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Numerical Analysis Report ~~11~~ 13/95

DEPARTMENT OF MATHEMATICS

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Abstract

Variational principles have been derived whose corresponding functionals are stationary for solutions of the shallow water equations of motion. In this paper a method is described which uses these variational principles to generate approximations to shallow water flows in channels of various breadth and fixed bed profiles. The approximations are derived as those functions in a finite dimensional subspace for which the functionals of the variational principles are stationary with respect to variations in that space. Approximations to both continuous and discontinuous flows are generated.

1 Introduction

The problem of fluid flow over uneven topography and through constricting channels has been of interest to hydraulic engineers and meteorologists for many years. The equations governing those flows are non-linear, as are the boundary conditions, and the position of the free surface is an unknown of the problem. Variational principles for such flows have previously been used by, for example, Ikegawa and Washizu (1973), to model flow over a spillway crest, and by Aitchison (1979), to study flow over a weir. In both of these cases finite element approximations to free surface flows are derived and the main issue is the calculation of the position of the free surface. By making the shallow water approximation, in which the motion of a vertical column of fluid is replaced by a representative motion in the horizontal spatial coordinates, the problem may be reduced to a two-dimensional problem in a domain which does not vary with the position of the free surface. Functionals which are stationary for the solutions of the shallow water equations have recently been derived in Wakelin (1993). In the present paper we describe a numerical algorithm in which variational principles based on those functionals are used to obtain approximations to the motion of an homogeneous, incompressible, inviscid fluid over a fixed bed profile, modelled by the shallow water equations.

The natural conditions of a variational principle are those which make the corresponding functional stationary. Luke (1967) showed that a variational principle in which the integrand (the Lagrangian density) is taken to be the fluid pressure, as given by Bernoulli's energy integral, has as its natural conditions the equations governing a free surface flow. There is a point of contact between incompressible free surface flows and compressible gas flows if the shallow water approximation is invoked (Stoker, 1957). This relationship is discussed more fully in Wakelin (1993). Bateman (1929) gave functionals which are stationary for solutions of the equations of motion for compressible gas flows. These functionals have been used by, for example, Lush and Cherry (1956) and Wixcey (1990) to obtain approximations to gas flows.

In this paper, shallow water theory at its lowest order is considered, which is the basic theory used in hydrostatics to model flows in open channels. The domain in which approximations are sought is here a channel of slowly varying breadth, so that, to a first approximation, the flow can be thought of as being quasi one-dimensional. In Wakelin (1993) the variational principle of Luke (1967) was used to derive a set of four functionals which are stationary for the solutions of the shallow water equations, each functional depending on a different subset of the variables of shallow water theory. In Section 2 below variational principles for time-independent quasi one-dimensional flow are presented. Both continuous and discontinuous flows are considered.

The Ritz method can be used to obtain approximations to the extrema of a variational principle by expanding the variables in terms of trial functions and using the variational principle to generate the parameters of the expansions. A finite element approach is implemented by dividing the region into elements and choosing the trial functions to be piecewise polynomial in each element and zero elsewhere. Here we shall use piecewise linear and piecewise constant trial functions. The method is to seek those functions in the finite dimensional subspace spanned by the set of trial functions for which the functional corresponding to a particular variational principle is stationary with respect to variations in that space. The parameters of the expansions are found by solving one or more sets of equations, at least one set of which is non-linear.

In Section 3 a constrained variational principle is used to generate approximations to the depth of flow in a channel for several different channel breadth functions and fixed bed profiles. In Section 4 is derived an error bound for the piecewise constant approximations to the depth, generated in Section 3. The unconstrained version of the variational principle is used in Section 5 to give approximations to the depth, mass flow and velocity potential functions. However, in this version, the approximations to the velocity potential (and therefore to the velocity) are generated using the approximations to the depth and mass flow already computed, which reduces their accuracy. In Section 6 a constrained version of a different variational principle, which depends on the velocity potential alone, is used to give more accurate approximations to the velocity potential.

In Section 7 approximations to discontinuous depth profiles are generated. The algorithm is based on forming separate approximations to the continuous parts of the depth function and coupling the approximations with a discontinuity by using the jump conditions. As a result of this process an approximation to the position of the discontinuity is also found.

2 The variational principles

Consider a channel which extends over the interval $[x_e, x_o]$ of the x-axis. Let $B(x)$ be the breadth of the channel, defined at each point x in $[x_e, x_o]$, and let $h(x)$ be the depth of undisturbed fluid, also defined on the interval $[x_e, x_o]$. Assume that the channel is of rectangular cross-section and that it is symmetric about the x-axis. Then, provided that B and h are slowly varying functions of x, the flow is quasi one-dimensional in the x direction, to a first approximation. The motion is also assumed to be time-independent.

Let $d = d(x)$ be the depth of fluid and let $v = v(x)$ be the velocity. Then the mass flow $Q = Q(x)$ is defined by

$$Q = dv \tag{2.1}$$

and the function $E = E(x)$, defined by

$$E = gd + \frac{1}{2}v^2, \tag{2.2}$$

is an energy per unit mass, referred to as the energy. Let $\phi = \phi(x)$ be the velocity potential.

The two variational principles for quasi one-dimensional shallow water flow, taken from Wakelin (1993), which are used in this paper, are given by

$$\delta I_1(v,Q,\phi) = \delta\left\{\int_{x_e}^{x_o} (p(v,E)+Q(v-\phi'))\,B\,dx + CB_e\,(\phi(x_o)-\phi(x_e))\right\} = 0, \tag{2.3}$$

$$\delta I_2(d,Q,\phi) = \delta\left\{\int_{x_e}^{x_o} (r(Q,d)+Ed-Q\phi')\,B\,dx + CB_e\,(\phi(x_o)-\phi(x_e))\right\} = 0, \tag{2.4}$$

where

$$p(v,E) = \frac{1}{2g}\left(E-\frac{1}{2}v^2\right)^2, \tag{2.5}$$

$$r(Q,d) = \frac{1}{2}\left(\frac{Q^2}{d}-gd^2\right), \tag{2.6}$$

C is a constant, to be defined, $B_e = B(x_e)$ and g is the acceleration due to gravity. The function $E(x)$ satisfies

$$E(x) = \tilde{E} + gh(x), \tag{2.7}$$

where $\tilde{E}$ is a given constant. The variational principle (2.3) is known as the 'p' principle and (2.4) as the 'r' principle.

The natural conditions of the 'p' principle (2.3) are given by

$$p_v + Q = 0\,,\ v-\phi' = 0\,,\ (BQ)' = 0 \text{ in } (x_e,x_o), \tag{2.8}$$

$$CB_e - QB = 0 \text{ at } x = x_e \text{ and } x = x_o. \tag{2.9}$$

Equation $(2.8)_1$ defines the mass flow Q in terms of v and E, while $(2.8)_2$ is the one-dimensional version of the equation that, in two-dimensional flow, is the irrotationality condition and $(2.8)_3$ is the equation of conservation of mass. The boundary conditions (2.9) are consistent with $(2.8)_3$, the constant C having the value of the mass flow at the channel inlet. The conservation of momentum equation is satisfied implicitly by forcing the energy E to satisfy (2.7).

The natural conditions of the 'r' principle (2.4) are given by

$$r_Q - \phi' = 0\,,\ r_d + E = 0\,,\ (BQ)' = 0 \text{ in } (x_e,x_o),$$

$$CB_e - QB = 0 \text{ at } x = x_e \text{ and } x = x_o,$$

which are identical to (2.8) and (2.9), if the definition (2.1) is invoked.

In practice the constants C and $\tilde{E}$ are calculated from given values of two of the three variables depth, velocity and mass flow at the inlet boundary. Given the values of two of these variables at $x = x_e$ the value of the third can be deduced from (2.1). Then $C = Q(x_e)$ and, using (2.2) and (2.7), $\tilde{E} = gd(x_e) + 1/2\,v(x_e)^2 - gh(x_e)$.

Using (2.2) to substitute for d in (2.1) gives $Q = (E - 1/2\,v^2)v/g$. By considering Q as a function of v for constant E it can be seen that Q has the maximum value $Q_* = (2E/3)^{\frac{3}{2}}/g$. Thus, for a continuous flow to exist, the constants C and $\tilde{E}$ must satisfy

$$CB_e \leq \frac{1}{g}\left(\frac{2}{3}(\tilde{E} + gh(x))\right)^{\frac{3}{2}} B(x) \text{ for } x \in [x_e, x_o], \tag{2.10}$$

using conservation of mass.

Hydraulic jumps may occur when conditions are imposed on the flow at the outlet boundary which cannot be achieved by a continuous flow. The value of the energy E is not conserved at a hydraulic jump, although values of mass flow Q and flow stress P, defined by $P = gd^2/2 + dv^2$, are conserved (Stoker, 1957).

Let $x_s \in (x_e, x_o)$ be the position of the discontinuity. Then the jump conditions may be written explicitly as

$$[Q]_{x_s} = 0\,,\; [P]_{x_s} = 0 \text{ and } [E]_{x_s} > 0, \tag{2.11}$$

where the brackets denote the jump in the value of the quantity on crossing the point x_s from the inlet side to the outlet side. The property $(2.11)_3$ is used to form variational principles for discontinuous flows.

The version of the 'r' principle (2.4) for flow containing a discontinuity, from Wakelin (1993), is

$$\begin{aligned} \delta I_3(d, Q, \phi, x_s) \;&=\; \delta\left\{\int_{x_e}^{x_s} (r(Q,d) + E_e d - Q\phi')\, B\, dx \right. \\ &\left. + \int_{x_s}^{x_o} (r(Q,d) + E_o d - Q\phi')\, B\, dx + CB_e\,(\phi(x_o) - \phi(x_e))\right\} = 0, \end{aligned} \tag{2.12}$$

where $E(x) = E_e(x) = \tilde{E}_e + gh(x)$ for $x \in (x_e, x_s)$ and $E(x) = E_o(x) = \tilde{E}_o + gh(x)$ for $x \in (x_s, x_o)$ for constants $\tilde{E}_e$ and $\tilde{E}_o$ such that $\tilde{E}_e > \tilde{E}_o$.

The natural conditions of the 'r' principle (2.12) are given by

$$r_Q - \phi' = 0\,,\; r_d + E = 0\,,\; (BQ)' = 0 \text{ in } (x_e, x_s) \text{ and } (x_s, x_o),$$

$$CB_e - QB = 0 \text{ at } x = x_e \text{ and } x = x_o,$$

$$[BQ]_{x_s} \;=\; 0, \tag{2.13}$$

$$[(r + Ed)\,B]_{x_s} \;=\; 0. \tag{2.14}$$

The last two natural conditions are obtained by assuming that the total variation in the velocity potential is continuous on crossing the point x_s, that is, $[\delta\phi + \phi'\delta x_s]_{x_s} = 0$, where $\delta\phi + \phi'\delta x_s$ is the first order term in the expansion of $(\phi + \delta\phi)|_{x_s+\delta x_s}$. By assumption $[B]_{x_s} = 0$ and so, using the definitions (2.1) and (2.2), equations (2.13) and (2.14) are equivalent to the required jump conditions $(2.11)_1$ and $(2.11)_2$.

For practical implementations it is more convenient to use variational principles which depend on just one variable. The variations of the two principles (2.3) and (2.4) for continuous flow can be constrained so that the resulting principles contain only one variable. Let the variations in (2.3) be constrained by $v = \phi'$ and let the variations in (2.4) be constrained to satisfy conservation of mass, that is, let $Q(x) = CB_e/B(x)$. Then the resulting 'p' principle depends on ϕ alone and the resulting 'r' principle depends on d alone. Similarly the 'r' principle for discontinuous flow (2.12) may be constrained, by conservation of mass, to give a principle which depends on the one flow variable d and on the position of the discontinuity x_s. This is described in the following sections.

3 Use of the constrained 'r' principle

The 'r' principle (2.4), with variations constrained to satisfy the conservation of mass equation, is now used to generate approximations to the depth of fluid for shallow water flows in a channel.

The constrained 'r' principle is given by

$$\delta J_1(d) = \delta \left\{ \int_{x_e}^{x_o} (r(Q,d) + Ed)\, B\, dx \right\} = 0, \tag{3.1}$$

where Q and E are known functions of x, namely,

$$Q(x) = \frac{CB_e}{B(x)} \text{ for } x \in [x_e, x_o], \tag{3.2}$$

from the conservation of mass constraint, and

$$E(x) = \tilde{E} + gh(x) \text{ for } x \in [x_e, x_o], \tag{3.3}$$

corresponding to conservation of momentum.

The method used here for generating approximations to d is to substitute finite element expansions for d into the functional of (3.1) and to find the parameters of the expansions for which J_1 is stationary with respect to variations in those parameters.

Let the interval $[x_e, x_o]$ be divided into $n-1$ regular intervals by the points $x_1, \ldots, x_n$ given by

$$x_i = \frac{(i-1)}{(n-1)}(x_o - x_e) + x_e \qquad i = 1, \ldots, n. \tag{3.4}$$

Let $\alpha_1(x), \ldots, \alpha_n(x)$ be finite element basis functions, defined on the grid given by (3.4), and let

$$d^h(x) = \sum_{i=1}^{n} d_i \alpha_i(x) \tag{3.5}$$

be the approximation to d, where the d_i $(i = 1, \ldots, n)$ are parameters of the solution, to be determined.

Consider the finite dimensional version of the functional in (3.1),

$$L(\mathbf{d}) = \int_{x_1}^{x_n} \left(r(Q, d^h) + Ed^h \right) B\, dx,$$

where $\mathbf{d} = (d_1, \ldots, d_n)^T$ and Q and E are given by (3.2) and (3.3).

The parameters $\mathbf{d}$ for which (3.5) is an approximation to d are taken to be those for which L is stationary with respect to variations in $\mathbf{d}$. They are found by solving the non-linear set of equations

$$F_i(\mathbf{d}) = \frac{\partial L}{\partial d_i} = \int_{x_1}^{x_n} (r_{d^h} + E)\,\alpha_i B\,dx = 0 \qquad i = 1, \ldots, n, \tag{3.6}$$

where $r_{d^h} = r_d(Q, d^h)$.

Generally there is more than one solution of the set of equations (3.6). One possible solution involves negative values of d_i and is not considered s-ince it has no physical meaning. In the case of approximations to non-critical flows there are two other solutions — one which approximates subcritical flow and one which approximates supercritical flow. In the case of flows which become critical at a point in the domain there is a further possibility, that is, an approximation to transitional flow.

In the present work (3.6) is solved using Newton's method. The Jacobian J is given by

$$J(\mathbf{d}) = \{J_{ij}\} = \left\{\frac{\partial F_i}{\partial d_j}\right\} = \left\{\frac{\partial^2 L}{\partial d_j\,\partial d_i}\right\} = \left\{\int_{x_1}^{x_n} r_{d^h d^h}\,\alpha_i \alpha_j B\,dx\right\}, \tag{3.7}$$

and is the Hessian of L.

Given an approximation $\mathbf{d}^k$ to the solution $\mathbf{d}$, Newton's method provides an updated approximation

$$\mathbf{d}^{k+1} = \mathbf{d}^k + \delta\mathbf{d}^k, \tag{3.8}$$

where

$$J(\mathbf{d}^k)\,\delta\mathbf{d}^k = -\mathbf{F}(\mathbf{d}^k). \tag{3.9}$$

This yields a sequence of approximations to $\mathbf{d}$. The process is repeated until

$$\max_i \left|\delta d_i^k\right| < \text{tolerance}. \tag{3.10}$$

Then $d_i = d_i^k$ for $i = 1, \ldots, n$ are taken to be the values of the parameters in the approximation (3.5) which make $L(\mathbf{d})$ stationary. The Jacobian J and the vector $\mathbf{F}$ are calculated using five point Gaussian quadrature, where it is assumed that the error introduced by the numerical integration is sufficiently small that the finite element solution, for the chosen tolerance in (3.10), is unaffected.

¿From (3.7) J has the form of a weighted mass matrix, where $Br_{d^h d^h}$ is the weight function. Using (2.6) it can be seen that $r_{dd} = Q^2/d^3 - g$. Thus, if the approximate solution in $[x_1, x_n]$ is subcritical throughout the Newton iteration, J is negative definite and the solution of (3.6) maximises L. Alternatively, if the approximate solution is supercritical in $[x_1, x_n]$ for all iterations, J is positive definite and the solution of (3.6) minimises L.

Thus, given values for $\tilde{E}$ and C, it is possible, using Newton's method, to generate finite element approximations to the depth of flow in a chan-nel for continuous flows which are either supercritical in the whole domain or subcritical in the whole domain. The success of the method relies on choosing the initial approximation $\mathbf{d}^0$ to $\mathbf{d}$ such that the approximations $\mathbf{d}^k$, calculated from (3.8) and (3.9), have either all subcritical components or all supercritical components. For each set of conditions, $\tilde{E}$ and C, two approximations will be generated — one corresponding to subcritical flow and the other to supercritical flow; the choice of $\mathbf{d}^0$ determines which solution is found by the algorithm.

If the flow for which an approximation is being sought includes both subcritical and supercritical motion or if an approximation at an iteration step has both subcritical and supercritical values, the Jacobian is indefinite and Newton's method will generally fail to converge to the solution.

The algorithm is implemented on the equi-spaced grid given by (3.4), with $x_e = 0$, $x_o = 10$ and $n = 21$. Two sets of basis functions are considered; the first, α_i^l for $i = 1, \ldots, n$, leads to continuous piecewise linear approximations and the second, α_i^c for $i = 1, \ldots, n-1$, gives discontinuous piecewise constant approximations. The basis functions α_i^l are chosen to be the linear functions

$$\begin{aligned}
\alpha_1^l(x) &= \begin{cases} \dfrac{x_2 - x}{x_2 - x_1} & x \in [x_1, x_2] \\ 0 & x \notin [x_1, x_2] \end{cases}, \\
\alpha_i^l(x) &= \begin{cases} \dfrac{x - x_{i-1}}{x_i - x_{i-1}} & x \in [x_{i-1}, x_i] \\ \dfrac{x_{i+1} - x}{x_{i+1} - x_i} & x \in [x_i, x_{i+1}] \\ 0 & x \notin [x_{i-1}, x_{i+1}] \end{cases} \qquad i = 2, \ldots, n-1 \quad (3.11) \\
\alpha_n^l(x) &= \begin{cases} \dfrac{x - x_{n-1}}{x_n - x_{n-1}} & x \in [x_{n-1}, x_n] \\ 0 & x \notin [x_{n-1}, x_n] \end{cases}
\end{aligned}$$

and α_i^c to be the piecewise constant functions

$$\alpha_i^c(x) = \begin{cases} 1 & x \in (x_i, x_{i+1}) \\ 0 & x \notin (x_i, x_{i+1}) \end{cases} \qquad i = 1, \ldots, n-1. \qquad (3.12)$$

For the basis functions defined by (3.11), J is tri-diagonal and (3.9) is solved quickly for $\delta\mathbf{d}^k$ using Gaussian elimination and back substitution. For the basis functions defined by (3.12), J is diagonal and (3.9) is easily solved.

The method is used to find approximations to flows in a number of different channels. Several breadth functions are considered. These are

$$B_{1,k}(x) = 6 + 4\left(1 - 2\frac{x - x_e}{x_o - x_e}\right)^k \quad \text{in } [x_e, x_o], \text{ for } k = 2, 4, 6, 8 \quad (3.13)$$

$$B_{2,\sigma}(x) = \begin{cases} 6 + 4\left(1 - \left(\frac{x - x_e}{\nu - x_e}\right)^\sigma\right) & \text{in } [x_e, \nu] \\ 6 + 4\left(1 - \left(\frac{x_o - x}{x_o - \nu}\right)^\sigma\right) & \text{in } [\nu, x_o] \end{cases}, \text{ for } \sigma = 2. \qquad (3.14)$$

Moving the reference level for potential energy from $z = 0$ to $z = -h(x_e)$ is equivalent to redefining the equilibrium depth to be $h(x) := h(x) - h(x_e)$, so that $h(x_e) = 0$, and the constant $\tilde{E}$ to be $\tilde{E} := \tilde{E} + gh(x_e)$. For convenience this is now assumed to be the case. The equilibrium depth functions considered here are

$$h_1(x) = 0 \quad \text{in } [x_e, x_o], \tag{3.15}$$

$$h_2(x) = H\frac{x - x_e}{x_o - x_e} \quad \text{in } [x_e, x_o]. \tag{3.16}$$

The energy $\tilde{E}$ is given the value 50. In order to guarantee that a continuous solution exists the value of mass flow at inlet C must satisfy

$$C \leq \frac{1}{g}\left(\frac{2\left(\tilde{E} + gh(x)\right)}{3}\right)^{\frac{3}{2}} \frac{B(x)}{B_e} \quad \text{in } [x_e, x_o],$$

from (2.10). For the case $h(x) = h_1$ this is just

$$C \leq \frac{1}{g}\left(\frac{2\tilde{E}}{3}\right)^{\frac{3}{2}} \frac{B_{\min}}{B_e},$$

where

$$B_{\min} = \min_{x\in[x_e,x_o]} B(x).$$

Thus, for the given breadth functions (3.13) and (3.14), C must have a value such that $C \leq C_*$, where

$$C_* = \frac{20}{\sqrt{3}}. \tag{3.17}$$

A value of $C = C_*$ yields flows which are critical at the point of minimum breadth. A value of $C = 10$ is used to give examples of non-critical flows.

The initial approximation $\mathbf{d}^0$ to the solution $\mathbf{d}$ determines whether the finite element solution is an approximation to subcritical or to supercritical flow. In practice subcritical approximations are obtained by specifying $d_i^0 > d_*$ for $i = 1, \ldots, n$, where d_* is the critical depth, that is, the depth of fluid at a point where $gd = v^2$. From (2.2) and (3.3) $d_* = 2(\tilde{E} + gh)/(3g)$. In this example, for $h(x) = h_1$, $d_* = \frac{100}{30} \approx 3.33$. Supercritical approximations are obtained by specifying $d_i^0 < d_*$ for $i = 1, \ldots, n$.

Let the tolerance on the Newton iteration be 10^{-3}. Consider the channel with breadth $B(x) = B_{1,6}$. Using the piecewise linear basis functions (3.11) Newton's method converges to the supercritical approximation from the initial approximation $d_i^0 = 1$ for $i = 1, \ldots, n$ in 15 iterations for critical flow and 7 iterations for non-critical flow. Subcritical approximations are obtained, using $d_i^0 = 4$ for $i = 1, \ldots, n$, in 10 iterations for critical flow and 3 iterations for non-critical flow. Figure 1*a* shows the linear finite element approximations to the depth for the critical and non-critical flows generated under these conditions. The top two lines are the approximations to the subcritical flows and the other two approximate the supercritical flows, for the two values of mass flow at inlet $C = 10$ and $C = C_*$, where C_* is defined by (3.17). Figure 1*b* shows a linear interpolation to the breadth function using the 21 grid points given by (3.4). The sides of the channel are almost parallel for part of its length at the narrowest part, so the depths of the two critical approximations are close to the critical depth value for some distance around the point $x = 5$.

Figure 2 shows corresponding results for $B(x) = B_{2,2}$ with $\nu = 7.5$.

Using the piecewise constant basis functions (3.12), in the channel with $B(x) = B_{1,6}$ and $h(x) = h_1$, the supercritical approximation, using $d_i^0 = 1$ for $i = 1, \ldots, n$, is found after 14 iterations for critical flow and 7 iterations for non-critical flow while the subcritical approximation, using $d_i^0 = 4$ for $i = 1, \ldots, n$, is found after 10 iterations for critical flow and 3 iterations for non-critical flow. Figure 3 shows these approximations.

Now consider the channel with $B(x) = B_{1,2}$ and $h(x) = h_2$ for $H = 0.2$. Figure 4*a* shows the subcritical and supercritical piecewise linear approximations for $C = 10$ and $C = 7.7$. The dashed line shows the position of the channel bed. Notice that the depth profiles are no longer symmetric about the line $x = 5$. The breadth $B_{1,2}$ is shown in Figure 4*b*. With $d_i^0 = 1$ for $i = 1, \ldots, n$ the supercritical approximations are found after 6 iterations in the $C = 10$ case and 5 iterations in the $C = 7.7$ case. Using $d_i^0 = 4$ for $i = 1, \ldots, n$ the subcritical approximations are found after 3 iterations in the $C = 10$ case and 3 iterations in the $C = 7.7$ case.

4 An error bound

Proposition The piecewise constant approximations, defined using (3.5) and (3.12) and generated from (3.6), converge linearly to the shallow water depth for wholly subcritical or wholly supercritical flows.

Proof The parameters of the approximation d^h are defined as those which satisfy (3.6), that is,

$$\int_{x_1}^{x_n} (r_{d^h} + E)\,\alpha_i^c B\,dx = 0 \qquad i = 1, \ldots, n-1, \tag{4.1}$$

where $r_{d^h} = r_d(Q, d^h)$. The exact depth d satisfies the equation

$$r_d + E = 0,$$

from the definitions of r (2.6), mass flow (2.1) and energy (2.2). Thus

$$\int_{x_1}^{x_n} (r_d + E)\,\alpha_i^c B\,dx = 0 \qquad i = 1, \ldots, n-1. \tag{4.2}$$

Subtracting (4.2) from (4.1) gives

$$\int_{x_1}^{x_n} (r_{d^h} - r_d)\,\alpha_i^c B\,dx = 0 \qquad i = 1, \ldots, n-1,$$

and so

$$\int_{x_i}^{x_{i+1}} (r_{d^h} - r_d)\,B\,dx = 0 \qquad i = 1, \ldots, n-1, \tag{4.3}$$

using (3.12).

Both d and d^h are differentiable on each interval $[x_i, x_{i+1}]$ and thus, using the Mean Value Theorem,

$$r_{d^h} - r_d = r_d(Q, d^h) - r_d(Q, d) = \left(d^h - d\right) r_{\psi\psi}(Q, \psi)\big|_{\psi=\theta}, \tag{4.4}$$

for θ between d^h and d (for each x), where $r_{\psi\psi} = Q^2/\psi^3 - g$, from (2.6). Thus if d^h and d are everywhere supercritical $r_{\psi\psi} > 0$ and if d^h and d are everywhere subcritical $r_{\psi\psi} < 0$ in $[x_i, x_{i+1}]$. Therefore, substituting (4.4) into (4.3) to give

$$\int_{x_i}^{x_{i+1}} \left(d^h - d\right) r_{\psi\psi}(Q, \psi)\big|_{\psi=\theta} B\, dx = 0,$$

implies that $d^h - d = 0$ at at least one point (say $x = \hat{x}$) in (x_i, x_{i+1}) for wholly subcritical or supercritical flows, since $B > 0$.

Now d^h is constant on $[x_i, x_{i+1}]$, so, for $x \in [x_i, x_{i+1}]$,

$$\int_{\hat{x}}^{x} d'(\sigma)\, d\sigma = \int_{\hat{x}}^{x} \left(d'(\sigma) - d^{h\prime}(\sigma)\right) d\sigma = \left[d(\sigma) - d^h(\sigma)\right]_{\hat{x}}^{x} = d(x) - d^h(x).$$

Thus

$$\begin{aligned}
\int_{x_i}^{x_{i+1}} \left(d(x) - d^h(x)\right)^2 dx &= \int_{x_i}^{x_{i+1}} \left(\int_{\hat{x}}^{x} d'(\sigma)\, d\sigma\right)^2 dx \\
&\leq \int_{x_i}^{x_{i+1}} \left((x_{i+1} - x_i) \max_{I_i} |d'|\right)^2 dx \\
&= (x_{i+1} - x_i)^3 \max_{I_i} |d'|^2,
\end{aligned}$$

where $\max_{I_i} |d'|^2 = \max_{x \in [x_i, x_{i+1}]} |d'(x)|^2$.

Therefore the square of the L_2 error is

$$\begin{aligned}
\left\|d - d^h\right\|^2 &= \int_{x_e}^{x_o} \left(d - d^h\right)^2 dx \qquad (4.5)\\
&= \sum_{i=1}^{n-1} \int_{x_i}^{x_{i+1}} \left(d - d^h\right)^2 dx \\
&\leq \sum_{i=1}^{n-1} (x_{i+1} - x_i)^3 \max_{I_i} |d'|^2 \\
&\leq \max_i (x_{i+1} - x_i)^2 \max_{x \in [x_e, x_o]} |d'|^2 \sum_{i=1}^{n-1} (x_{i+1} - x_i) \\
&= \max_i (x_{i+1} - x_i)^2 \max_{x \in [x_e, x_o]} |d'|^2 (x_o - x_e),
\end{aligned}$$

that is, for a uniform grid,

$$\left\|d - d^h\right\| \leq \Delta x \max_{x \in [x_e, x_o]} |d'| (x_o - x_e)^{\frac{1}{2}},$$

where $\Delta x = (x_e - x_o)/(n - 1)$.

n	$\frac{\Delta x}{10}$	critical flows		non-critical flows	
		subcritical	supercritical	subcritical	supercritical
3	2^{-1}	6.028×10^{-1}	8.608×10^{-1}	2.687×10^{-1}	5.097×10^{-1}
5	2^{-2}	1.870×10^{-1}	3.521×10^{-1}	1.188×10^{-1}	2.654×10^{-1}
9	2^{-3}	7.249×10^{-2}	1.550×10^{-1}	4.882×10^{-2}	1.218×10^{-1}
17	2^{-4}	3.198×10^{-2}	7.249×10^{-2}	2.192×10^{-2}	5.772×10^{-2}
33	2^{-5}	1.504×10^{-2}	3.504×10^{-2}	1.038×10^{-2}	2.805×10^{-2}
65	2^{-6}	7.293×10^{-3}	1.722×10^{-2}	5.050×10^{-3}	1.382×10^{-2}
129	2^{-7}	3.592×10^{-3}	8.539×10^{-3}	2.491×10^{-3}	6.860×10^{-3}
257	2^{-8}	1.782×10^{-3}	4.252×10^{-3}	1.237×10^{-3}	3.417×10^{-3}
513	2^{-9}	8.878×10^{-4}	2.121×10^{-3}	6.164×10^{-4}	1.705×10^{-3}
1025	2^{-10}	4.431×10^{-4}	1.060×10^{-3}	3.077×10^{-4}	8.520×10^{-4}

Table 1: L_2 errors for piecewise constant depth approximations.

n	$\frac{\Delta x}{10}$	critical flows		non-critical flows	
		subcritical	supercritical	subcritical	supercritical
3	2^{-1}	1.178×10^{-1}	9.217×10^{-2}	1.087×10^{-2}	3.975×10^{-2}
5	2^{-2}	2.087×10^{-2}	2.084×10^{-2}	6.606×10^{-3}	9.668×10^{-3}
9	2^{-3}	4.395×10^{-3}	5.122×10^{-3}	1.155×10^{-3}	1.769×10^{-3}
17	2^{-4}	9.825×10^{-4}	1.235×10^{-3}	2.842×10^{-4}	4.714×10^{-4}
33	2^{-5}	2.304×10^{-4}	3.135×10^{-4}	6.858×10^{-5}	1.249×10^{-4}
65	2^{-6}	5.595×10^{-5}	7.657×10^{-5}	1.651×10^{-5}	3.976×10^{-5}
129	2^{-7}	1.280×10^{-5}	2.234×10^{-5}	4.401×10^{-6}	1.453×10^{-5}

Table 2: L_2 errors for piecewise linear depth approximations.

Thus for wholly subcritical or wholly supercritical flow the piecewise constant depth approximation converges linearly with n to the solution d.

The L_2 error is calculated for piecewise constant approximations on grids with different numbers of nodes for the example $B(x) = B_{1,2}$, defined by (3.13), and $h(x) = h_1$, defined by (3.15). The energy $\tilde{E} = 50$ and both $C = C_*$, defined by (3.17), and $C = 10$ are considered. The results are given in Table 1, from which it can be seen, more especially for larger n, that as the interval length Δx halves the L_2 error also halves.

The L_2 errors for the corresponding piecewise linear approximations are given in Table 2. It can be seen that the convergence is almost quadratic.

5 Use of the unconstrained 'r' principle

More generally, finite element expansions for the mass flow and the velocity potential, as well as for the fluid depth, can be obtained using the unconstrained 'r' principle (2.4). The method used here is a simple extension of the algorithm in Section 3.

Consider the grid defined by the points (3.4), with $x_e = 0$, $x_o = 10$ and $n = 21$. Let

$$Q^h(x) = \sum_{i=1}^{n} Q_i\alpha_i(x)\,,\ d^h(x) = \sum_{i=1}^{n} d_i\alpha_i(x)\,,\ \phi^h(x) = \sum_{i=1}^{n} \phi_i\alpha_i(x) \qquad (5.1)$$

be approximations to the mass flow, depth and velocity potential, respectively, where the α_i ($i = 1,\ldots,n$) are finite element basis functions. Substituting (5.1) into the functional of (2.4) yields the finite dimensional version

$$L(\mathbf{Q},\mathbf{d},\boldsymbol{\phi}) = \int_{x_1}^{x_n} \left(r(Q^h,d^h) + Ed^h - \phi^{h\prime}Q^h\right) B\,dx + CB_e\left(\phi^h(x_n) - \phi^h(x_1)\right), \qquad (5.2)$$

where $\mathbf{Q} = (Q_1,\ldots,Q_n)^T$, $\mathbf{d} = (d_1,\ldots,d_n)^T$, $\boldsymbol{\phi}=(\phi_1,\ldots,\phi_n)^T$ and $E(x) = \tilde{E} + gh(x)$. The parameters $\mathbf{Q}$, $\mathbf{d}$ and $\boldsymbol{\phi}$ are calculated by solving

$$\frac{\partial L}{\partial Q_i} = 0\,,\ \frac{\partial L}{\partial d_i} = 0\,,\ \frac{\partial L}{\partial \phi_i} = 0 \quad \text{for } i = 1,\ldots,n. \qquad (5.3)$$

Let the α_i be the piecewise linear basis functions α_i^l defined by (3.11). Then equations $(5.3)_3$ yield

$$-\int_{x_1}^{x_n} \alpha_i' Q^h B\,dx + CB_e(\alpha_i(x_n) - \alpha_i(x_1)) = 0 \quad i = 1,\ldots,n,$$

which may be rewritten as

$$\begin{aligned}
\sum_{j=1}^{2} Q_j \int_{x_1}^{x_2} \alpha_1'\alpha_j B\,dx &= -CB_e, \\
\sum_{j=i-1}^{i+1} Q_j \int_{x_{i-1}}^{x_{i+1}} \alpha_i'\alpha_j B\,dx &= 0 \qquad i = 2,\ldots,n-1, \\
\sum_{j=n-1}^{n} Q_j \int_{x_{n-1}}^{x_n} \alpha_n'\alpha_j B\,dx &= CB_e,
\end{aligned}$$

or as

$$A_Q\mathbf{Q} = C_Q, \qquad (5.4)$$

where A_Q is a constant $n \times n$ matrix and C_Q is a constant $n \times 1$ vector with only first and last entries non-zero. The matrix A_Q is singular of rank $n-1$ but, using the boundary condition $Q_1 = C$, the solution of (5.4) is unique. A_Q is tri-diagonal and $\mathbf{Q}$ is calculated using Gaussian elimination and back substitution.

Equations $(5.3)_2$ yield

$$\int_{x_1}^{x_n} (r_{d^h} + E)\,\alpha_i B\,dx = 0 \quad i = 1,\ldots,n,$$

which, once Q^h is known, can be solved for d^h by the method of Section 3.

Equations $(5.3)_1$ give

$$\int_{x_1}^{x_n} \left(r_{Q^h} - \phi^{h'} \right) \alpha_i B \, dx = 0 \quad i = 1, \ldots, n,$$

which may be written as

$$\begin{aligned}
\sum_{j=1}^{2} \phi_j \int_{x_1}^{x_2} \alpha_1 \alpha_j' B \, dx &= \int_{x_1}^{x_2} r_{Q^h} \alpha_1 B \, dx, \\
\sum_{j=i-1}^{i+1} \phi_j \int_{x_{i-1}}^{x_{i+1}} \alpha_i \alpha_j' B \, dx &= \int_{x_{i-1}}^{x_{i+1}} r_{Q^h} \alpha_i B \, dx \quad i = 2, \ldots, n-1, \\
\sum_{j=n-1}^{n} \phi_j \int_{x_{n-1}}^{x_n} \alpha_n \alpha_j' B \, dx &= \int_{x_{n-1}}^{x_n} r_{Q^h} \alpha_n B \, dx,
\end{aligned}$$

or as

$$A_\phi \phi = C_\phi, \tag{5.5}$$

where A_ϕ is an $n \times n$ matrix and C_ϕ is an $n \times 1$ vector. Once Q^h and d^h are known ϕ can be calculated directly. The matrix A_ϕ is of rank $n-1$ and singular but ϕ is a potential function and the important quantity is its gradient, so one of the values, say ϕ_1, is specified arbitrarily. This procedure is equivalent to setting the arbitrary constant in ϕ by assigning its value at the boundary.

Results for critical flow in a channel with $B(x) = B_{1,4}$, defined by (3.13), and $h(x) = h_1$, defined by (3.15), are shown in Figure 5. The energy $\bar{E}$ is taken to be 50. The piecewise linear approximation to the mass flow is shown in Figure 5*a*. The piecewise linear approximations to the velocity potential and depth for a supercritical flow are given in Figures 5*b* and 5*c*, respectively. Figure 5*d* shows the piecewise constant approximation to the supercritical velocity derived by taking the gradient of the piecewise linear velocity potential approximation in each interval $[x_i, x_{i+1}]$ for $i = 1, \ldots, n-1$. The Newton iteration to find d^h converges after 13 iterations, using $d_i^0 = 1$ for $i = 1, \ldots, n$, with a tolerance of 10^{-3}.

Corresponding results for the subcritical flow are given in Figure 6. The Newton iteration converges from $d_i^0 = 4$ for $i = 1, \ldots, n$ in 8 iterations.

Notice that the velocity approximation is not quite symmetric about the line $x = 5$, even though the breadth and equilibrium fluid depth functions are. This is probably a consequence of using approximations to mass flow and depth in (5.5). By increasing the number of grid points the symmetry of the approximations can be improved — Figure 7 shows the supercritical solutions for $n = 61$.

Thus, although approximations to all of the variables can be generated using the unconstrained 'r' principle, in the case of the velocity potential (and therefore the velocity) the procedure is not ideal. However another variational principle exists which depends on the velocity potential alone, that is, the 'p' principle (2.3), constrained by $v = \phi'$. In using this constrained principle to seek an approximation to ϕ (and therefore v) no other approximations are made and more accurate results might be expected, as well as the procedure being more direct and cheaper.

6 Use of the constrained 'p' principle

The 'p' principle (2.3), constrained to satisfy $v = \phi'$, is given by

$$\delta J_2(\phi) = \delta \left\{ \int_{x_e}^{x_o} p(\phi', E) B\, dx + C B_e \left(\phi(x_o) - \phi(x_e) \right) \right\} = 0, \tag{6.1}$$

where $E(x) = \tilde{E} + gh(x)$ and the constants $\tilde{E}$ and C are prescribed.

The velocity potential of a shallow water flow is the function ϕ which satisfies $\delta J_2 = 0$. The algorithm for generating an approximation to the velocity potential using (6.1) is similar to that of Section 3.

Let the x_i $(i = 1, \ldots, n)$, given by (3.4), define the grid. Let the finite element approximation to the velocity potential be given by

$$\phi^h(x) = \sum_{i=1}^{n} \phi_i \alpha_i(x),$$

where the α_i are the piecewise linear basis functions α_i^l of (3.11) and the ϕ_i are parameters of the solution. Thus the finite dimensional version of the functional of the constrained 'p' principle is given by

$$L(\boldsymbol{\phi}) = \int_{x_1}^{x_n} p(\phi^{h'}, E) B\, dx + C B_e \left(\phi^h(x_n) - \phi^h(x_1) \right),$$

where $E(x) = \tilde{E} + gh(x)$ and $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_n)^T$. The approximation to the velocity potential is determined by the $\boldsymbol{\phi}$ which causes L to be stationary, that is, the $\boldsymbol{\phi}$ which satisfies

$$F_i(\boldsymbol{\phi}) = \frac{\partial L}{\partial \phi_i} = \int_{x_1}^{x_n} p_{\phi^{h'}} \alpha_i' B\, dx + C B_e \left(\alpha_i(x_n) - \alpha_i(x_1) \right) = 0 \quad i = 1, \ldots, n, \tag{6.2}$$

where $p_{\phi^{h'}} = p_{\phi'}(\phi^{h'}, E)$. The solution of the non-linear set of equations (6.2) is found using Newton's method. The Jacobian is given by

$$J(\boldsymbol{\phi}) = \{J_{ij}\} = \left\{ \frac{\partial F_i}{\partial \phi_j} \right\} = \left\{ \frac{\partial^2 L}{\partial \phi_j \partial \phi_i} \right\} = \left\{ \int_{x_1}^{x_n} p_{\phi^{h'} \phi^{h'}} \alpha_i' \alpha_j' B\, dx \right\},$$

which is the Hessian of L and has the form of a weighted mass matrix, with weight $p_{\phi^{h'} \phi^{h'}} B$. From (2.5) $p_{\phi' \phi'} = (3\phi'^2/2 - E)/g$, so that, J is negative definite for wholly subcritical flows and positive definite for wholly supercritical flows.

Given an initial approximation $\boldsymbol{\phi}^0$ to the solution $\boldsymbol{\phi}$, Newton's method produces a sequence of approximations $\boldsymbol{\phi}^k$ from

$$\boldsymbol{\phi}^{k+1} = \boldsymbol{\phi}^k + \delta \boldsymbol{\phi}^k, \tag{6.3}$$

where

$$J(\boldsymbol{\phi}^k)\, \delta \boldsymbol{\phi}^k = -\mathbf{F}(\boldsymbol{\phi}^k). \tag{6.4}$$

The sequence ends when

$$\max_i \left| \delta \phi_i^k \right| < \text{tolerance}. \tag{6.5}$$

The Jacobian and the vector $\mathbf{F}$ are integrated exactly. The Jacobian is tri-diagonal and (6.4) is solved by Gaussian elimination and back substitution. The initial approximation ϕ^0 is given by

$$\phi_i^0 = (i-1)\, v^0 \qquad i = 1, \ldots, n,$$

where v^0 is assigned a value which determines whether the approximation being calculated is an approximation to subcritical or to supercritical flow. Let $c_*^{\min} = \min_{x \in [x_1, x_n]} c_*$, where c_* is the critical velocity, $c_* = \sqrt{g d_*} = \sqrt{2E/3}$. Then, if $v^0 < c_*^{\min}(x_n - x_1)/(n-1)$, the approximation will be subcritical. Let $c_*^{\max} = \max_{x \in [x_1, x_n]} c_*$. Then, if $v^0 > c_*^{\max}(x_n - x_1)/(n-1)$, the approximation will be supercritical.

The algorithm is implemented on the grid (3.4), with $x_e = 0$, $x_o = 10$ and $n = 21$. The energy $\tilde{E}$ is again taken to be 50. Approximations to flows in channels with breadths given by (3.13) and (3.14) and fluid depths below the level $z = 0$ given by (3.15) and (3.16) are considered.

For $h(x) = h_1$ the value of mass flow at inlet $C = C_*$, where C_* is given by (3.17), is used to give examples of critical flows and $C = 10$ is used to give examples of non-critical flows.

Consider the channel with breadth $B(x) = B_{1,6}$ and let the tolerance in (6.5) be 10^{-3}. The method converges to the subcritical approximation in 4 iterations, using $v^0 = 1$, and to the supercritical approximation in 5 iterations, using $v^0 = 5$, for non-critical flows. For critical flows the method converges to the subcritical approximation in 7 iterations, using $v^0 = 1$, and to the supercritical approximation in 8 iterations, using $v^0 = 5$. Results for the critical flows are shown in Figure 8. Figure 8*a* shows the piecewise linear velocity potential approximations, the top line corresponding to supercritical flow and the bottom line to subcritical flow. Figure 8*b* shows the piecewise constant velocity approximations derived from the gradients of the velocity potential approximations in each element. Notice that this time the velocity approximations are approximately symmetric about the line $x = 5$ as is expected for flows in a channel whose breadth and equilibrium depth functions are symmetric about this line.

For $B(x) = B_{1,2}$ and $h(x) = h_2$ with $H = 0.2$ Figure 9*a* shows the piecewise linear approximations to the velocity potential for subcritical and supercritical flows with $C = 10$. The corresponding piecewise constant approximations to the velocity are given in Figure 9*b*.

The L_2 error of the approximations to the velocity is defined by

$$\left\| v - \phi^{h'} \right\| = \left(\int_{x_e}^{x_o} \left(v - \phi^{h'} \right)^2 dx \right)^{\frac{1}{2}}.$$

Table 3 shows the L_2 errors for piecewise constant velocity approximations in the channel with $B(x) = B_{1,2}$ and $h(x) = h_1$. The energy $\tilde{E}$ is given the value 50, $C = C_*$, defined by (3.17), is used to derive the critical approximations and $C = 10$ is used to give the non-critical approximations. It can be seen that, as the grid is refined, the convergence of $\phi^{h'}$ to v is linear.

n	$\frac{\Delta x}{10}$	critical flows		non-critical flows	
		subcritical	supercritical	subcritical	supercritical
3	2^{-1}	3.266×10^{0}	2.744×10^{0}	1.933×10^{0}	1.444×10^{0}
5	2^{-2}	1.582×10^{0}	1.352×10^{0}	8.914×10^{-1}	6.549×10^{-1}
9	2^{-3}	7.782×10^{-1}	6.653×10^{-1}	4.449×10^{-1}	3.261×10^{-1}
17	2^{-4}	3.861×10^{-1}	3.296×10^{-1}	2.227×10^{-1}	1.632×10^{-1}
33	2^{-5}	1.923×10^{-1}	1.640×10^{-1}	1.114×10^{-1}	8.164×10^{-2}
65	2^{-6}	9.597×10^{-2}	8.179×10^{-2}	5.569×10^{-2}	4.082×10^{-2}
129	2^{-7}	4.794×10^{-2}	4.084×10^{-2}	2.784×10^{-2}	2.041×10^{-2}
257	2^{-8}	2.396×10^{-2}	2.041×10^{-2}	1.392×10^{-2}	1.021×10^{-2}
513	2^{-9}	1.198×10^{-2}	1.020×10^{-2}	6.961×10^{-3}	5.103×10^{-3}
1025	2^{-10}	5.987×10^{-3}	5.100×10^{-3}	3.481×10^{-3}	2.552×10^{-3}

Table 3: L_2 errors for piecewise constant velocity approximations.

7 Use of the constrained 'r' principle for discontinuous flows

In this section the 'r' principle (2.12), constrained to satisfy the conservation of mass equation, is used to generate approximations to the depths in discontinuous shallow water flows. In order to achieve an accurate finite element approximation to the depth, one of the grid nodes must be positioned at the point of discontinuity; this requires the use of irregular grids.

The functional of the 'r' principle for discontinuous flow (2.12), constrained to satisfy conservation of mass, is

$$J_3(d, x_s) = \int_{x_e}^{x_s} (r(Q,d) + E_e d)\, B\, dx + \int_{x_s}^{x_o} (r(Q,d) + E_o d)\, B\, dx, \qquad (7.1)$$

where $Q(x) = CB_e/B(x)$. The equilibrium fluid depth h is assumed constant so that the energy E, defined by (2.2), has the constant value E_e in $[x_e, x_s)$ and the constant value E_o in $(x_s, x_o]$. The values of E_e and E_o are deduced from boundary conditions and, from $(2.11)_3$, are such that $E_e > E_o$. The natural conditions of the first variation of J_3 are

$$\begin{aligned} r_d + E_e &= 0 \quad \text{in } (x_e, x_s), \\ r_d + E_o &= 0 \quad \text{in } (x_s, x_o), \\ [r + Ed]_{x_s} &= 0. \end{aligned} \qquad (7.2)$$

The method of finding approximations is based on that of Section 3 in that finite element expansions for d in the regions of the domain before and after the discontinuity are substituted into a finite dimensional version of (7.1). As the iteration proceeds the node which separates the pre- and post-discontinuity approximations is repositioned in order to satisfy $(7.2)_3$.

Let the domain of the problem $[x_e, x_o]$ be divided initially into $n-1$ adjacent regular intervals $[x_i, x_{i+1}]$ by the points x_i $(i = 1, \ldots, n)$ defined by

(3.4). One of these nodes must be chosen as being the initial approximation to the position of the discontinuity and the number of the node nearest to the actual position of the hydraulic jump is needed for this purpose (see below). Let x_N be the initial guess for the jump position.

The method requires that approximations to the flow in front of and behind the jump are generated separately and coupled, by means of a discontinuity, at the position of the hydraulic jump.

Let the approximation to the depth in the pre-jump region $[x_1, x_N]$ be

$$d^e(x) = \sum_{i=1}^{N} d_i^e \alpha_i^e(x),$$

where

$$\alpha_1^e(x) = \begin{cases} \dfrac{x_2 - x}{x_2 - x_1} & x \in [x_1, x_2] \\ 0 & x \notin [x_1, x_2] \end{cases},$$

$$\alpha_i^e(x) = \begin{cases} \dfrac{x - x_{i-1}}{x_i - x_{i-1}} & x \in [x_{i-1}, x_i] \\ \dfrac{x_{i+1} - x}{x_{i+1} - x_i} & x \in [x_i, x_{i+1}] \\ 0 & x \notin [x_{i-1}, x_{i+1}] \end{cases} \qquad i = 2, \ldots, N-1,$$

$$\alpha_N^e(x) = \begin{cases} \dfrac{x - x_{N-1}}{x_N - x_{N-1}} & x \in [x_{N-1}, x_N] \\ 0 & x \notin [x_{N-1}, x_N] \end{cases},$$

and let the approximation to the depth in the post-jump region $[x_N, x_n]$ be

$$d^o(x) = \sum_{i=N}^{n} d_i^o \alpha_i^o(x),$$

where

$$\alpha_N^o(x) = \begin{cases} \dfrac{x_{N+1} - x}{x_{N+1} - x_N} & x \in [x_N, x_{N+1}] \\ 0 & x \notin [x_N, x_{N+1}] \end{cases},$$

$$\alpha_i^o(x) = \begin{cases} \dfrac{x - x_{i-1}}{x_i - x_{i-1}} & x \in [x_{i-1}, x_i] \\ \dfrac{x_{i+1} - x}{x_{i+1} - x_i} & x \in [x_i, x_{i+1}] \\ 0 & x \notin [x_{i-1}, x_{i+1}] \end{cases} \qquad i = N+1, \ldots, n-1,$$

$$\alpha_n^o(x) = \begin{cases} \dfrac{x - x_{n-1}}{x_n - x_{n-1}} & x \in [x_{n-1}, x_n] \\ 0 & x \notin [x_{n-1}, x_n] \end{cases}.$$

The algorithm is in two parts. Firstly the two finite element approximations d^e and d^o are derived by finding the values of $\mathbf{d}^e = (d_1^e, \ldots, d_N^e)^T$ and $\mathbf{d}^o = (d_N^o, \ldots, d_n^o)^T$ such that

$$L(\mathbf{d}^e, \mathbf{d}^o) = \int_{x_1}^{x_N} (r(Q, d^e) + E_e d^e)\, B\, dx + \int_{x_N}^{x_n} (r(Q, d^o) + E_o d^o)\, B\, dx$$

is stationary with respect to variations in $\mathbf{d}^e$ and $\mathbf{d}^o$. This requires solving the two sets of equations

$$\frac{\partial L}{\partial d_i^e} = 0 \;\; i = 1, \ldots, N \quad \text{and} \quad \frac{\partial L}{\partial d_i^o} = 0 \;\; i = N, \ldots, n,$$

using Newton's method, as described in Section 3. The initial approximation to $\mathbf{d}^e$ must be supercritical in order that the supercritical flow in the region before the jump is approximated and the initial approximation to $\mathbf{d}^o$ must be subcritical.

The second stage of the algorithm is to alter the position of x_N by employing the jump condition $(7.2)_3$. If x_s is the exact position of the jump and d is the exact solution then, from $(7.2)_3$,

$$(r(Q,d) + E_e d)|_{x_{s-}} - (r(Q,d) + E_o d)|_{x_{s+}} = 0,$$

where $-$ denotes the x_e side of x_s and $+$ the x_o side of x_s. If the approximation satisfies

$$\left| (r(Q,d^e) + E_e d^e)|_{x_N} - (r(Q,d^o) + E_o d^o)|_{x_N} \right| < \text{tolerance}, \tag{7.3}$$

for some specified tolerance, then the approximate solution has been found and x_N is the approximate position of the hydraulic jump. If (7.3) is not satisfied then a new approximation to the jump position is found using the jump condition, as follows.

The equation

$$r(Q_s, d_N^e) + E_e d_N^e - r(Q_s, d_N^o) - E_o d_N^o = 0 \tag{7.4}$$

is solved for Q_s, the value of the mass flow which would occur at the jump if d_N^e and d_N^o were the actual depths of the flow before and after the jump. The conservation of mass constraint gives

$$Q(x)B(x) = CB_e \qquad x \in [x_1, x_n]$$

and, since $B(x)$ and C are specified, this can be used to find the point x_s^N in the channel where the mass flow is Q_s. Only flows which are critical at the channel throat will be considered so that

$$B(x_s^N) = \frac{CB_e}{Q_s} \tag{7.5}$$

can be solved, by bisection, to give a unique value for x_s^N.

The process which occurs on solving (7.4) is explained in detail in Wakelin (1993). The main point is that the iteration procedure of repeatedly solving (7.4) and (7.5) and generating depth approximations can be shown to converge, under certain conditions, as follows.

Let $\hat{d^e}$ be the exact supercritical depth corresponding to the energy $E = E_e$ and the mass flow $Q = Q(x_N) = CB_e/B(x_N)$ and let $\hat{d^o}$ be the exact subcritical depth corresponding to $E = E_o$ and $Q = Q(x_N)$. Then, if

$d^e_N = \hat{d}^e$ and $d^o_N = \hat{d}^o$ and provided that the value of Q_s obtained on first solving (7.4) has a solution x^N_s in the domain, the iteration for x^N_s can be shown to converge to give the position of the discontinuity. The result also holds if d^e_N and d^o_N are sufficiently accurate approximations to $\hat{d}^e$ and $\hat{d}^o$, respectively.

The algorithm for positioning a node at the jump is in two parts. Firstly, beginning with $N = n-1$, the corresponding value of x^{n-1}_s is found using (7.4) and (7.5). Then, stepping backwards along the channel to the $n-2$ th node, the value of x^{n-2}_s is found. If $(x_{n-1} - x^{n-1}_s)(x_{n-2} - x^{n-2}_s) < 0$ then x_s lies between x_{n-1} and x_{n-2}. Otherwise the process is repeated until the node j is found, where $(x_j - x^j_s)(x_{j-1} - x^{j-1}_s) < 0$. Then, if $|x_j - x^j_s| < |x_{j-1} - x^{j-1}_s|$, the number N of the node to be moved to the jump position is j; otherwise $N = j-1$.

Once the number of the node to be moved to the jump position has been established in this way, x_N is moved to x^N_s. The finite element approximations d^e and d^o are recalculated on the modified grid and, if (7.3) is still not satisfied, (7.4) and (7.5) are used to reposition x_N and the process is repeated until (7.3) is satisfied. The approximate solution has then been found and x_N is an approximation to the jump position.

The algorithm is applied to a grid with $x_e = 0$, $x_o = 10$ and $n = 21$. The energy at inlet E_e is given the value 50 and the mass flow at inlet $C = C_*$, where C_* is defined by (3.17), to give a critical flow in a channel with breadth $B(x) = B_{1,k}$, defined by (3.13). The depth at outlet d_o is given for each case and is used to deduce the value of E_o, using the definitions of mass flow (2.1) and energy (2.2). ¿From the conservation of mass equation $Q(x_o) = CB_e/B(x_o)$, which yields $E_o = gd_o + (CB_e/(B(x_o)d_o))^2/2$.

The piecewise linear approximation to the discontinuous depth profile with $d_o = 4.69$ and breadth $B(x) = B_{1,6}$ is given in Figure 10*a*. For a tolerance on the Newton iteration of 10^{-3} and on the jump condition (7.3) of 10^{-3}, the method converges in 3 iterations on the position of the discontinuity, once the node to be placed at the discontinuity has been found; in this case it is node number 16. These iterations require 15, 8 and 8 Newton iterations. The initial approximation on the original regular grid is given the values $d^e_i = 1$ $(i = 1, \ldots, N)$ and $d^o_i = 4.69$ $(i = N, \ldots, n)$. Once the number of the node to approximate the jump position is found subsequent approximations to the finite element solutions use the approximation on the previous grid as the first guess in Newton's method to find the approximation on the new grid.

The piecewise linear approximation for $d_o = 3.86$ is shown in Figure 10*b*. This converges in 3 iterations on the position of node 20, which is selected by the algorithm to be moved to approximate the jump position, requiring 15, 4 and 4 Newton iterations.

8 Conclusions

In this paper new variational principles for shallow water flows developed in Wakelin (1993) are used to determine finite element approximations to depth and/or velocity functions in several examples of channel flows with variable cross-sections. Approximations to both continuous and discontinuous flows are given and convergence of the procedure is demonstrated by reference to a typical case.

The variational principles considered in this paper are only a subset of the variational principles for shallow water flows described in Wakelin (1993). In particular, two further variational principles exist whose natural conditions are the equations of steady quasi one-dimensional motion. However, those principles do not have the useful property of the 'p' and 'r' principles, (2.3) and (2.4), where the variations can be constrained to obtain principles which depend on just one variable. Other sets of variational principles exist for the cases of steady shallow water flow in two dimensions and for both quasi one-dimensional and two-dimensional unsteady flows.

The variational principles for steady two-dimensional motion and unsteady quasi one-dimensional motion are used in Wakelin (1993) to obtain approximate solutions of the shallow water equations. The method in the case of two-dimensional steady motion is an extension of the method given in this paper, that is, the variables of the motion are taken to be those functions for which the functionals of the variational principles are stationary with respect to variations in a finite dimensional space spanned by a set of piecewise linear basis functions defined on a triangular grid. In this way, piecewise linear approximations to the depth and velocity potential and piecewise constant approximations to the velocity are obtained.

The same method is also applied to the variational principles for quasi one-dimensional unsteady motion but is only of limited use. This is because the structures of the variational principles for unsteady flow require that the solutions at the beginning and end of the time interval are given functions, thus the solution at the final time needs to be known before the problem can be solved using this method. One possibility is to consider a very long time interval and to set the boundary functions at the final time equal to the asymptotic solutions, given as the time tends to infinity. However, this would be computationally expensive because of the size of the domain which would be necessary to accommodate a sufficiently long time.

Acknowledgement

SLW acknowledges the support of the SERC.

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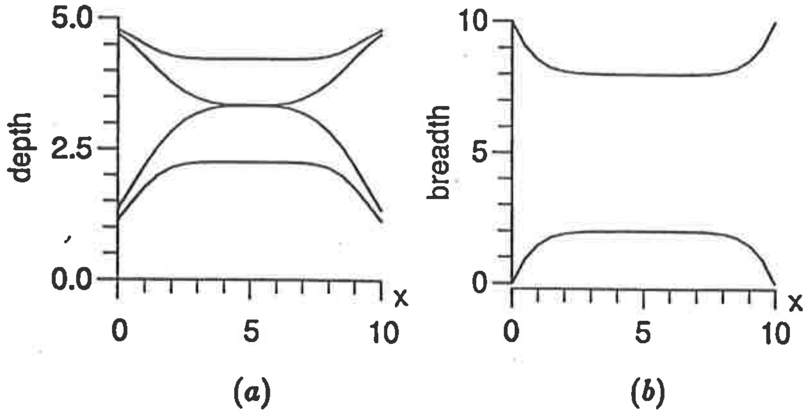

Figure 1: *a*) Piecewise linear depth approximations for *b*) $B_{1,6}(x)$.

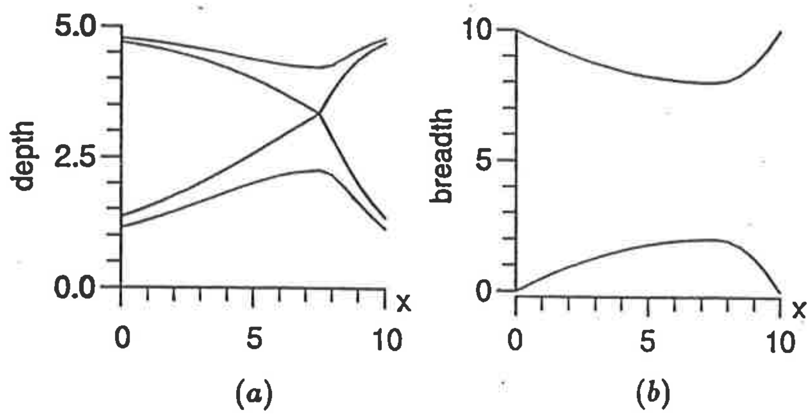

Figure 2: *a*) Piecewise linear depth approximations for *b*) $B_{2,2}(x)$ with $\nu = 7.5$.

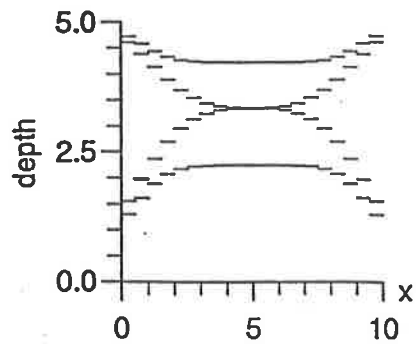

Figure 3: Piecewise constant depth approximations for $B_{1,6}(x)$.

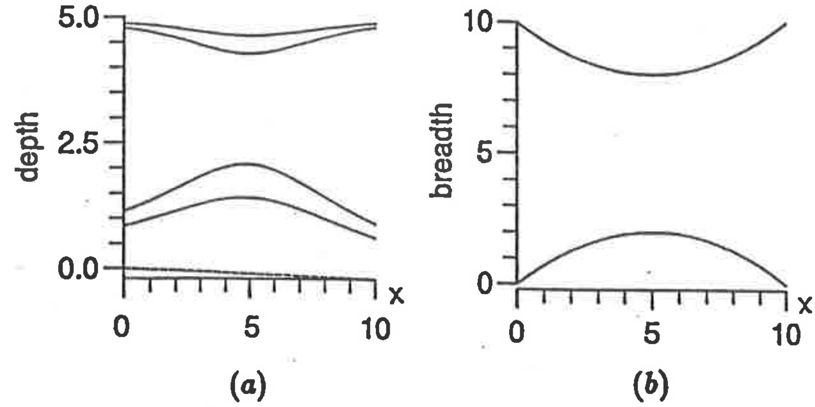

Figure 4: *a*) Piecewise linear depth approximations for $B(x) = B_{1,2}$ and $h(x) = h_2$ with $H = 0.2$ and *b*) $B_{1,2}(x)$.

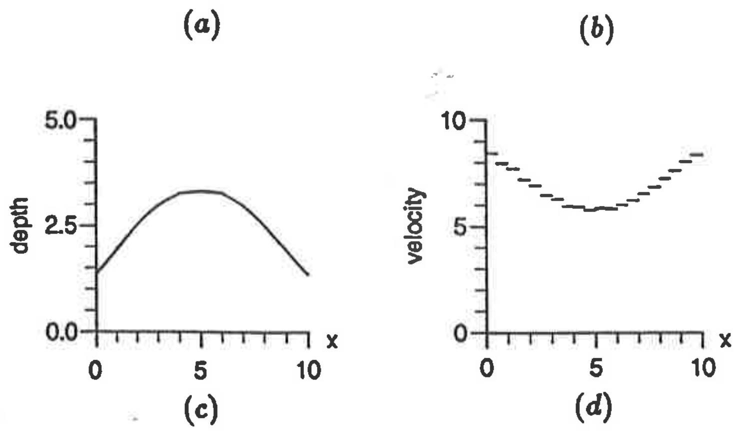

Figure 5: *a*) Mass flow, *b*) velocity potential, *c*) depth and *d*) velocity approximations for $B(x) = B_{1,4}$ — supercritical case.

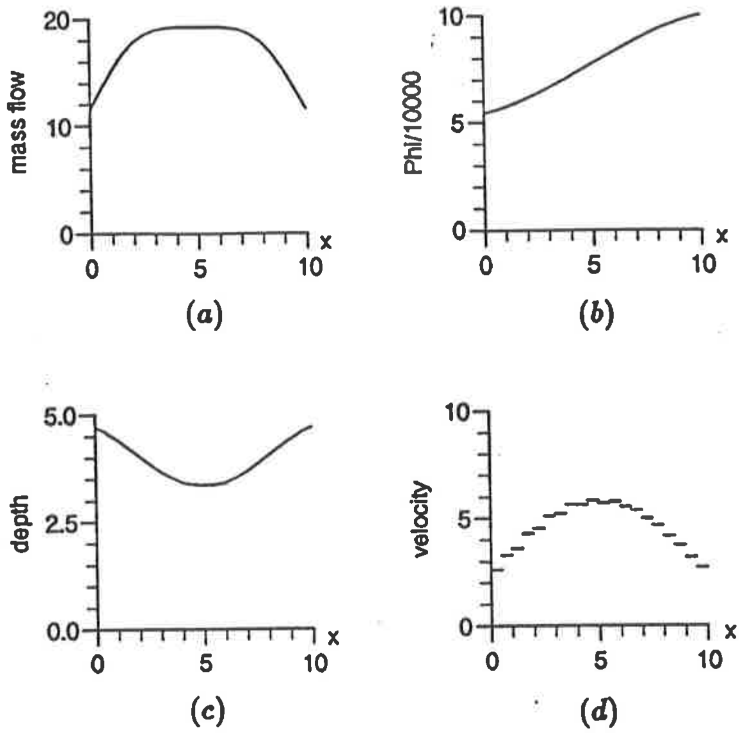

Figure 6: *a*) **Mass flow,** *b*) **velocity potential,** *c*) **depth and** *d*) **velocity approximations for** $B(x) = B_{1,4}$ **— subcritical case.**

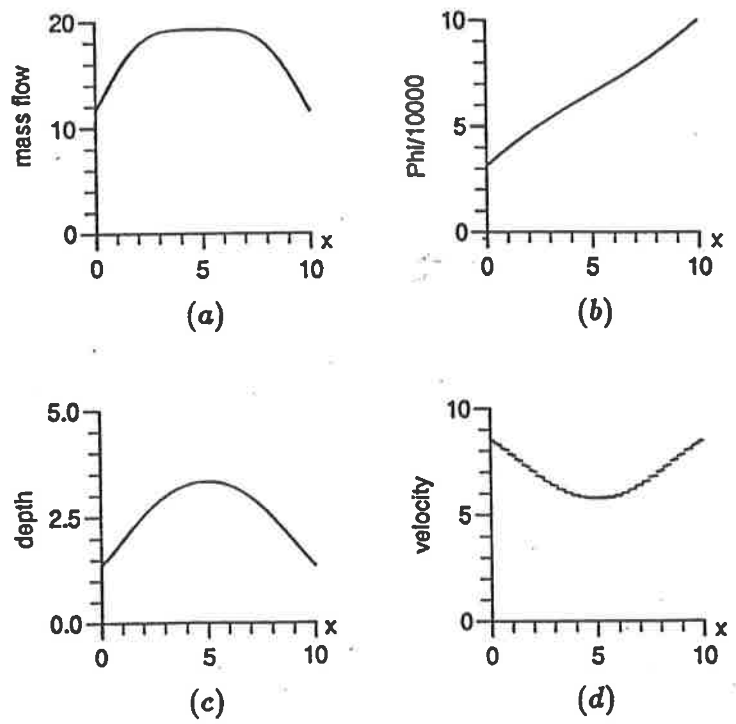

Figure 7: As Figure 5 for $n = 61$**.**

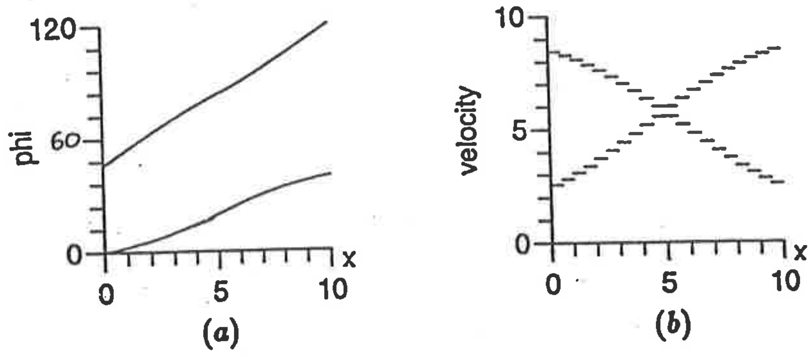

Figure 8: *a*) **Velocity potential and** *b*) **velocity approximations for** $B(x) = B_{1,6}$ **and** $h(x) = h_1$.

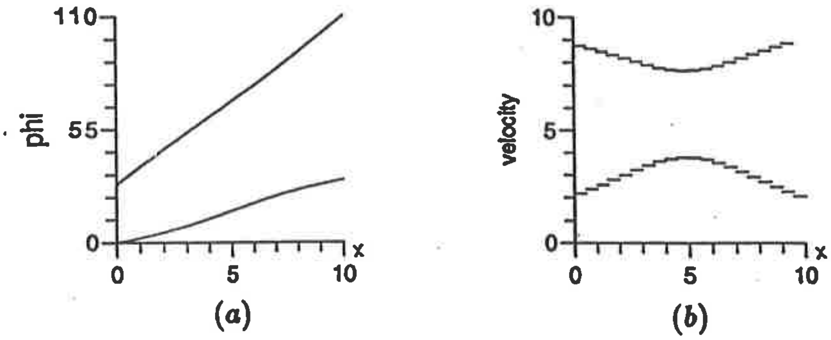

Figure 9: *a*) **Velocity potential and** *b*) **velocity approximations for** $B(x) = B_{1,2}$ **and** $h(x) = h_2$ **with** $H = 0.2$.

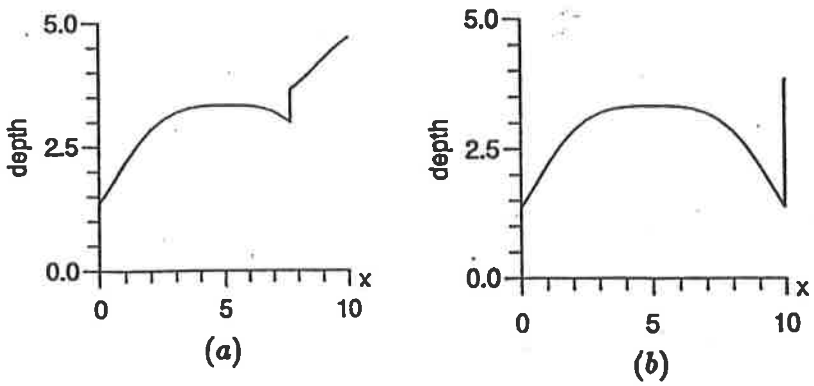

Figure 10: Piecewise linear depth approximations for *a*) $d_o = 4.69$ **and** *b*) $d_o = 3.86$.