Generalizations of the Bauer-Fike Theorem

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Abstract

The Bauer-Fike Theorem for diagonalizable matrices is generalized to cases where (i) non-diagonalizable matrices or (ii) only part of the spectrum, is considered.

Keywords: condition number, defective matrix, eigenvalue problem, perturbation.

§1. Introduction.

For a non-defective matrix A (a matrix with only linear elementary divisors, and thus scalar Jordan blocks) the Bauer-Fike Theorem [1] states that

Theorem 1 [Bauer-Fike]:-

Let $A = X \Lambda X^{-1}$, $\Lambda = \text{diag}(\lambda_i)$; i.e. the columns x_j or X are eigenvectors of A corresponding to λ_j . Assume that the eigenvalues of $(A + \varepsilon B)$ are $\{\tilde{\lambda}\}$. Then

$$\min_i |\tilde{\lambda} - \lambda_i| \leq \varepsilon \|B\| \cdot \kappa(X) \tag{1}$$

where $\kappa(X) = \|X\| \cdot \|X^{-1}\|$ and $\|\cdot\|$ is any norm which satisfies

$$\|(\tilde{\lambda}I - \Lambda)^{-1}\| = \left(\min_i |\tilde{\lambda} - \lambda_i|\right)^{-1} . \tag{2}$$

$\|\cdot\|$ can be the 1, 2 or ∞ norm, and ε does not have to be small.

From Theorem 1, one can investigate the condition of the eigenvalues of A by looking at $\kappa(X)$, the condition number of the eigenvector matrix. In some applications [2] [3] [6], the eigenvectors x_j can be chosen and the conditioning is thus improved by minimizing $\kappa(X)$.

For defective matrices A , a generalization involving the condition number does not exist. A close analogy to Theorem 1 is as follows:- [4]

Theorem 2:- Let $Q^H AQ = D + N$, where Q is unitary, D is diagonal and N upper triangular with zero diagonal, with $(\cdot)^H$ denoting the Hermitian transpose. Then for $\tilde{\lambda} \varepsilon \lambda(A+\varepsilon B)$ $\left(\equiv \text{spectrum of } (A+\varepsilon B)\right)$,

$$\min_i |\tilde{\lambda} - \lambda_i| \leq \max\{\theta_1, \theta_1^{1/p}\} \tag{3}$$

where $\theta_1 = \|\varepsilon B\|_2 \cdot \sum_{k=0}^{p-1} \|N\|_2^k$, and $N^p = 0$ with $N^{p-1} \neq 0$.

Note that the columns of Q are the Schur vectors of A.

From the proof of Theorem 2 [4], ε again does not have to be small. The theorem also holds for other norms which satisfy (2).

Theorem 2 indicates that a small $\|N\|$ will ensure the well-conditioning of the eigenvalues of A. However, in terms of some applications [2] [3] [6], the minimization of $\|N\|$ when choosing the eigen- and principal-vectors x_j of the matrix A is most inconvenient.

Nevertheless, inequalities of the types in (1) and (3) are important in various applications. In §2, a generalization of Theorem 1, involving $\kappa(X)$ where columns x_j of X are the eigen- and principal-vectors of the matrix A, is presented in Theorem 3. Similar to (3), one has

$$\min_i |\tilde{\lambda} - \lambda_i| \leq \max\{\theta_2, \theta_2^{1/p}\} \tag{4}$$

where $\theta_2 = C\varepsilon \|B\| \cdot \kappa(X)$, with p similar to that in (3) and C a constant specified in Theorem 3. ε does not have to be small.

In §3, a brief comparison of the bounds on the RHS's of (3) and (4) is carried out, by experimenting on some trivial examples of A. Some comments on the numerical aspects are included.

In §4, Theorem 1 is further generalized to cope with the situation when

$$A(X_1,X_2) = (X_1,X_2)\begin{pmatrix}\Lambda_1 & 0\\ 0 & \Lambda_2\end{pmatrix}, \quad \Lambda_1 = \mathrm{diag}(\lambda_{1i})$$

and one is only interested in the behaviour of Λ_1 under perturbation, or when only X_1 is available. If the eigenvalues of Λ_1 and Λ_2 are disjoint and the perturbation εB to A is small (i.e. ε small), then it can be proved that, similar to (4),

$$\min_i |\tilde{\lambda} - \lambda_{1i}| \leq \theta_3^{1/p_1} + O(\varepsilon^{2/p_1}) \tag{18}$$

with $\kappa(X)$ in θ_2 replaced by $\|X_1\|\cdot\|Y_1\|$ in θ_3, and columns of Y_1 are the left eigen- and principal-vectors corresponding to Λ_1 and X_1. p_1 is the dimension of the biggest Jordan Block in Λ_1.

Thus (18) indicates that $\|X\|\cdot\|Y\|$ reflects the conditioning of the eigenvalues λ_{1i} of Λ_1. The result in (18) is important especially in the case when only part of the spectrum (Λ_1 in this case) is sensitive to perturbation.

§5 concludes the paper.

This paper is written with applications in mind, in particular the eigensystem assignment for defective matrices [7], with the deadbeat control problem [9] a notable example. Given the set of eigenvalues $\{\lambda_i\}$, $\{x_i\}$ have to be chosen from various subspaces, and it will be sensible to choose the $\{x_i\}$ to improve the conditioning of the eigenvalue problem, if the degrees of freedom allow. Applications of results in this paper to the deadbeat control problem will appear in [3].

§2. The Main Theorem.

Let $X^{-1}AX = \mathrm{diag}(J_{\lambda_j}) = J$, where columns x_j of X are eigenvectors or principal vectors of A, and J_{λ_j} the Jordan blocks corresponding to λ_j.

We state the main result of this paper as follows:-

Theorem 3:- For $\tilde{\lambda} \in \lambda(A+\varepsilon B)$, one has

$$\min_i |\tilde{\lambda} - \lambda_i| \leqq \max\{\theta_2, \theta_2^{1/p}\} \tag{4}$$

where $\theta_2 = C\varepsilon\|B\| \cdot \kappa(X)$. p is the largest dimension of all J_{λ_j} (i.e. the smallest integer such that $[J-\mathrm{diag}(\lambda_i)]^p = 0.$)

with (i) $C = \sqrt{\frac{p(p+1)}{2}}$ for the 2- and F- (Frobenius) norms;

(ii) $C = p$ for the 1- and ∞- norms.

Note that Theorem 1 is a special case of Theorem 3, when $p = 1$.

(Proof):- Consider only the 1-, 2- and ∞- norms.

One has

$$X^{-1}(A + \varepsilon B - \tilde{\lambda}I)X = J + \varepsilon X^{-1}BX - \tilde{\lambda}I .$$

If $\tilde{\lambda}$ is an eigenvalue of J, then (4) is trivial, as the LHS of (4) vanishes. Thus assume that $(J - \tilde{\lambda}I)$ is non-singular. Then

$$X^{-1}(A + \varepsilon B - \tilde{\lambda}I)X = (J - \tilde{\lambda}I)\{I + \varepsilon(J - \tilde{\lambda}I)^{-1} \cdot X^{-1}BX\}$$

$$= (J - \tilde{\lambda}I)(I + M) .$$

As the LHS of the previous equation is singular, $(I + M)$ has to be singular and thus $\|M\| \geq 1$, i.e.

$$\| \varepsilon (J - \tilde{\lambda} I)^{-1} \cdot X^{-1} B X \| \geq 1 .$$

$$\Rightarrow \varepsilon \| (J - \tilde{\lambda} I)^{-1} \| \cdot \| X^{-1} \| \cdot \| X \| \cdot \| B \| \geq 1$$

$$\Rightarrow \varepsilon \cdot \|B\| \cdot \kappa(X) \cdot \geq \frac{1}{\| (J - \tilde{\lambda} I)^{-1} \|} . \tag{5}$$

Obviously for the norms we are considering,

$$\| (J - \tilde{\lambda} I)^{-1} \| = \max_i \| (J_{\lambda_i} - \tilde{\lambda})^{-1} \| . \tag{6}$$

Assume that the maximum occurs at λ_i .

Let $z = \lambda_i - \tilde{\lambda}$

$$(J_{\lambda_i} - \tilde{\lambda})^{-1} = \begin{bmatrix} z & 1 & & 0 \\ & z & 1 & \\ & & \ddots & \ddots \\ & & & z & 1 \\ 0 & & & & z \end{bmatrix}^{-1} = \begin{bmatrix} z^{-1} & -z^{-2} & z^{-3} & \cdots & -(-z)^{-p_i} \\ & z^{-1} & -z^{-2} & & \vdots \\ & & z^{-1} & & -z^{-2} \\ 0 & & & \ddots & z^{-1} \end{bmatrix} . \tag{7}$$

Note that $(J_{\lambda_i} - \tilde{\lambda})$ is $p_i \times p_i$.

From (6) and (7), $\| (J - \tilde{\lambda} I)^{-1} \|_2^2 \leq \| (J - \tilde{\lambda} I)^{-1} \|_F^2$

$$= p z^{-2} + (p - 1) z^{-4} + \ldots + z^{-2p_i}$$

$$\leq (p_i + (p_i - 1) + \ldots + 1) \phi^2$$

$$= \frac{p_i (p_i + 1)}{2} \phi^2 , \tag{8}$$

where $\phi = \max(|z^{-1}|, |z^{-p}|)$. (An estimation of $\|(J - \tilde{\lambda}I)^{-1}\|_2^2$ directly using the Gerschgorin Theorem yields the same result.) For the 1- or ∞-norm, similarly

$$\|(J - \tilde{\lambda}I)^{-1}\|_1 = \|(J - \tilde{\lambda}I)^{-1}\|_\infty$$

$$= \sum_{j=1}^{p} |z|^{-j} \leq p\phi . \qquad (9)$$

If $\|(J - \tilde{\lambda}I)^{-1}\| = \max_i \|(J_{\lambda_i} - \tilde{\lambda})^{-1}\|$ occurs simultaneously as $\min_i |\tilde{\lambda} - \lambda_i|$, (5) to (9) prove the theorem. p in (4) will then be the dimension of J_{λ_i} or the largest one if λ_i has more than one Jordan block .

If $\max_i \|(J_{\lambda_i} - \tilde{\lambda})^{-1}\|$ occurs at λ_k , with $|\tilde{\lambda} - \lambda_k| \geq \min_i |\tilde{\lambda} - \lambda_i|$, (5) to (9) still imply

$$|\tilde{\lambda} - \lambda_k| \leq \max(\theta_2, \theta_2^{1/p})$$

and thus the theorem.

However, p is now the largest dimension of the Jordan blocks corresponding to λ_k .

Q.E.D.

Note that ε does not have to be small.

§3. Some Numerical Experiments.

In this section, we look at the quantities

$$B_2 = \sum_{k=0}^{p-1} \|N\|^k \qquad \text{(from (3))}$$

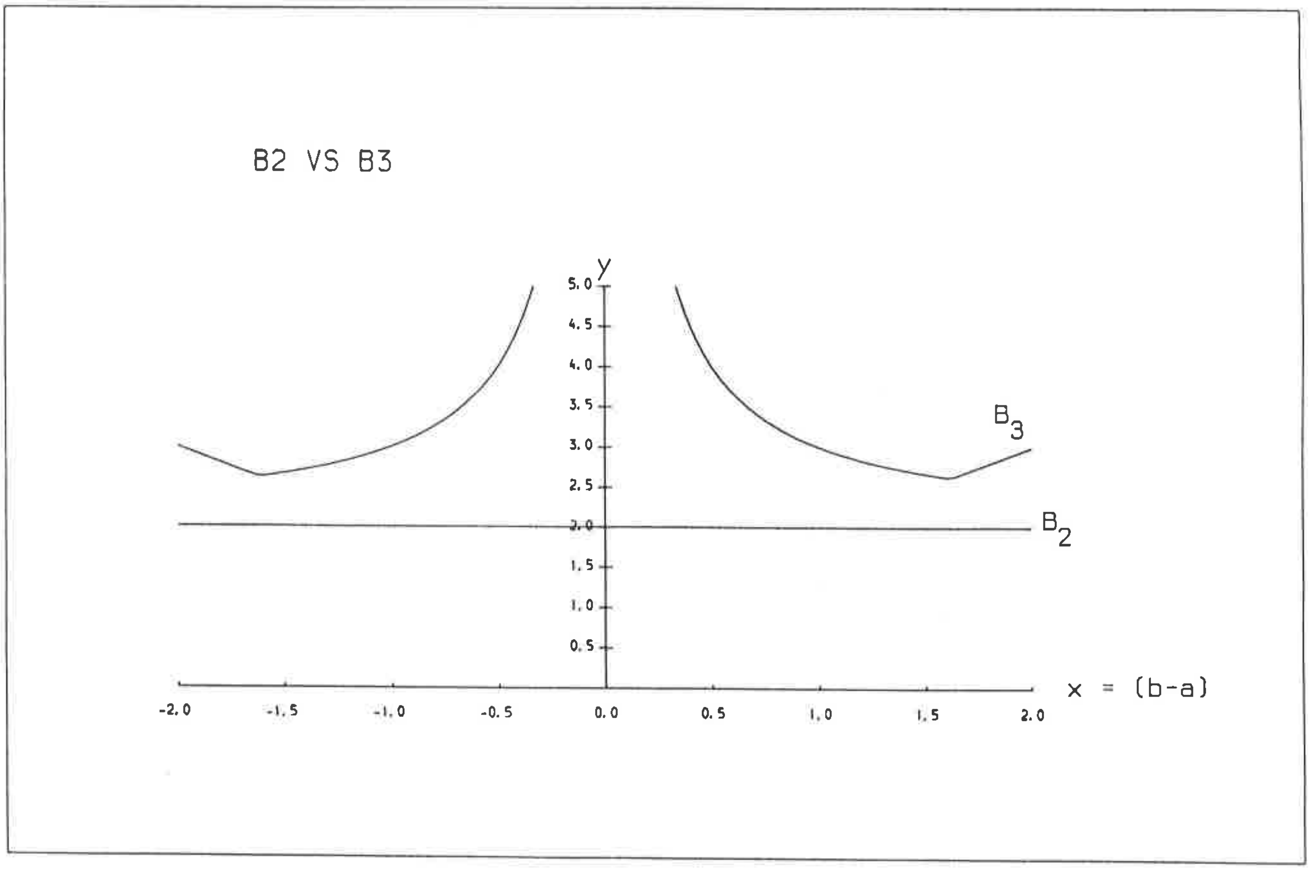

figure 1 (1-norm)

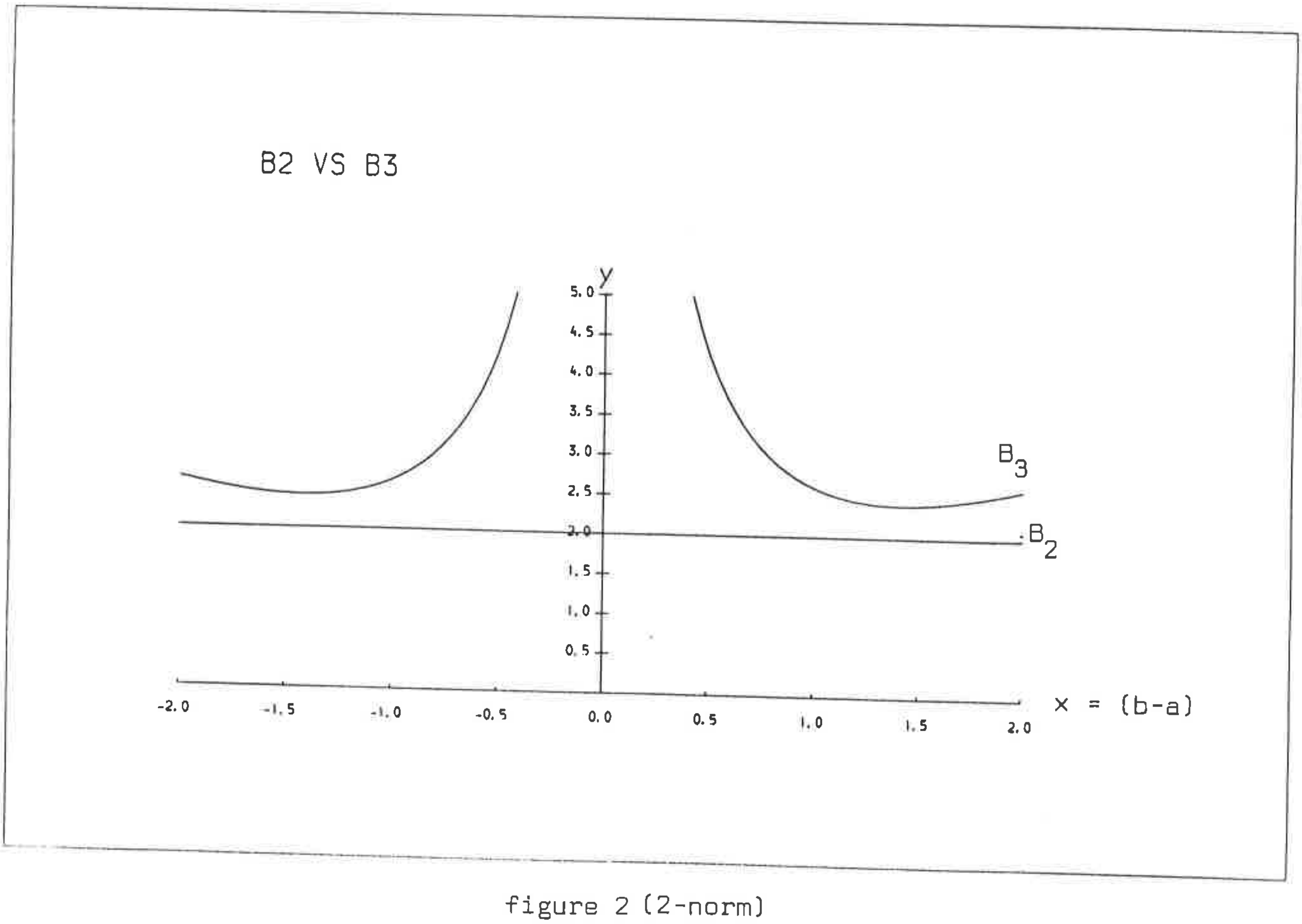

figure 2 (2-norm)

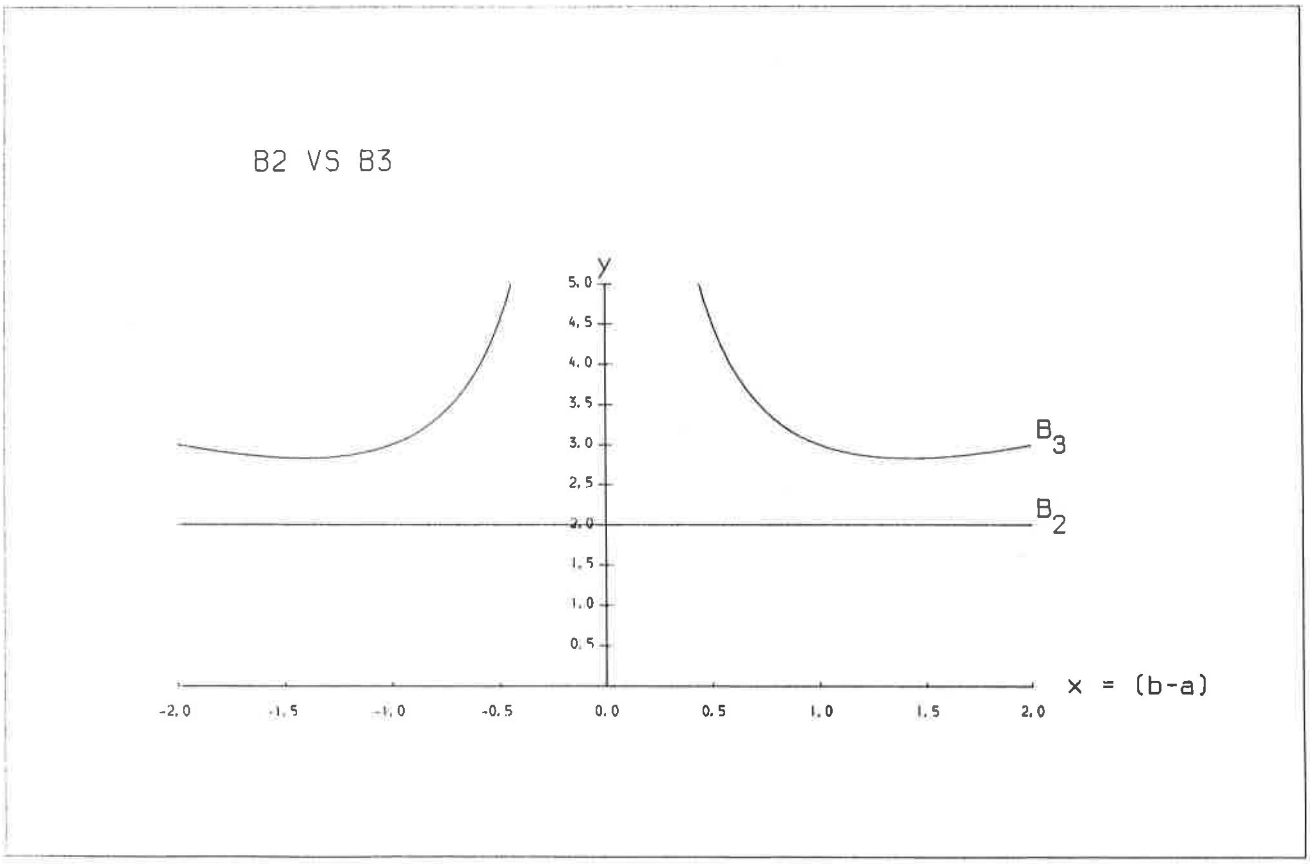

figure 3 (F-norm)

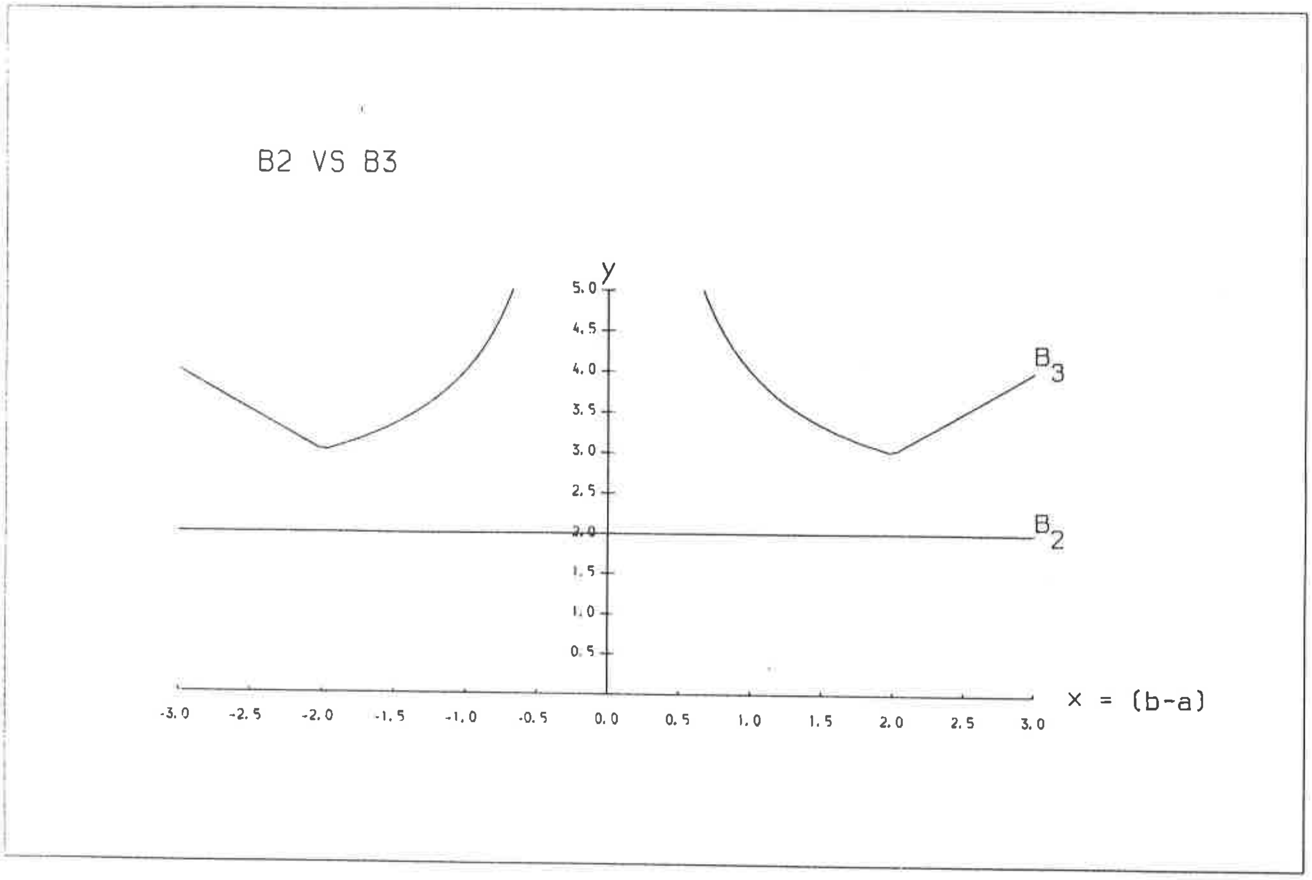

figure 4 (∞-norm)

and $$B_3 = C\cdot\kappa(X) \qquad \left(\text{from } (3)\right)$$

for some simple matrices.

Note that $B_3 = B_1 = \kappa(X)$ in (1), for non-defective cases.

(i) For diagonal matrices, it is obvious that

$B_2 = 1$ and $B_3 = 1$ (for $1,2,\infty$ and F norms.)

(ii) For a Jordan block $A = \begin{pmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix}$ which is $n \times n$, one has

	1-norm	2-norm	∞-norm	F-norm
B_2	n	n	n	$n\sqrt{n-1}$
B_3	n	$\sqrt{\frac{n(n+1)}{2}}$	n	$\frac{n}{\sqrt{2}}\sqrt{n+1}$

Thus, for large n, $B_2 \geq B_3$.

(iii) Consider $A = \begin{pmatrix} a & 1 \\ 0 & b \end{pmatrix}$.

$$B_2 = 2 \quad \text{for } 1,2,\infty \text{ and F norms}$$

and

$$B_3 = \begin{cases} |b-a| + 2|b-a|^{-1}, & \text{for the F-norm} \\ \max\{1+|b-a|,\ 2+|b-a|^{-1}\}, & \text{1-norm} \\ \max\{2(1+|b-a|^{-1}),\ 1+|b-a|\}, & \infty\text{-norm} \\ \sqrt{1+\frac{(b-a)^2}{2}+\frac{2}{(b-a)^2}+\left(\frac{1}{2}+\frac{1}{(b-a)^2}\right)\sqrt{(b-a)^4+4}}, & \text{2-norm}. \end{cases}$$

Figures 1-4 indicate that, B_3 is always greater than B_2, and $B_3 \to \infty$ as $(b-a) \to 0$. This is not surprising, as A becomes nearly defective when $b \to a$ and the eigenvectors in $\kappa(X)$ in B_3 converge to each other. Thus X is nearly rank 1 and $B_3 \to \infty$. When $(b-a) \to 0$, it will be more sensible to consider A to be $\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix} + O(b-a)$ and get the bound as in (ii), i.e.

	1-norm	2-norm	∞-norm	F-norm
B_2	2	2	2	2
B_3	2	$\sqrt{3}$	2	$\sqrt{6}$

Then $B_3 \leqq B_2$ for the norms we considered.

This example indicates the problem of using the Jordan form in numerical analysis. Care has to be taken when estimating the principal vectors in X, if they are not available. (See [5], [8]).

(iv) Consider $A = \begin{pmatrix} 1 & \cdots & 1 \\ & \ddots & \vdots \\ 0 & & 1 \end{pmatrix}$; i.e. all entries in A are one except the lower triangular part.

Then $B_2 = \sum_{k=0}^{n-1} \|N\|^k$, with $\|N\|_2$ increasing with respect to increasing n.

e.g. $n = 10$, $B_2 \cong 1.3 \times 10^7$, with $\|N\|_2 = 6.0548$.

For B_3, $X = \begin{pmatrix} 1 & 0 & & & \\ & 1 & -1 & & \\ & & 1 & -1 & \\ & & & \ddots & -1 \\ & & & & 1 \end{pmatrix}$ and $X^{-1} = \begin{pmatrix} 1 & 0 & \cdots & & & 0 \\ 0 & 1 & -1 & 1 & -1 & \cdots \\ \vdots & 0 & 1 & -1 & 1 & \cdots \\ \vdots & & & \ddots & \ddots & 1 \\ \vdots & & O & & \ddots & -1 \\ \vdots & & & & & 1 \\ 0 & \cdots & & 0 & & 1 \end{pmatrix}$

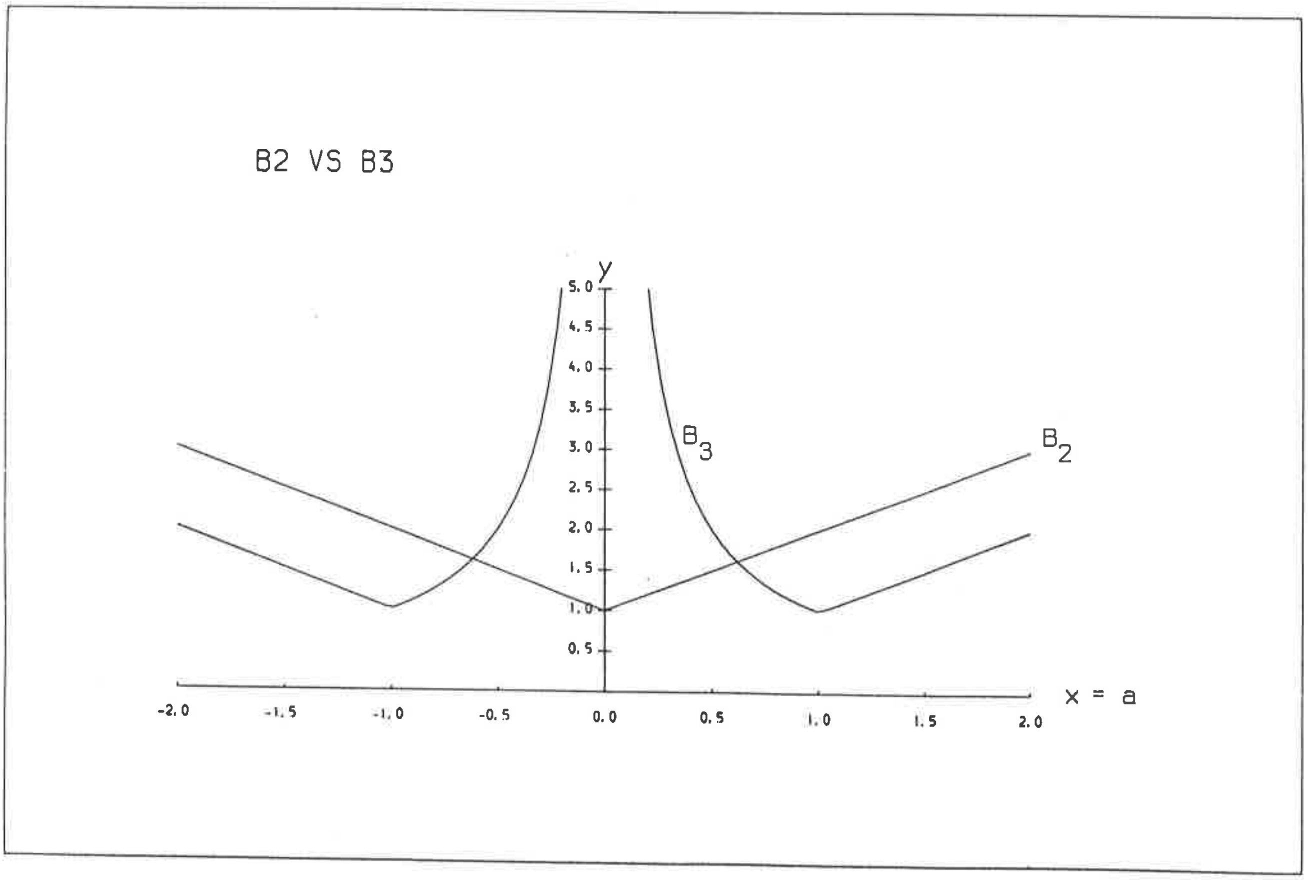

figure 5 (1,2,∞-norm)

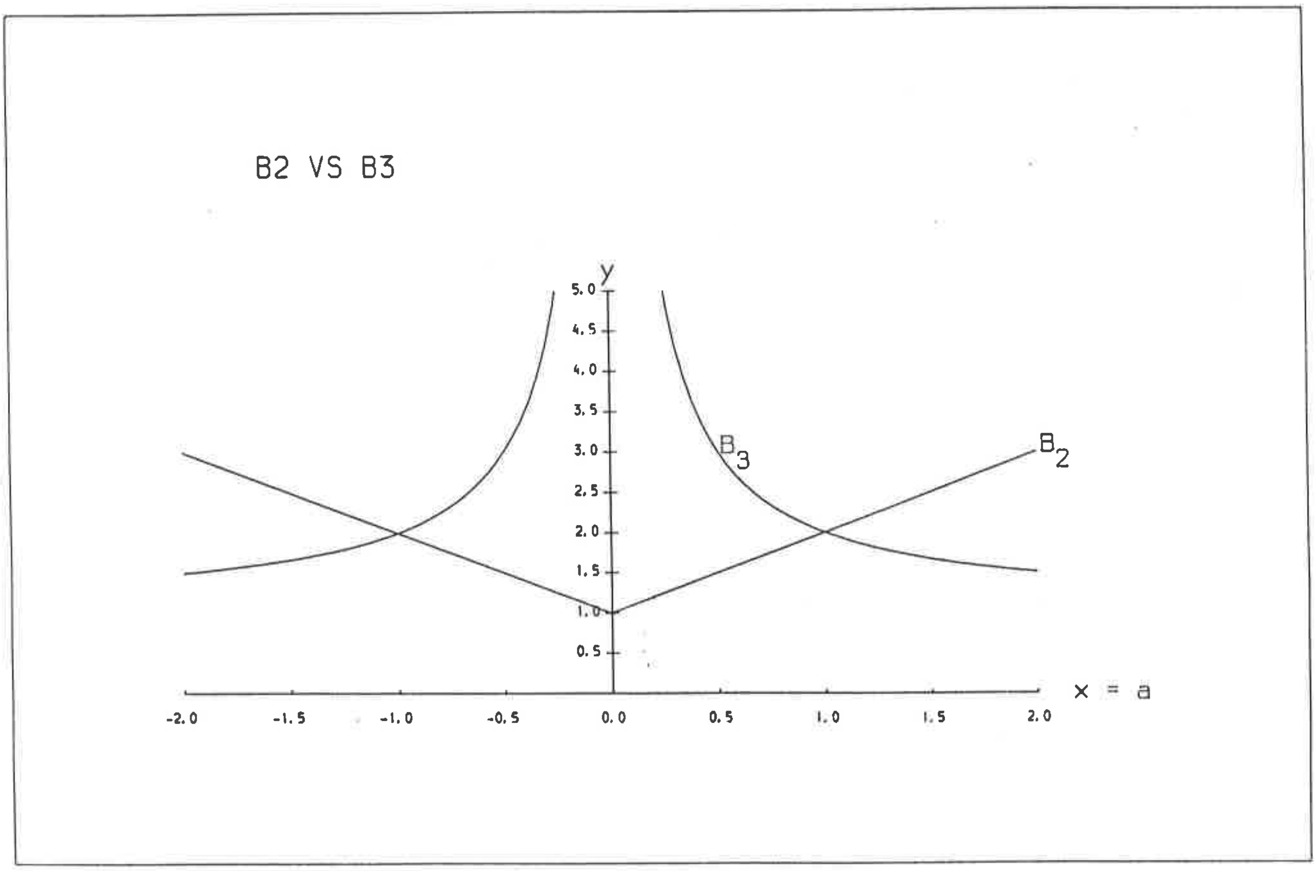

figure 6 (F-norm)

One has

(B_2, B_3)	1-norm	2-norm	∞-norm	F-norm
n = 2	2, 2	2, 1.7	2, 2	2, 3.5
3	7, 1.2(1)	5.2, 6.4	7, 1.2(1)	5.7, 9.8
4	4.0 (1), 2.4(1)	2.0 (1), 1.3(1)	4.0 (1), 2.4(1)	2.4 (1), 2.0(1)
5	3.4 (2), 4.0(1)	1.0 (2), 2.1(1)	3.4 (2), 4.0(1)	1.5 (2), 3.6(1)
6	3.9 (1), 6.0(1)	7.5 (2), 3.1(1)	3.9 (3), 6.0(1)	1.2 (3), 5.8(1)
7	5.6 (4), 8.4(1)	6.7 (3), 4.3(1)	5.6 (4), 8.4(1)	1.2 (4), 8.6(1)
8	9.6 (5), 1.1(2)	7.2 (4), 5.6(1)	9.6 (5), 1.1(2)	1.4 (5), 1.2(2)
9	1.9 (7), 1.4(2)	9.1 (5), 7.1(1)	1.9 (7), 1.4(2)	2.0 (6), 1.6(2)
10	4.4 (8), 1.8(2)	1.3 (7), 8.9(1)	4.4 (8), 1.8(2)	3.2 (7), 2.1(2)
11	1.1(10), 2.2(2)	2.1 (8), 1.1(2)	1.1 (10), 2.2(2)	5.8 (8), 2.7(2)
12	3.1(11), 2.6(2)	3.8 (9), 1.3(2)	3.14(11), 2.6(2)	1.2(10), 3.4(2)
13	9.7(12), 3.1(2)	7.4(10), 1.5(2)	9.7 (12), 3.1(2)	2.5(11), 4.2(2)
14	3.3(14), 3.6(2)	1.6(12), 1.8(2)	3.3 (14), 3.6(2)	6.1(12), 5.0(2)
15	1.2(16), 4.2(2)	3.7(13), 2.0(2)	1.2 (16), 4.2(2)	1.6(14), 6.0(2)
16	4.7(17), 4.8(2)	9.2(14), 2.3(2)	4.7 (17), 4.8(2)	4.3(15), 7.0(2)
17	2.0(19), 5.4(2)	2.4(16), 2.6(2)	2.0 (19), 5.4(2)	1.3(17), 8.2(2)
18	8.9(20), 6.1(2)	6.9(17), 2.9(2)	8.8 (20),6.12(2)	4.0(18), 9.5(2)
19	4.17(22),6.8(2)	2.1(19), 3.2(2)	4.2 (22), 6.8(2)	1.4(20), 1.1(3)
20	2.1(24), 7.6(2)	6.7(20), 3.6(2)	2.1 (24), 7.6(2)	4.8(21), 1.2(3)

(a(b) denotes $a \times 10^b$.)

Contrary to example (iii), B_3 is far better than B_2 as $B_2 \to \infty$ quite rapidly.

(v) $A = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$. $N = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$; $X = \begin{pmatrix} 1 & 0 \\ 0 & a^{-1} \end{pmatrix}$; $X^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}$.

One has

	1-norm	2-norm	∞-norm	F-norm
B_2	$1 + \lvert a\rvert$	$1 + \lvert a\rvert$	$1 + \lvert a\rvert$	$1 + \lvert a\rvert$
B_3	$\max\{1, \lvert a\rvert, \lvert a\rvert^{-1}\}$	$\max\{1, \lvert a\rvert, \lvert a\rvert^{-1}\}$	$\max\{1, \lvert a\rvert, \lvert a\rvert^{-1}\}$	$\lvert a\rvert + \lvert a\rvert^{-1}$.

When $|a|$ is large, $B_2 = 1 + |a| > |a| = B_3$, but they are only different by 1. (For F-norm, $B_3 \cong |a|$.)

When $|a|$ is small, $B_2 \cong 1$ and $B_3 \cong |a|^{-1}$. Similar to example (ii), A is better to be treated as I_2 for very small $|a|$.

Figures 5-6 indicate that B_3 provides a tighter bound than B_2 when

$|a| \geqq 0.7$ for the 1,2 and ∞-norms

or $|a| \geqq 1$ for the F-norm.

Otherwise, B_2 increases to infinity quite rapidly as $a \to 0$.

This example, like example (iii), highlights the difficulties in using Jordan Canonical form in numerical analysis.

This section only sketches out the weaknesses of the inequalities in (3) and (4) using a few trivial examples. More work is required for a detail study.

§4. Further Generalizations.

Consider the case where

$$A(X_1, X_2) = (X_1, X_2)\begin{pmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{pmatrix}, \tag{10a}$$

$$\begin{pmatrix} Y_1^H \\ Y_2^H \end{pmatrix} A = \begin{pmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{pmatrix} \begin{pmatrix} Y_1^H \\ Y_2^H \end{pmatrix} \tag{10b}$$

and $\Lambda_1 = \mathrm{diag}(\lambda_{1i})$, which is $n_1 \times n_1$.

Apply Theorem 3 to the matrix $Y_1^H A X_1 = \Lambda_1$, one has

$$\min_i |\hat{\lambda} - \lambda_{1i}| \leq \theta_3^{1/p_1} , \quad \text{for small } \varepsilon , \tag{11}$$

where $\hat{\lambda}$ is an eigenvalue of $Y_1^H(A + \varepsilon B)X_1 = \Lambda_1 + \varepsilon Y_1^H B X_1$,

p_1 is the dimension of the biggest Jordan block in Λ_1 ,

$$\theta_3 = C\varepsilon \|B\| \cdot \| X_1 \| \cdot \|Y_1\| \cdot \kappa(I_{n_1}) ,$$

$$C = \sqrt{\frac{p_1(p_1 + 1)}{2}} \quad \text{for 2 or F-norm}$$

$$= p_1 \quad \text{for 1 or } \infty\text{-norm} . \tag{12}$$

Note that the eigen- and principal-vectors of Λ_1 are columns of I_{n_1} .

However one is interested in the eigenvalues of $(A + \varepsilon B)$, instead of that of $Y_1^H(A + \varepsilon B)X_1$. One can prove the following Lemma:- (c.f. [8])

<u>Lemma 4</u>:- Let ε be a small perturbation parameter and $\tilde{\Lambda}_1 = \text{diag}(\tilde{\lambda}_{1i})$, with λ_{1i} of A perturbed to $\tilde{\lambda}_{1i}$ of $A + \varepsilon B$. If

$$\lambda(\tilde{\Lambda}_1) \cap \lambda(\Lambda_2 + \varepsilon Y_2^H B X_2) = \emptyset \tag{13a}$$

and

$$\| \Phi^{-1} \| = O(1) \tag{13b}$$

where $\Phi(\cdot) = \tilde{\Lambda}_1(\cdot) - (\cdot)(\Lambda_2 + \varepsilon Y_2^H B X_2)$, then

$$\tilde{\lambda}_{1i} = \hat{\lambda}_{1i} + O(\varepsilon^{2/p_1}) \tag{14}$$

where $\hat{\lambda}_{1i}$ are the eigenvalues of $(\Lambda_1 + \varepsilon Y_1^H B X_1)$ with the same ordering as for $\tilde{\lambda}_{1i}$.

Note that (13) is equivalent to

$$\lambda(\Lambda_1) \cap \lambda(\Lambda_2) = \emptyset \tag{15a}$$

$$\left[\min_{i,j} |\lambda_{1i} - \lambda_{2j}|\right]^{-1} = O(1) \tag{15b}$$

and ε small enough.

(Proof):- Consider

$$\begin{pmatrix} Y_1^H \\ Y_2^H \end{pmatrix} (A + \varepsilon B)(X_1, X_2) = \begin{pmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{pmatrix} + \begin{pmatrix} E_1 & E_2 \\ E_3 & E_4 \end{pmatrix}$$

with $E_i = O(\varepsilon)$.

Then $$\left[\begin{pmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{pmatrix} + \begin{pmatrix} E_1 & E_2 \\ E_3 & E_4 \end{pmatrix}\right]\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}\tilde{\Lambda}_1 .$$

$$\Longleftrightarrow \quad \begin{cases} (\Lambda_1 + E_1)Z_1 + E_2 Z_2 = Z_1 \tilde{\Lambda}_1 & \text{(16a)} \\ E_3 Z_1 + (\Lambda_2 + E_4)Z_2 = Z_2 \tilde{\Lambda}_1 . & \text{(16b)} \end{cases}$$

(13) and (16b) $\Rightarrow$ $Z_2 = \Phi^{-1}(E_3 Z_1)$, with

$$\Phi(\cdot) = \tilde{\Lambda}_1(\cdot) - (\cdot)(\Lambda_2 + E_4) . \tag{17}$$

(13) ensures that Φ is invertible.

Substitute (17) into (16a) implies (14), after some simple perturbation analysis ([8][10].)

Q.E.D.

Lemma 4 and (11) imply the following generalization of Theorem 1.

Theorem 5:- If conditions in Lemma 4 as well as (10) hold, one has

$$\min_i |\tilde{\lambda} - \lambda_{1i}| \leq \theta_3^{1/p_1} + O(\varepsilon^{2/p_1}) \tag{18}$$

with $\tilde{\lambda} \in \lambda(\tilde{\Lambda}_1)$, $\lambda_{1i} \in \lambda(\Lambda_1)$ and θ_3 specified in (11) and (12).

Note that Theorem 5 holds for all norms for which Theorem 3 holds. For symmetric cases, $Y_1^H = X_1^H = X_1^\dagger$ and $\|X_1\| \cdot \|Y_1\|$ in θ_3 in (18) can be replaced by $\kappa(X_1)$. In addition, Y_1^H is a left inverse of X_1 and $X_1^\dagger$ is the left inverse of X_1 with minimum norm (in 2 and F-norms). $\kappa(X)$ can be used as a rough estimate of $\|X_1\| \cdot \|Y_1\|$. As $\kappa(X)$ is a lower bound for $\|X_1\| \cdot \|Y_1\|$ in the 2- and F-norms, a large $\kappa(X)$ will indicate a large $\|X_1\| \cdot \|Y_1\|$, and thus ill-conditioning. (Of course, we cannot be sure of a small $\|X_1\| \cdot \|Y_1\|$ when $\kappa(X)$ is small.) For non-defective cases, $p_1 = 1$ and (18) collapses to

$$\begin{aligned} \min_i |\hat{\lambda} - \lambda_{1i}| &\leq \theta_3 + O(\varepsilon^2) \\ &= \tilde{C} \cdot \varepsilon \cdot \|B\| \cdot \|X_1\| \cdot \|Y_1\| + O(\varepsilon^2) , \end{aligned} \tag{19}$$

with $\tilde{C} = \|I_{n_1}\|^2$.

The result in (19) is trivial for $n_1 = 1$, as $\|x_1\| \cdot \|y_1\|$ is just the usual condition number related to the individual eigenvalues, in this case λ_1. (c.f. s_i in [10].)

By applying Theorem 5, one can break up the spectrum of a matrix into ℓ subgroups $(1 \leq \ell \leq n)$ and ℓ condition numbers of the form $\|X_\ell\| \cdot \|Y_\ell\|$ can be used to represent the conditioning of the eigenvalue problem, instead of using one condition number $\kappa(X)$ as in Theorems 1 and 3, or using n condition numbers s_i as in [10]. An obvious choice for ℓ will be to group the multiple eigenvalues together.

§5. Conclusions.

This paper generalizes the classic Bauer-Fike Theorem for diagonalizable or non-defective matrices, to cases where (i) the matrix is defective, or (ii) only part of spectrum is considered. A few trivial examples have also been considered to compare the results in this paper with a closely-related one in [4].

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