Comments on the Numerical Integration of a Class of Singular Perturbation Problems

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Abstract

A three-point difference scheme recently proposed in [2] for the numerical solution of a class of linear, singularly perturbed, two-point boundary value problem is investigated. The scheme is derived from a first-order approximation to the original problem with a small deviating argument. It is shown here that in the limit, as the deviating argument tends to zero, the difference scheme converges to a one-sided approximation to the original singularly perturbed equation in conservation form. The limiting scheme is shown to be stable on any uniform grid. No advantage arises, therefore, from using the deviating argument, and the most accurate and efficient results are obtained with the deviation at its zero limit.

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1. INTRODUCTION

Recently a novel three-term difference scheme for the numerical integration of a class of linear, singularly perturbed, two-point boundary value problems with a boundary layer at the left end of the interval has been proposed in [2]. The scheme is obtained by approximating the original problem by a first-order differential equation with a small deviating argument. The authors state that the method does not require a very fine mesh size and that iteration on the deviating argument may be used. Numerical experiments are presented to substantiate these results.

In this paper we examine the proposed difference approximation in more detail and show that, in the limiting value of the deviating argument, it reduces to a one-sided scheme for the original singularly-perturbed problem in conservation form and that this scheme is stable and accurate to the order of the step size for any mesh. No advantage arises, therefore, from using the deviating argument and the most accurate results are obtained with the deviation at zero.

2. NUMERICAL SCHEME

In [2] the following singular perturbation problem (SPP) is considered:

$$\epsilon y''(x) + a(x)y'(x) - b(x)y(x) = f(x), \qquad 0 \le x \le 1, \tag{1}$$

$$y(0) = \alpha \qquad y(1) = \beta. \tag{2}$$

Here ϵ is a small positive parameter such that $0 < \epsilon < 1$; α, β are given constants; $a(x)$, $b(x)$, $f(x)$ are sufficiently continuously differentiable functions on $[0,1]$; and $a(x) \ge M > 0$ on $[0,1]$, where M is some positive constant. The assumption

$$b(x) \ge 0, \tag{3}$$

is also made in [2]. Under these conditions, the SPP problem (1)-(2) has a unique solution $y(x) \in C^2[0,1]$ which, in general, displays a boundary layer of width $O(\epsilon)$ at $x = 0$ for small values of ϵ. It can be shown, moreover, that the solution is uniformly bounded in the L_∞-norm as $\epsilon \to 0$, [5]. If it is assumed, alternatively, that

$$b(x) + a'(x) \ge 0, \tag{4}$$

then the same results can be shown to hold, by arguments similar to those in [5] and [1]. (See Appendix I for proof.) Assumption (4) is the natural condition for the equation (1) in conservation form (where the term ay' is replaced by $(ay)' - a'y$).

The original differential equation is approximated in [2] by a first-order equation with a small deviating argument, given by

$$y'(x) - p(x)y'(x - \delta) - q(x)y(x) = r(x), \tag{5}$$

for $\delta \le x \le 1$, where δ is a small positive constant such that $0 < \delta << 1$ and

$$\begin{aligned} p(x) &= \epsilon/[\epsilon + \delta a(x)], \\ q(x) &= \delta b(x)/[\epsilon + \delta a(x)], \\ r(x) &= \delta f(x)/[\epsilon + \delta a(x)]. \end{aligned} \tag{6}$$

The difference equation is then obtained by integrating (5) over the interval

$[x_j, x_{j+1}]$ using the trapezoidal quadrature formula together with the approximation

$$y(x_j - \delta) \approx y_j + \delta[y_j - y_{j-1}]/h \ .$$

The three-point scheme found by this technique is given by

$$E_j y_{j-1} - F_j y_j + G_j y_{j+1} = H_j \tag{7}$$

where

$$E_j = (\delta/h)[p_j + (h/2)p_j']$$

$$F_j = 1 + (\delta/h)[p_{j+1} - (h/2)p_{j+1}']$$
$$\qquad - (1 - \delta/h)[p_j + (h/2)p_j'] + (h/2)q_j,$$

$$G_j = 1 - (1 - \delta/h)[p_{j+1} - (h/2)p_{j+1}'] - (h/2)q_{j+1},$$

$$H_j = (h/2)[r_{j+1} + r_j],$$

and

$$p_j = p(x_j), \quad q_j = q(x_j), \quad r_j = r(x_j)$$

$$h = 1/N, \quad x_j = jh, \quad j = 0,1,2,\ldots,N.$$

In the next section we show that in the limit, as the deviating argument $\delta \to 0$, this scheme reduces to a one-sided difference approximation to the original equation (1).

We remark here that, in the case $a(x) \leq -M < 0$ holds on $[0,1]$, a unique solution to the singular perturbation problem (1)-(2) exists if either (3) or (4) is assumed, but it has, in general, a boundary layer of width $O(\varepsilon)$ at the right hand end of the interval (i.e., at $x = 1$). In this case a corresponding stable three-term difference equation can be found by the method of [2], provided the integration is taken over the interval $[x_{j-1}, x_j]$ and the approximation

$$y(x_j - \delta) \approx y_j - \delta[y_{j+1} - y_j]/h$$

is used to obtain the recurrence relation. The scheme found by this procedure is given by

$$\tilde{E}_j y_{j-1} - \tilde{F}_j y_j + \tilde{G}_j y_{j+1} = \tilde{H}_j, \tag{8}$$

where

$$\tilde{E}_j = -1 + (1 + \delta/h)[p_{j-1} + (h/2)p'_{j-1}] - (h/2)q_{j-1},$$
$$\tilde{F}_j = -1 + (1 - \delta/h)[p_j - (h/2)p'_j]$$
$$\qquad + (\delta/h)[p_{j-1} + (h/2)p'_{j-1}] + (h/2)q_j,$$
$$\tilde{G}_j = (\delta/h)[p_j - (h/2)p'_j],$$
$$\tilde{H}_j = (h/2)[r_j + r_{j-1}], \quad j = 1,2,\ldots,N-1.$$

In the limit, as the deviation $\delta \to 0$, this scheme also converges to a stable one-sided approximation to (1), by arguments given in the next section.

3. ANALYSIS

We now establish the behavior of the difference scheme (7) in the limit as the deviating argument $\delta \to 0$. This analysis is necessary to validate the use of iteration on δ. We show that the recurrence relation (7) reduces to a one-sided difference approximation to the original singularly perturbed differential equation (1).

We begin by observing that the first-order differential approximation (5), (after division by δ), converges in the limit, as $\delta \to 0$, to the equation

$$y''(x) + [a(x)y(x)]'/\epsilon - [a'(x) + b(x)]\, y(x)/\epsilon = f(x)/\epsilon, \tag{9}$$

which is easily seen to be equivalent to equation (1) in conservation form. This result follows from

$$\lim_{\delta\to 0} [y'(x) - p(x)y'(x-\delta)]/\delta = y''(x) + a(x)y'(x)/\epsilon,$$

$$\lim_{\delta\to 0} q(x)/\delta = b(x)/\epsilon,$$

$$\lim_{\delta\to 0} r(x)/\delta = f(x)/\epsilon,$$

together with the relation

$$a(x)y'(x) = [a(x)y(x)]' - a'(x)y(x).$$

It is easily shown, furthermore, that

$$\lim_{\delta\to 0} [1 - p(x)]/\delta = a(x)/\epsilon,$$

$$\lim_{\delta\to 0} p'(x)/\delta = \lim_{\delta\to 0} - \epsilon a'(x)/[\epsilon + \delta a(x)]^2 = - a'(x)/\epsilon,$$

and that

$$\lim_{\delta\to 0} p(x) = 1, \qquad \lim_{\delta\to 0} p'(x) = 0.$$

From these results it follows that the coefficients of the difference scheme (7) (divided by δ) converge to the limits

$$\lim_{\delta\to 0} E_j/\delta = (1/h),$$

$$\lim_{\delta\to 0} F_j/\delta = (2/h) + a_j/\epsilon + (h/2)(a_j' + b_j)/\epsilon,$$

$$\lim_{\delta\to 0} G_j/\delta = (1/h) + a_{j+1}/\epsilon - (h/2)(a_{j+1}' + b_{j+1})/\epsilon,$$

$$\lim_{\delta\to 0} H_j/\delta = (h/2)[f_j + f_{j+1}]/\epsilon.$$

The approximation (after division by δ and h) thus converges, as $\delta \to 0$, to the difference equation

$$[y_{j-1} - 2y_j + y_{j+1}]/h^2 + [a_{j+1}y_{j+1} - a_jy_j]/\epsilon h - \frac{1}{2}[a_j'y_j + a_{j+1}'y_{j+1} + b_jy_j + b_{j+1}y_{j+1}]/\epsilon = \frac{1}{2}[f_j + f_{j+1}]/\epsilon. \quad (10)$$

Equation (10) is clearly a one-sided approximation to the original differential equation (1) written in conservation form (9).

The recurrence (8), for right-end boundary layers, converges similarly to the equation

$$[y_{j-1} - 2y_j + y_{j+1}]/h^2 + [a_j y_j - a_{j-1} y_{j-1}]/\epsilon h$$
$$- \frac{1}{2} [a'_{j-1} y_{j-1} + a'_j y_j + b_{j-1} y_{j-1} + b_j y_j]/\epsilon = \frac{1}{2} [f_{j-1} + f_j]/\epsilon \quad (11)$$

which also approximates (9).

We remark that, provided both the schemes (7) and (10) (or (8) and (11)) are stable for all δ, h (and ϵ) and are convergent, then, as the step size h tends to zero, the accuracy of the approximations given by (7) (or (8)) will be dominated by the error between the solution of the approximate differential equation (5) and that of the true equation (9); that is, the error will be determined by the size of the deviating argument δ, for all h sufficiently small. For larger step-sizes the error in both methods is expected to be of the order of the step-size, since both can be shown to be methods of order one. There is no advantage, therefore, in using scheme (7) (or (8)) in preference to scheme (10) (or (11)), and in fact, the latter schemes are computationally less expensive, as well as being more accurate.

In order to establish stability and convergence, it is necessary to show that the difference equations, together with the boundary conditions $y_o = \alpha$, $y_N = \beta$, have a uniqe solution and that this solution is <u>uniformly bounded</u> for all h (and ϵ). In the next section we investigate the stability of the schemes (7), (10) (and (8),(11)) and demonstrate that, if condition (4) is assumed, then the scheme (10) (or (11)) is stable for all choices of the step-size h.

4. STABILITY AND CONVERGENCE

The linear difference equations (7) or (10) (or (8) or (11)) together with the boundary conditions $y_0 = \alpha$, $y_N = \beta$, form a tridiagonal system of N-1 algebraic equations for the N-1 unknowns y_j, $j = 1,2,\ldots,N-1$. It is well-known that the coefficient matrix of such a system is non-singular if it is either strictly diagonally dominant or irreducibly diagonally dominant [6]. Moreover, if these conditions hold, the method of LU decomposition (or Gaussian Elimination) provides a numerically stable technique for solving the system. Written in recurrence form [3], this method is exactly equivalent to the discrete invariant embedding algorithm described in [2].

It is also well-known that if, in addition to strict or irreducible diagonal dominance, it is assumed that the diagonal elements of the matrix are strictly negative, the off-diagonal elements are non-negative, and the graph of the matrix is strongly connected, then the coefficient matrix is an M-matrix and has a negative inverse. From this result a maximum principle can be established for the difference equations and it can be shown that the solutions not only exist and are unique, but are also uniformly bounded for all N (i.e., for all $h = 1/N$) [7].

In [2] it is shown that under the following conditions the solution algorithm for (7) is numerically stable:

$$E_j > 0, \quad G_j > 0, \quad F_j \geq E_j + G_j \text{ and } |E_j| \leq |G_j|.$$

It is easily seen that these conditions are also sufficient to establish the existence and uniqueness of solutions and the stability of the difference scheme. Under the initial assumptions made on the SPP(1)-(2), however, these relations cannot be guaranteed to hold for all h and δ; in particular, for the diagonal dominance property to hold, it is necessary that

$$[p_{j+1} - p_j]/h + \frac{1}{2} [- p'_{j+1} - p'_j + q_{j+1} + q_j] \geq 0, \tag{12}$$

which requires, in general, that h be sufficiently small. The stability of the difference scheme (7) cannot, therefore, be established by this argument for all choices of the step-size.

The limiting difference equations (10) can similarly be shown to have unique solutions and to be stable provided the inequalities

$$1 + ha_j/\epsilon + h^2[a'_j + b_j]/2\epsilon \geq$$
$$1 + ha_{j+1}/\epsilon - h^2[a'_{j+1} + b_{j+1}]/2\epsilon \geq 0 \tag{13}$$

are satisfied. These relations hold, in general, only if h is sufficiently small to guarantee that

$$a_j - (a'_j + b_j)h/2 \geq 0 \tag{14}$$

and

$$- [a_{j+1} - a_j]/h + \frac{1}{2}[a'_{j+1} + a'_j + b_{j+1} + b_j] \geq 0 \tag{15}$$

We cannot show, therefore, that the difference scheme (10) is stable for all choices of h by this argument either. (Similar conclusions hold for the schemes (8) and (11) in the case $a(x) \leq - M < 0$.)

We observe that under certain special conditions the inequalities (12), (15) are automatically satisfied for all h. In particular, if $[a'(x) + b(x)] \geq 0$ and $a'(x) \leq 0$, $\forall x \in [0,1]$, as in the examples in [2], then (12) and (15) hold for any step-size. In these cases, to achieve stability only a relatively unrestrictive condition of form (14) need be satisfied for $h \leq \frac{1}{2}$. For general coefficients $a(x)$, $b(x)$, however, the inequalities (12), (15) cannot be expected to be satisfied for all choices of the step-length.

The difference schemes (7), (10) and (8), (11) are all one-sided schemes, however, and the results of [5], [1] and [4] strongly suggest that these schemes may be expected to be stable independently of the grid-size. We now establish that under assumption (4) (plus the initial assumptions on the SPP (1)-(2)), the solution of the difference equations (10), together with the boundary conditions $y_0 = \alpha$, $y_N = \beta$, exists, is unique and is uniformly bounded for all h (and ϵ) and, hence, is always stable. (A similar result holds for the difference scheme (11) in the case $a(x) \le - M < 0$.) Since (10) (or (11)) is the preferred scheme if it is stable for all grid sizes, we do not attempt to derive results for (7) (or (8)).

We have the following:

<u>Theorem 1</u>. Given that $a(x) \ge M > 0$ and $\tilde{b}(x) = a'(x) + b(x) \ge 0$, $\forall x \in [0,1]$, the solution to the difference equations (10), together with the boundary conditions $y_0 = \alpha$, $y_N = \beta$, exists, is unique, and satisfies

$$\|y\|_{h,\infty} \le M^{-1}\|f\|_{h,1} + (1 + M^{-1}C)(|\alpha| + |\beta|), \tag{16}$$

where C is a universal constant, independent of h, ϵ, and α, β, f.

(Here $\|\cdot\|_{h,\infty}$ and $\|\cdot\|_{h,1}$ are the discrete ℓ_∞ and ℓ_1 norms given by

$$\|v\|_{h,\infty} = \max_{0\le j\le N-1} |v_j|, \quad \|v\|_{h,1} = \tfrac{1}{2}h \sum_{j=0}^{N-1} |v_j| + |v_{j+1}|.)$$

<u>Proof</u>: The proof follows similar arguments to those of [5] and [1], but the result is more general. Let $L_h(\cdot)$ denote the difference operator on the

left-hand side of equation (10) after multiplication by ϵ, and let w_j be any mesh function satisfying $L_h(w_j) = \tilde{f}_j$. By rearranging the difference scheme and using the non-negativity of the coefficients we obtain

$$\left(\frac{2\epsilon}{h^2} + \frac{a_j}{h} + \frac{1}{2}\tilde{b}_j\right)|w_j| \le |\tilde{f}_j| + \left(\frac{\epsilon}{h^2} + \frac{a_{j+1}}{h} + \frac{1}{2}\tilde{b}_{j+1}\right)|w_{j+1}| + \frac{\epsilon}{h^2}|w_{j-1}|.$$

It follows that

$$0 \le \epsilon(|w_{j+1}| - |w_j|)/h^2 - \epsilon(|w_j| - |w_{j-1}|)/h^2 + (a_{j+1}|w_{j+1}| - a_j|w_j|)/h + \frac{1}{2}(\tilde{b}_{j+1}|w_{j+1}| - \tilde{b}_j|w_j|) + |\tilde{f}_j| . \qquad (17)$$

(i) To show existence and uniqueness, let $\{u_j\}$, $\{v_j\}$ be two solutions to the difference equations (10) satisfying the boundary conditions. Then $w_j = u_j - v_j$ satisfies $L_h(w_j) = \tilde{f}_j$, where $\tilde{f}_j = 0$, and $w_o = 0 = w_N$. Summing (17) over $j = 1$ to $N-1$ then gives

$$0 \le -\epsilon|w_1|/h^2 - \epsilon|w_{N-1}|/h^2 - a_1|w_1|/h - \frac{1}{2}\tilde{b}_1|w_1|,$$

which implies $w_1 = 0 = w_{N-1}$. Repeating the argument gives $w_j = 0$, $j = 0,1,\ldots,N-1$, which ensures the uniqueness of the solutions. For linear equations existence is implied by uniqueness.

(ii) To establish the estimate (16), let $w_j = y_j - \ell_j$, where y_j satisfies (10) and the boundary conditions $y_o = \alpha$, $y_N = \beta$, and $\ell_j = \beta(jh) + \alpha(1 - jh)$. Then $w_o = 0 = w_N$ and w_j, $j = 1,,2,\ldots,N-1$, satisfies

$$L_h(w_j) = \tilde{f}_j \equiv L_h(y_j) - L_h(\ell_j) = \tfrac{1}{2}[f_j + f_{j+1}] - L_h(\ell_j).$$

Now let $|w_n| = \|w\|_{h,\infty} \ge |w_j|$, $j = 0,1,\ldots,N$; then, summing (17) from $j = n$ to $N - 1$ gives

$$0 \le -\epsilon(|w_n| - |w_{n-1}|)/h^2 - \epsilon|w_{N-1}|/h^2$$
$$- a_n|w_n|/h - \frac{1}{2}\tilde{b}_n|w_n| + \sum_{n}^{N-1} |\tilde{f}_j|. \qquad (18)$$

It follows from (18) and the conditions on a(x) and $\tilde{b}(x)$ that

$$M|w_n| \le \sum_{j=0}^{N-1} h|\tilde{f}_j| \le \|f\|_{h,1} + \sum_{j=0}^{N-1} h|L_h(\ell_j)|.$$

To complete the result, it is not difficult to show that

$$\|\ell\|_{h,\infty} = \max(|\alpha|, |\beta|), \qquad \|\ell\|_{h,1} = \frac{1}{2}(|\alpha| + |\beta|)$$

and

$$\sum_{j=0}^{N-1} h|L_h(\ell_j)| \le \|a\|_\infty(|\alpha| + |\beta|) + (\|a'\|_\infty + \|\tilde{b}\|_\infty)\|\ell\|_{h,1}.$$

The estimate (16) then follows immediately from $\|y\|_{h,\infty} \le \|w\|_{h,\infty} + \|\ell\|_{h,\infty}$, with C taken to be a positive constant which bounds $\|a\|_\infty + \frac{1}{2}(\|a'\|_\infty + \|\tilde{b}\|_\infty)$. □

This Theorem, together with the relation $\|f\|_{h,1} \le \|f\|_\infty$, implies that the solutions to the difference equations (10) are uniformly bounded, independently of the grid-size h and the parameter ϵ, and we conclude that the scheme is stable for all step-sizes.

From Theorem 1 we can immediately establish an error bound. We have:

<u>Corollary 1</u>. Under the conditions of Theorem 1, the error $e_j = y(x_j) - y_j$ between the solution $y(x_j)$ of the singular perturbation problem (1)-(2) and the solution y_j of the difference equations (10), with boundary conditions $y_o = \alpha$, $y_N = \beta$, satisfies the estimate

$$\|e\|_{h,\infty} \le M^{-1}\|\tau\|_{h,1} \qquad (19)$$

where

$$|\tau_j| \le \frac{1}{2} h \{ \max_{x_{j-1}\le x\le x_{j+1}} |((a' + b)y)'| + \max_{x_{j-1}\le x\le x_{j+1}} |(ay)''| + \max_{x_{j-1}\le x\le x_{j+1}} 2\epsilon |y'''|/3\} . \quad (20)$$

(It is assumed that the coefficients a(x), b(x) are sufficiently differentiable to ensure that the solution y(x) belongs to $C^3[0,1]$).

<u>Proof</u>: Using the notation of Theorem 1, it is easy to show that $L_h(y(x_j)) = \tau_j$, where the truncation error τ_j satisfies (20), and, therefore, the error e_j satisfies $e_o = 0 = e_N$ and

$$L_h(e_j) = L_h(y(x_j)) - L_h(y_j) = \tau_j, \qquad j = 1,2,\ldots,N-1.$$

Defining $\tau_o = 0 = \tau_N$, the estimate (19) then follows directly from Theorem 1. □

We remark that the estimate (19) demonstrates that for hM ≥ 1 the error behaves essentially like $O(h^2)$ and hence the scheme (10) is satisfactory for large step-sizes. If hM << 1, on the other hand, the accuracy may be expected to be relatively poor.

The estimate (19) also establishes the convergence of the difference scheme (10) for fixed values of the small parameter $\epsilon > 0$. To establish <u>uniform</u> convergence using this error estimate, uniform bounds on the derivatives of the solution to the differential equation are required for all ϵ. Such bounds can be determined from results found in [5]. Alternatively, it is sufficient to show that the solutions to the homogeneous difference scheme are of bounded variation uniformly for any grid and all ϵ. Then the convergence result follows by the arguments of [5]. We now establish the property of bounded variation for the difference scheme (10). (We remark that the results of [5] and [1] are not directly applicable here, due to the fact that the coefficient a(x) is variable with x.)

We have the following

<u>Theorem 2</u>. Given that $a(x) \geq M > 0$ and $\tilde{b}(x) \equiv a'(x) + b(x) > 0$, $\forall x \in [0,1]$, the solution to the difference equations (10) for $f_j = 0$, $j = 0,1,\ldots,N$, together with the boundary conditions $y_o = \alpha$, $y_N = \beta$, is of bounded variation and satisfies the estimate

$$\sum_{j=1}^{N} |y_j - y_{j-1}| \leq M^{-1}(\|a'\|_\infty + 2\|\tilde{b}\|_\infty)\|y\|_{h,\infty} + \|a\|_\infty (|\alpha| + |\beta|)). \tag{21}$$

<u>Proof</u>. If we define $\Delta_j = y_j - y_{j-1}$, then the homogeneous difference equation (10) may be written

$$\epsilon(\Delta_{j+1} - \Delta_j)/h^2 + (a_{j+1}y_{j+1} - a_jy_j)/h - \tfrac{1}{2}(\tilde{b}_{j+1}y_{j+1} + \tilde{b}_jy_j) = 0. \tag{22}$$

Rearranging (22) gives

$$\begin{aligned}(1 + ha_{j+1}/\epsilon)\Delta_{j+1} = \Delta_j &- (h^2/\epsilon)((a_{j+1} - a_j)/h)y_j \\ &+ (h^2/2\epsilon)(\tilde{b}_{j+1}y_{j+1} + \tilde{b}_jy_j)\end{aligned} \tag{23}$$

From (23) it follows that

$$(1 + hM/\epsilon)|\Delta_{j+1}| \leq |\Delta_j| + (Mh/\epsilon)(hC/M), \tag{24}$$

where $C = (\|a'\|_\infty + \|\tilde{b}\|_\infty)\|y\|_{h,\infty}$. Then, denoting $\kappa = (1 + hM/\epsilon)^{-1}$, we find from (24) that the difference $|\Delta_j|$, for $j = 1,2,\ldots,N-1$ satisfies the recurrence

$$|\Delta_{j+1}| \leq \kappa|\Delta_j| + (1 - \kappa)(hC/M),$$

which has the solution

$$|\Delta_j| \leq A\kappa^{j-1} + (hC/M) \tag{25}$$

where $A = |\Delta_1| - (hC/M)$. Summing (25) over $j = 1$ to N then gives

$$\sum_{1}^{N}|\Delta_j| \leq (1 - \kappa^N)A/(1 - \kappa) + C/M. \tag{26}$$

To complete the estimate we must, in essence, find an O(h) bound on $|\Delta_1|$ or, equivalently, on A. We observe that if we sum (22) from j = 1 to N - 1 and rearrange, we obtain

$$(1 + ha_1/\epsilon)\Delta_1 = \Delta_N + (Mh/\epsilon)M^{-1}(a_N y_N - a_1 y_0 - \frac{1}{2} h \sum_j (\tilde{b}_{j+1} y_{j+1} + \tilde{b}_j y_j)) \; .$$

It follows that

$$|\Delta_1| \le \kappa |\Delta_N| + (1 - \kappa)C_0/M, \tag{27}$$

where $C_0 = \|a\|_\infty(|\alpha| + |\beta|) + \|\tilde{b}\|_\infty \|y\|_{h,\infty}$. Then, using (25) with j = N and the definition of A, we find

$$(1 - \kappa^N) \; |\Delta_1| \le (\kappa - \kappa^N)hC/M + (1 - \kappa)C_0/M,$$

since $\kappa < 1$, and we have

$$(1 - \kappa^N)A/(1 - \kappa) \le C_0/M - hC/M. \tag{28}$$

The estimate (21) then follows directly from (26) and (28). □

Theorem 1 together with the estimate (21) ensures the bounded variation of the solutions $\{y_j\}_0^N$ to the difference scheme (10), independently of the grid-size h and the parameter ϵ. Following the arguments of [5], it can then be shown that if, for given h and ϵ, $y^{h,\epsilon}(x)$ is the unique piecewise linear continuous function on $x \in [0,1]$ agreeing with $\{y_j\}_0^N$ at the grid points and having jumps in its derivative only at points $x = x_j$, then for any pair of positive sequences $h_\nu \longrightarrow 0$ and $\epsilon_\nu \longrightarrow \epsilon$, there exists a subsequence of functions y^{h_ν,ϵ_ν} which converge, in the case $\epsilon \ne 0$, to $y^{0,\epsilon} = y(x)$, the unique solution to the singular perturbation problem (1)-(2) and, in the case $\epsilon = 0$, to $y^{0,0} = y(x)$, a weak solution of (1).

We conclude that the difference scheme (10) (and similarly scheme (11)) is stable for all choices of the grid-size h and is convergent in the limit as $h \to 0$

to the solution of the SPP (1)-(2) for all $\epsilon > 0$. Furthermore, from Theorem 2, it follows that continuation on the parameter ϵ may be safely used. The accuracy of the scheme is satisfactory for large grid-sizes, but if it is required to resolve the boundary layer accurately, it is advisable to use a non-uniform grid or a more sophisticated approximation.

We remark that the condition $a(x) \geq M > 0$ (or, correspondingly, $a(x) \leq -M < 0$) excludes the possibility of turning points in the solution. Many other schemes are available for solving such problems. For cases where turning points may occur and for certain classes of nonlinear problems, approximate difference schemes are given in [4], [5] and [1].

5. CONCLUSIONS

A three-point finite difference scheme for solving a singularly perturbed linear, two-point boundary value problem with a boundary layer at one end of the region is investigated. The scheme is derived in [2] and depends on a small deviating argument. It is shown here that in the limit, as the deviating argument tends to zero, the seheme converges to a simple one-sided approximation to the original singular perturbation equation in conservation form. The solution to the limiting scheme is shown, furthermore, to be uniformly bounded in the ℓ_∞-norm for all values of the small parameter on any uniform grid and, thus, to be uniformly stable. It is proved also that the solution is of bounded variation for all meshes and all values of the small parameter, and, hence, is uniformly convergent.

It is concluded that no advantage arises from using the deviating argument, and that the most accurate and efficient results are obtained using the limiting finite difference approximation with the deviation at zero. On a large uniform grid the one-sided limiting scheme is found to give reasonable precision, but to resolve the boundary layer accurately, a non-uniform grid or a more sophisticated approximation is recommended.

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Appendix

We show here that under the assumptions $a(x) \geq M > 0$ and $\tilde{b}(x) \equiv a'(x) + b(x) \geq 0$, the solution to the differential equation (1) with boundary conditions (2) is uniformly bounded in the maximum norm for all ϵ, and that the following estimate holds:

$$\|y\|_\infty \leq M^{-1}\|f\|_1 + (1 + M^{-1}C)(|\alpha| + |\beta|), \tag{29}$$

where C is a constant independent of ϵ, α, β and f.

Proof Let $L(\cdot)$ denote the differential operator on the left-hand side of equation (1), and let $w(x) = y(x) - \ell(x)$, where $y(x)$ satisfies the SPP (1)-(2) and $\ell(x) = \beta x + \alpha(1 - x)$. Then $w(0) = 0 = w(1)$ and

$$L(w) = L(y) - L(\ell) = f - L(\ell) \equiv \tilde{f}. \tag{30}$$

Furthermore, let $|w(x_n)| \equiv \|w\|_\infty$, and let $x_m > x_n$ be the first point to the right of x_n where $w(x)$ changes sign. Then $w'(x_n) = 0$, $w(x_m) = 0$ and

$$w'(x_m) \leq 0 \quad \text{if} \quad w(x_n) > 0$$
$$\geq 0 \quad\quad\quad w(x_n) < 0.$$

Integrating (30) from x_n to x_m then gives

$$\int_{x_n}^{x_m} \epsilon w'' + (aw)' - \tilde{b}w \, dx = \epsilon w'(x_m) - a_n w(x_n) - \int_{x_n}^{x_m} \tilde{b}w \, dx = \int_{x_n}^{x_m} \tilde{f} \, dx, \tag{31}$$

and multiplying both sides of (31) by $-\mathrm{sgn}(w(x_n))$ and rearranging leads to the result

$$M|w(x_n)| \leq \left| \int_{x_n}^{x_m} \tilde{f} \, dx \right|.$$

Finally, from

$$\|\ell\|_{\infty} = \max\{|\alpha|, |\beta|\}, \quad \|\ell\|_1 = \frac{1}{2}(|\alpha| + |\beta|)$$

and

$$\left|\int_{x_n}^{x_m} L(\ell)dx\right| \le \int_0^1 |L(\ell)|dx \le \|a\|_{\infty}(|\alpha|+|\beta|) + \|b\|_{\infty}\|\ell\|_1$$

it follows that

$$\left|\int_{x_n}^{x_m} \tilde{f}\, dx\right| \le \|f\|_1 + C(|\alpha| + |\beta|)$$

where C is a constant which bounds $\|a\|_{\infty} + \frac{1}{2}\|b\|_{\infty}$ and is independent of h, ϵ, α, β and f. The estimate (29) then follows directly from

$$\|y\|_{\infty} \le \|w\|_{\infty} + \|\ell\|_{\infty}. \qquad \square$$