D E P A R T M E N T O F M A T H E M A T I C S

THE TAYLOR GALERKIN METHOD FOR ADVECTION ON THE SPHERE

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0. Abstract

In this paper we consider a method that has already made its mark in a number of fields in computational fluid dynamics (CFD) but has yet to make it in environmental or meteorological flows. The Taylor-Galerkin finite element method provides the accuracy of the Lax-Wendroff method, although with fewer of the well-known problems of that method, together with the flexibility of mesh refinement associated with the finite element method in general. Here we will formulate the method in a form suitable for advection on a sphere and will proceed to demonstrate its power on a test problem.

1. Introduction

The Taylor-Galerkin method introduced by Donea (1984) and further expounded by Donea et al (1984, 1987, 1988), for example, has already proved to be a very powerful method. See Löhner et al (1986) and references therein for its application to high speed gas flows.

Basically the Taylor-Galerkin method is the finite element version of the Lax-Wendroff scheme. The Lax-Wendroff method is still popular despite its well-known misbehaviour at discontinuities. For smooth flows Lax-Wendroff is second order accurate in space and time whilst at shocks artificial viscosity can help to give a reasonable solution. However, the finite element approach not only gives us the flexibility of the meshes that can be used but also improves the phase accuracy, Donea et al

(1987). Although this does not eliminate the problems caused by strong gradients it does substantially reduce them.

In the rest of this section the Taylor-Galerkin method is derived in its usual form for a conservation law, followed by the much more convenient two-stage version. It will then be shown how the scheme can be written for advection equations of the type that are of interest to us here.

In Section 2 a test problem for advection on the sphere, due to Ritchie (1987), is presented and the application of the scheme to this problem is explained.

Finally, in Section 3, the results are given and analyzed and we draw our conclusions.

Firstly let us consider the single non-linear conservation law in one dimension,

$$u_t + F(u)_x = 0 \ . \tag{1.1}$$

A Taylor series in time is now performed, the spatial discretization being momentarily left aside, with a time-step of Δt giving an approximation to u at the new time-level of

$$u^{n+1} = u^n + \Delta t u_t^n + \frac{1}{2}(\Delta t)^2 u_{tt}^n + \frac{1}{6}(\Delta t)^3 u_{ttt}^n + \ldots \ . \tag{1.2}$$

Into this semi-discretized equation we can substitute for the temporal derivatives, i.e.

$$u_t^n = -F(u^n)_x$$
$$u_{tt}^n = -F(u^n)_{x_t} = -(a(u^n)u_t^n)_x = [a(u^n)F(u^n)_x]_x$$

to second order, where $a(u) = \partial F(u)/\partial u$.

This then leads to the semi-discretized scheme

$$\frac{(u^{n+1} - u^n)}{\Delta t} = -F(u^n)_x + \frac{\Delta t}{2}(a(u^n)F(u^n)_x)_x \ . \qquad (1.3)$$

From this point we can either apply central differencing to the right hand side of (1.3) to obtain the Lax-Wendroff method or we can expand u in terms of some finite element basis functions and then take the weak form of equation (1.3). That is we write

$$u^n(x) = \sum_{i=1}^{\text{number of nodes}} U_i^n \phi_i(x)$$

where $\phi_i(x)$ is the finite element basis function. Throughout the rest of this paper we will assume $\phi_i(x)$ to be the usual piecewise linear 'hat' function and we will use $\phi_e(x)$ to represent piecewise constant functions that take a value of 1 on element e and 0 everywhere else.

Substituting into equation (1.3) and multiplying through by ϕ_j we have the second order accurate Taylor-Galerkin scheme

$$\int_{\Omega}(U^{n+1} - U^{n})\phi_j(x)d\Omega - -\Delta t\int_{\Omega}F(U^{n})_x\phi_j(x)d\Omega$$

$$- \frac{\Delta t^2}{2}\Big[\int_{\Omega}(\phi_j(x))_x a(U^{n})F(U^{n})_x d\Omega$$

$$- \int_{\Gamma}\phi_j(x)a(U^{n})F(U^{n})d\Gamma\Big] \quad \forall_j \ .$$

Here we denote the solution domain by Ω and its boundary by Γ .

If we now let $A(U^n)$ or A^n denote the Jacobian of $\underline{F}$, i.e.

$$A = \frac{\partial \underline{F}}{\partial \underline{u}}$$

then we can write down the Taylor-Galerkin scheme for the system of conservation laws

$$\underline{u}_t + \frac{d\underline{F}_j}{dx_j} = 0$$

as

$$\int_{\Omega}(\underline{U}^{n+1} - \underline{U}^{n})\phi_j d\Omega = -\Delta t\int_{\Omega}\frac{\partial \underline{F}_j}{\partial x_j}\phi_j d\Omega - \frac{\Delta t^2}{2}\int_{\Omega}A(\underline{U}^{n})\frac{\partial \underline{F}_j}{\partial x_j}.\underline{\nabla}\phi_j d\Omega$$

$$+ \frac{\Delta t^2}{2}\int_{\Gamma}\underline{n}.A(\underline{U}^{n})\frac{\partial \underline{F}_j}{\partial x_j}\phi_j d\Gamma, \qquad (1.4)$$

where $\underline{n}$ is the unit outward normal of the boundary Γ .

Just as the one-step Lax-Wendroff method is rarely used because of the expense of calculating the matrix A and of then multiplying it by the vector $\frac{\partial \underline{F}_j}{\partial x_j}$, a two-stage version of the Taylor-Galerkin method has been developed. Following Löhner et al (1986) we introduce a new function $\underline{U}^{n+\frac{1}{2}}$ which is a piecewise constant approximation to $\underline{u}$ at the half time level and is given by

$$\int_\Omega \underline{U}^{n+\frac{1}{2}} \phi_e d\Omega = \int_\Omega \underline{U}^n \phi_e d\Omega - \frac{\Delta t}{2} \int_\Omega \frac{\partial \underline{F}_j}{\partial x_j} \phi_e d\Omega \ . \qquad (1.5a)$$

To proceed we note that, to within first order accuracy,

$$\underline{F}^{n+\frac{1}{2}} = \underline{F}^n + \frac{\Delta t}{2} \frac{\partial \underline{F}^n}{\partial t} = \underline{F}^n - \frac{\Delta t}{2} A^n \frac{\partial \underline{F}_j}{\partial x_j}$$

and hence that

$$A^n \frac{\partial \underline{F}_j}{\partial x_j} = - \frac{2(\underline{F}^{n+\frac{1}{2}} - \underline{F}^n)}{\Delta t} \ .$$

Equation (1.4) can now be replaced by

$$\int_{\Omega}(\underline{U}^{n+1} - \underline{U}^{n})\phi_i \, d\Omega = \Delta t \int_{\Omega} \underline{F}^{n+\frac{1}{2}} \frac{\partial \phi_i}{\partial x_j} \, d\Omega$$

$$- \Delta t \int_{\Gamma} \underline{F}^{n}\phi_i d\Gamma - \Delta t \int_{\Gamma}(\underline{F}_n^{n+\frac{1}{2}} - \bar{\underline{F}}_n^{n})\phi_i d\Gamma \; \forall_i , \qquad (1.5b)$$

where $\underline{F}_n$ denotes the outward normal flux and the overbar represents an element-averaged quantity.

We now specialize this method to advection on a sphere. We have no boundaries and so equation (1.5b) simplifies somewhat.

A problem though is that we will not in general be solving a conservation law. This means that the integration by parts used to transfer the derivative operator from the piecewise constant $\underline{F}^{n+\frac{1}{2}}$ to the differentiable basis function may not be as clear as before.

Consider the equation

$$\underline{u}_t + \underline{L}(\underline{u}) = \underline{0} \; . \qquad (1.6)$$

The first stage of the Taylor-Galerkin procedure goes through entirely as before in that we write

$$\int_{\Omega} \underline{U}^{n+\frac{1}{2}} \phi_e d\Omega = \int_{\Omega} \underline{U}^{n}\phi_e d\Omega - \frac{\Delta t}{2} \int_{\Omega} \underline{L}(\underline{U}^{n})\phi_e d\Omega \; . \qquad (1.7a)$$

The second stage is now

$$\int_{\Omega}(\underline{U}^{n+1} - \underline{U}^{n})\Phi_i d\Omega = - \Delta t \int_{\Omega} \underline{L}(\underline{U}^{n+\frac{1}{2}})\phi_i d\Omega \ . \qquad (1.7b)$$

Consider

$$L(u) = \underline{a}.\underline{\nabla}u,$$

the extension to a system being obvious. The velocity field, $\underline{a}$, will be a function of position, $\underline{x}$, alone in the example to be tested later, but will generally be a function of the solution u. Hence we will consider $\underline{a} = \underline{a}(\underline{x}, u)$.

Now

$$\int_{\text{Element} = E} L(U^{n+\frac{1}{2}})\phi_i dE = \int_{E} \underline{a}(\underline{x}, U^{n+\frac{1}{2}})\underline{\nabla}U^{n+\frac{1}{2}}\phi_i dE$$

$$= - \int_{E} U^{n+\frac{1}{2}} \underline{\nabla}.(\phi_i \underline{a}(\underline{x}, U^{n+\frac{1}{2}}))dE$$

$$= - U^{n+\frac{1}{2}} \int_{E} \underline{\nabla}.(\phi_i \underline{a}(\underline{x}, U^{n+\frac{1}{2}}))dE$$

$$= - U^{n+\frac{1}{2}} \int_{S} \phi_i \underline{a}(\underline{x}, U^{n+\frac{1}{2}}).d\underline{S},$$

where S is the surface of the element and $d\underline{S} = \underline{n}dS$, where $\underline{n}$ is again the outward normal. Given that ϕ_i is a linear function on each element and that $\underline{a}(\underline{x}, U^{n+\frac{1}{2}})$ is most likely to be given as a piecewise constant (if a function of U) or a piecewise linear (if a function of $\underline{x}$), the surface integral is then very simple to evaluate exactly.

2. The Test Problem

This problem, due to Ritchie (1987), involves the advection of a Gaussian hill around a sphere. Although the problem will be fully defined here the reader is advised to read Ritchie (1987) if more details are required.

Four co-ordinate systems will be used. First, Cartesian (x, y, z) and latitude and longitude (λ, θ) which are related by

$$x = \cos\lambda \cos\theta$$

$$y = \sin\lambda \cos\theta$$

$$z = \sin\theta \ .$$

A second set of latitude, longitude co-ordinates (λ', θ') is then introduced. These are related to the first by defining a new north pole, P', to be at position (λ_0, θ_0) . We then have the following relationships:-

$$\lambda'(\lambda, \theta) = \tan^{-1}\left\{\frac{\sin(\lambda - \lambda_0)}{\sin\theta_0 \cos(\lambda - \lambda_0) - \cos\theta_0 \tan\theta}\right\}$$

and

$$\theta'(\lambda, \theta) = \sin^{-1}(\sin\theta \sin\theta_0 + \cos\theta \cos\theta_0 \cos(\lambda - \lambda_0)) \ .$$

These co-ordinates will be used to define the velocity field.

The fourth set of co-ordinates is used purely to define the initial data and to visualize the results. We introduce a plane which is tangent to the earth's surface at the new north pole P' . A stereographic projection, true at P', is then defined as

$$X = \frac{2a\ \cos\lambda'\ \cos\theta'}{1 + \sin\theta'} + X_{P'}$$

$$Y = \frac{2a\ \sin\lambda'\ \cos\theta'}{1 + \sin\theta'} + Y_{P'} \ .$$

where a is the radius of the earth and $(X_{P'}, Y_{P'})$ is the position in (X, Y) space of the point P'.

The velocity field is designed to rotate uniformly around the displaced north pole P'. That is, in the (λ', θ') system

$$\left.\begin{aligned} \frac{d\lambda'}{dt} &= w \\ \text{and} \qquad \frac{d\theta'}{dt} &= 0 \ , \end{aligned}\right\} \tag{2.1}$$

where w is a constant designed to give one complete rotation in 20 days.

In the (λ, θ) system it can then be shown that the zonal and meridional wind components are given by

$$u = aw[\cos\theta\ \sin\theta_0 - \cos(\lambda - \lambda_0)\sin\theta\ \cos\theta_0]$$

and

$$v = aw\ \sin(\lambda - \lambda_0)\cos\theta_0 \ .$$

If we define $\tilde{u} = u/a$ and $\tilde{v} = v/a$ and similarly $\tilde{x}$, $\tilde{y}$, $\tilde{z}$ as x/a, y/a, and z/a, then the normalized Cartesian velocity components are

$$\dot{\tilde{x}} = -\tilde{u}\ \sin\lambda - \tilde{v}\ \cos\lambda\ \sin\theta$$

$$\dot{\tilde{y}} = \tilde{u}\ \cos\lambda - \tilde{v}\ \sin\lambda\ \sin\theta$$

$$\dot{\tilde{z}} = \tilde{v}\ \cos\theta \ .$$

From equation (2.1) it is easy to see that if we have initial data $u_0(\lambda', \theta')$ then the solution at time t is given by

$$u(\lambda', \theta', t) = u_0(\lambda' - wt, \theta')$$

or in (λ, θ) co-ordinates

$$u(\lambda, \theta, t) = u_0(\lambda'(\lambda, \theta) - wt, \theta'(\lambda, \theta)) \ .$$

We shall define our initial data by choosing

$$(\lambda_0, \theta_0) = (0^o, 45^o)$$

with the Gaussian hill centred on the point

$$(\lambda, \theta) = (0^o, 0^o) \ .$$

The initial Gaussian hill is given by

$$G(r) = 100e^{-\pi^2 r^2/L^2} \ ,$$

where r is the distance from the chosen centre as measured in the (X, Y) plane. L is the wave length of the field for which we take a value of 10,000 km.

The reasons for the choice of P' and the centering of the cone are that after half a revolution it will pass directly over

the north pole. This has traditionally been a stumbling block when solving on the sphere, usually due to the types of grid being used which, naturally enough, have been defined in (λ, θ) space. This leads, though, to a great clustering of points around the poles and hence to much smaller Δx's there. For explicit schemes with a CFL limit, as spectral methods usually are, this leads to a greatly reduced time-step caused by this unnecessary and unwanted refinement.

The problem of the reduced CFL limit is overcome by the semi-Lagrangian scheme (see Ritchie's paper or references therein), because this method has no CFL limit. There is still a problem at the poles, though, because the accurate calculation of trajectories, an essential part of this method, is made tricky by the polar singularity. Some care must therefore be taken with the semi-Lagrangian method at the poles.

The Taylor-Galerkin finite element method suffers from neither of the above complaints. Firstly, because it is not a Lagrangian scheme there are no trajectory calculations involved and secondly, because it is a finite element method, we have a lot of freedom in the mesh we use as opposed to spectral or finite difference methods. This means that we can define the grids in Cartesian (x, y, z) space and the poles require no special attention. The Taylor-Galerkin method is still subject to a CFL limit, but this limit is not now drastically lowered by the grid refinement at the poles.

3. Results and Conclusions

Three grids will be used. The 3rd refinement has 386 nodes and 768 elements, the 4th refinement has 1538 nodes and 3072 elements and the 5th refinement has 6146 nodes and 12,288 elements. The 4th and 5th refinements of the grid, as viewed from infinity, are shown in Figures 1 and 2, whilst in Figures 3 and 4 we show the stereographic projections of the 3rd and 4th grid refinements. The initial data is shown in Figure 5.

The results are all taken after a full 20 day rotation. The maxima and minima are given to indicate the success of the solution and an ℓ_2 error provides a more objective, comparative measure. The maximum and minimum should, of course, be 100 and 0 . No pictures are given of the solution as they only confirm the impression gained from looking at the extrema, i.e. the solution was obviously degrading or clearly doing very well. There was no noticeable phase error. Results are given for various time-steps and meshes to assess temporal and spatial convergence.

Δt(secs)	Max.	Min.	ℓ_2 error
8000	61.33	-106.5	0.699
4000	72.24	- 28.69	0.289
2000	79.28	- 12.80	0.2
1000	83.62	- 7.87	0.175
500	86.29	- 7.52	0.176
250	87.87	- 10.2	0.185
125	88.74	- 11.95	0.195

Table 1 : Results for 3rd grid

From Table 1 we see that there is no point reducing the time-step below 1,000 seconds as the results have time-converged at that point and to take more time-steps is to invite more projection error to creep in, as indeed happens. If we subtract off the purely spatial error, i.e. that error that remains in the time-converged solution then we can calculate temporal orders of convergence to be 2.2 and 2.19.

Δt	Max.	Min.	ℓ_2 error
1000	91.59	- 39.28	0.144
500	95.37	- 9.74	0.0409
250	97.61	- 3.93	0.0189
125	98.79	- 1.85	0.01043
62.5	99.4	- 0.963	0.00672
31.25	99.69	- 0.507	0.00508

Table 2 : Results for 4th grid

If again we subtract off what is assumed to be the time-converged solution we can calculate the order of convergence to be 1.955, 1.38, 1.37 and 1.7. It is worth noting here that the 4th grid has resolved the Gaussian hill much more adequately than on the previous grid. This is shown in the vast improvements, apart from with the very large time-steps, in the maximum and minimum values. If a strictly non-negative quantity needs transporting, for example humidity, then it is clear from these results that, provided the mesh is capable of representing the solution and a moderate time-step is chosen, then very little post-processing of the solution will need to be done to maintain positivity.

The final grid we shall look at is the 5th refinement.

Δt	Max.	Min.	ℓ_2 error
125	98.77	- 6.77	0.01135
62.5	99.39	- 2.39	0.00482
31.25	99.71	- 1.04	0.00252

Table 3 : Results for 5th grid

Rather surprisingly these results show little further improvement than was obtained on the 4th grid. This is due to the fact that most wavelengths present could be adequately resolved on the 4th grid. Due to the lack of data we cannot give figures for the spatial convergence rate but we can at least say it is not slow.

In conclusion, then, we can say that the Taylor-Galerkin method copes with advection on the sphere very well; the poles present no problems, as opposed to methods based on Gaussian grids, and temporal and spatial accuracy are both very good.

It is hoped in the near future to look at problems involving the solution of the shallow water equations on the sphere and, whilst the regular grid ably demonstrated there was no need to treat the poles specially, this is not an optimal way of solving the problem with finite elements and it is hoped to include solution dependent adaptivity before long.

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