

A SEMI-ROBUST POLE ASSIGNMENT ALGORITHM FOR
LINEAR STATE FEEDBACK

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Abstract

A New Algorithm for the pole assignment problem of a controllable, time-invariant, linear, multivariable system with linear state feedback, is presented. The resulting feedback matrix is a least square solution and can be shown to be robust in some sense. The method is based on the controllability canonical (staircase) form and amounts to a new proof for the existence of a solution of the pole assignment problem.

Keywords

Pole Assignment, Linear System, Controllability, Stabilizability, Robustness.

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1. INTRODUCTION

The linear state-feedback pole assignment problem (PAP) of a time-invariant, linear, multivariable system is one of the most studied problems in control system design. (See [1],[3]-[6],[11],[14], and many general texts on control theory, and the references therein). There are essentially four types of methods for the solution of the problem:

- (1) Classical methods - transforming the system into one or several SISO systems or canonical forms (Frobenius, Luenberger, Jordan), or involving the controllability matrix; (e.g. [3][14]).
- (2) Direct methods - transforming the system into canonical form using stable unitary matrices, (e.g. Schur form [6][11]).
- (3) Matrix Equation methods - solving directly the equation

$$\begin{cases} AX - X\Lambda = BG & , & (1a) \\ FX = G & ; & (1b) \end{cases}$$

for the matrix F , (e.g. [1]),

and

- (4) Eigenvector methods - selecting the eigenvectors x_j , the columns in the matrix X in equation (1), from some admissible subspaces, (e.g. [4][5]), and then recover the matrix F from equation (1b).

The methods in class (1) are usually inefficient or numerically unstable. The class (3) methods usually require the solution of the Sylvester equation (1a) and cannot reassign eigenvalues of the system matrix A . In [1], the equation (1) may have to be solved more than once for different guesses of the matrix G . The most efficient and numerically stable methods to date are those in classes (2) and (4), with class (4) methods look for the most robust solution using some sort of iterative searching algorithms. Class (4) methods can thus be more expensive. In contrast, class (2) methods do not utilize fully the available degrees of freedom (when one has more than a single input) and do not tackle the problem of robustness of the solution.

The algorithm presented in this paper falls between the class (2) and (4) methods. It is based on the staircase or controllability canonical form [6] [7] [9] [10] [12], a stable canonical form resulting from unitary transformations, and the direct method involving equation (1) produces a least square solution. It can be shown that the least square solution implies some sort of robustness [4] [5] in a loose or intermediate sense. In addition, any composition of the spectrum can be assigned.

Finally, this paper is the result of applying the techniques by Van Dooren [10], for reduced order observer design, to our problem.

2. TRANSFORMING THE PROBLEM

Consider the time-invariant, linear, multivariable, completely controllable system defined by

$$\mathcal{D}x = Ax + Bu \quad (2)$$

where x and u are n - and m -dimensional real vectors, A, B are constant real matrices of appropriate orders, and \mathcal{D} denotes the differential operator for continuous-time systems or the delay operator for discrete-time systems.

For the PAP, one requires to find a real feedback matrix F such that the closed-loop system matrix $(A + BF)$ has eigenvalues equal to $L = \{\lambda_1, \dots, \lambda_n\}$, a given set which is closed under complex conjugation. It is well-known that the problem has a solution if the system defined by equation (2) is completely controllable [14].

An equivalent problem will be to find the matrices X and G such that equation (1a) is satisfied for some matrix A with spectrum $\rho(A) = L$. If X is invertible, the solution F can then be retrieved from equation (1b).

Transforming equation (1a) to the equivalent form

$$P^H A P \cdot P^H X Q - P^H X Q \cdot Q^H \Lambda Q = P^H B \cdot G Q \quad (3)$$

with unitary matrices P and Q , where $(\cdot)^H$ denotes the hermitian.

Note that the use of unitary matrices ensures that the numerical stability of the equation (1a) is the same as that of equation (3) and thus not worse [2]-[4] [6]-[13].

Let P be selected such that (A,B) is transformed into the staircase form [6]-[10], [12], where

$$(P^H A P, P^H B) = \left(\begin{array}{cccccccc|c} A_{11} & A_{12} & 0 & \cdot & \cdot & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ A_{k-1,1} & A_{k-1,2} & \cdot & \cdot & A_{k-1,k-1} & \cdot & \cdot & A_{k-1,k} & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ A_{k,1} & A_{k,2} & \cdot & \cdot & A_{k,k-1} & \cdot & \cdot & A_{k,k} & B_k \end{array} \right), \quad (4)$$

with A_{ij} being $r_i \times r_j$ and B_k being $r_k \times m$.

Moreover, the off diagonal blocks $A_{i,i+1}$ can be chosen to be in full-ranked lower triangular echelon form [6] [10].

$$A_{i,i+1} = \left(\begin{array}{cccccccc} x & 0 & 0 & \cdot & \cdot & \cdot & 0 & \cdot & \cdot & \cdot & 0 \\ * & x & 0 & & & & & & & & \cdot \\ \cdot & * & x & \cdot & & & & & & & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & & & & & & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & & & & & & \cdot \\ \cdot & \cdot & * & \cdot & \cdot & \cdot & & & & & \cdot \\ * & * & \cdot & \cdot & \cdot & * & x & 0 & \cdot & \cdot & 0 \end{array} \right), \quad i=1, \dots, k-1, \quad (5)$$

as the system defined in equation (2) is assumed to be completely controllable. Here the x 's denote some non-zero components, and $*$'s some arbitrary ones.

The matrix X is obviously nonsingular, and will be better conditioned if the strictly lower triangular part is minimized in some norm [2] [10].

Assume that all the eigenvalues λ_i are real. (The complex case will be discussed in section 5).

Denote the j -th columns of X , Λ and G by, respectively

$$\begin{pmatrix} 0 \\ 1 \\ \underline{x}_j \end{pmatrix}, \quad \begin{pmatrix} \underline{z}_j \\ \lambda_j \\ 0 \end{pmatrix} \quad \text{and} \quad \underline{g}_j \quad . \quad (9)$$

Also denote the matrix which contains the j_1 - to j_2 -th columns of the matrix M by $M_{j_1:j_2}$, and the j -th column by M_j .

With the notation in equation (9), one can prove that a solution of the form defined in equation (8) exists. Consider the j -th column of the equation (1a):-

$$\underline{j=1:-} \quad [(\lambda_1-A)_{2:n}, B] \begin{pmatrix} \underline{x}_1 \\ \underline{g}_1 \end{pmatrix} = (A-\lambda_1)_1 \quad ; \quad (10a)$$

$$\underline{1 < j < n:-} \quad [X_{1:j-1}, (\lambda_j-A)_{j+1:n}, B] \begin{pmatrix} \underline{z}_j \\ \underline{x}_j \\ \underline{g}_j \end{pmatrix} = (A-\lambda_j)_j \quad ; \quad (10b)$$

and

$$\underline{j=n:-} \quad [X_{1:n-1}, B] \begin{pmatrix} \underline{z}_n \\ \underline{g}_n \end{pmatrix} = (A-\lambda_n)_n \quad . \quad (10c)$$

The matrices on the LHS of equations (10), $M(j)$, are $n \times (n+m-1)$.

If the matrix $M(j)$ is of full-row rank, equations (10) can then be solved, as the matrices are then right-invertible. It is easy to see that the matrix $M(j)$, once constructed, is of full-row rank row-echelon form, as a result of the elaborate choices of the forms of the matrices A , B and X .

Consider a typical example, with $r_1 = 2$, $r_2 = 2$, $r_3 = 3$, $m = 3$ and $n = 10$, and (A,B) of the form

$$(A,B) = \left[\begin{array}{ccc|ccc|ccc|ccc} * & x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & x & x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & * & x & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & * & x & 0 & 0 & 0 & 0 & 0 & 0 \\ * & & & & * & * & x & 0 & 0 & 0 & 0 & 0 \\ & & & * & & & & x & 0 & 0 & 0 & 0 \\ & & & & & & * & * & x & 0 & 0 & 0 \\ & & & & & & & & & x & 0 & 0 \\ & & & & & & & & & * & x & 0 \\ & & & & & & & & & * & * & x \end{array} \right] \quad (11)$$

The matrices $M(1)$, $M(5)$, $M(10)$ are of the forms, respectively,

$$M(1) = \left[\begin{array}{ccc|ccc|ccc|ccc} * & x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & x & x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & * & x & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & * & x & 0 & 0 & 0 & 0 & 0 & 0 \\ * & & & & * & * & x & 0 & 0 & 0 & 0 & 0 \\ & & & * & & & & x & 0 & 0 & 0 & 0 \\ & & & & & & * & * & x & 0 & 0 & 0 \\ & & & & & & & & & x & 0 & 0 \\ & & & & & & * & * & x & 0 & 0 & 0 \\ & & & & & & & & & * & * & x \end{array} \right] \quad (12a)$$

$$M(5) = \left[\begin{array}{cccc|ccc|ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & 1 & x & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & & x & 0 & 0 & 0 & 0 & 0 \\ & & & & & & * & x & 0 & 0 & 0 & 0 \\ * & & & & & & * & * & x & 0 & 0 & 0 \\ & & & & & & & & & x & 0 & 0 \\ & & & & & & * & * & x & 0 & 0 & 0 \\ & & & & & & & & & * & * & x \end{array} \right] \quad (12b)$$

and

$$M(10) = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & & & \\ * & 1 & \dots & & & \\ & * & \dots & & & \\ & & * & & & \\ & & & \dots & & \\ & & & & 1 & \\ & & & & & * \\ & & & & & * \end{array} \right] , \quad (12c)$$

(with nine one's on the diagonal).

The columns X_j of X are solved recursively in the order $j = 1, \dots, n$; for some $\{\lambda_j\}$. The equations in (10) can be solved using the usual least square technique, e.g. QR [2] [13]. The solutions \underline{z}_j , \underline{x}_j and \underline{g}_j are then minimum norm solutions for the 2- or F-norm. Note that the unitary transformation P in equations (3), (6) and (7) will not affect the minimum norm nature of the solution.

The above algorithm amounts to a constructive proof for the existence of a solution to the PAP, for completely controllable systems and any given spectrum L .

Note that for the case $m = 1$, the matrices $M(j)$ will be $n \times n$ and non-singular. As a result, a unique solution is obtained.

4. SEMI-ROBUSTNESS

Recall that the unknowns \underline{z}_j , \underline{x}_j and \underline{g}_j in equation (10) are least square solutions, with minimum 2- or F-norm values. The vectors \underline{z}_j and \underline{x}_j are in the off-diagonal parts of the triangular matrices Λ and X respectively, and their minimizations imply "better" symmetry, and thus conditioning, for the matrix X and the eigenvalue problem involving the matrix Λ ([2]). In addition, equation (1b) implies that

$$\|F\| \leq \|X^{-1}\| \cdot \|G\| . \quad (13)$$

With X well-conditioned enough and \underline{g}_j , (and thus the matrix G) minimized, inequality (13) implies that the solution feedback matrix F is of reasonably

small size, as the upper bound of the feedback in inequality (13) is minimized. Note also that the solution of equation (1b) will be numerically stable with a well-conditioned matrix X ([2][13]).

The resulted well-conditioning of the matrix X and the eigenvalue problem involving A also implies the minimization of an upper bound of the transient response of the closed-loop system, and the maximization of lower bound of the stability margin. The details can be found in [4].

As the condition numbers and sensitivity measures are not directly optimized as in [4] [5], the solution of the PAP by the algorithm in this paper can only be called semi-robust. It is thus a trade-off between the optimization of conditioning and the efficiency of algorithms.

5. COMPLEX EIGENVALUE ASSIGNMENT

Similar to section 3, assume that the matrices A and B are in the staircase controllability canonical form. Assume that the matrix is in real Schur form, with 2×2 block on the diagonal. Let the complex conjugate pair of eigenvalues λ_j and $\lambda_{j+1} = \bar{\lambda}_j$ be represented by the 2×2 blocks $\Lambda_j = \begin{bmatrix} a_j & -b_j \\ b_j & a_j \end{bmatrix}$ in Λ . Similarly, as in section 3, assume that

$$\Lambda_{j:j+1} = \begin{bmatrix} \underline{z}_j & \underline{z}_{j+1} \\ a_j & -b_j \\ b_j & a_j \\ \underline{0} & \underline{0} \end{bmatrix} \quad (14)$$

Consider the j -th and $(j+1)$ -th columns of equation (1a), with $1 < j < n-1$, one has

$$AX_{j:j+1} - X_{j:j+1} \Lambda_j = BG_{j:j+1} + X_{1:j-1} (\underline{z}_j, \underline{z}_{j+1}) \quad (15)$$

Using the Kronecker product \otimes , equation (15) can be written as,

Again, the x's denote non-zero entries. The indicated elements in the matrix $M(5,6)$ in equation (17) divide the matrix into three parts, with the upper "triangle" part containing only zero components. A lot of elements in the low "triangular" part are zero because of the Kronecker product \otimes in equation (16) involving the identity matrix I_2 .

6. OPERATION COUNTS

In this section, an operation count is presented for the numerical procedures discussed in sections 3 and 5. The procedures can be summarized into three steps:-

Step 1 : Transforming the matrices A and B into staircase form, which requires approximately $(3n + m)n^2$ flops [10].

Step 2 : Constructing and solving the equations in (10) and (16). Because of the echelon form of the matrices involved, the required QR decompositions [2] [13] can be obtained efficiently, using Householder transformations [8] [10]. It requires $(m + \frac{1}{2})n^3$ and $(4m + 2)n^3$ flops for the real and complex cases respectively. Note that one "complex" flop is equivalent to four real ones.

Step 3 : Recovering the feedback matrix F from equation (7), which involves a backsubstitution using the structure of the matrix \tilde{X} , and a back-transformation P^H . It requires approximately $n^3 + n^2/2$ flops.

Note that the operation count will be dominated by that of Step 2 if $m > 3$, and amounts to $O(n^4)$ if $m \cong n$. The operation count can be reduced to $O(n^3)$, if step 2 is performed by backsubstitution using the echelon structure of the matrices $M(j)$ or $M(j,j+1)$, instead of the QR decomposition [10].

Note that the procedure of backsubstitution without pivoting is, in general, numerically unstable for the solution of least square problems.

7. STABILIZABLE AND UNCONTROLLABLE SYSTEMS

In case of the system defined by equation (2) being uncontrollable, the controllable modes can still be assigned, with uncontrollable ones reassigned. It will be particularly useful if one can generalise the methods in sections 3 and 5 to cope with such systems, e.g. stabilizable systems, where all uncontrollable modes are stable [14]. In addition, one may consider a system uncontrollable, if the matrix $A_{k-1,k}$ in equation (4) is "nearly" rank deficient, to avoid numerical ill-conditioning.

Note that one can modify the eigenstructure of the reassigned controllable modes [4].

Consider equation (4) for an uncontrollable system, the matrix $A_{k-1,k}$ will be a zero matrix. Rewrite equation (4) as

$$(P^H A P, P^H B) = \left(\begin{array}{cc|c} A_{11} & 0 & 0 \\ A_{21} & A_{22} & B_2 \end{array} \right) \quad (18)$$

The staircase controllability canonical form essentially decomposes the system defined by matrices A and B into two parts: the controllable part in A_{22} , and the uncontrollable part in A_{11} . (c.f. [6]-[10], [12]).

Note that for a feedback matrix $F = (F_1, F_2)$, the closed-loop system matrix will be in the form

$$\left(\begin{array}{cc|c} A_{11} & & 0 \\ \hline A_{21} + B_2 F_1 & & A_{22} + B_2 F_2 \end{array} \right)$$

and thus any feedback will not influence the spectrum of the uncontrollable part A_{11} . Note also that the reduced system defined by matrices A_{22} and B_2 is controllable (otherwise one just increases the size of A_{11}).

Assume that the matrices A and B are in the form as in equation (18). Assume that the matrices X and Λ in equation (1a) are in the form

$$X = \begin{bmatrix} X_{11} & 0 \\ X_{21} & X_{22} \end{bmatrix}, \quad \Lambda = \begin{bmatrix} \Lambda_{11} & 0 \\ \Lambda_{21} & \Lambda_{22} \end{bmatrix} \quad (19)$$

Equation (1) can be broken up into three parts: (a fourth equation degenerates into $0 = 0$)

$$\begin{cases} A_{11}X_{11} - X_{11}\Lambda_{11} = 0 & (20a) \\ (A_{22} + B_2F_2)X_{22} - X_{22}\Lambda_{22} = 0, & (20b) \\ T(X_{21}) \frac{\Delta}{=} (A_{22} + B_2F_2)X_{21} - X_{21}\Lambda_{11} = X_{22}\Lambda_{21} - (A_{21} + B_2F_1)X_{11}. & (20c) \end{cases}$$

Equation (20a) represents the uncontrollable subsystem and so long if the matrix Λ_{11} has spectrum equal to the uncontrollable one of the matrix A_{11} , a matrix X_{11} can be chosen easily. The easiest choice will be to choose $X_{11}\Lambda_{11}X_{11}^{-1}$ to be the Schur decomposition of A_{11} [2]. Equation (20b) indicates that the pole assignment problem for the controllable subsystem has to be solved, so that the matrices F_2, X_{22} and Λ_{22} can be chosen. It can be done by the algorithms in sections 3 and 5.

For any arbitrary matrices F_1 and Λ_{21} , the matrix X_{21} can be chosen to modify the eigenvectors of the uncontrollable modes, if equation (20c) is satisfied. The operator T on the LHS of equation (20c) is invertible, if the spectra of Λ_{11} and Λ_{22} have an empty intersection [1]. In such a case, the matrix X_{21} will be given by

$$X_{21} = T^{-1}[X_{22}\Lambda_{21} - (A_{21} + B_2F_1)X_{11}] \quad (21)$$

If the operator T is not invertible, the matrix Λ_{21} can be chosen such that the RHS of the equation (20c) is in the span of the operator T .

It is possible, as the matrix X_{22} is nonsingular. Alternatively, one can expand equation (20c) using the Kronecker product and solve the resulting linear equation in the least sense. The simplest thing to do will be, from equation (20c), choose $\Lambda_{21} = X_{22}^{-1} \cdot \{(A_{21} + B_2F_1)X_{11} + T(X_{21})\}$ for some F_1 and X_{21} .

8. CONCLUSIONS

In this paper, we presented a new algorithm for the pole assignment problem of a controllable, time-invariant, linear, multivariable system with linear state feedback. The algorithm is numerically stable, non-iterative and produces a semi-robust least square solution.

Although the method is based on the Sylvester type equation (1), no restriction is necessary on the composition of the spectrum. Furthermore, no restrictions on the eigenstructure are required. (In [4], the multiplicity of any eigenvalue has to be less than $m+1$). A "minus" for the method will be the lack of control over the eigenstructure.

The method can also cope with complex eigenvalues with ease. Uncontrollable system can be tackled, if decomposed into subsystems.

More work has to be done to compare the algorithm in this paper with others numerically.

Finally, a very interesting generalization of the techniques in this paper to the solution of the pole assignment problem for descriptor or singular systems, defined by

$$EDx = Ax + Bu, \quad (E \text{ possibly singular})$$

is possible, and the result will be reported elsewhere.

REFERENCES

- [1] S.P. Bhattacharyya & E. de Sousa, "Pole Assignment via Sylvester's Equation", Systems and Control Lett. 1(4)(1982)261-263.
- [2] G. Golub & C. Van Loan, "Matrix Computations", (The John Hopkins University Press, Baltimore, 1983).
- [3] S.K. Katti, "Pole Placement in Multi-Input Systems via Elementary Transformations", Int. J. Control 37(2)(1983) 315-347.
- [4] J. Kautsky, N.K. Nichols & P. Van Dooren, "Robust Pole Assignment in Linear State Feedback", Int. J. Control 41(5) (1985) 1129-1155.
- [5] G. Klein & B.C. Moore, "Feedback Design of Linear Systems by Eigenvector Selection", Department of Mechanical Engineering, Columbia University (1982), (preprint).
- [6] G.S. Minimis & C.C. Paige, "An Algorithm for Pole Assignment of Time Invariant Multi-Input Linear Systems", Proc. I.E.E.E. Conf. on Decision and Control (1982) 62-67.
- [7] C.C. Paige, "Properties of Numerical Algorithms Related to Computing Controllability", I.E.E.E. Trans. Autom. Contr. AC-26(1)(1981) 130-138.
- [8] P. Van Dooren & M. Verhaegen, "The Use of Condensed Forms in Linear Systems Theory", Proc. I.E.E.E. Workshop on Reliable Numerical Methods in Control Systems Design, London (1983) 4/1-4/6.
- [9] P. Van Dooren, "The Generalized Eigenstructure Problem in Linear System Theory", I.E.E.E. Trans. Autom. Contr. AC-26 (1)(1981), 111-129.
- [10] P. Van Dooren, "Reduced Order Observers: A New Algorithm and Proof", Systems and Control Lett. (1985), (to appear).
- [11] A. Varga, "A Schur Method for Pole Assignment", I.E.E.E. Trans. Autom. Contr. AC-26 (2) (1981) 517-519.
- [12] A. Varga, "Numerically Stable Algorithm for Standard Controllability Form Determination", Electronic Lett. 17 (1981) 74-75.
- [13] J.H. Wilkinson, "The Algebraic Eigenvalue Problem", (Clarendon Press, Oxford, 1965).
- [14] W.M. Wonham, "Linear Multivariable Control: A Geometric Approach", 2nd Edition, (Springer-Verlag, Berlin, 1979).