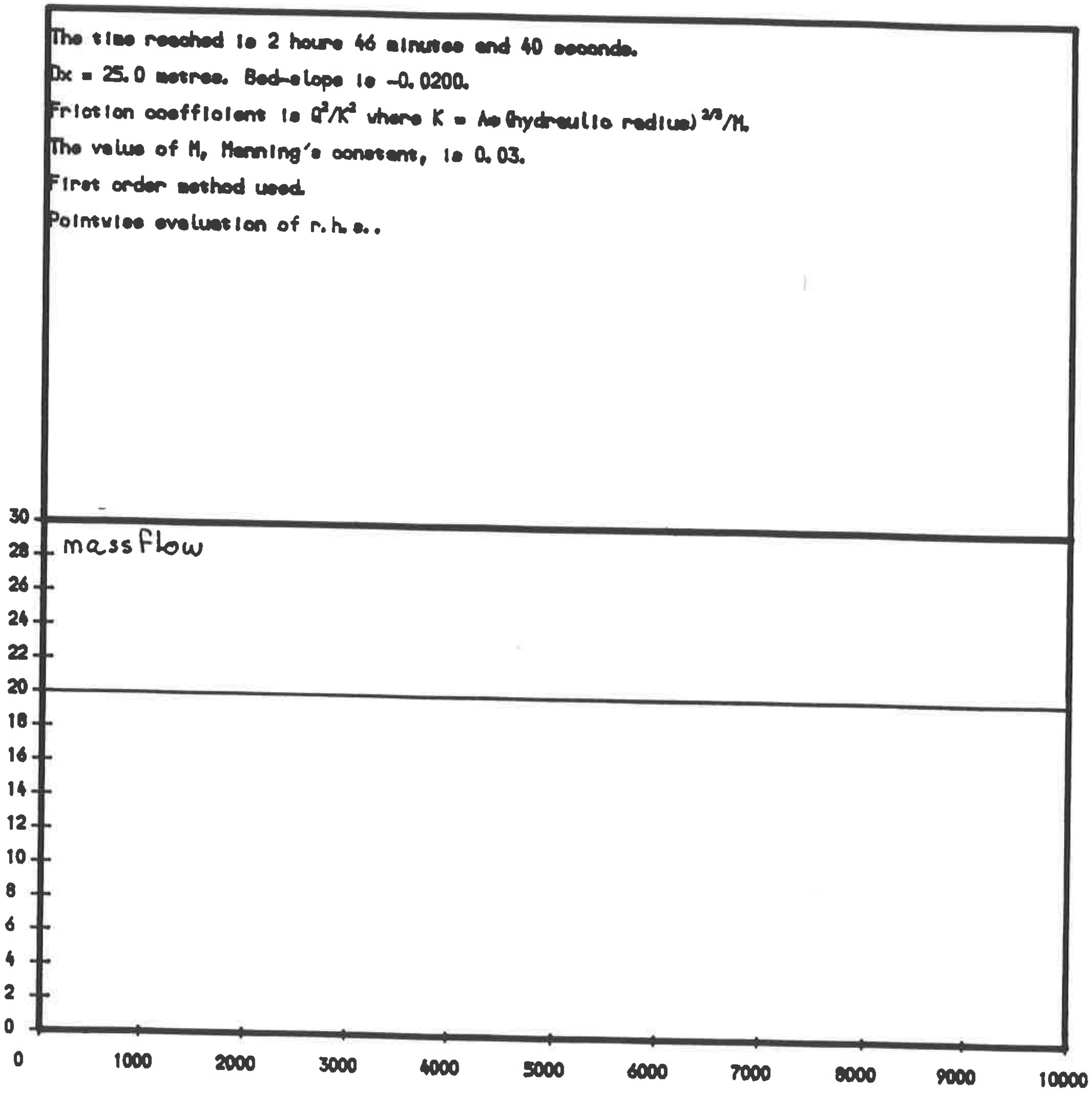

The time reached is 2 hours 46 minutes and 40 seconds.
Dx = 25.0 metres. Bed-slope is -0.0200.
Friction coefficient is Q^2/K^2 where K = A×(hydraulic radius)$^{2/3}$/M.
The value of M, Manning's constant, is 0.03.
First order method used.
Pointwise evaluation of r.h.s..
mass flow
30
28
26
24
22
20
18
16
14
12
10
8
6
4
2
0
0
1000
2000
3000
4000
5000
6000
7000
8000
9000
10000

Figure 16

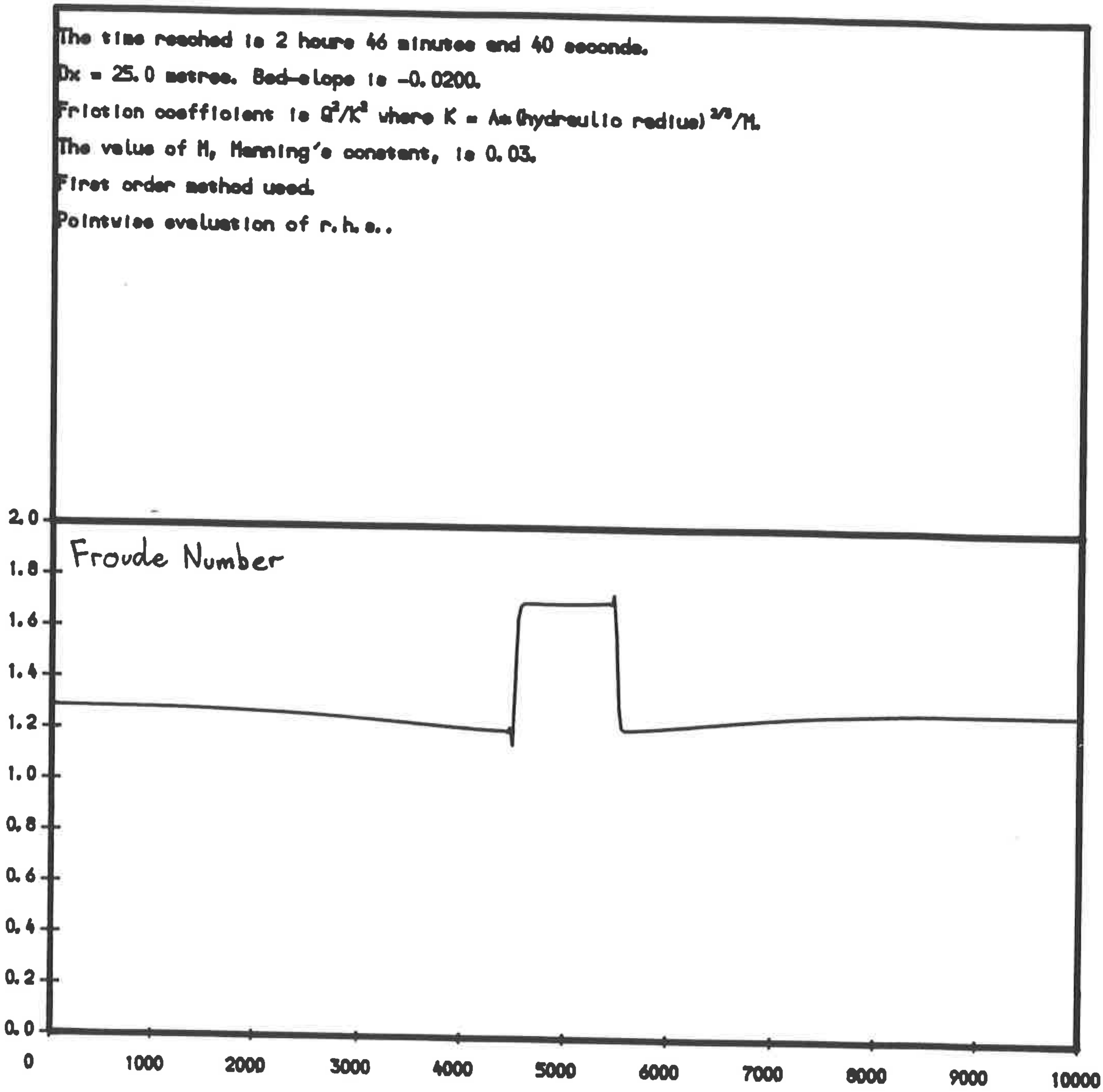

The time reached is 2 hours 46 minutes and 40 seconds.
Dx = 25.0 metres. Bed-slope is -0.0200.
Friction coefficient is Q^2/K^2 where K = A×(hydraulic radius)$^{2/3}$/M.
The value of M, Manning's constant, is 0.03.
First order method used.
Pointwise evaluation of r.h.s..
Froude Number
2.0
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
0
1000
2000
3000
4000
5000
6000
7000
8000
9000
10000

Figure 17