

U N I V E R S I T Y O F R E A D I N G

A QUASI-RIEMANN METHOD FOR THE SOLUTION OF
ONE-DIMENSIONAL SHALLOW WATER FLOW

A. PRIESTLEY

Numerical Analysis Report 5/90

University of Reading
Department of Mathematics
P.O. Box 220
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Abstract

In this report we consider the flow in an open channel where super-critical flow may be induced by the geometry of the channel. The scheme presented calculates the Riemann invariants along characteristic curves and hence has no CFL limit. The presence of source terms may lead to a restriction on the time-step though to ensure the stability of the ODE solver used. This is not an issue in the results presented here.

1. Introduction

The Preissman scheme, Preissman (1961), has long been a favourite of hydraulic engineers. It is simple, accurate and due to its implicitness large time-steps can be taken. However, it does not perform well at discontinuities in the solution. In Priestley (1989b) a flux limited scheme was applied to the equations of one-dimensional river flow. This was a Roe type scheme, Roe (1981), and although it deals with shocks very well, the source terms did cause a slight under and overshoot at discontinuities. Roe's scheme is not cheap. It relies upon a characteristic decomposition and upwinding. Second-order forms require even more logical switching. Whilst this expense can be justified in situations where there are jumps in the solution, it seems rather inefficient for entirely smooth flows.

In this report we present a method that is efficient when the flow is smooth and yet can give monotone solutions even at shocks.

2. The St. Venant Equations and the Quasi-Riemann Scheme

The St. Venant equations for rough-turbulent flow in an open channel are:-

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = 0 \quad (1a)$$

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x} \left[\frac{Q^2}{A} \right] + gA \left[\frac{\partial h}{\partial x} + \frac{Q|Q|}{K^2} \right] = 0 \quad (1b)$$

Where

A = cross-sectional area = breadth \times depth
= $B \times d$ (only rectangular channels are considered here)

Q = massflow

u = velocity

g = gravity

h = height = $d + z$ where z is the height of the river bed

$\frac{Q|Q|}{K^2}$ is the friction term chosen to give the fully rough-turbulent form in this case

$K = \frac{A}{M}$ (hydraulic radius)^{2/3} where the hydraulic radius = $A/\text{wetted perimeter}$,
 M is Manning's constant for which we take a value of 0.03

β = bed-slope = $\partial z / \partial x$

c = Froude number = $\sqrt{gA/B}$.

To apply our chosen scheme we need to rewrite equation (1). Equation (1b) is put in terms of the velocity u and equation (1a) in terms of the Froude number c .

The St. Venant equations become

$$u_t + uu_x + 2cc_x = g \left[-\frac{Q|Q|}{K^2} - \beta \right] \quad (2a)$$

$$c_t + \frac{cu_x}{2} + uc_x = -\frac{ucB_x}{2B} \quad (2b)$$

We then write equation (2) in the form

$$\underline{u}_t + E\underline{u}_x = \underline{b} \quad (3)$$

where

$$\underline{u} = \begin{bmatrix} u \\ c \end{bmatrix} \quad (4a)$$

$$E = \begin{bmatrix} u & 2c \\ c/2 & u \end{bmatrix} \quad (4b)$$

$$\underline{b} = \begin{bmatrix} g \left[-\frac{Q|Q|}{k^2} - \beta \right] \\ -\frac{ucB_x}{2B} \end{bmatrix} \quad (4c)$$

The eigenvalues of E are

$$\lambda^+ = u + c$$
$$\lambda^- = u - c .$$

The associated eigenvectors are then

$$\underline{e}^- = (2, -1)^T \text{ and } \underline{e}^+ = (2, 1)^T .$$

Defining the matrix H to be given by

$$H = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$$

and the new vector

$$\underline{v} = H^{-1}\underline{u}$$

we get

$$\underline{v}_t + H^{-1}EH\underline{v}_x = H^{-1}\underline{b} .$$

Expanded this becomes, putting $F^+ = u + 2c$ and $F^- = u - 2c$,

$$F_t^+ + \lambda^+ F_x^+ = g \left[-\frac{Q|Q|}{K^2} - \beta \right] - \frac{ucB}{B} x$$

$$F_t^- + \lambda^- F_x^- = g \left[-\frac{Q|Q|}{K^2} - \beta \right] + \frac{ucB}{B} x .$$

The equations (1) can now be rewritten as

$$\frac{d^+ F^+}{dt} = b^1 \tag{5a}$$

$$\frac{d^- F^-}{dt} = b^2 \tag{5b}$$

with $\frac{d^\pm}{dt} = \frac{\partial}{\partial t} + \lambda^\pm \frac{\partial}{\partial x} .$

Equations (5) are now in a form suitable to apply a Lagrangian type scheme to. Perhaps, though, we should call this a Riemannian scheme to avoid confusion with schemes that use the fluid velocity u , as is normally the case with Lagrangian methods.

The use of the Riemann invariants has been suggested before by Goussebaile & Lepeintre (1989). However, we believe that our method is better suited to these problems for a number of reasons, stemming from the way that the Lagrangian/Riemannian derivatives are treated.

In Goussebaile & Lepeintre a finite element method is used for the advection step introduced in Berqué et al (1982). See also Morton,

Priestley & Süli (1988) for a discussion of this and related schemes. Being a least squares fit this Lagrangian method suffers from oscillations at shocks. Therefore a flux-limited scheme had to be used with the predicted values from the Riemannian step to keep monotonicity. Although the flux-limited scheme undoubtedly benefits from the use of these predicted upwinded values it seems rather inefficient. Even more so since the flux-limited step imposes a CFL restriction of 1 on the method whereas the Riemannian part of the scheme would have no such limit.

Here we suggest a scheme that is Lagrangian and monotone. In the meteorological literature it is called the semi- or quasi-Lagrangian method. See Robert (1981, 1982), Bates (1985), Staniforth & Temperton (1986), Ritchie (1987) and Temperton & Ritchie (1987) for a flavour of this work.

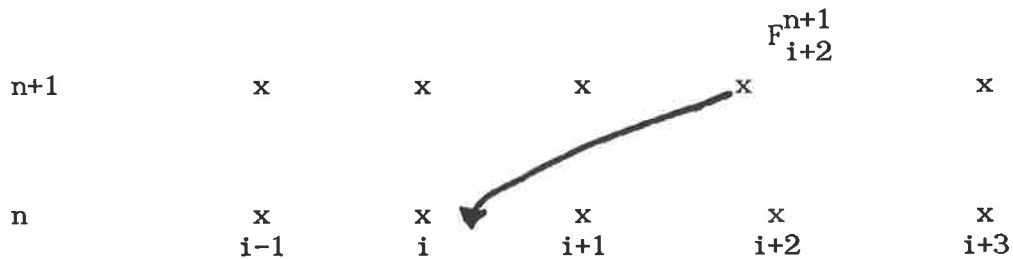


Figure 1

Suppose we require the value of F_{i+2}^{n+1} . The trajectory, characteristic curve, or whatever, is traced back to the old time-level. We now know that $F_{i+2}^{n+1} = F^n(x)$ and so we compute $F^n(x)$ by interpolation from the neighbouring points $F_{i-1}^n, F_i^n, F_{i+1}^n$ etc. A cubic

interpolation is chosen to provide accuracy but we also require the cubic polynomial to be monotone to avoid problems at large gradients. This question has been addressed by Fritsch & Carlson (1980) but since it is a fundamental part of the method we repeat their results for clarity.

Consider a cubic polynomial function $p(x)$ on the interval $[x_i, x_{i+1}]$ such that $p(x)$ is monotone and

$$p(x_i) = F_i$$

$$p(x_{i+1}) = F_{i+1}$$

We can write $p(x)$ on each sub-interval in terms of the cubic Hermite basis functions to obtain

$$p(x) = F_i H_1(x) + F_{i+1} H_2(x) + d_i H_3(x) + d_{i+1} H_4(x)$$

where

$$d_j = p'(x_j) \quad j = i, i+1$$

$$H_1(x) = \phi((x_{i+1}-x)/h_i) ,$$

$$H_2(x) = \phi((x-x_i)/h_i) ,$$

$$H_3(x) = -h_i \psi((x_{i+1}-x)/h_i) ,$$

$$H_4(x) = h_i \psi((x-x_i)/h_i) ,$$

with

$$h_i = x_{i+1} - x_i ,$$

$$\phi(t) = 3t^2 - 2t^3 ,$$

and

$$\psi(t) = t^3 - t^2 .$$

Letting $\Delta_i = (F_{i+1} - F_i)/h_i$ we can rewrite the Hermite cubic polynomial as

$$p(x) = \left[\frac{d_i + d_{i+1} - 2\Delta_i}{h_i^2} \right] (x-x_i)^3 + \left[\frac{-2d_i - d_{i+1} + 3\Delta_i}{h_i} \right] (x-x_i)^2 + d_i(x-x_i) + F_i . \quad (6)$$

As it stands (6) will not be monotone in general. Monotonicity is ensured by limiting the values of d_i and d_{i+1} , c.f. Sweby (1985).

An obvious necessary condition for monotonicity is that

$$\text{sign}(d_i) = \text{sign}(d_{i+1}) = \text{sign}(\Delta_i) . \quad (7)$$

Writing $\alpha = d_i/\Delta_i$ and $\beta = d_{i+1}/\Delta_i$, $\Delta_i \neq 0$, Fritsch & Carlson (1980) were able to prove that (6) is always monotone if and only if (7) holds in conjunction with one or both of the following conditions on (α, β) :-

$$0 \leq \alpha \leq 3, \quad 0 \leq \beta \leq 3 \quad (8a)$$

and

$$\phi(\alpha, \beta) \leq 0 \quad (8b)$$

where

$$\begin{aligned} \phi(\alpha, \beta) = & (\alpha-1)^2 + (\alpha-1)(\beta-1) + (\beta-1)^2 \\ & - 3(\alpha+\beta-2) . \end{aligned}$$

For obvious reasons (8a) is the more usually applied constraint.

This still leaves us with the question of how to choose the estimates of the derivatives. Rasch & Williamson (1989) have performed an excellent series of tests on the choices of derivative and on the type of polynomial. In their tests the Hyman derivative

$$d_i = \frac{-\Delta_{i-2} + 7\Delta_{i-1} + 7\Delta_i - \Delta_{i+1}}{12}$$

consistently performed well in relation to the monotonic cubic polynomial.

The derivative that we will use here was first described by Priestley (1989) and is given by

$$d_i = \frac{-3\Delta_{i-2} + 19\Delta_{i-1} + 19\Delta_i - 3\Delta_{i+1}}{32} .$$

Needless to say I believe this to be superior, although others may well feel inclined to disagree.

3. The Problem and the Results

We consider, only, flow in a rectangular channel. This is not a restriction on the scheme which can equally well be applied to non-rectangular channels or pipes. Three flow regimes are considered. Entirely sub-critical, entirely super-critical and sub-critical \rightarrow super-critical \rightarrow sub-critical. At the left-hand end we fix the massflow, Q , in the super-critical case we also fix the depth, d , here by extrapolation from the interior. In the sub-critical case the Riemann invariant F^- is fixed at the right-hand boundary by extrapolation from the interior.

The parameters available to us in choosing the channel are its breadth and slope. The channel is 10,000 metres long and has a smooth constriction that goes from a breadth of 10 metres \rightarrow 5 metres \rightarrow 10 metres, see Figure 2.

The bed-slope was taken to be a constant value except between 4500 and 5500 metres where twice this value was taken. See Figure 3 for a typical cross-section.

Depth, massflow and Froude number have been plotted out for the following slopes:-

$\frac{1}{10,000}$	Figures 4-6
$\frac{1}{1,000}$	Figures 7-9
$\frac{1}{500}$	Figures 10-12
$\frac{1}{100}$	Figures 13-15
$\frac{1}{50}$	Figures 16-18 .

These results show that this Riemannian scheme copes well with both the smooth and discontinuous flows. Solutions obtained with much larger time-steps were just as good.

It is worth noting that the solution method always solved the steady-state problem to machine zero, not only for the results presented here but with other time-steps and grids as well. Slight aberrations in some of the results are therefore entirely due to discretization errors and the ODE solution technique. As grids were refined these errors dissipated so perhaps the discretization error is the dominant one.

4. Conclusion

We have introduced, or perhaps more accurately, brought together various ideas that result in a very accurate scheme that is explicit and

yet can use large time-steps and is monotonic. The scheme cannot only be used for river flow but also pipe-flow and in gas dynamics.

Here we just solved the ODE's for the Riemann invariants, equation (5), using Euler's method. This was perfectly adequate for what we wanted to demonstrate here but may prove to be too inaccurate with very large time-steps. In the future, then, we hope to use more sophisticated ODE solvers and to attack other problem areas.

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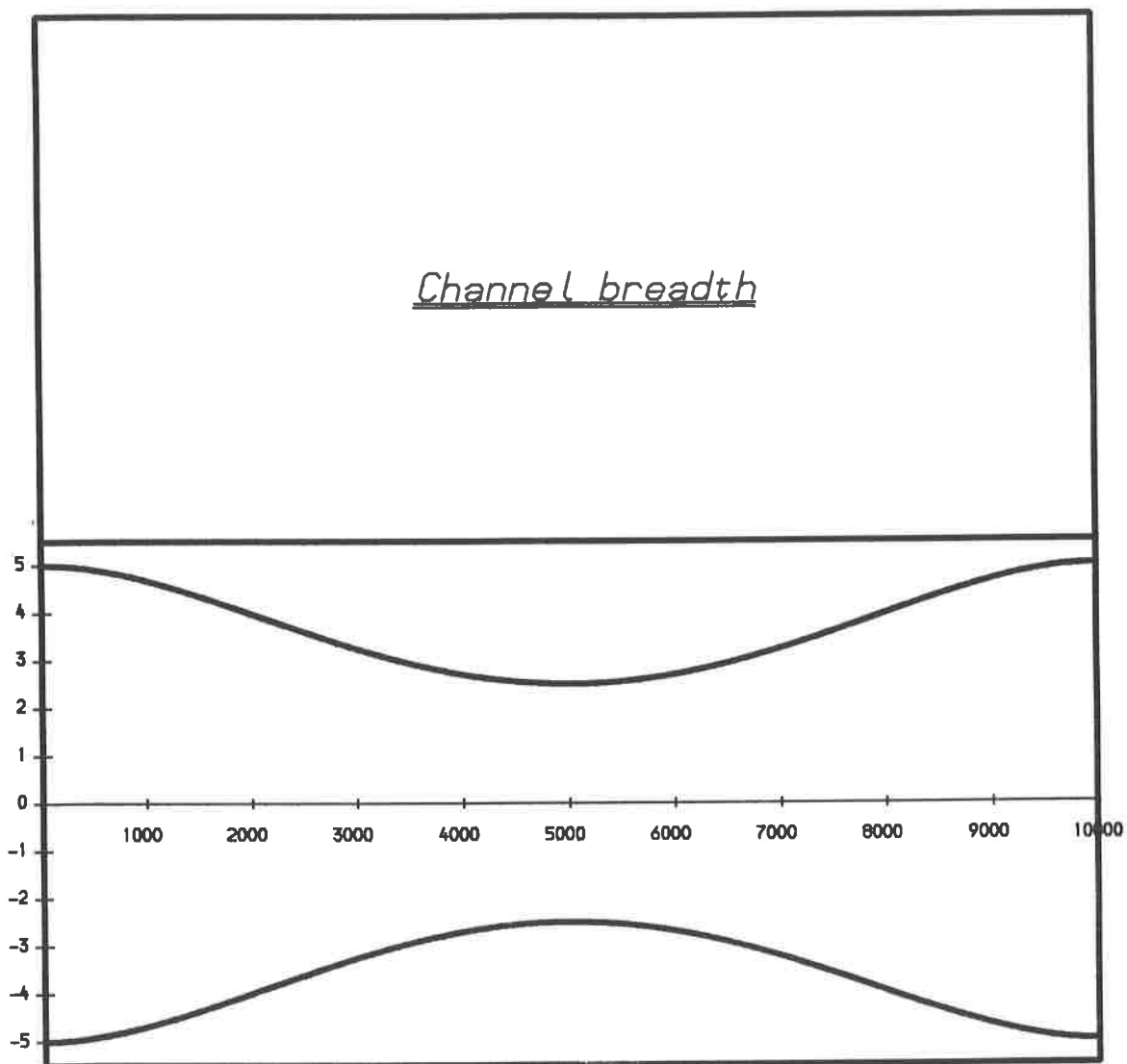


Figure 2

River-bed cross-section

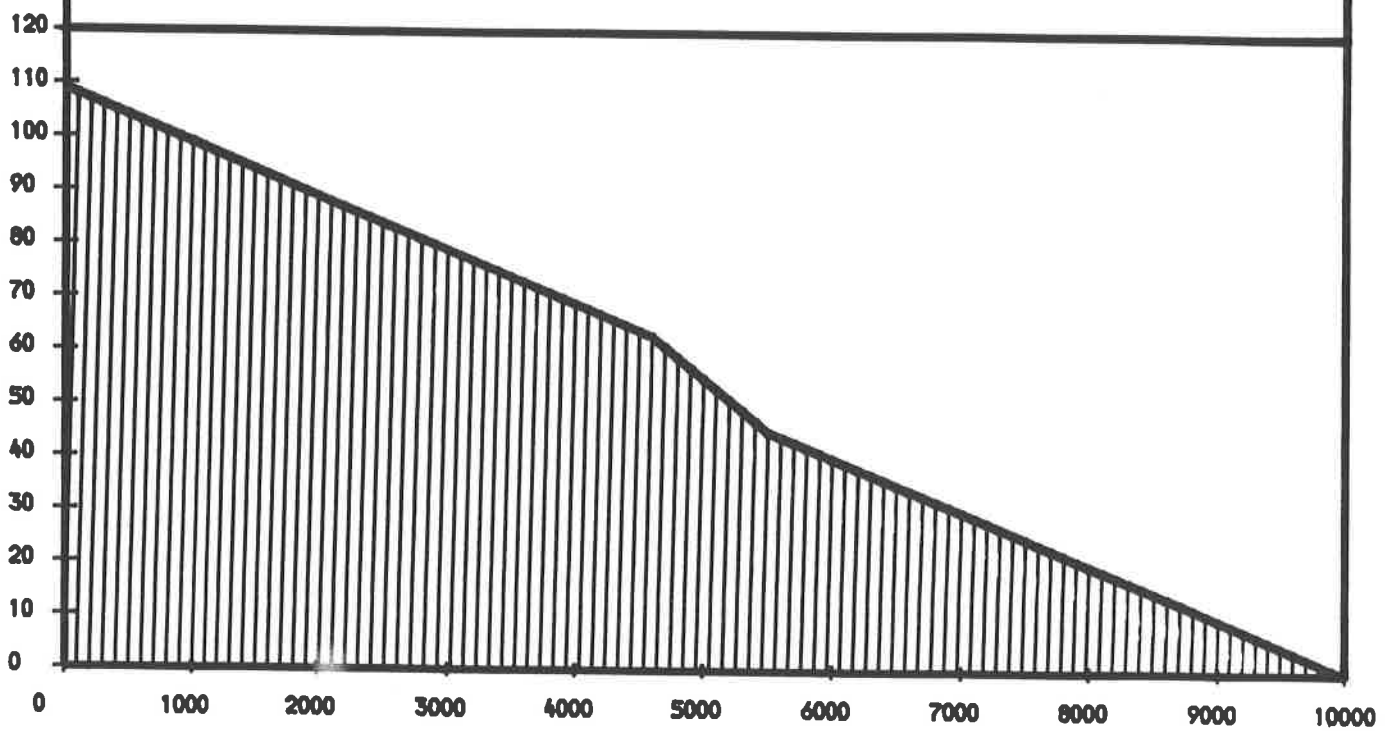


Figure 3

The time reached is 27 hours 46 minutes and 40 seconds.

$Dt = 10.0$.

$Dx = 25.0$ metres. Bed-slope is -0.0001 .

Friction coefficient is Q^2/K^2 where $K = A * (\text{hydraulic radius})^{2/3} / M$.

The value of M , Manning's constant, is 0.03 .

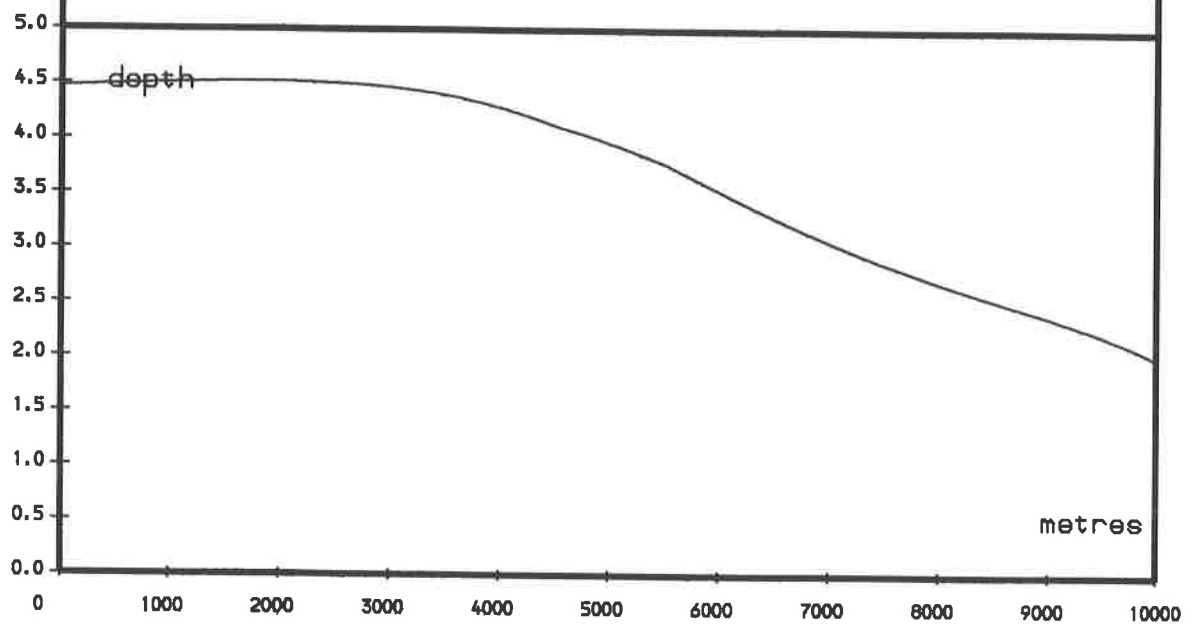


Figure 4

The time reached is 27 hours 46 minutes and 40 seconds.

$\Delta t = 10.0$.

$\Delta x = 25.0$ metres. Bed-slope is -0.0001 .

Friction coefficient is Q^2/K^2 where $K = A \cdot (\text{hydraulic radius})^{2/3}/M$.

The value of M , Manning's constant, is 0.03 .

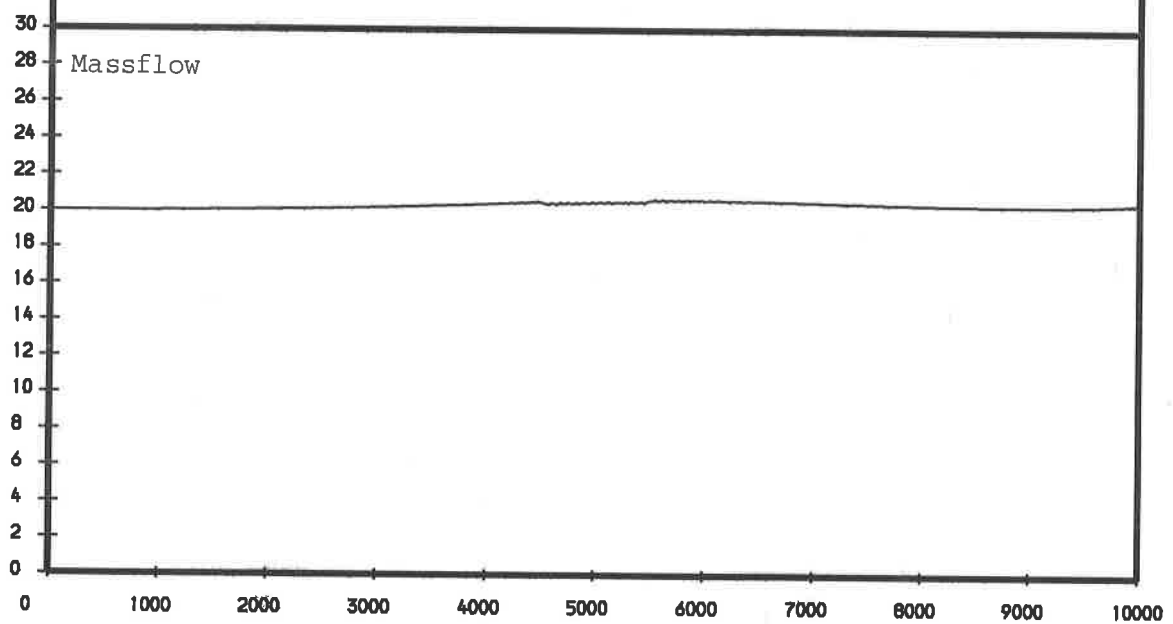


Figure 5

The time reached is 8 hours 20 minutes and 0 seconds.

$\Delta t = 10.0$.

$\Delta x = 25.0$ metres. Bed-slope is -0.0010 .

Friction coefficient is Q^2/K^2 where $K = \lambda * (\text{hydraulic radius})^{2/5}/M$.

The value of M , Manning's constant, is 0.03 .

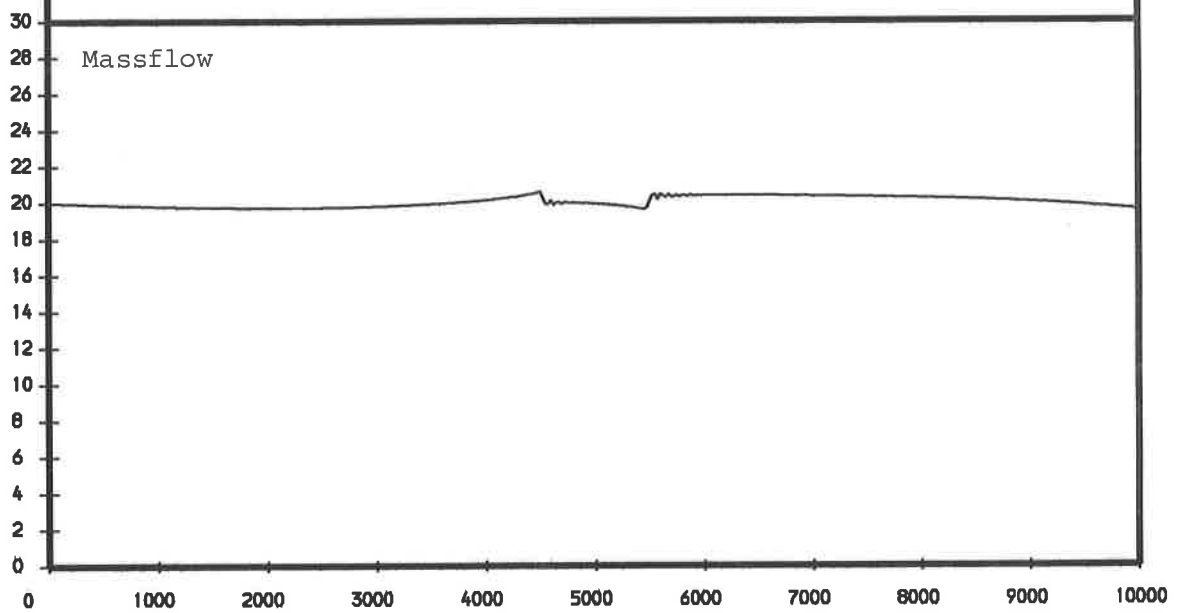


Figure 8

The time reached is 5 hours 0 minutes and 0 seconds.

$\Delta t = 10.0$.

$\Delta x = 25.0$ metres. Bed-slope is -0.0020 .

Friction coefficient is Q^2/K^2 where $K = A \cdot (\text{hydraulic radius})^{2/3}/M$.

The value of M , Manning's constant, is 0.03 .

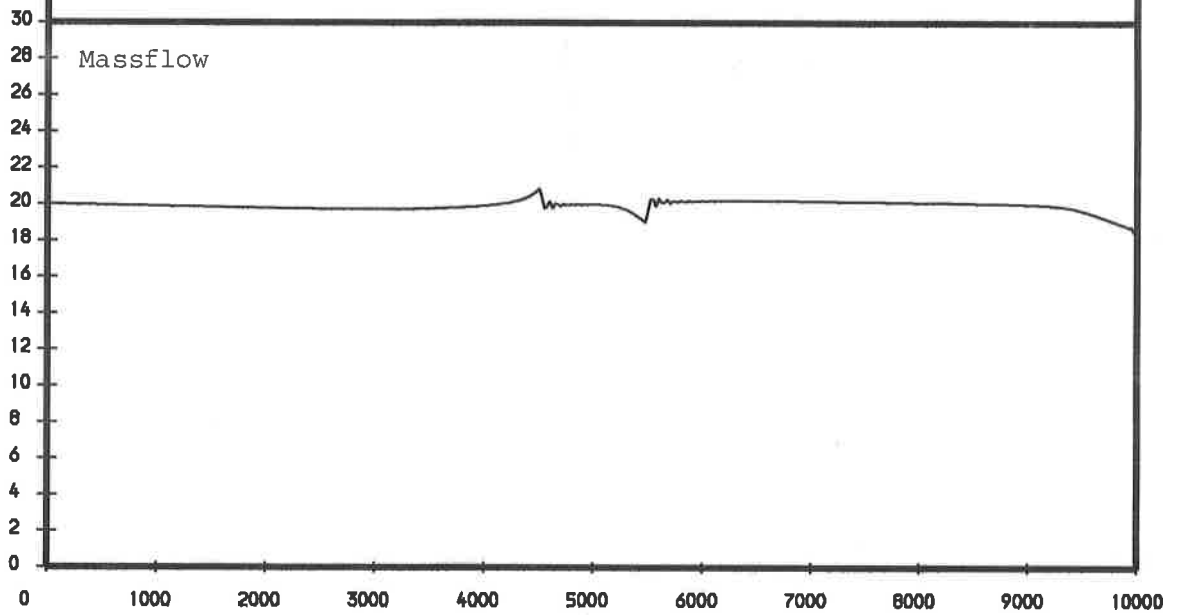


Figure 11

The time reached is 2 hours 0 minutes and 0 seconds.

$Dt = 1.0$.

$Dx = 25.0$ metres. Bed-slope is -0.0100 .

Friction coefficient is Q^2/K^2 where $K = A \cdot (\text{hydraulic radius})^{2/3}/M$.

The value of M , Manning's constant, is 0.03 .

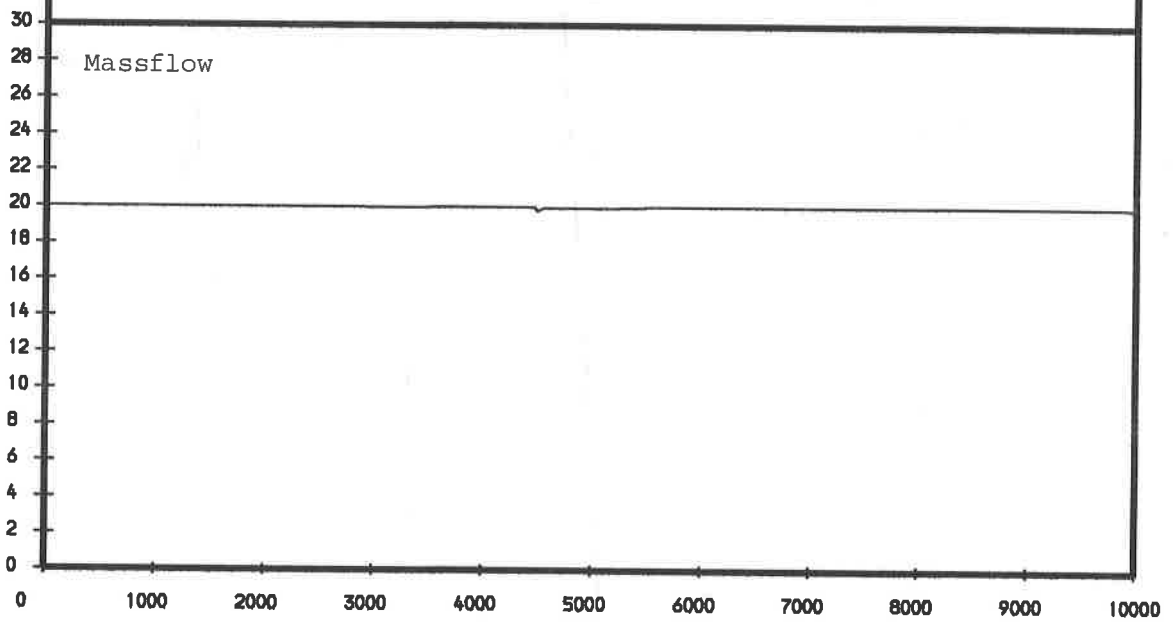


Figure 14

The time reached is 1 hours 0 minutes and 0 seconds.

$\Delta t = 1.0$.

$\Delta x = 25.0$ metres. Bed-slope is -0.0200 .

Friction coefficient is G^2/K^2 where $K = \lambda \cdot (\text{hydraulic radius})^{2/3}/M$.

The value of M , Manning's constant, is 0.03 .

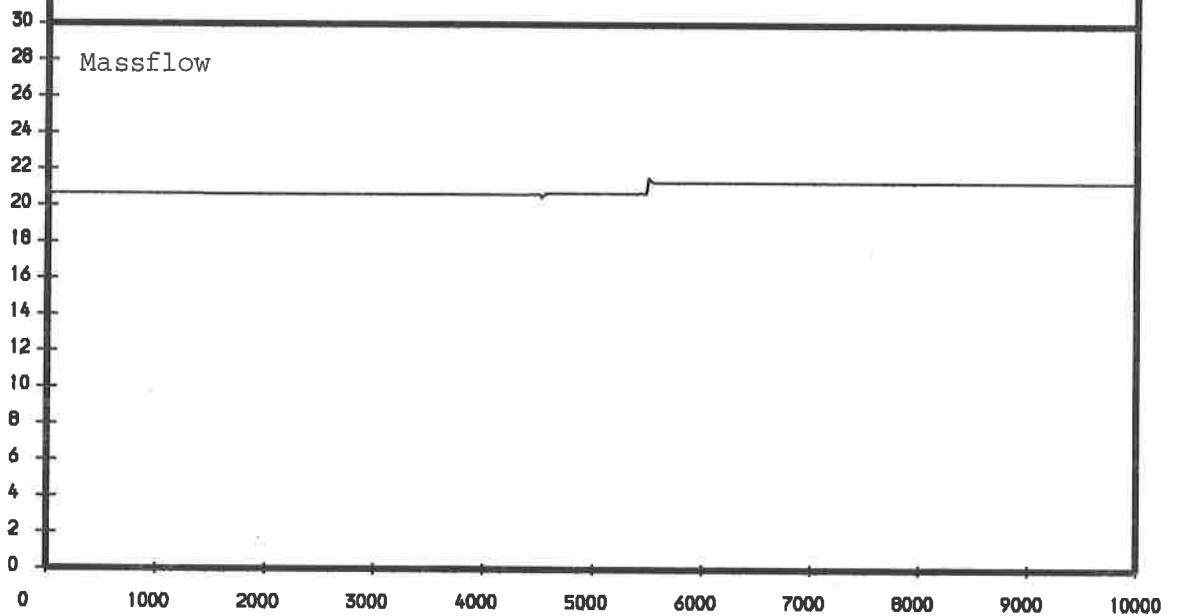


Figure 17

The time reached is 27 hours 46 minutes and 40 seconds.

$Dt = 10.0$.

$Dx = 25.0$ metres. Bed-slope is -0.0001 .

Friction coefficient is Q^2/K^2 where $K = A \times (\text{hydraulic radius})^{2/3}/M$.

The value of M , Manning's constant, is 0.03 .

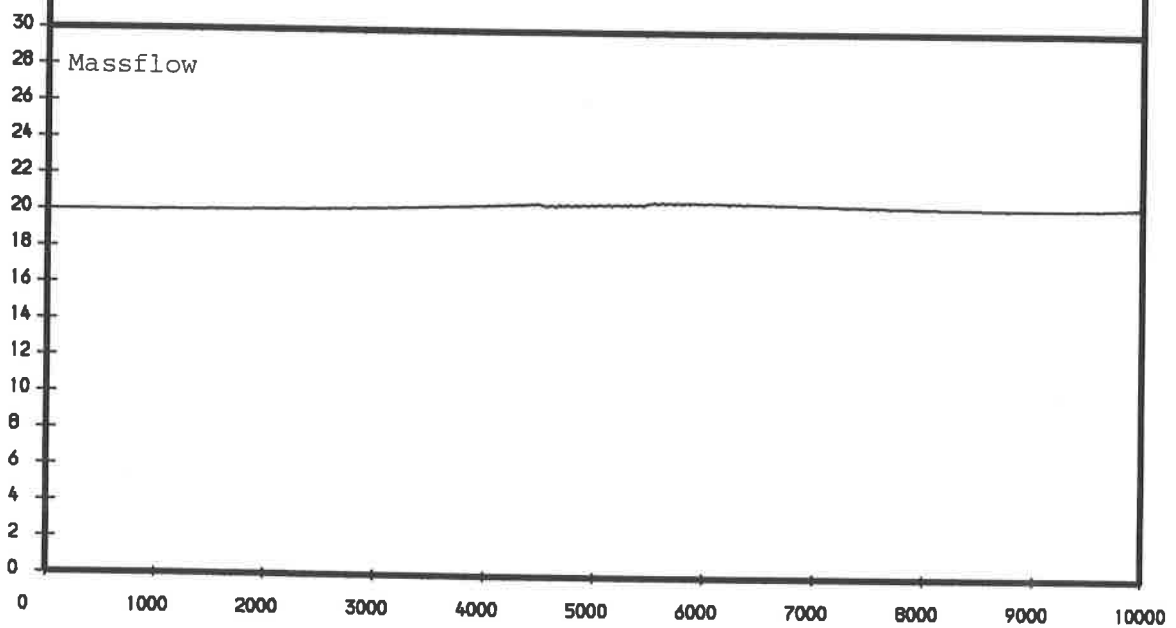


Figure 5

The time reached is 27 hours 46 minutes and 40 seconds.

$Dt = 10.0$.

$Dx = 25.0$ metres. Bed-slope is -0.0001 .

Friction coefficient is Q^2/K^2 where $K = A*(hydraulic\ radius)^{2/3}/M$.

The value of M , Manning's constant, is 0.03 .

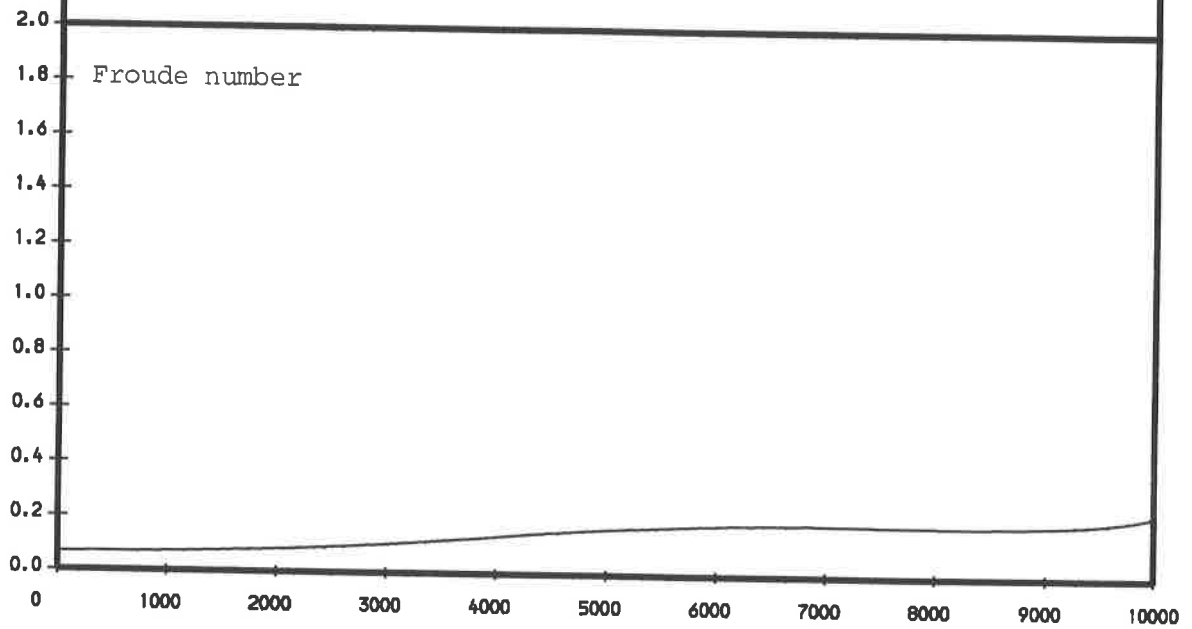


Figure 6

The time reached is 8 hours 20 minutes and 0 seconds.

$Dt = 10.0$.

$Dx = 25.0$ metres. Bed-slope is -0.0010 .

Friction coefficient is Q^2/K^2 where $K = A \cdot (\text{hydraulic radius})^{2/3}/M$.

The value of M , Manning's constant, is 0.03 .

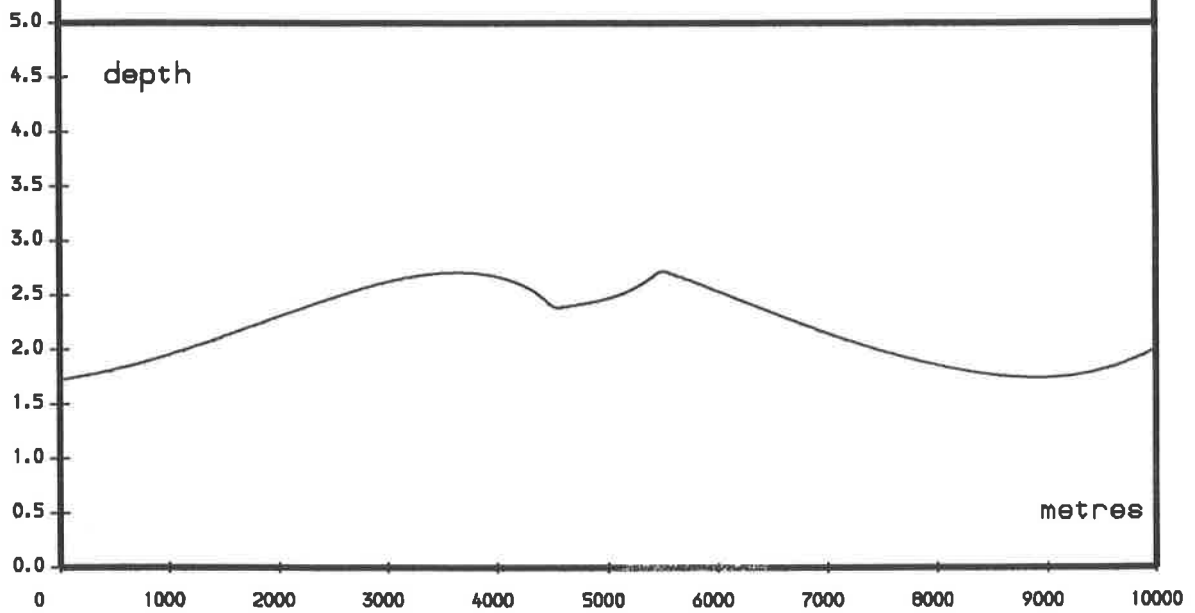


Figure 7

The time reached is 8 hours 20 minutes and 0 seconds.

$\Delta t = 10.0$.

$\Delta x = 25.0$ metres. Bed-slope is -0.0010 .

Friction coefficient is Q^2/K^2 where $K = \lambda * (\text{hydraulic radius})^{2/3}/M$.

The value of M , Manning's constant, is 0.03 .

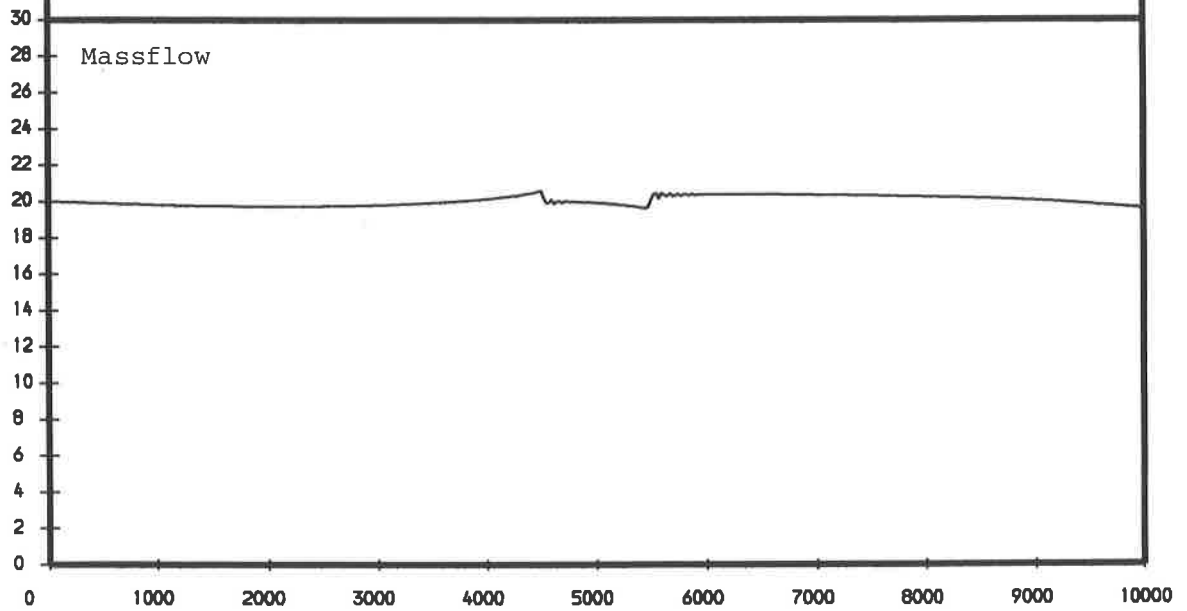


Figure 8

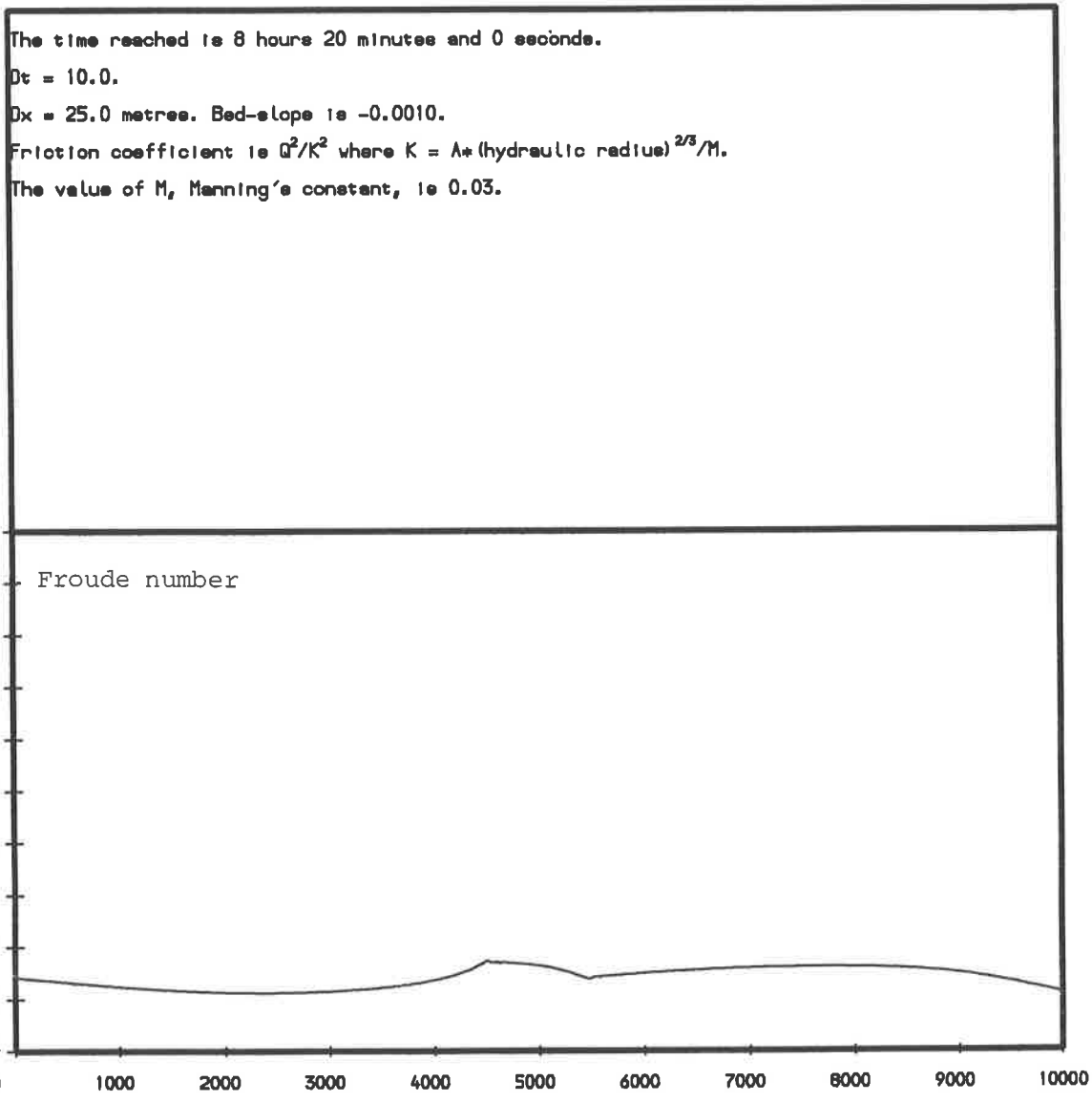


Figure 9

The time reached is 5 hours 0 minutes and 0 seconds.

$Dt = 10.0$.

$Dx = 25.0$ metres. Bed-slope is -0.0020 .

Friction coefficient is Q^2/K^2 where $K = A * (\text{hydraulic radius})^{2/3} / M$.

The value of M , Manning's constant, is 0.03 .

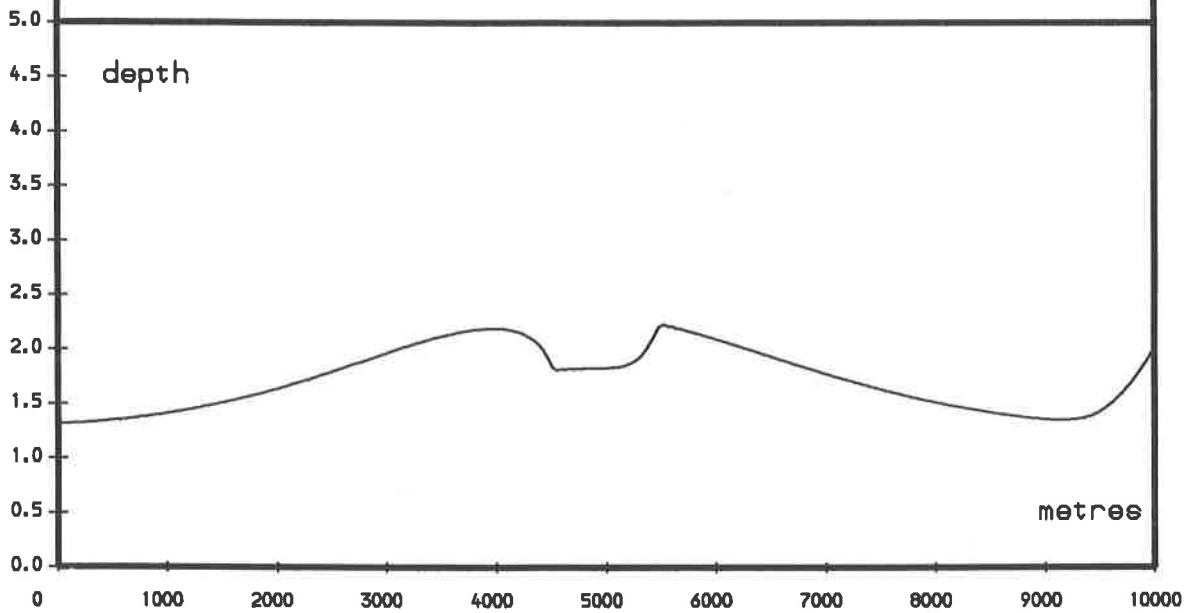


Figure 10

The time reached is 5 hours 0 minutes and 0 seconds.

$Dt = 10.0$.

$Dx = 25.0$ metres. Bed-slope is -0.0020 .

Friction coefficient is Q^2/K^2 where $K = A*(\text{hydraulic radius})^{2/3}/M$.

The value of M , Manning's constant, is 0.03 .

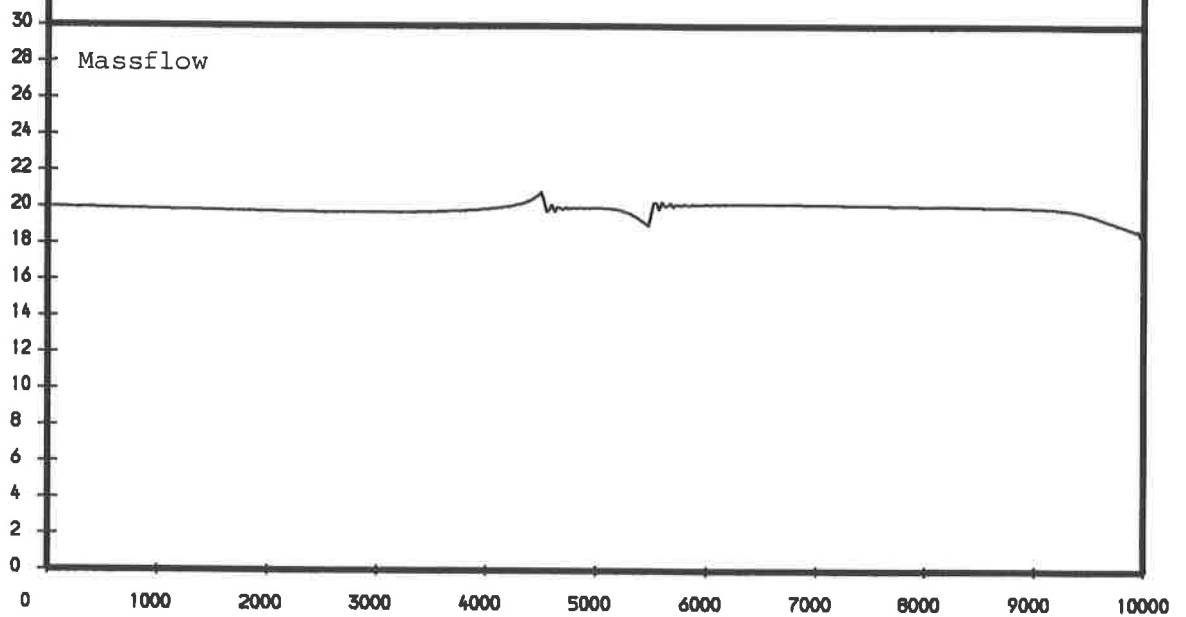


Figure 11

The time reached is 5 hours 0 minutes and 0 seconds.
Dt = 10.0.
Dx = 25.0 metres. Bed-slope is -0.0020.
Friction coefficient is Q^2/K^2 where $K = \lambda * (\text{hydraulic radius})^{2/3}/M$.
The value of M, Manning's constant, is 0.03.

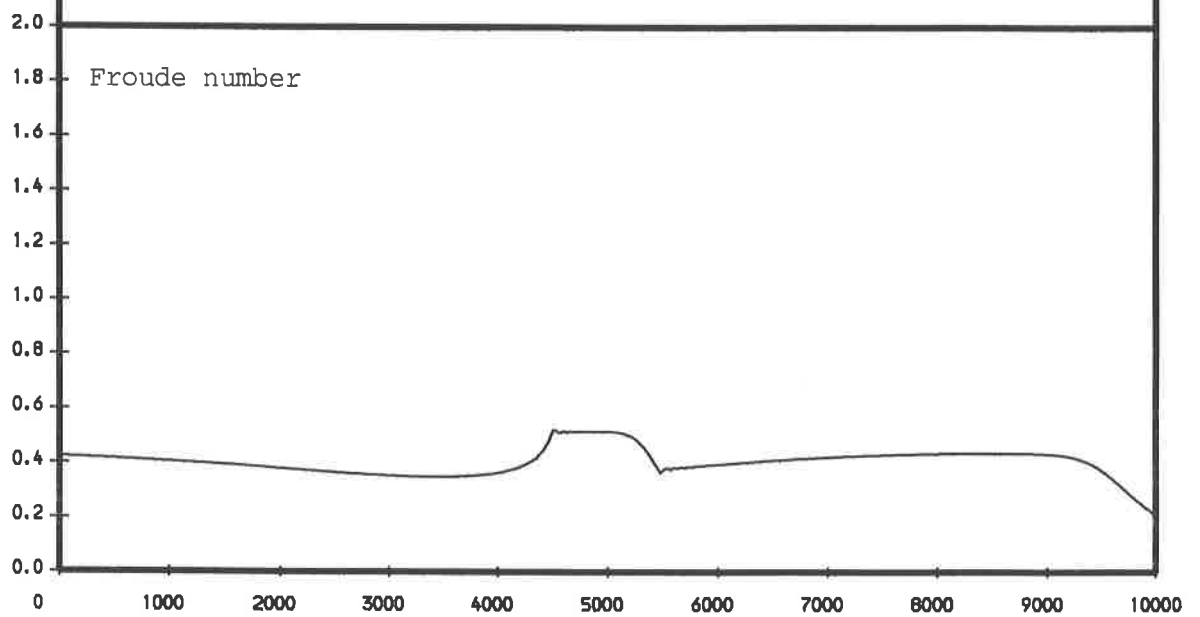


Figure 12

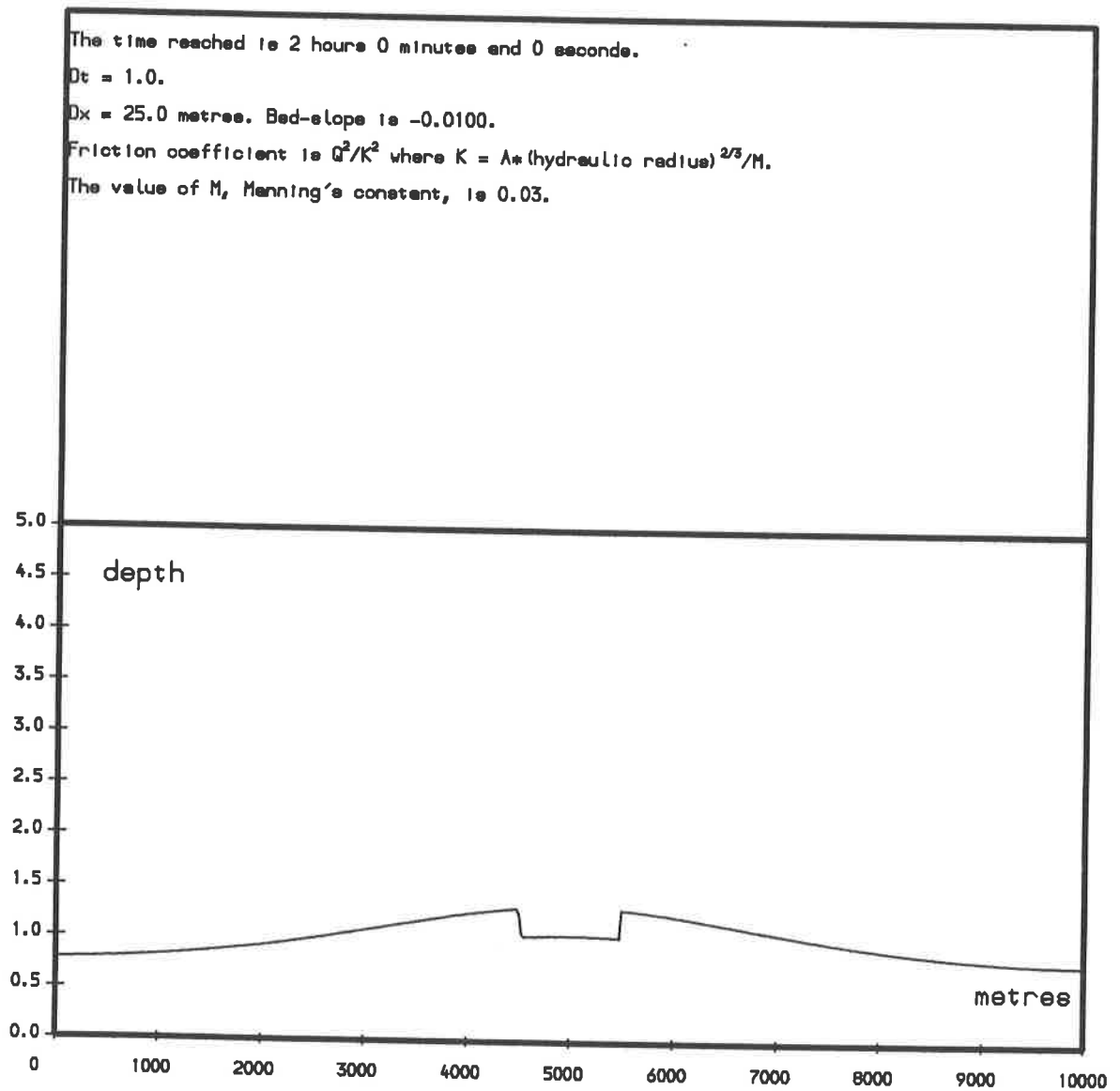


Figure 13

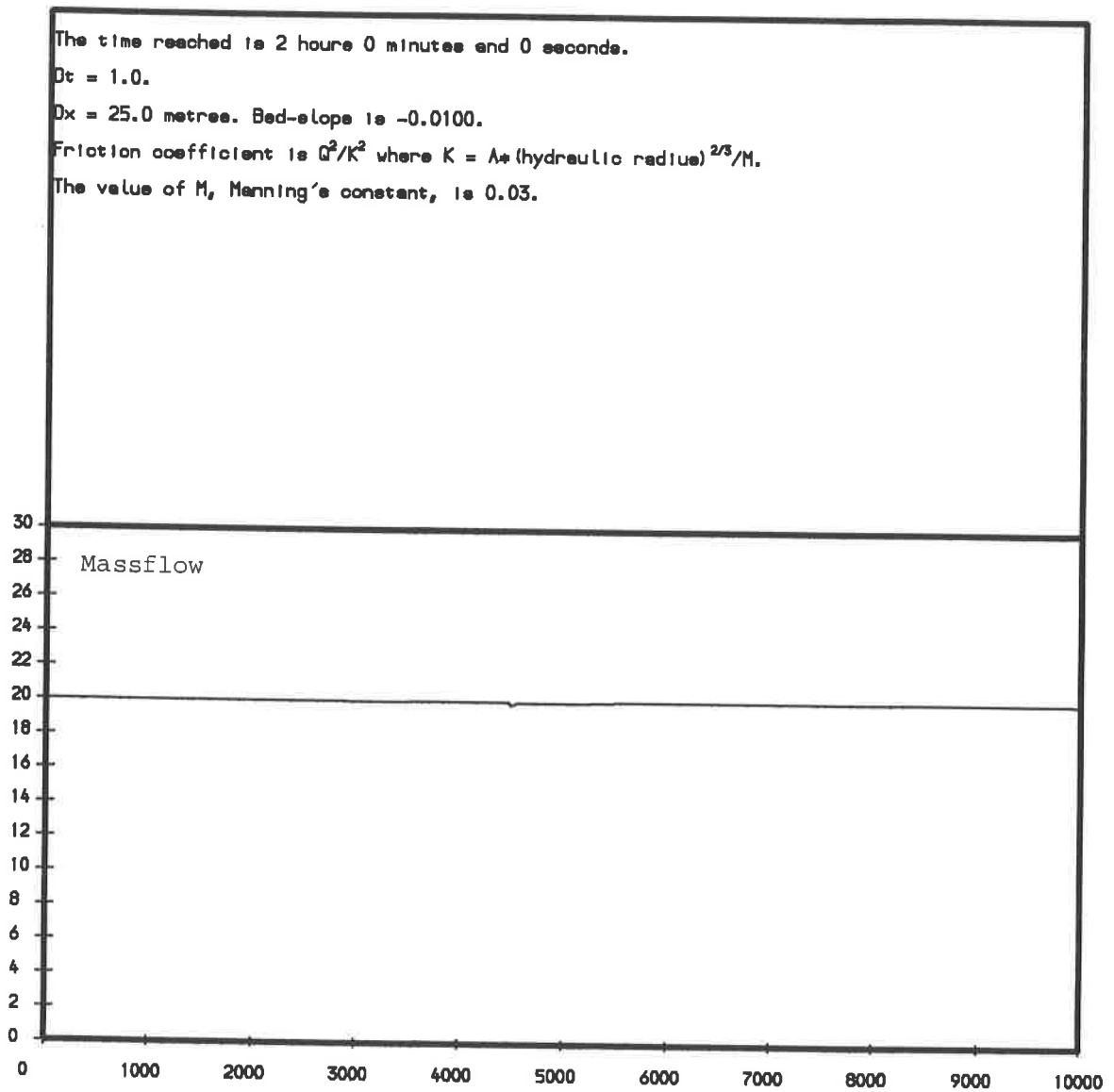


Figure 14

The time reached is 2 hours 0 minutes and 0 seconds.

$\Delta t = 1.0$.

$\Delta x = 25.0$ metres. Bed-slope is -0.0100 .

Friction coefficient is Q^2/K^2 where $K = A \cdot (\text{hydraulic radius})^{2/3}/M$.

The value of M , Manning's constant, is 0.03 .

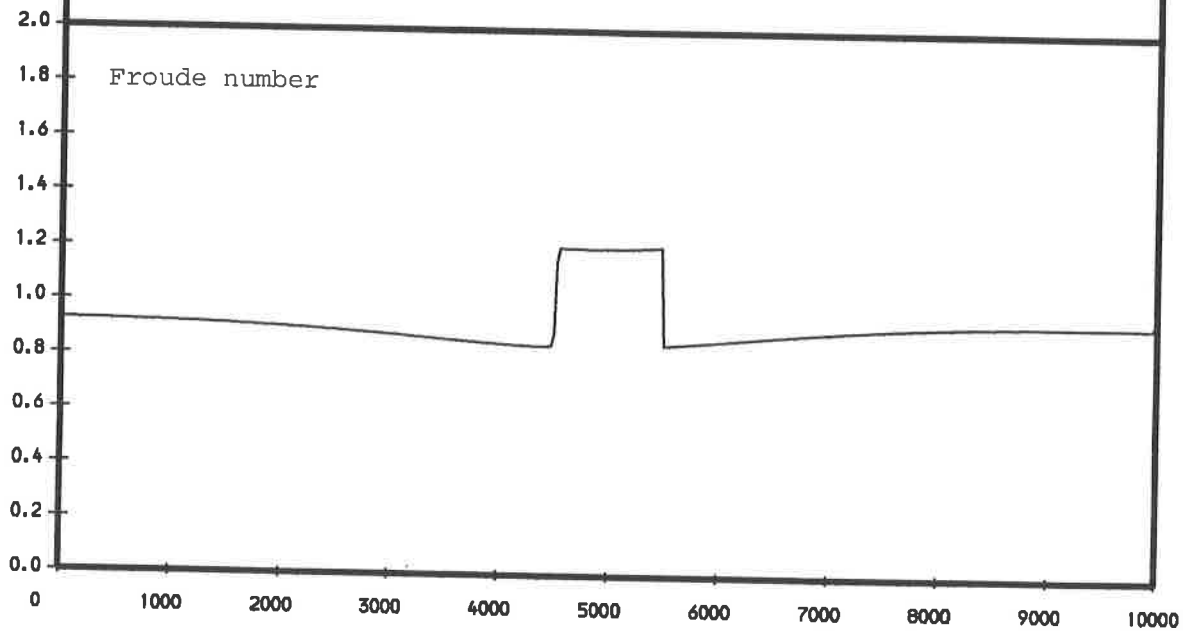


Figure 15

The time reached is 1 hour 0 minutes and 0 seconds.

$Dt = 1.0$.

$Dx = 25.0$ metres. Bed-slope is -0.0200 .

Friction coefficient is Q^2/K^2 where $K = \lambda * (\text{hydraulic radius})^{2/3}/M$.

The value of M , Manning's constant, is 0.03 .

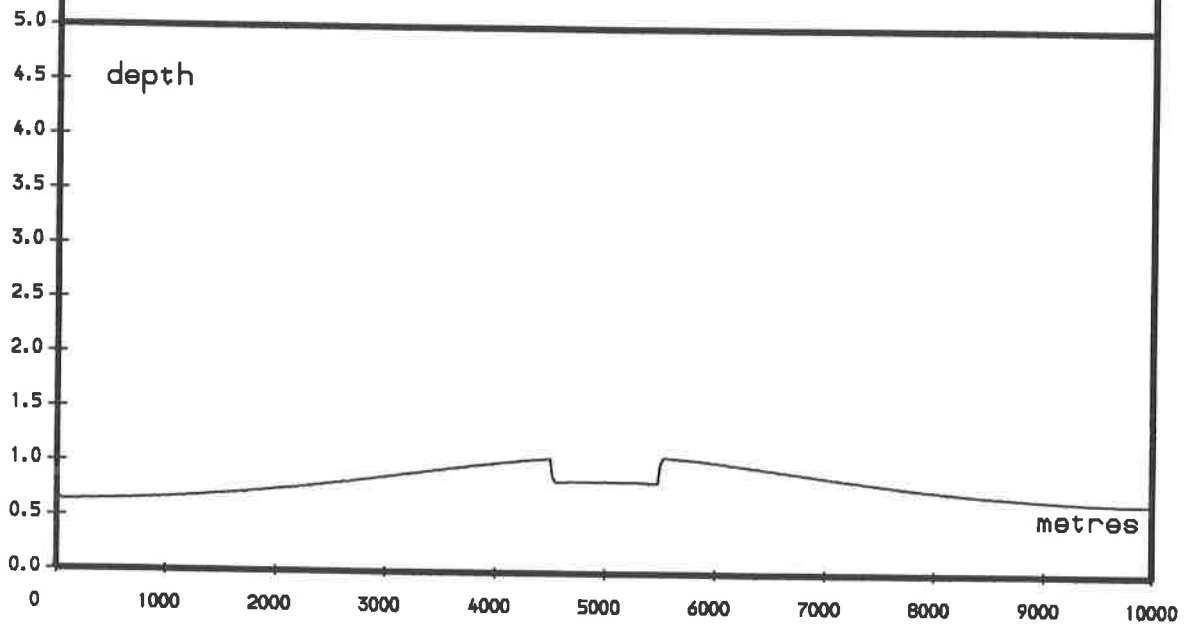


Figure 16

The time reached is 1 hours 0 minutes and 0 seconds.

$\Delta t = 1.0$.

$\Delta x = 25.0$ metres. Bed-slope is -0.0200 .

Friction coefficient is Q^2/K^2 where $K = A \cdot (\text{hydraulic radius})^{2/3}/M$.

The value of M , Manning's constant, is 0.03 .

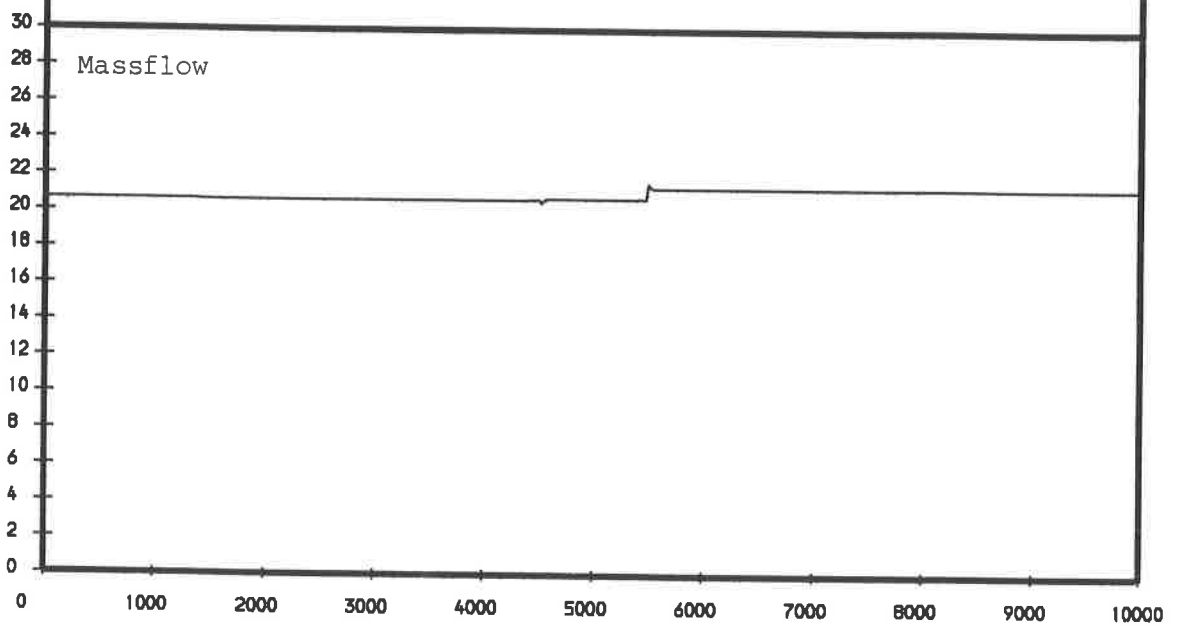


Figure 17

The time reached is 1 hours 0 minutes and 0 seconds.

$Dt = 1.0$.

$Dx = 25.0$ metres. Bed-slope is -0.0200 .

Friction coefficient is Q^2/K^2 where $K = A * (\text{hydraulic radius})^{2/3}/M$.

The value of M , Manning's constant, is 0.03 .

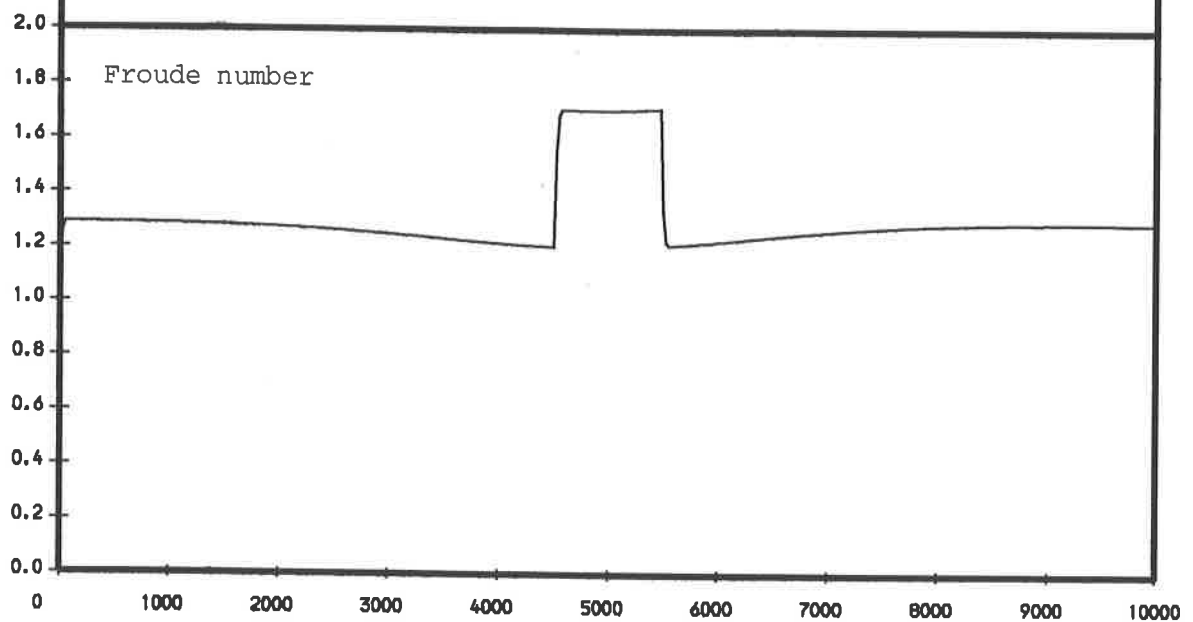


Figure 18