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SMOOTH REGRADING OF DISCRETIZED DATA

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Abstract

Methods for producing non-uniform regradings of discrete data are discussed. Regradings which "equidistribute" the histogram of the data, that is, which transform it into a constant weight function, are obtained. Techniques for smoothing the regrading, dependent upon a continuously variable parameter, are also presented. Numerical algorithms for implementing the procedures and applications to specific examples are described.

1. Introduction

Experimental and statistical data are frequently graded into a finite set of classes. Each class, or grade, may be represented by an integer value, and to each data point, the value of its grade may be attached. The histogram of total occurrences of data points belonging to each grade may also be formed, and with each grade, we may thus associate a weight equal to its number of occurrences.

In practical applications, the original choice of grades may not represent the required information satisfactorily, and a regrading of the data is necessary. A regrading is essentially just a transformation from one set of integers into another. The most natural transformation is the uniform regrading, in which the original grades are divided as equally as possible among the new grades. In this case, the histogram of the regraded data is basically just a scaling of the original histogram. In some situations, however, it is advantageous to determine a non-uniform regrading, such that the histogram of the regraded data is as uniform, i.e. constant, as possible. In this case the total occurrences of the old grades are as equally distributed over the new grades as possible, and such a regrading is called equidistributing. Obviously we may also consider a range of regradings between these two extreme cases: uniform and equidistributing. In this report we present efficient numerical procedures for determining such regradings explicitly.

2. Statement of the Problem

We consider a set of grades represented by the integers $\{1, 2, \ldots, n\}$ and a set of non-negative weights $\{f_1, f_2, \ldots, f_n\}$ associated with these grades. We define a regrading to be a non-decreasing map from the given set of integers into another set. In particular we have:

Definition 1. If $\mathbb{Z}_k = \{1, 2, \ldots, k\}$ is the set of integers from 1 to k, and if m, n are integers, then

$$\underline{\tau} : \mathbb{Z}_n \to \mathbb{Z}_m$$

is a regrading if $1 \le i < j \le n$ implies

$$1 \le \tau_i \le \tau_j \le m \qquad \forall\, i, j .$$

(We represent the mapping $\underline{\tau}$ by the integer valued vector $(\tau_1, \tau_2, \ldots, \tau_n)$.)

For each regrading $\underline{\tau}$, a new (transformed) set of weights $\{g_1, g_2, \ldots, g_m\}$ defined by

$$g_i = \sum_{j \ni \tau_j = i} f_j, \qquad i = 1, 2, \ldots, m, \tag{1}$$

is associated with the new grades $\{1, 2, \ldots, m\}$; that is, the new weights g_i are the sum of all the f_j for which $\tau_j = i$. Denoting

$$N = \sum_{j=1}^{n} f_j, \quad \text{and} \quad M = \max_{1\le j\le n} f_j, \tag{2}$$

we observe the following obvious relations:

$$N = \sum_{1}^{m} g_i, \tag{3}$$

$$M \le M_\tau \equiv \max_{1\le i\le m} g_i, \tag{4}$$

$$N \le mM_\tau. \tag{5}$$

We wish to determine a regrading in which the transformed weights are equally distributed over the new grades. In that case, we must have, by (3):

$$g_i = N/m \qquad \forall\ i = 1, 2, \ldots, m. \tag{6}$$

Unfortunately, because of the discrete character of our transformations, we cannot expect such a regrading to be possible, except in rare cases. In particular, if $M > N/m$, it follows from (4) that (6) cannot be satisfied. We could, of course, consider regradings $\underline{\tau}$ for which M is minimal, which would give the required equi-distribution when possible. In this paper we introduce, instead, another concept of equidistribution, which corresponds to the theory discussed in [1] for continuous weight functions. In section 3 we define a regrading which is "approximately" equidistributing, and give a procedure for its construction. In the following section we consider methods for smoothing the regrading, dependent upon a continuously variable parameter. In section 5 we describe the implementation of such procedures and give results for specific examples. A more general concept of weighted regrading is discussed in the final section.

3. Equidistributing regradings

The grades $\{1, 2, \ldots, n\}$ and the corresponding weights $\{f_1, f_2, \ldots, f_n\}$ can be represented graphically on the region $[0, n]$ by a piecewise constant function, taking values f_j on the intervals $(j - 1, j)$, $j = 1, 2, \ldots, n$. (This is the histogram associated with the grading). This function may be equidistributed over m points, as in [1], simply by finding sub-intervals (x_{j-1}, x_j), $j = 1, 2, \ldots, m$ such that the integral of the function over each sub-interval is equal to a constant (given by the total integral divided by m). Then by associating each of the old grades with a specific sub-interval, a transformation is defined which gives a regrading of the discrete data. We construct the transformation explicitly as follows.

Given integer n and weights $\{f_1, f_2, \ldots, f_n\}$, $f_j \geq 0$, $\forall j$, let $F(t)$ be a piecewise-linear continuous function for $t \in [0, n]$, such that

$$F(0) = 0, \quad \text{and} \quad F(j) = \sum_{k=1}^{j} f_k, \qquad j = 1, 2, \ldots, n, \tag{7}$$

with corners at points $1, 2, \ldots, n - 1$. Obviously F is non-decreasing and F' is a piecewise constant function such that

$$F'(t) = f_j \quad \text{for} \quad t \in (j - 1, j), \qquad j = 1, 2, \ldots, n.$$

Thus F' represents the weights f_j and we consider the mid-points $t_j = j - \frac{1}{2}$ of the intervals $(j - 1, j)$ as points representing the grades j, $j = 1, 2, \ldots, n$. Now we define breakpoints x_k, $k = 0, 1, \ldots, m$ as the largest values such that

$$F(x_k) = kF(n)/m. \tag{8}$$

Obviously $x_0 = 0$, $x_m = n$, and the other breakpoints x_k, $k = 1, 2, \ldots, m - 1$, can easily be determined by inverse linear interpolation, as the function F is piecewise-linear.

By simple inspection, a regrading which equi-distributes the weights exactly is possible if (and only if) $x_1, x_2, \ldots, x_{m-1}$ are all integers. To satisfy (6) it is then necessary and sufficient to choose

$$\tau_j = k, \qquad j \text{ such that } x_{k-1} < j - \tfrac{1}{2} < x_k. \tag{9}$$

Although the cases where the exact equidistribution is possible are expected to be very few, (9) leads to a general procedure for constructing a regrading which is exact whenever possible. We make the following definition:

Definition 2. Given integers n and m, and non-negative weights $\{f_1, f_2, \ldots, f_n\}$, let the breakpoints $x_0, x_1, \ldots, x_m$ be given by (8) where F is the piecewise linear function specified by (7). The regrading $\underline{\tau}$ satisfying

$$\tau_j = k \quad \text{for all } j \text{ such that } x_{k-1} < j - \tfrac{1}{2} \leq x_k,$$

is said to "equidistribute" the weights f_j, $j = 1, 2, \ldots, n$.

Remarks: We note that another regrading where $\tau_j = k$, for all j such that $x_k \leq j - \frac{1}{2} < x_k$, could equally well be defined, and our choice is somewhat arbitrary. Similarly, if some weights f_j vanish, $F(t)$ may be constant over some interval and the equation (8) need not have a unique solution. In our definition we have explicitly specified the breakpoint to be the largest solution of (8); however, this choice is essentially irrelevant, since the original grades in this case have no weight attached to them and hence have no effect on the transformations anyway.

Also of interest is a uniform regrading, in which the old grades are evenly distributed among the new grades. Such a regrading arises from equidistributing the weights $f_j \equiv f$, $\forall\, j = 1, 2, \ldots, n$, where f is an arbitrary positive constant. In this case, a simple calculation leads to an explicit formula for the regrading constructed by Definition 2. We obtain

$$\tau_j = \left[\!\left[1 + (j - \tfrac{1}{2})\frac{m}{n} \right]\!\right]', \qquad j = 1, 2, \ldots, n, \tag{10}$$

where $[\![x]\!]'$ denotes the largest integer less than x. We observe that in the case $m = n$, the regrading given by Defintion 2 thus gives

$$\tau_j = j, \qquad j = 1, 2, \ldots, n,$$

as we would expect. Similarly, in the case $n = km$, we find that exactly k old grades are mapped into each new grade by this definition. Finally in the case $n = m + 1$, we obtain a "symmetric" transformation, which maps each old grade into one new grade, except at the centre of the region, where two old grades are joined to form one new one. We thus verify the suitability of our definition and conclude that the behaviour of the equidistributing regrading given by Definition 2 is reasonable in these special cases.

4. Smoothing the regrading

Problems where the weights are extremely unevenly distributed are difficult to regrade sensibly, and any attempt to equidistribute the data merely shifts the largest weights fairly arbitrarily into new grades. Therefore, it is natural that we now examine procedures which smooth the data in the extreme cases, in order to produce consistent and reasonable regradings. The methods we aim to produce will depend continuously on a parameter p, $0 \le p \le 1$, which determines the amount of smoothing to be applied. At one end of the scale, say $p = 0$, the method will give the equidistributing regrading of Definition 2 without smoothing, and at the other end, $p = 1$, it will give a uniform regrading.
To realize such procedures, we use the concepts developed in [1] and [2] for constrained equidistributing meshes.

There are essentially two approaches for smoothing the regrading:

1) For some $K_1 \ge 1$, enforce the restriction

$$\frac{\max_k (x_k - x_{k-1})}{\min_k (x_k - x_{k-1})} \le K_1, \tag{11}$$

giving a quasi-uniform distribution of breakpoints.

2) For some $K_2 \ge 1$, enforce the constraint

$$\frac{1}{K_2} \le \frac{x_{k+1} - x_k}{x_k - x_{k-1}} \le K_2, \tag{12}$$

giving a locally-bounded distribution of breakpoints.

Obviously, $K_i = 1$ implies the uniform distribution of the breakpoints in either case $(i = 1, 2)$, and thus a uniform regrading. On the other hand, very large K_i implies no constraint on the breakpoints. In both cases, the required constraints are enforced by a suitable change in the data, called padding (see [1]), which is essentially a smoothing of the weight function. We state here the necessary results and derive two basic, parameter dependent procedures for the padding.

Padding Procedure 1

Let the vector $\underline{f}$ represent the weights $\{f_1, f_2, \ldots, f_n\}$. To enforce restriction (11), the given weights are replaced by new (padded) weights $P_p(\underline{f})$, defined by

$$\left(P_p(\underline{f})\right)_j = \max\left(f_j, c(p)\right) \tag{13}$$

where $c(p)$ is taken to be

$$c(p) = \frac{1}{K_1} M \equiv \frac{1}{K_1} \max_i f_i . \tag{14}$$

For a gradual smoothing we may choose, for example,

$$c(p) = pM.$$

However, when the average value N/n of the data is high or low, this choice may not achieve the required result. Therefore we use, instead, an alternative definition

$$c(p) = p(pM + 2(1 - p)N/n), \tag{15}$$

which takes into account both the maximal weight and the average weight. We consider this choice to give a more natural dependence of the smoothing on the parameter p.

Padding Procedure 2

In [1] it is shown that constraint (12) is enforced by replacing the given weights $\underline{f}$ by new (padded) weights $P_p(\underline{f})$ defined as

$$\left(P_p(\underline{f})\right)_j = \max_i \frac{f_i}{1 + \lambda_p f_i |i - j|} , \tag{16}$$

where λ_p satisfies

$$m \log K_2 = \lambda_p \sum_{j=1}^{n} \left(P_p(\underline{f})\right)_j. \tag{17}$$

It is not difficult to see that for λ_p sufficiently large, (16) gives

$$\left(P_p(\underline{f})\right)_j = f_j,$$

while, for $\lambda_p = 0$ (or sufficiently small)

$$\left(P_p(\underline{f})\right)_j = \max_i f_i \equiv M.$$

To obtain a gradual smoothing, as in Procedure 1, we may define λ_p as a function of p by replacing K_2 in (17) by $1/p$. However, as the sum of the $\left(P_p(\underline{f})\right)_j$ is not available until after λ_p has been chosen, we also replace this sum by $N = \sum_{j=1}^{n} f_j$. The effect of this replacement is merely to shift the parameter inside the interval [0, 1]. The padded data is then defined by (16) with λ_p given by

$$\lambda_p = \frac{m}{N} \log \frac{1}{p}, \qquad 0 < p \le 1. \tag{18}$$

This padding procedure takes into account both the maximum of the weights, through (16), and the average weight, through (18). We observe that this padding is of a more sophisticated form than that of Procedure 1, and we expect the resulting regrading to be essentially smoother and more well-behaved.

5. Some examples

In this section we present and discuss numerical results of the equidistributing and smoothing procedures for several examples. Smooth regradings are obtained by padding the given weight function using Padding Procedure 1 or 2 described in section 4, and then equidistributing the padded weights by the method described in section 3. Solutions are obtained for values of the padding parameter p belonging to $[0, 1]$; when $p = 0$, the equidistributing regrading of Definition 2 is produced, and when $p = 1$, a uniform regrading results. The algorithms are implemented in a simple FORTRAN subroutine. The program code is given in Appendix I.

In Tables 1a, b, c we summarize the results of regrading data from 20 grades into 8 grades, where the original grades have a V-shaped weight function given explicitly by

$$f_j = |j - 10|, \qquad j = 1, 2, \ldots, 20.$$

The regradings obtained with the two padding procedures and various values of p are shown in Table 1a. We observe that the regradings change gradually from the equidistributing (E) to the uniform (U) distribution of grades. The uniform grading (U) is as uniform as possible, given that 20 old grades cannot be equally divided among the 8 new grades. (We note that the particular solution obtained is a consequence of the inequalities specified in Definition 2, and that if these are altered as discussed in section 3, then a different "uniform" solution is produced, namely: {1, 1, 2, 2, 2, 3, 3, 4, 4, 4, ...}.)

The new transformed weights for the regradings of Table 1a are presented in Table 1b, and the breakpoints used to define the regradings are shown in Table 1c. The weights of the equidistributing regrading are seen to be as constant as possible in this discrete case, while those of the uniform regrading remain approximately V-shaped. (It should also be noted that, except in the uniform case, the choice of inequalities in Definition 2 is irrelevant, since none of the breakpoints is equal to $j - \frac{1}{2}$ for any j.)

In Table 2 the results of another application are summarized. For this example the weight function is such that the equidisributing regrading from 20 to 8 grades is exact. We observe that the weights of the new grades are all equal and that the breakpoints are indeed integer-valued.

To demonstrate the difficulties in equidistributing extreme weight functions, we consider some examples where the histogram of the original grades contains large separated peaks. In the case of a single peak, where the weight function is $f_j = c\delta_{jk}$, $c > 0$, $j = 1, 2, \ldots, n$ for some k, a direct application of Definition 2 shows that $\tau_k = m/2$; that is, the regrading always places the old grade k in the middle of the new scale, regardless of its original position. Similarly, equidistributing a weight function with a few large peaks produces a regrading dependent only on the relative size and order of the peaks, but independent of their particular positions. Illustrations are shown in Table 3, which summarizes the results of equidistributing 10 grades, with a variety of simple extreme weight functions, into 5 grades. For a single peak the regrading always shifts the peak weight into grade 3, and for two equal peaks, the peak weights are always regraded into grades 2 and 4. If there are two unequal peaks, their new positions depend on their relative sizes, but not their original positions. Similar results are seen to hold in the case of weight functions with three peaks.

In Table 4 we show the affect of smoothing an extreme weight function containing three peaks. The equidistributing regrading for this function is given in the last column of Table 3. We observe that the weight originally in grade one is shifted into new grade 2. The smooth regradings, however, place this peak weight, more reasonably, in grade one. As the parameter p increases from zero, the padding increases, and the equi-distributing grading is transformed into the uniform grading. The peak weight in grade 4 is thus eventually moved into grade 5. We note that the padding of Procedure 1 is constant between the peaks, covering the central peak for $p = \frac{1}{2}, \frac{3}{4}$, and that with $p = \frac{3}{4}$ the regrading is already uniform in this case. For Procedure 2, the padding drops away smoothly from each peak, the central peak being covered by the padding only for $p = \frac{3}{4}$. In this case the uniform regrading is only produced when p is close to unity.

The difference in the behaviour of the two smoothing procedures is more explicitly illustrated in Figures 1 and 2. A weight function with three sharp peaks on 95 grades is regraded into 9 grades. On a scale from [0, 1], the peak weights occur at points 0, 0.3 and 1.0. For Padding Procedures 1 and 2, the positions of the 10 (scaled) breakpoints $0 = x_0 < x_1 \ldots < x_9 = 1.0$ are shown in the Figures as continuous functions of p, for p from 0 to 1.0.

In the smoothing produced by Padding Procedure 1, (Figures 1) there are fairly sharp changes in the positions of the breakpoints. For small p, each breakpoint changes very little as p increases, until the constant padding reaches some critical level; then the breakpoint moves rapidly into its position in the uniform distribution. By contrast, the behaviour of the breakpoints produced by Padding Procedure 2 (Figure 2) is essentially smoother. All the breakpoints move gradually from the equidistributing distribution towards positions close to the uniform distribution.

With Padding Procedure 1, we also observe that the uniform regrading is achieved fairly quickly (in this case for $p \geq 0.5$); while with Padding Procedure 2, the character of the initial distribution of the points is retained for fairly large values of p ($p \leq 0.8$, here), and the final transition to the uniform regrading is somewhat abrupt. These characteristics are accentuated by the sharpness of the peaks, and for smooth weight functions the behaviour of both procedures is more modulated. However, we may conclude, that for a small, smooth transition away from the equidistributing regrading, Procedure 2 is preferable to Procedure 1.

Complete results for the examples described in this section are given in Appendix II.

Table 1a

Old Grades	Weights	New Grades: E	Padding Procedure 1			Padding Procedure 2			U
		p = 0	p = $\frac{1}{4}$	p = $\frac{1}{2}$	p = $\frac{3}{4}$	p = $\frac{1}{4}$	p = $\frac{1}{2}$	p = $\frac{3}{4}$	p = 1
1	9.0	1	1	1	1	1	1	1	1
2	8.0	2	1	1	1	2	1	1	1
3	7.0	2	2	2	2	2	2	2	1
4	6.0	3	3	2	2	3	3	2	2
5	5.0	3	3	3	2	3	3	3	2
6	4.0	3	3	3	3	3	3	3	3
7	3.0	4	4	3	3	4	4	3	3
8	2.0	4	4	4	3	4	4	4	3
9	1.0	4	4	4	4	4	4	4	4
10	0.0	4	4	4	4	4	4	4	4
11	1.0	4	4	4	5	4	4	4	5
12	2.0	4	4	5	5	4	4	5	5
13	3.0	4	5	5	5	5	5	5	5
14	4.0	5	5	5	6	5	5	5	6
15	5.0	5	5	6	6	5	5	6	6
16	6.0	6	6	6	7	6	6	6	7
17	7.0	6	6	7	7	6	6	6	7
18	8.0	7	7	7	7	7	7	7	7
19	9.0	7	7	8	8	7	7	8	8
20	10.0	8	8	8	8	8	8	8	8

Table 1b

New Grades	New Weights							
	E	Padding Procedure 1			Padding Procedure 2			U
	p = 0	p = ¼	p = ½	p = ¾	p = ¼	p = ½	p = ¾	p = 1
1	9.0	17.0	17.0	17.0	9.0	17.0	17.0	24.0
2	15.0	7.0	13.0	18.0	15.0	7.0	13.0	11.0
3	15.0	15.0	12.0	9.0	15.0	15.0	12.0	9.0
4	12.0	9.0	4.0	1.0	9.0	9.0	4.0	1.0
5	9.0	12.0	9.0	6.0	12.0	12.0	9.0	6.0
6	13.0	13.0	11.0	9.0	13.0	13.0	18.0	9.0
7	17.0	17.0	15.0	21.0	17.0	17.0	8.0	21.0
8	10.0	10.0	19.0	19.0	10.0	10.0	19.0	19.0

Table 1c

Breakpoints

E	Padding Procedure 1			Padding Procedure 2			U
p = 0	p = ¼	p = ½	p = ¾	p = ¼	p = ½	p = ¾	p = 1
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
1.44	1.54	1.83	2.34	1.49	1.55	1.70	2.5
3.17	3.44	4.25	4.95	3.32	3.46	3.86	5.0
5.63	6.31	7.38	7.56	5.96	6.35	6.88	7.5
12.67	11.50	10.50	10.17	12.07	11.56	10.94	10.0
15.42	15.01	13.63	12.78	15.19	14.97	14.29	12.5
17.25	17.05	16.39	15.38	17.14	17.03	16.69	15.0
18.72	18.63	18.38	17.93	18.67	18.63	18.49	17.5
20.0	20.0	20.0	20.0	20.0	20.0	20.0	20.0

Table 2

Old Grades	Weights	New Grades	New Weights	Breakpoints
1	13.0	1	13	1.0
2	7.0	2		
3	6.0	2	13	3.0
4	5.0	3		
5	4.0	3		
6	4.0	3	13	6.0
7	4.0	4		
8	4.0	4		
9	3.0	4		
10	2.0	4	13	10.0
11	1.0	5		
12	2.0	5		
13	3.0	5		
14	3.0	5		
15	4.0	5	13	15.0
16	6.0	6		
17	7.0	6	13	17.0
18	6.0	7		
19	7.0	7	13	19.0
20	13.0	8	13	

Table 3

W = weights   NG = new grades

Old Grades	One Peak						Two Equal Peaks						Two Unequal Peaks								Three Peaks			
	W	NG	W	NG	W	NG	W	NG	W	NG	W	NG	W	NG	W	NG	W	NG	W	NG	W	NG	W	NG
1	10	3	0	1	0	1	10	2	10	2	0	1	10	2	10	2	5	1	0	1	20	2	20	2
2	0	5	0	1	0	1	0	3	0	3	0	1	0	4	0	4	0	2	0	1	4	3	0	3
3	0	5	0	1	0	1	0	3	0	3	0	1	0	4	0	4	0	2	0	1	0	3	0	3
4	0	5	0	1	10	3	0	3	0	3	0	1	0	4	0	4	0	2	0	1	0	3	0	3
5	0	5	0	1	0	5	0	3	10	4	0	1	0	4	5	5	0	2	5	1	20	4	0	3
6	0	5	0	1	0	5	0	3	0	5	0	1	0	4	0	5	0	2	10	4	0	5	4	3
7	0	5	0	1	0	5	0	3	0	5	0	1	0	4	0	5	0	2	0	5	0	5	0	3
8	0	5	0	1	0	5	0	3	0	5	0	1	0	4	0	5	0	2	0	5	0	5	0	3
9	0	5	0	1	0	5	0	3	0	5	10	2	0	4	0	5	0	2	0	5	0	5	20	4
10	0	5	10	3	0	5	10	4	0	5	10	4	5	5	0	5	10	4	0	5	0	5	0	5

Table 4

PW = padded weights NG = new grades

		Padding Procedure 1						Padding Procedure 2					
		$p = \frac{1}{4}$		$p = \frac{1}{2}$		$p = \frac{3}{4}$		$p = \frac{1}{4}$		$p = \frac{1}{2}$		$p = \frac{3}{4}$	
Old Grades	Weights	PW	NG	PW	NG	PW	NG	PW	NG	PW	NG	PW	NG
1	20.0	20.0	1	20.0	1	20.0	1	20.0	1	20.0	1	20.0	1
2	0.0	2.9	2	7.2	2	12.9	1	4.82	2	7.77	2	12.09	2
3	0.0	2.9	2	7.2	2	12.9	2	2.74	2	4.82	2	8.67	2
4	0.0	2.9	3	7.2	2	12.9	2	1.91	3	3.49	3	6.75	2
5	0.0	2.9	3	7.2	3	12.9	3	2.45	3	3.04	3	5.53	3
6	4.0	4.0	3	7.2	3	12.9	3	4.00	3	4.00	3	6.75	3
7	0.0	2.9	3	7.2	4	12.9	4	2.74	3	4.82	3	8.67	3
8	0.0	2.9	4	7.2	4	12.0	4	4.82	4	7.77	4	12.09	4
9	20.0	20.0	4	20.0	5	20.0	5	20.00	4	20.00	4	20.00	5
10	0.0	2.9	5	7.2	5	12.9	5	4.82	5	7.77	5	12.07	5

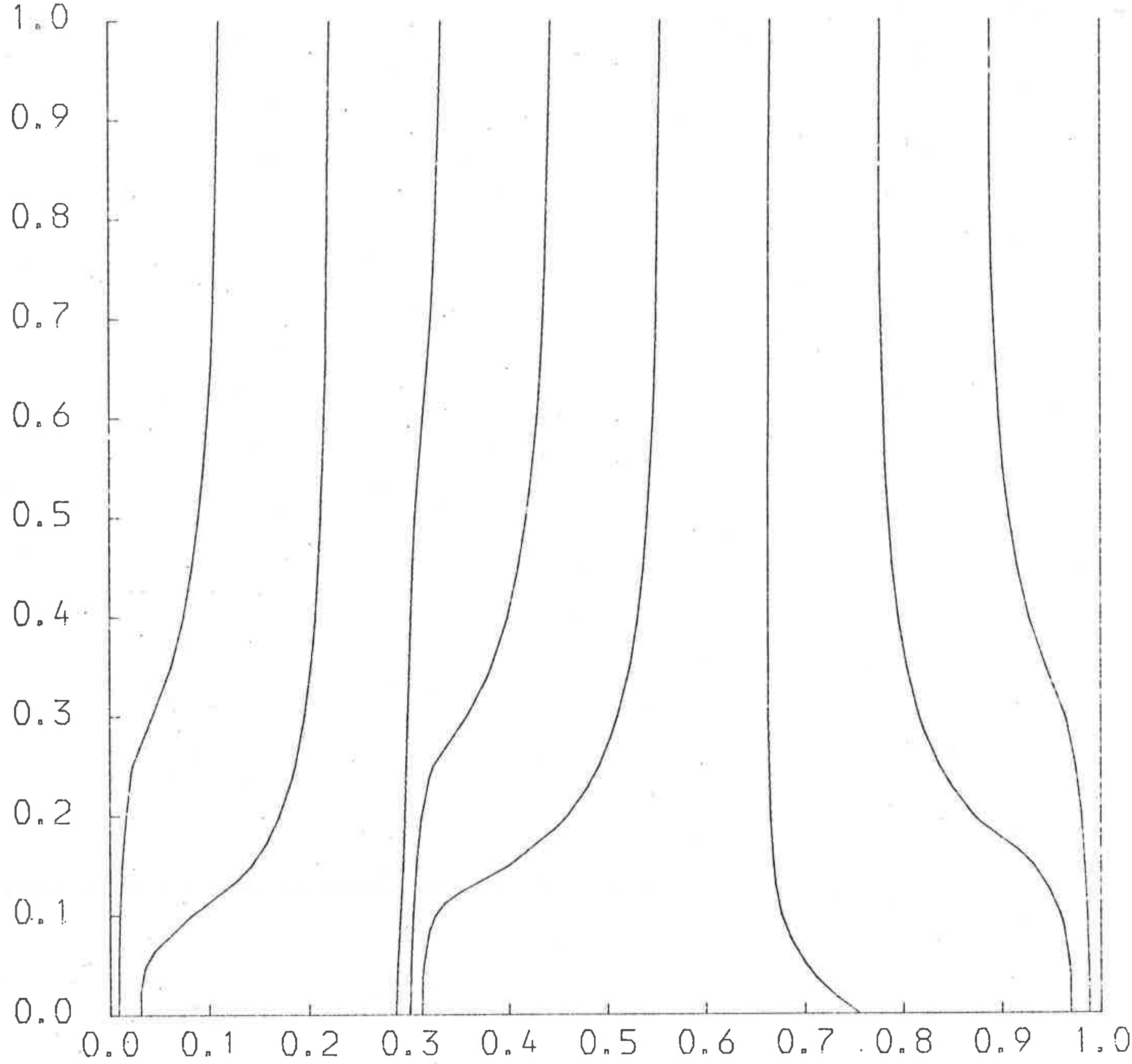

Figure 1.

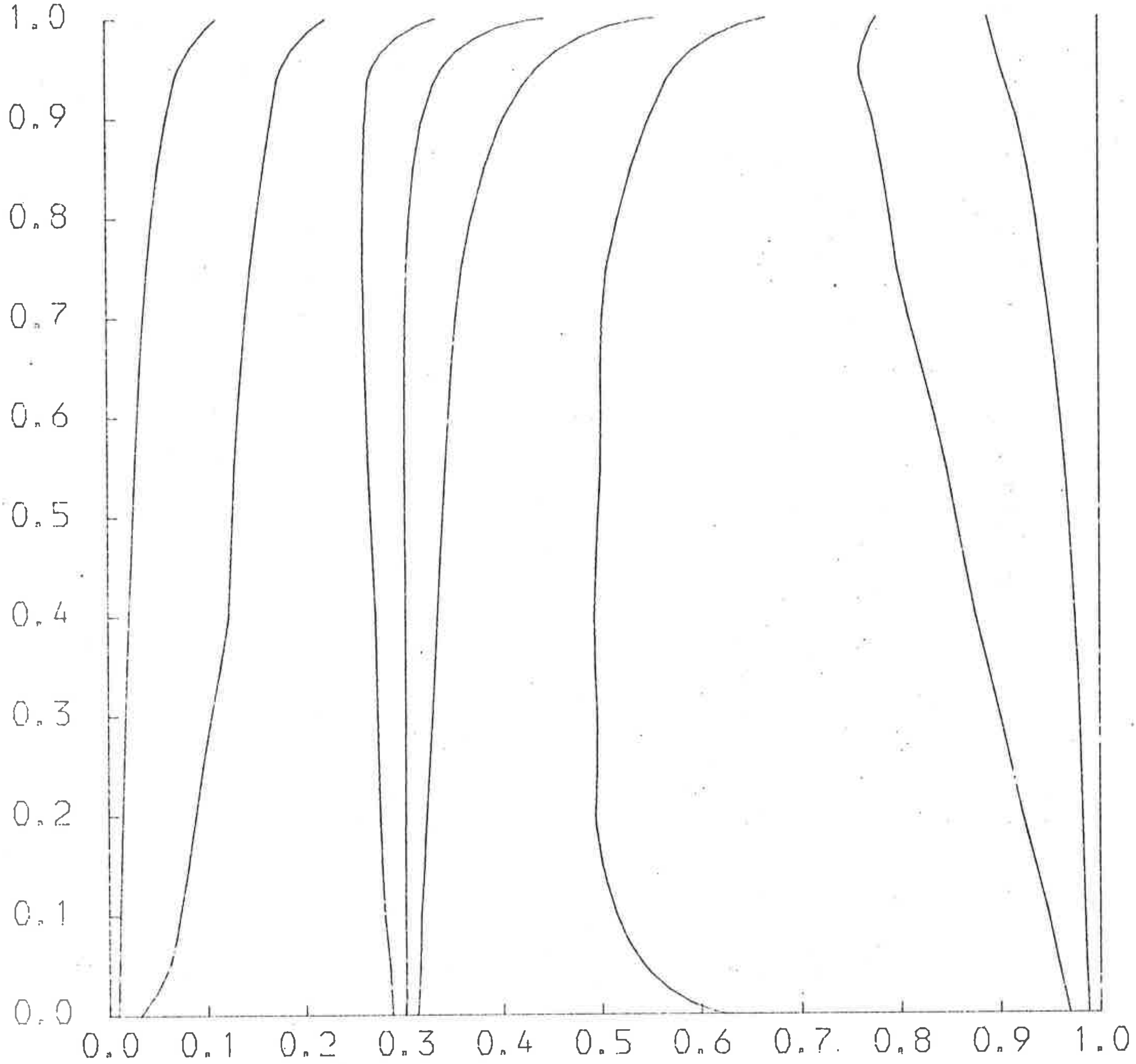

Figure 2.

6. Weighted regradings

In this section we describe an application of the procedures given in sections 3 and 4 to a more general regrading problem. In section 2 we introduced the concept of an equidistributing regrading as a regrading which transforms a given weight function, or histogram, into an (approximately) constant weight function. We now define a weighted regrading as one which transforms a given weight function into another prescribed weight function called a reference function. The equidistributing regrading is obviously a special case where the prescribed reference function is (approximately) constant. Generally, the reference function may be prescribed on a different number of grades from that of the data being regraded.

For a precise statement of the problem, let m, n q be integers, and let $\{f_1, f_2, \ldots, f_n\}$ and $\{w_1, w_2, \ldots, w_q\}$ be non-negative weights associated with sets of n and q grades, respectively. Also let $\{w_1^*, w_2^*, \ldots, w_m^*\}$ be the set of transformed weights obtained from weights w_j, $j = 1, 2, \ldots, q$, by a uniform regrading from q into m grades. We seek a regrading $\underline{\tau}$ from n into m grades with respect to weights f_j, $j = 1, 2, \ldots, n$, such that the transformed weights $\{f_1^*, f_2^*, \ldots, f_m^*\}$ are approximately equal to w_j^*, $j = 1, 2, \ldots, m$.

To give an explicit definition of $\underline{\tau}$ we proceed as in section 3. We let F be a piecewise linear function on $[0, n]$ such that

$$F(0) = 0, \qquad F(j) = \sum_{i=1}^{j} f_i, \qquad j = 1, 2, \ldots, n. \tag{19}$$

and

$$F'(t) = f_j \quad \text{for} \quad t \in (j - 1, j).$$

Similarly we define W to be a piecewise linear function on $[0, q]$ such that

$$W(0) = 0, \qquad W(j) = \sum_{i=1}^{j} w_i, \quad j = 1, 2, \ldots, q. \tag{20}$$

and

$$W'(t) = w_j \quad \text{for} \quad t \in (j-1, j).$$

Breakpoints $x_k \in [0, n]$ are then determined by

$$F(x_k) = W(kq/m)F(n)/W(q), \quad k = 0, 1, \ldots, m, \tag{21}$$

and we have the following definition:

Definition 3. Given integers n, q, and m, and non-negative weights $\{f_1, f_2, \ldots, f_n\}$ and $\{w_1, w_2, \ldots, w_q\}$, let the breakpoints x_0, x_1, $\ldots$, x_m be given by (21), where F and W are specified by (19) and (20) respectively. Then the regrading $\underline{\tau}$ satisfying

$$\tau_j = k, \quad \text{for all} \quad j \quad \text{such that} \quad x_{k-1} < j - \tfrac{1}{2} \le x_k$$

is said to distribute weights f_j, $j = 1, 2, \ldots, n$ with respect to reference weights w_j, $j = 1, 2, \ldots, q$, and $\underline{\tau}$ is called a weighted regrading.

A direct procedure for constructing the weighted regrading $\underline{\tau}$ is easy to implement, but a technique for smoothing such a regrading is less obvious to devise. With the following theorem, however, we may transform the weighted distribution problem into an equi-distribution problem and then apply the procedures of sections 3 and 4 to determine weighted regradings.

Theorem. Let $w_j > 0$, $j = 1, 2, \ldots, q$ and define

$$H(x) = W^{-1}\big(F(x)W(q)/F(n)\big), \qquad x \in [0, n]. \tag{22}$$

Then there exists $g_j \ge 0$, $j = 1, 2, \ldots, 3n$, such that

$$\sum_{i=1}^{3n} g_i = q \tag{23}$$

and $\quad \tfrac{1}{2}g_{3j-1} + \sum_{i=1}^{3j-2} g_i = H(j - \tfrac{1}{2}), \quad j = 1, 2, \ldots, n.$

Let $\underline{\sigma}$ be a $3n$ to m regrading which equidistributes weights g_j,

$j = 1, 2, \ldots, 3n$. Then $\underline{\tau}$, given by

$$\tau_j = \sigma_{3j-1}, \quad j = 1, 2, \ldots, n \tag{24}$$

is the <u>weighted</u> regrading which distributes the weights f_j, $j = 1, 2, \ldots, n$ with respect to reference weights w_j, $j = 1, 2, \ldots, q$.

<u>Proof:</u> As $w_j > 0$, the inverse function of W exists and the definition (21) of the breakpoints x_k is equivalent to

$$H(x_k) = kq/m, \qquad k = 0, 1, \ldots, m. \tag{25}$$

The function H is a non-decreasing piecewise linear function mapping $[0, n]$ into $[0, q]$ with corners at points j, $j = 1, 2, \ldots, n - 1$ and at points t_i where $H(t_i) = i$, $i = 1, 2, \ldots, q - 1$. The weighted regrading $\underline{\tau}$ of Definition 3 is uniquely determined by the relative positions of points x_k and points $j - \frac{1}{2}$, but is independent of the actual values of the breakpoints x_k as long as the relative ordering of the points is preserved.

Thus the same regrading $\underline{\tau}$ is obtained with the set of breakpoints x_k which satisfy

$$G(3x_k) = kq/m, \qquad k = 0, 1, 2, \ldots, m, \tag{26}$$

where G is <u>any</u> non-decreasing function from $[0, 3n]$ to $[0, q]$ such that

$$G(0) = 0, \qquad G(3n) = q,$$

and
$$G\big(3(j - \tfrac{1}{2})\big) = H(j - \tfrac{1}{2}), \qquad j = 1, 2, \ldots, n. \tag{27}$$

In particular, if G is the piecewise linear function with corners at points $1, 2, \ldots, 3n - 1$ such that

$$G'(t) = g_j, \ t \in (j - 1, j), \quad j = 1, 2, \ldots, 3n, \tag{28}$$

then G satisfies (27) by the definition of the weights g_j given in the theorem. Furthermore, the regrading $\underline{\sigma}$ from $[0, 3n]$ to $[0, m]$ which equidistributes the weights g_j, $j = 1, 2, \ldots, 3n$ is determined by breakpoints $3x_k$ satisfying (26). A direct application of Definition 2 then gives that $x_{k-1} < j - \frac{1}{2} \le x_k$ implies $\tau_j = \sigma_{3j-1} = k$, and hence $\underline{\tau}$ is the required weighted regrading.

The existence of the weights g_j, $j = 1, 2, \ldots, 3n$ is shown by explicit construction. We may choose, for example,

$$g_1 = H(\tfrac{1}{2}), \quad g_{3n} = q - H(n - \tfrac{1}{2})$$

$$g_{3j-1} = 0, \quad j = 1, 2, \ldots, n$$

$$g_{3j} = g_{3j+1} = \tfrac{1}{2}\left(H(j + \tfrac{1}{2}) - H(j - \tfrac{1}{2})\right), \quad j = 1, 2, \ldots, n - 1,$$

and the theorem is proved.

We conclude that a generalized weighted regrading may be obtained by the equidistributing procedure of section 3, and a variety of smooth regradings determined by applying the padding procedures of section 4 to the weights g_j, $j = 1, 2, \ldots, 3n$ defined in (23). Unfortunately the weighted regradings cannot be determined exactly by an equidistributing regrading from n to m grades. However, we expect that good approximations to the weighted regradings can be obtained by equidistributing an appropriate choice of n weights, based on the Theorem. Experiments are currently being carried out and will be reported elsewhere. A different approach to the problem of determining smooth weighted regradings will also be discussed in another paper.

Acknowledgements

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References

[1] Kautsky, J. and Nichols, N.K. Equidistributing meshes with constraints. Stanford University Computer Science Department Rept. Stan-CS-79-746, 1979.

[2] Pereyra, V. and Sewell, E.G. Mesh selection for discrete solution of boundary problems in ordinary differential equations. Num. Math. 23, 1975, 261.

Appendix I

The procedures described in sections 3 and 4, for finding equidistributing regradings, are implemented in a FORTRAN subroutine DEQUI with the following specifications:

```
CALL DEQUI(N, M, F, NG, IP, P)
```

Parameters:

Input: N - number of old grades

M - number of new grades

F - array of dimension N containing the non-negative weights

IP - type of smoothing required -

IP = 0 .. no smoothing

IP = 1 .. smoothing as in Padding Procedure 1

IP = 2 .. smoothing as in Padding Procedure 2

P - smoothing parameter, $0 \leq P \leq 1$

P = 0 .. no padding (as for IP = 0)

P = 1 .. uniform regrading results

P is irrelevant if IP = 0

Output: NG - integer array of dimension N contaning the regrading τ_j.

Auxiliary subroutines: S/R CORW, as described in [1] is required if DEQUI is called with IP = 2.

Comments: The working fields X, W, Z, must have dimension at least N + IP, and the dimension of X must also be at least M.

A complete listing of the subroutines DEQUI and CORW follows.

```
      SUBROUTINE PEQUI(N,M,Y,NK,IP,P)
      DIMENSION Y(N),NK(N),X(100),W(100),Z(100)
      COMMON INC,W,X,Z
      INC=0
      W(1)=Y(1)
      Z(1)=Y(1)
      YM=Y(1)
      DO 1 J=2,N
      W(J)=Y(J)
      IF (Y(J).GT.YM) YM=Y(J)
1     Z(J)=Z(J-1)+Y(J)
      ZZ=Z(N)
      IF (IP.EQ.0.OR.P.EQ.0) GOTO 2
      IF(IP.EQ.1) GOTO 5
      DO 6 J=1,N
      X(J+2)=J+0.5
6     Z(J+1)=Y(J)
      Z(1)=0
      Z(N+2)=0
      X(1)=-0.5
      X(2)=0.5
      PA=-M*ALOG(P)/ZZ
      CALL CORW (X,Z,W,N+2,PA)
      INC=1
      GOTO 7
5     INC =0
      PA=P*(P*YM+2.*(1.-P)*ZZ/N)
      DO 3 I=1,N
      IF (W(I).LT.PA) W(I)=PA
3     CONTINUE
7     Z(1)=W(1+INC)
      DO 4 J=2,N
4     Z(J)=Z(J-1)+W(INC+J)
2     CONTINUE
      D=Z(N)/M
      I=1
      J=1
12    IF (Z(I).GT.J*D) GOTO 11
      I=I+1
      IF (I.GT.N) GOTO 20
      GOTO 12
11    CONTINUE
      ZZ=0
      IF (I.GT.1) ZZ=Z(I-1)
      X(J)=I+(J*D-ZZ)/(Z(I)-ZZ) -1
      J=J+1
      IF (J.LT.M) GOTO 12
20    X(M)=N
      K=1
      DO 18 J=1,N
16    IF (J-0.5.LE.X(K)) GOTO 17
      K=K+1
      GOTO 16
17    NK(J)=K
18    CONTINUE
      RETURN
      END
```

```
      SUBROUTINE CORH(X,Y,Z,N,CL)
      DIMENSION X(N),Y(N),Z(N)
      NTF=0
      J=0
1     J=J+1
      Z(J)=Y(J)
      IF (J.GE.N) GOTO 20
      IF (Y(J+1).GE.Z(J)) GOTO 1
      IF (Y(J+1).GE.Z(J)) GOTO 1
12    T=1.D0/(CL*(X(J+1)-X(J))+1.D0/Z(J))
      IF (Y(J+1).GE.T) GOTO 1
      TC =1.D0/Z(J) - CL*X(J)
13    J=J+1
      Z(J)=T
      IF (J.GE.N) GOTO 20
      T=1.D0/(CL*X(J+1)+TC)
      IF (Y(J+1).GT.T) GOTO 1
      GOTO 13
20    J=N+1
21    J=J-1
      IF (J.LE.1) RETURN
      IF (Z(J-1).GE.Z(J)) GOTO 21
      T=1.D0/(1.D0/Z(J) + CL*(X(J)-X(J-1)))
      IF (Z(J-1).GE.T) GOTO 21
      TC=1.D0/Z(J) + CL*X(J)
23    J=J-1
      Z(J)=T
      IF (J.LE.1) RETURN
      T=1.D0/(TC-CL*X(J-1))
      IF (Z(J-1).GT.T) GOTO 21
      GOTO 23
      END
     FINISH
****
```

Appendix II

Example 1.

20 OLD GRADES, 8 NEW GRADES, PADDING TYPE 0, SIZE 0.00

GRADES OLD	GRADES NEW	WEIGHTS GIVEN	WEIGHTS PADDED	PARTIAL SUMS
1	1	9.000	9.000	9.000
2	2	8.000	8.000	17.000
3	2	7.000	7.000	24.000
4	3	6.000	6.000	30.000
5	3	5.000	5.000	35.000
6	3	4.000	4.000	39.000
7	4	3.000	3.000	42.000
8	4	2.000	2.000	44.000
9	4	1.000	1.000	45.000
10	4	0.000	0.000	45.000
11	4	1.000	1.000	46.000
12	4	2.000	2.000	48.000
13	4	3.000	3.000	51.000
14	5	4.000	4.000	55.000
15	5	5.000	5.000	60.000
16	6	6.000	6.000	66.000
17	6	7.000	7.000	73.000
18	7	8.000	8.000	81.000
19	7	9.000	9.000	90.000
20	8	10.000	10.000	100.000

NEW GRADES	NEW WEIGHTS	BREAK POINTS
1	9.000	1.437
2	15.000	3.167
3	15.000	5.625
4	12.000	12.667
5	9.000	15.417
6	13.000	17.250
7	17.000	18.722
8	10.000	20.000

20 OLD GRADES, 8 NEW GRADES, PADDING TYPE 1, SIZE 0.25

GRADES		WEIGHTS		PARTIAL
OLD	NEW	GIVEN	PADDED	SUMS
1	1	9.000	9.000	9.000
2	1	8.000	8.000	17.000
3	2	7.000	7.000	24.000
4	3	6.000	6.000	30.000
5	3	5.000	5.000	35.000
6	3	4.000	4.000	39.000
7	4	3.000	3.000	42.000
8	4	2.000	2.500	44.500
9	4	1.000	2.500	47.000
10	4	0.000	2.500	49.500
11	4	1.000	2.500	52.000
12	4	2.000	2.500	54.500
13	5	3.000	3.000	57.500
14	5	4.000	4.000	61.500
15	5	5.000	5.000	66.500
16	6	6.000	6.000	72.500
17	6	7.000	7.000	79.500
18	7	8.000	8.000	87.500
19	7	9.000	9.000	96.500
20	8	10.000	10.000	106.500

	NEW	
GRADES	WEIGHTS	BREAK POINTS
1	17.000	1.539
2	7.000	3.437
3	15.000	6.312
4	9.000	11.500
5	12.000	15.010
6	13.000	17.047
7	17.000	18.632
8	10.000	20.000

20 OLD GRADES, 8 NEW GRADES, PADDING TYPE 1, SIZE 0.50

GRADES		WEIGHTS		PARTIAL
OLD	NEW	GIVEN	PADDED	SUMS
1	1	9.000	9.000	9.000
2	1	8.000	8.000	17.000
3	2	7.000	7.000	24.000
4	2	6.000	6.000	30.000
5	3	5.000	5.000	35.000
6	3	4.000	5.000	40.000
7	3	3.000	5.000	45.000
8	4	2.000	5.000	50.000
9	4	1.000	5.000	55.000
10	4	0.000	5.000	60.000
11	4	1.000	5.000	65.000
12	5	2.000	5.000	70.000
13	5	3.000	5.000	75.000
14	5	4.000	5.000	80.000
15	6	5.000	5.000	85.000
16	6	6.000	6.000	91.000
17	7	7.000	7.000	98.000
18	7	8.000	8.000	106.000
19	8	9.000	9.000	115.000
20	8	10.000	10.000	125.000

NEW GRADES	WEIGHTS	BREAK POINTS
1	17.000	1.828
2	13.000	4.250
3	12.000	7.375
4	4.000	10.500
5	9.000	13.625
6	11.000	16.393
7	15.000	18.375
8	19.000	20.000

----

20 OLD GRADES, 8 NEW GRADES, PADDING TYPE 1, SIZE 0.75

GRADES OLD	GRADES NEW	WEIGHTS GIVEN	WEIGHTS PADDED	PARTIAL SUMS
1	1	9.000	9.000	9.000
2	1	8.000	8.000	17.000
3	2	7.000	7.500	24.500
4	2	6.000	7.500	32.000
5	2	5.000	7.500	39.500
6	3	4.000	7.500	47.000
7	3	3.000	7.500	54.500
8	3	2.000	7.500	62.000
9	4	1.000	7.500	69.500
10	4	0.000	7.500	77.000
11	5	1.000	7.500	84.500
12	5	2.000	7.500	92.000
13	5	3.000	7.500	99.500
14	6	4.000	7.500	107.000
15	6	5.000	7.500	114.500
16	7	6.000	7.500	122.000
17	7	7.000	7.500	129.500
18	7	8.000	8.000	137.500
19	8	9.000	9.000	146.500
20	8	10.000	10.000	156.500

NEW GRADES	NEW WEIGHTS	BREAK POINTS
1	17.000	2.342
2	18.000	4.950
3	9.000	7.558
4	1.000	10.167
5	6.000	12.775
6	9.000	15.383
7	21.000	17.930
8	19.000	20.000

20 OLD GRADES, 8 NEW GRADES, PADDING TYPE 2, SIZE 0.25

GRADES OLD	GRADES NEW	WEIGHTS GIVEN	WEIGHTS PADDED	PARTIAL SUMS
1	1	9.000	9.000	9.000
2	2	8.000	8.000	17.000
3	2	7.000	7.000	24.000
4	3	6.000	6.000	30.000
5	3	5.000	5.000	35.000
6	3	4.000	4.000	39.000
7	4	3.000	3.000	42.000
8	4	2.000	2.251	44.251
9	4	1.000	1.801	46.052
10	4	0.000	1.501	47.554
11	4	1.000	1.801	49.355
12	4	2.000	2.251	51.606
13	5	3.000	3.000	54.606
14	5	4.000	4.000	58.606
15	5	5.000	5.000	63.606
16	6	6.000	6.000	69.606
17	6	7.000	7.000	76.606
18	7	8.000	8.000	84.606
19	7	9.000	9.000	93.606
20	8	10.000	10.000	103.606

NEW GRADES	WEIGHTS	BREAK POINTS
1	9.000	1.494
2	15.000	3.317
3	15.000	5.963
4	9.000	12.066
5	12.000	15.191
6	13.000	17.137
7	17.000	18.672
8	10.000	20.000

20 OLD GRADES, 8 NEW GRADES, PADDING TYPE 2, SIZE 0.50

GRADES		WEIGHTS		PARTIAL
OLD	NEW	GIVEN	PADDED	SUMS
1	1	9.000	9.000	9.000
2	1	8.000	8.000	17.000
3	2	7.000	7.000	24.000
4	3	6.000	6.000	30.000
5	3	5.000	5.000	35.000
6	3	4.000	4.000	39.000
7	4	3.000	3.274	42.274
8	4	2.000	2.771	45.045
9	4	1.000	2.402	47.446
10	4	0.000	2.120	49.566
11	4	1.000	2.402	51.968
12	4	2.000	2.771	54.739
13	5	3.000	3.274	58.012
14	5	4.000	4.000	62.012
15	5	5.000	5.000	67.012
16	6	6.000	6.000	73.012
17	6	7.000	7.000	80.012
18	7	8.000	8.000	88.012
19	7	9.000	9.000	97.012
20	8	10.000	10.000	107.012

NEW		BREAK POINTS
GRADES	WEIGHTS	
1	17.000	1.547
2	7.000	3.459
3	15.000	6.345
4	9.000	11.555
5	12.000	14.974
6	13.000	17.031
7	17.000	18.625
8	10.000	20.000

20 OLD GRADES, 8 NEW GRADES, PADDING TYPE 2, SIZE 0.75

GRADES OLD	GRADES NEW	WEIGHTS GIVEN	WEIGHTS PADDED	PARTIAL SUMS
1	1	9.000	9.000	9.000
2	1	8.000	8.000	17.000
3	2	7.000	7.000	24.000
4	2	6.000	6.029	30.029
5	3	5.000	5.294	35.323
6	3	4.000	4.719	40.042
7	3	3.000	4.257	44.299
8	4	2.000	3.877	48.176
9	4	1.000	3.559	51.735
10	4	0.000	3.290	55.025
11	4	1.000	3.559	58.585
12	5	2.000	3.877	62.462
13	5	3.000	4.257	66.719
14	5	4.000	4.719	71.438
15	6	5.000	5.294	76.732
16	6	6.000	6.029	82.761
17	6	7.000	7.000	89.761
18	7	8.000	8.000	97.761
19	8	9.000	9.000	106.761
20	8	10.000	10.000	116.761

NEW GRADES	NEW WEIGHTS	BREAK POINTS
1	17.000	1.699
2	13.000	3.861
3	12.000	6.879
4	4.000	10.943
5	9.000	14.290
6	18.000	16.687
7	8.000	18.489
8	19.000	20.000

20 OLD GRADES, 8 NEW GRADES, PADDING TYPE 2, SIZE 1.00

GRADES OLD	GRADES NEW	WEIGHTS GIVEN	WEIGHTS PADDED	PARTIAL SUMS
1	1	9.000	10.000	10.000
2	1	8.000	10.000	20.000
3	1	7.000	10.000	30.000
4	2	6.000	10.000	40.000
5	2	5.000	10.000	50.000
6	3	4.000	10.000	60.000
7	3	3.000	10.000	70.000
8	3	2.000	10.000	80.000
9	4	1.000	10.000	90.000
10	4	0.000	10.000	100.000
11	5	1.000	10.000	110.000
12	5	2.000	10.000	120.000
13	5	3.000	10.000	130.000
14	6	4.000	10.000	140.000
15	6	5.000	10.000	150.000
16	7	6.000	10.000	160.000
17	7	7.000	10.000	170.000
18	7	8.000	10.000	180.000
19	8	9.000	10.000	190.000
20	8	10.000	10.000	200.000

NEW GRADES	WEIGHTS	BREAK POINTS
1	24.000	2.500
2	11.000	5.000
3	9.000	7.500
4	1.000	10.000
5	6.000	12.500
6	9.000	15.000
7	21.000	17.500
8	19.000	20.000

Example 2.

20 OLD GRADES, 8 NEW GRADES, PADDING TYPE 0, SIZE 0.00

GRADES		WEIGHTS		PARTIAL
OLD	NEW	GIVEN	PADDED	SUMS
1	1	13.000	13.000	13.000
2	2	7.000	7.000	20.000
3	2	6.000	6.000	26.000
4	3	5.000	5.000	31.000
5	3	4.000	4.000	35.000
6	3	4.000	4.000	39.000
7	4	4.000	4.000	43.000
8	4	4.000	4.000	47.000
9	4	3.000	3.000	50.000
10	4	2.000	2.000	52.000
11	5	1.000	1.000	53.000
12	5	2.000	2.000	55.000
13	5	3.000	3.000	58.000
14	5	3.000	3.000	61.000
15	5	4.000	4.000	65.000
16	6	6.000	6.000	71.000
17	6	7.000	7.000	78.000
18	7	6.000	6.000	84.000
19	7	7.000	7.000	91.000
20	8	13.000	13.000	104.000

NEW		BREAK POINTS
GRADES	WEIGHTS	
1	13.000	1.000
2	13.000	3.000
3	13.000	6.000
4	13.000	10.000
5	13.000	15.000
6	13.000	17.000
7	13.000	19.000
8	13.000	20.000

Example 3.

(a) One Peak.

10 OLD GRADES, 5 NEW GRADES, PADDING TYPE 0, SIZE 0.00

GRADES OLD	GRADES NEW	WEIGHTS GIVEN	WEIGHTS PADDED	PARTIAL SUMS
1	3	100.000	100.000	100.000
2	5	0.000	0.000	100.000
3	5	0.000	0.000	100.000
4	5	0.000	0.000	100.000
5	5	0.000	0.000	100.000
6	5	0.000	0.000	100.000
7	5	0.000	0.000	100.000
8	5	0.000	0.000	100.000
9	5	0.000	0.000	100.000
10	5	0.000	0.000	100.000

NEW GRADES	WEIGHTS	BREAK POINTS
1	0.000	0.200
2	0.000	0.400
3	100.000	0.600
4	0.000	0.800
5	0.000	10.000

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10 OLD GRADES, 5 NEW GRADES, PADDING TYPE 0, SIZE 0.00

GRADES OLD	GRADES NEW	WEIGHTS GIVEN	WEIGHTS PADDED	PARTIAL SUMS
1	1	0.000	0.000	0.000
2	1	0.000	0.000	0.000
3	1	0.000	0.000	0.000
4	1	0.000	0.000	0.000
5	3	100.000	100.000	100.000
6	5	0.000	0.000	100.000
7	5	0.000	0.000	100.000
8	5	0.000	0.000	100.000
9	5	0.000	0.000	100.000
10	5	0.000	0.000	100.000

NEW GRADES	WEIGHTS	BREAK POINTS
1	0.000	4.200
2	0.000	4.400
3	100.000	4.600
4	0.000	4.800
5	0.000	10.000

10 OLD GRADES, 5 NEW GRADES, PADDING TYPE 0, SIZE 0.00

GRADES OLD	GRADES NEW	WEIGHTS GIVEN	WEIGHTS PADDED	PARTIAL SUMS
1	1	0.000	0.000	0.000
2	1	0.000	0.000	0.000
3	1	0.000	0.000	0.000
4	1	0.000	0.000	0.000
5	1	0.000	0.000	0.000
6	1	0.000	0.000	0.000
7	1	0.000	0.000	0.000
8	1	0.000	0.000	0.000
9	1	0.000	0.000	0.000
10	3	100.000	100.000	100.000

NEW GRADES	NEW WEIGHTS	BREAK POINTS
1	0.000	9.200
2	0.000	9.400
3	100.000	9.600
4	0.000	9.800
5	0.000	10.000

(b) Two Equal Peaks.

10 OLD GRADES, 5 NEW GRADES, PADDING TYPE 0, SIZE 0.00

GRADES		WEIGHTS		PARTIAL
OLD	NEW	GIVEN	PADDED	SUMS
1	2	100.000	100.000	100.000
2	3	0.000	0.000	100.000
3	3	0.000	0.000	100.000
4	3	0.000	0.000	100.000
5	4	100.000	100.000	200.000
6	5	0.000	0.000	200.000
7	5	0.000	0.000	200.000
8	5	0.000	0.000	200.000
9	5	0.000	0.000	200.000
10	5	0.000	0.000	200.000

NEW GRADES	WEIGHTS	BREAK POINTS
1	0.000	0.400
2	100.000	0.800
3	0.000	4.200
4	100.000	4.600
5	0.000	10.000

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10 OLD GRADES, 5 NEW GRADES, PADDING TYPE 0, SIZE 0.00

GRADES		WEIGHTS		PARTIAL
OLD	NEW	GIVEN	PADDED	SUMS
1	2	100.000	100.000	100.000
2	4	100.000	100.000	200.000
3	5	0.000	0.000	200.000
4	5	0.000	0.000	200.000
5	5	0.000	0.000	200.000
6	5	0.000	0.000	200.000
7	5	0.000	0.000	200.000
8	5	0.000	0.000	200.000
9	5	0.000	0.000	200.000
10	5	0.000	0.000	200.000

NEW GRADES	WEIGHTS	BREAK POINTS
1	0.000	0.400
2	100.000	0.800
3	0.000	1.200
4	100.000	1.600
5	0.000	10.000

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10 OLD GRADES, 5 NEW GRADES, PADDING TYPE 0, SIZE 0.00

GRADES OLD	GRADES NEW	WEIGHTS GIVEN	WEIGHTS PADDED	PARTIAL SUMS
1	1	0.000	0.000	0.000
2	1	0.000	0.000	0.000
3	1	0.000	0.000	0.000
4	1	0.000	0.000	0.000
5	2	100.000	100.000	100.000
6	4	100.000	100.000	200.000
7	5	0.000	0.000	200.000
8	5	0.000	0.000	200.000
9	5	0.000	0.000	200.000
10	5	0.000	0.000	200.000

NEW GRADES	WEIGHTS	BREAK POINTS
1	0.000	4.400
2	100.000	4.800
3	0.000	5.200
4	100.000	5.600
5	0.000	10.000

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10 OLD GRADES, 5 NEW GRADES, PADDING TYPE 0, SIZE 0.00

GRADES OLD	GRADES NEW	WEIGHTS GIVEN	WEIGHTS PADDED	PARTIAL SUMS
1	1	0.000	0.000	0.000
2	1	0.000	0.000	0.000
3	1	0.000	0.000	0.000
4	1	0.000	0.000	0.000
5	1	0.000	0.000	0.000
6	1	0.000	0.000	0.000
7	1	0.000	0.000	0.000
8	1	0.000	0.000	0.000
9	2	100.000	100.000	100.000
10	4	100.000	100.000	200.000

NEW GRADES	WEIGHTS	BREAK POINTS
1	0.000	8.400
2	100.000	8.800
3	0.000	9.200
4	100.000	9.600
5	0.000	10.000

(c) Two Unequal Peaks.

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10 OLD GRADES, 5 NEW GRADES, PADDING TYPE 0, SIZE 0.00

GRADES OLD	GRADES NEW	WEIGHTS GIVEN	WEIGHTS PADDED	PARTIAL SUMS
1	2	100.000	100.000	100.000
2	4	0.000	0.000	100.000
3	4	0.000	0.000	100.000
4	4	0.000	0.000	100.000
5	4	0.000	0.000	100.000
6	4	0.000	0.000	100.000
7	4	0.000	0.000	100.000
8	4	0.000	0.000	100.000
9	4	0.000	0.000	100.000
10	5	50.000	50.000	150.000

NEW GRADES	WEIGHTS	BREAK POINTS
1	0.000	0.300
2	100.000	0.600
3	0.000	0.900
4	0.000	9.400
5	50.000	10.000

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10 OLD GRADES, 5 NEW GRADES, PADDING TYPE 0, SIZE 0.00

GRADES OLD	GRADES NEW	WEIGHTS GIVEN	WEIGHTS PADDED	PARTIAL SUMS
1	1	50.000	50.000	50.000
2	2	0.000	0.000	50.000
3	2	0.000	0.000	50.000
4	2	0.000	0.000	50.000
5	2	0.000	0.000	50.000
6	2	0.000	0.000	50.000
7	2	0.000	0.000	50.000
8	2	0.000	0.000	50.000
9	2	0.000	0.000	50.000
10	4	100.000	100.000	150.000

NEW GRADES	WEIGHTS	BREAK POINTS
1	50.000	0.600
2	0.000	9.100
3	0.000	9.400
4	100.000	9.700
5	0.000	10.000

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10 OLD GRADES, 5 NEW GRADES, PADDING TYPE 0, SIZE 0.00

GRADES		WEIGHTS		PARTIAL
OLD	NEW	GIVEN	PADDED	SUMS
1	2	100.000	100.000	100.000
2	4	0.000	0.000	100.000
3	4	0.000	0.000	100.000
4	4	0.000	0.000	100.000
5	5	50.000	50.000	150.000
6	5	0.000	0.000	150.000
7	5	0.000	0.000	150.000
8	5	0.000	0.000	150.000
9	5	0.000	0.000	150.000
10	5	0.000	0.000	150.000

NEW GRADES	WEIGHTS	BREAK POINTS
1	0.000	0.300
2	100.000	0.600
3	0.000	0.900
4	0.000	4.400
5	50.000	10.000

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10 OLD GRADES, 5 NEW GRADES, PADDING TYPE 0, SIZE 0.00

GRADES		WEIGHTS		PARTIAL
OLD	NEW	GIVEN	PADDED	SUMS
1	1	50.000	50.000	50.000
2	2	0.000	0.000	50.000
3	2	0.000	0.000	50.000
4	2	0.000	0.000	50.000
5	4	100.000	100.000	150.000
6	5	0.000	0.000	150.000
7	5	0.000	0.000	150.000
8	5	0.000	0.000	150.000
9	5	0.000	0.000	150.000
10	5	0.000	0.000	150.000

NEW GRADES	WEIGHTS	BREAK POINTS
1	50.000	0.600
2	0.000	4.100
3	0.000	4.400
4	100.000	4.700
5	0.000	10.000

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10 OLD GRADES, 5 NEW GRADES, PADDING TYPE 0, SIZE 0.00

GRADES OLD	GRADES NEW	WEIGHTS GIVEN	WEIGHTS PADDED	PARTIAL SUMS
1	2	100.000	100.000	100.000
2	5	50.000	50.000	150.000
3	5	0.000	0.000	150.000
4	5	0.000	0.000	150.000
5	5	0.000	0.000	150.000
6	5	0.000	0.000	150.000
7	5	0.000	0.000	150.000
8	5	0.000	0.000	150.000
9	5	0.000	0.000	150.000
10	5	0.000	0.000	150.000

NEW GRADES	WEIGHTS	BREAK POINTS
1	0.000	0.300
2	100.000	0.600
3	0.000	0.900
4	0.000	1.400
5	50.000	10.000

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10 OLD GRADES, 5 NEW GRADES, PADDING TYPE 0, SIZE 0.00

GRADES OLD	GRADES NEW	WEIGHTS GIVEN	WEIGHTS PADDED	PARTIAL SUMS
1	1	50.000	50.000	50.000
2	4	100.000	100.000	150.000
3	5	0.000	0.000	150.000
4	5	0.000	0.000	150.000
5	5	0.000	0.000	150.000
6	5	0.000	0.000	150.000
7	5	0.000	0.000	150.000
8	5	0.000	0.000	150.000
9	5	0.000	0.000	150.000
10	5	0.000	0.000	150.000

NEW GRADES	WEIGHTS	BREAK POINTS
1	50.000	0.600
2	0.000	1.100
3	0.000	1.400
4	100.000	1.700
5	0.000	10.000

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10 OLD GRADES, 5 NEW GRADES, PADDING TYPE 0, SIZE 0.00

GRADES OLD	GRADES NEW	WEIGHTS GIVEN	WEIGHTS PADDED	PARTIAL SUMS
1	1	0.000	0.000	0.000
2	1	0.000	0.000	0.000
3	1	0.000	0.000	0.000
4	1	0.000	0.000	0.000
5	2	100.000	100.000	100.000
6	5	50.000	50.000	150.000
7	5	0.000	0.000	150.000
8	5	0.000	0.000	150.000
9	5	0.000	0.000	150.000
10	5	0.000	0.000	150.000

NEW GRADES	NEW WEIGHTS	BREAK POINTS
1	0.000	4.300
2	100.000	4.600
3	0.000	4.900
4	0.000	5.400
5	50.000	10.000

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10 OLD GRADES, 5 NEW GRADES, PADDING TYPE 0, SIZE 0.00

GRADES OLD	GRADES NEW	WEIGHTS GIVEN	WEIGHTS PADDED	PARTIAL SUMS
1	1	0.000	0.000	0.000
2	1	0.000	0.000	0.000
3	1	0.000	0.000	0.000
4	1	0.000	0.000	0.000
5	1	50.000	50.000	50.000
6	4	100.000	100.000	150.000
7	5	0.000	0.000	150.000
8	5	0.000	0.000	150.000
9	5	0.000	0.000	150.000
10	5	0.000	0.000	150.000

NEW GRADES	NEW WEIGHTS	BREAK POINTS
1	50.000	4.600
2	0.000	5.100
3	0.000	5.400
4	100.000	5.700
5	0.000	10.000

(d) Three Peaks.

10 OLD GRADES, 5 NEW GRADES, PADDING TYPE 0, SIZE 0.00

GRADES OLD	GRADES NEW	WEIGHTS GIVEN	WEIGHTS PADDED	PARTIAL SUMS
1	2	100.000	100.000	100.000
2	3	20.000	20.000	120.000
3	3	0.000	0.000	120.000
4	3	0.000	0.000	120.000
5	4	100.000	100.000	220.000
6	5	0.000	0.000	220.000
7	5	0.000	0.000	220.000
8	5	0.000	0.000	220.000
9	5	0.000	0.000	220.000
10	5	0.000	0.000	220.000

NEW GRADES	WEIGHTS	BREAK POINTS
1	0.000	0.440
2	100.000	0.880
3	20.000	4.120
4	100.000	4.560
5	0.000	10.000

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10 OLD GRADES, 5 NEW GRADES, PADDING TYPE 0, SIZE 0.00

GRADES OLD	GRADES NEW	WEIGHTS GIVEN	WEIGHTS PADDED	PARTIAL SUMS
1	2	100.000	100.000	100.000
2	3	0.000	0.000	100.000
3	3	0.000	0.000	100.000
4	3	0.000	0.000	100.000
5	3	0.000	0.000	100.000
6	3	20.000	20.000	120.000
7	4	100.000	100.000	220.000
8	5	0.000	0.000	220.000
9	5	0.000	0.000	220.000
10	5	0.000	0.000	220.000

NEW GRADES	WEIGHTS	BREAK POINTS
1	0.000	0.440
2	100.000	0.880
3	20.000	6.120
4	100.000	6.560
5	0.000	10.000

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10 OLD GRADES, 5 NEW GRADES, PADDING TYPE 0, SIZE 0.00

GRADES OLD	GRADES NEW	WEIGHTS GIVEN	WEIGHTS PADDED	PARTIAL SUMS
1	2	100.000	100.000	100.000
2	3	0.000	0.000	100.000
3	3	0.000	0.000	100.000
4	3	0.000	0.000	100.000
5	3	0.000	0.000	100.000
6	3	20.000	20.000	120.000
7	3	0.000	0.000	120.000
8	3	0.000	0.000	120.000
9	3	0.000	0.000	120.000
10	4	100.000	100.000	220.000

NEW GRADES	NEW WEIGHTS	BREAK POINTS
1	0.000	0.440
2	100.000	0.880
3	20.000	9.120
4	100.000	9.560
5	0.000	10.000

Example 4.

10 OLD GRADES, 5 NEW GRADES, PADDING TYPE 0, SIZE 0.00

GRADES OLD	NEW	WEIGHTS GIVEN	PADDED	PARTIAL SUMS
1	2	100.000	100.000	100.000
2	3	0.000	0.000	100.000
3	3	0.000	0.000	100.000
4	3	0.000	0.000	100.000
5	3	0.000	0.000	100.000
6	3	20.000	20.000	120.000
7	3	0.000	0.000	120.000
8	3	0.000	0.000	120.000
9	4	100.000	100.000	220.000
10	5	0.000	0.000	220.000

BREAK POINTS
0.440 0.880 8.120 8.560 10.000

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10 OLD GRADES, 5 NEW GRADES, PADDING TYPE 1, SIZE 0.25

GRADES OLD	NEW	WEIGHTS GIVEN	PADDED	PARTIAL SUMS
1	1	100.000	100.000	100.000
2	2	0.000	14.500	114.500
3	2	0.000	14.500	129.000
4	3	0.000	14.500	143.500
5	3	0.000	14.500	158.000
6	3	20.000	20.000	178.000
7	3	0.000	14.500	192.500
8	4	0.000	14.500	207.000
9	4	100.000	100.000	307.000
10	5	0.000	14.500	321.500

BREAK POINTS
0.643 2.972 7.028 8.502 10.000

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10 OLD GRADES, 5 NEW GRADES, PADDING TYPE 1, SIZE 0.50

GRADES OLD	GRADES NEW	WEIGHTS GIVEN	WEIGHTS PADDED	PARTIAL SUMS
1	1	100.000	100.000	100.000
2	2	0.000	36.000	136.000
3	2	0.000	36.000	172.000
4	2	0.000	36.000	208.000
5	3	0.000	36.000	244.000
6	3	20.000	36.000	280.000
7	4	0.000	36.000	316.000
8	4	0.000	36.000	352.000
9	5	100.000	100.000	452.000
10	5	0.000	36.000	488.000

BREAK POINTS

0.976	3.644	6.356	8.384	10.000

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10 OLD GRADES, 5 NEW GRADES, PADDING TYPE 1, SIZE 0.75

GRADES OLD	GRADES NEW	WEIGHTS GIVEN	WEIGHTS PADDED	PARTIAL SUMS
1	1	100.000	100.000	100.000
2	1	0.000	64.500	164.500
3	2	0.000	64.500	229.000
4	2	0.000	64.500	293.500
5	3	0.000	64.500	358.000
6	3	20.000	64.500	422.500
7	4	0.000	64.500	487.000
8	4	0.000	64.500	551.500
9	5	100.000	100.000	651.500
10	5	0.000	64.500	716.000

BREAK POINTS

1.670	3.890	6.110	8.213	10.000

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10 OLD GRADES, 5 NEW GRADES, PADDING TYPE 2, SIZE 0.25

GRADES OLD	GRADES NEW	WEIGHTS GIVEN	WEIGHTS PADDED	PARTIAL SUMS
1	1	100.000	100.000	100.000
2	2	0.000	24.093	124.093
3	2	0.000	13.696	137.789
4	3	0.000	9.568	147.356
5	3	0.000	12.269	159.625
6	3	20.000	20.000	179.625
7	3	0.000	13.696	193.321
8	4	0.000	24.093	217.414
9	4	100.000	100.000	317.414
10	5	0.000	24.093	341.506

BREAK POINTS

0.683 2.913 7.481 8.558 10.000

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10 OLD GRADES, 5 NEW GRADES, PADDING TYPE 2, SIZE 0.50

GRADES OLD	GRADES NEW	WEIGHTS GIVEN	WEIGHTS PADDED	PARTIAL SUMS
1	1	100.000	100.000	100.000
2	2	0.000	38.830	138.830
3	2	0.000	24.093	162.922
4	3	0.000	17.464	180.387
5	3	0.000	15.208	195.595
6	3	20.000	20.000	215.595
7	3	0.000	24.093	239.687
8	4	0.000	38.830	278.517
9	4	100.000	100.000	378.517
10	5	0.000	38.830	417.347

BREAK POINTS

0.835 3.230 7.276 8.554 10.000

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10 OLD GRADES, 5 NEW GRADES, PADDING TYPE 2, SIZE 0.75

GRADES OLD	GRADES NEW	WEIGHTS GIVEN	WEIGHTS PADDED	PARTIAL SUMS
1	1	100.000	100.000	100.000
2	2	0.000	60.466	160.466
3	2	0.000	43.334	203.800
4	2	0.000	33.767	237.567
5	3	0.000	27.660	265.227
6	3	20.000	33.767	298.995
7	3	0.000	43.334	342.329
8	4	0.000	60.466	402.795
9	5	100.000	100.000	502.795
10	5	0.000	60.466	563.261

BREAK POINTS

1.209 3.637 6.899 8.478 10.000

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10 OLD GRADES, 5 NEW GRADES, PADDING TYPE 1, SIZE 1.00

GRADES OLD	GRADES NEW	WEIGHTS GIVEN	WEIGHTS PADDED	PARTIAL SUMS
1	1	100.000	100.000	100.000
2	1	0.000	100.000	200.000
3	2	0.000	100.000	300.000
4	2	0.000	100.000	400.000
5	3	0.000	100.000	500.000
6	3	20.000	100.000	600.000
7	4	0.000	100.000	700.000
8	4	0.000	100.000	800.000
9	5	100.000	100.000	900.000
10	5	0.000	100.000	1000.000

BREAK POINTS

2.000 4.000 6.000 8.000 10.000
