SECOND ORDER DIFFERENCE SCHEMES FOR HYPERBOLIC CONSERVATION LAWS WITH SOURCE TERMS

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ABSTRACT

A second order scheme based on upwind differencing is presented for hyperbolic systems derived from conservation laws with source terms. The special cases of compressible flow in a duct of variable cross section and incompressible flow in a channel are discussed.

1. INTRODUCTION

In [1] Roe proposed a linearised approximate Riemann solver for the solution of the Euler equations of compressible flow. Associated scalar schemes which are second order accurate have also been proposed by a number of authors, (see [2], [3]). The Euler equations have no source terms but for systems of conservation laws with source terms Roe [4] has also proposed a method of "upwinding" the source terms. However, there remains the question of where the source terms should be evaluated. By analysing the scalar case we seek to answer this question in such a way that the resulting scheme for systems is second order accurate.

In section 2 we describe first and second order schemes for a scalar conservation law without source terms and in section 3 apply these schemes to a system of hyperbolic conservation laws. In section 4 we derive first and second order schemes for a scalar conservation law with a source term and in section 5 we extend these schemes to a system of conservation laws with source terms. Finally, in section 6 we discuss the scheme of section 5 when applied to unsteady compressible flow in a duct of smoothly varying cross section and to incompressible flow in a channel. This includes the special cases of cylindrically and spherically symmetric compressible flows.

2. SCALAR CONSERVATION LAW

In this section we describe the first order upwind scheme and the second order Lax-Wendroff scheme for a single conservation law.

Consider the hyperbolic problem

$$u_t + f_x = 0 \qquad (x,t) \in (-\infty,\infty) \times [0,T] \tag{2.1}$$

with initial data

$$u(x,0) = u_0(x) \tag{2.2}$$

for the function $u = u(x,t)$ with a convex flux function $f = f(u)$. We define $a(u) = f'(u)$ so that equation (2.1) can be written as

$$u_t + a(u)u_x = 0 \ . \tag{2.3}$$

Define a grid $x_j = x_{j-1} + \Delta x$ in the x-direction with constant mesh spacing Δx, and a grid in the t-direction $t_n = t_{n-1} + \Delta t$ with mesh spacing Δt and denote by u_j^n an approximation to $u(x_j,t_n)$. We shall assume the solution at time level n to consist of a set of piecewise constant states

$$u(x,t_n) = u_j^n \ , \qquad x \in (x_j - \tfrac{\Delta x}{2}, x_j + \tfrac{\Delta x}{2}) \ . \tag{2.4}$$

2.1 First-order scheme

A first order upwind scheme for the solution of equations (2.1)-(2.2) can be written in the form

$$u_j^{n+1} = u_j^n - a_{j-\frac{1}{2}}^{+} \frac{\Delta t}{\Delta x}(u_j^n - u_{j-1}^n) - a_{j+\frac{1}{2}}^{-} \frac{\Delta t}{\Delta x}(u_{j+1}^n - u_j^n) \tag{2.5}$$

where we define

$$v^{+} = \tfrac{1}{2}(v + |v|) , \qquad v^{-} = \tfrac{1}{2}(v - |v|) \tag{2.6a-b}$$

and $a_{j-\frac{1}{2}}$ represents an approximation to $a(u) = f'(u)$ at $x_{j-\frac{1}{2}} = \frac{1}{2}(x_{j-1} + x_j)$, i.e. at the midpoint of the interval $[x_{j-1}, x_j]$. The approximation we use for $a_{j-\frac{1}{2}}$ is

$$a_{j-\frac{1}{2}} = \begin{cases} \dfrac{f(u_j^n) - f(u_{j-1}^n)}{u_j^n - u_{j-1}^n} & u_{j-1}^n \neq u_j^n \\ f'(\frac{1}{2}(u_{j-1}^n + u_j^n)) & u_{j-1}^n = u_j^n \end{cases} \tag{2.7a-b}$$

The first order scheme given by equation (2.5) can be thought of as being centred at the point x_j .

Alternatively, we can write the scheme in a form based on the cell $[x_{j-1}, x_j]$, i.e. for each cell we carry out the update:

$$u_{j-1}^{n+1} = u_{j-1}^n - \nu_{j-\frac{1}{2}}^{-}(u_j^n - u_{j-1}^n) \tag{2.8a}$$

$$u_j^{n+1} = u_j^n - \nu_{j-\frac{1}{2}}^{+}(u_j^n - u_{j-1}^n) \tag{2.8b}$$

which can also be rewritten as

$$\left.\begin{aligned} u_{j-1}^{n+1} &= u_{j-1}^{n} - \nu_{j-\frac{1}{2}}(u_j^n - u_{j-1}^n) \\ u_j^{n+1} &= u_j^n \end{aligned}\right\} \nu_{j-\frac{1}{2}} < 0 \qquad (2.9a)$$

$$\left.\begin{aligned} u_{j-1}^{n+1} &= u_{j-1}^{n} \\ u_j^{n+1} &= u_j^n - \nu_{j-\frac{1}{2}}(u_j^n - u_{j-1}^n) \end{aligned}\right\} \nu_{j-\frac{1}{2}} > 0\ , \qquad (2.9b)$$

where we have introduced the notation $\nu_{j-\frac{1}{2}} = a_{j-\frac{1}{2}}\frac{\Delta t}{\Delta x}$ as an approximation to the CFL number in $[x_{j-1},x_j]$, and $\nu^{\pm}_{j-\frac{1}{2}} = a^{\pm}_{j-\frac{1}{2}}\frac{\Delta t}{\Delta x}$ (see figure 1). (N.B. if $\nu_{j-\frac{1}{2}} = 0$ then either of equations (2.9a-b) apply.)

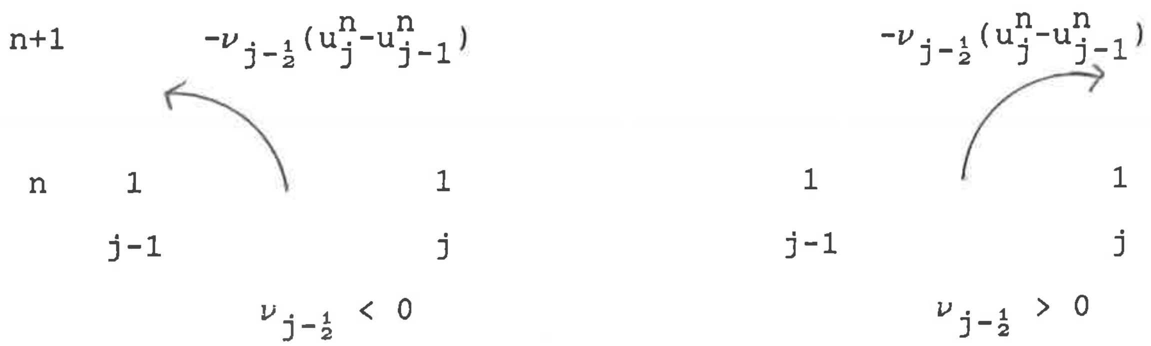


Figure 1

2.2 Second order scheme

Consider now the Taylor series expansion of $u(x_j,t_n+\Delta t)$ about the point (x_j,t_n) to second order,

$$u(x_j,t_n+\Delta t) \simeq u + \Delta t u_t + \frac{\Delta t^2}{2} u_{tt} \tag{2.10}$$

where the terms on the right hand side of the equation are evaluated at (x_j,t_n). If $u(x,t)$ satisfies equation (2.1) then we may write

$$u_t = - f_x = - f'u_x = - a(u)u_x \tag{2.11}$$

as before, and

$$u_{tt} = - f_{xt} = - (f_t)_x = - (f'u_t)_x = - (a(u)u_t)_x \tag{2.12}$$

which can be rewritten as

$$u_{tt} = (a^2(u)u_x)_x \tag{2.13}$$

using equation (2.11). Thus equation (2.10) becomes

$$u(x_j,t_n+\Delta t) \simeq u - \Delta t a(u)u_x + \frac{\Delta t^2}{2}(a^2(u)u_x)_x \tag{2.14}$$

where the right hand side of equation (2.14) is again evaluated at (x_j,t_n). If we approximate

$$a(u)u_x \simeq \tfrac{1}{2}\left[a_{j-\frac{1}{2}}\frac{\Delta_- u_j^n}{\Delta x} + a_{j+\frac{1}{2}}\frac{\Delta_+ u_j^n}{}\right] \tag{2.15}$$

and

$$(a^2(u)u_x)_x \cong \frac{1}{\Delta x}\left[a^2_{j+\frac{1}{2}}\frac{\Delta_+ u_j^n}{\Delta x} - a^2_{j-\frac{1}{2}}\frac{\Delta_- u_j^n}{\Delta x}\right] \tag{2.16}$$

where $a_{j-\frac{1}{2}}$ are as before, and

$$\Delta_+ u_j^n = u_{j+1}^n - u_j^n, \qquad \Delta_- u_j^n = u_j^n - u_{j-1}^n , \tag{2.17a-b}$$

we obtain the following second order centrally based scheme for the solution of equations (2.1)-(2.2)

$$\begin{aligned} u_j^{n+1} = u_j^n &- \tfrac{1}{2}\nu_{j-\frac{1}{2}}\Delta_- u_j^n - \tfrac{1}{2}\nu_{j+\frac{1}{2}}\Delta_+ u_j^n \\ &+ \tfrac{1}{2}\nu_{j+\frac{1}{2}}^2\Delta_+ u_j^n - \tfrac{1}{2}\nu_{j-\frac{1}{2}}^2\Delta_- u_j^n . \end{aligned} \tag{2.18}$$

We have used the notation $\nu_{j\pm\frac{1}{2}} = a_{j\pm\frac{1}{2}}\frac{\Delta t}{\Delta x}$ as before.

The scheme given by equation (2.18) is usually referred to as the Lax-Wendroff scheme. We can compare this second order scheme with the first order scheme given by equations (2.9a-b) by noticing that equation (2.18) can be written as

$$\begin{aligned} u_j^{n+1} = u_j^n &- \nu_{j-\frac{1}{2}}\Delta_- u_j \\ &+ \tfrac{1}{2}\nu_{j-\frac{1}{2}}(1 - \nu_{j-\frac{1}{2}})\Delta_- u_j^n \\ &- \tfrac{1}{2}\nu_{j+\frac{1}{2}}(1 - \nu_{j+\frac{1}{2}})\Delta_+ u_j^n \end{aligned} \tag{2.19a}$$

or

$$\begin{aligned} u_j^{n+1} = u_j^n &- \nu_{j+\frac{1}{2}}\Delta_+ u_j^n \\ &+ \tfrac{1}{2}\nu_{j+\frac{1}{2}}(1 + \nu_{j+\frac{1}{2}})\Delta_+ u_j^n \\ &- \tfrac{1}{2}\nu_{j-\frac{1}{2}}(1 + \nu_{j-\frac{1}{2}})\Delta_- u_j^n . \end{aligned} \tag{2.19b}$$

If we compare equations (2.19a-b) with equations (2.9a-b) we can rewrite the scheme given by equation (2.18) in a form based on the cell $[x_{j-1}, x_j]$. Moreover, we can consider the scheme as consisting of a first order increment together with a second order transfer:

$$\left.\begin{aligned} u_{j-1}^{n+1} &= u_{j-1}^{n} + b_{j-\frac{1}{2}} \\ u_{j}^{n+1} &= u_{j}^{n} + \phi_{j-\frac{1}{2}} - b_{j-\frac{1}{2}} \end{aligned}\right\} \quad \nu_{j-\frac{1}{2}} > 0 \tag{2.20a}$$

$$\left.\begin{aligned} u_{j-1}^{n+1} &= u_{j-1}^{n} + \phi_{j-\frac{1}{2}} - b_{j-\frac{1}{2}} \\ u_{j}^{n+1} &= u_{j}^{n} + b_{j-\frac{1}{2}} \end{aligned}\right\} \quad \nu_{j-\frac{1}{2}} < 0 \tag{2.20b}$$

where

$$\phi_{j-\frac{1}{2}} = -\nu_{j-\frac{1}{2}} \Delta_{-} u_{j}^{n} \tag{2.21}$$

is the first order increment and

$$b_{j-\frac{1}{2}} = \tfrac{1}{2}(1 - |\nu_{j-\frac{1}{2}}|)\phi_{j-\frac{1}{2}} , \tag{2.22}$$

is the second order transfer (see figure 2). (It is possible to limit the transfers $b_{j-\frac{1}{2}}$ to avoid non-physical oscillations created by the scheme, see [2], [3].)

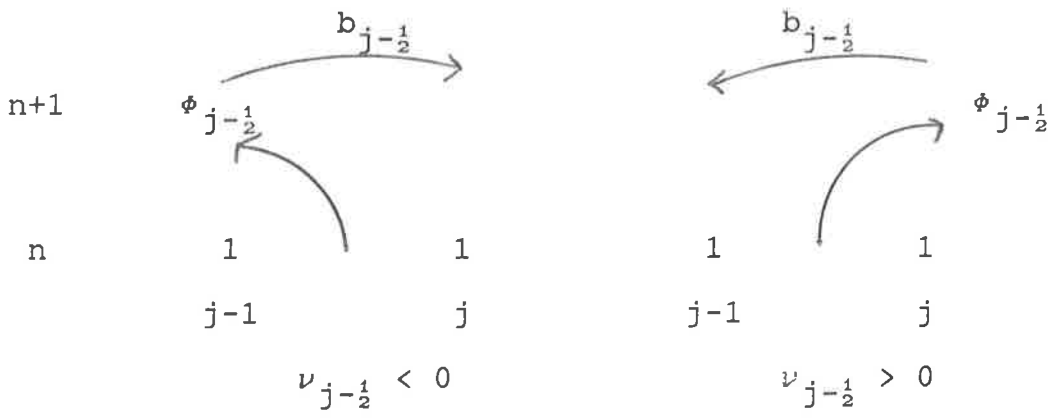


Figure 2

We can also write equations (2.20a)-(2.22) in a form similar to the first order scheme (2.5), i.e.

$$u_j^{n+1} = u_j^n - \nu^+_{j-\frac{1}{2}}(u_j^n-u_{j-1}^n) - \nu^-_{j+\frac{1}{2}}(u_{j+1}^n-u_j^n)$$
$$+ \tfrac{1}{2}\nu^+_{j-\frac{1}{2}}(1-|\nu_{j-\frac{1}{2}}|)(u_j^n-u_{j-1}^n) - \tfrac{1}{2}\nu^-_{j-\frac{1}{2}}(1-|\nu_{j-\frac{1}{2}}|)(u_j^n-u_{j-1}^n)$$
$$+ \tfrac{1}{2}\nu^-_{j+\frac{1}{2}}(1-|\nu_{j+\frac{1}{2}}|)(u_{j+1}^n-u_j^n) - \tfrac{1}{2}\nu^+_{j+\frac{1}{2}}(1-|\nu_{j+\frac{1}{2}}|)(u_{j+1}^n-u_j^n).$$

(2.23)

by taking account of equations (2.19a-b).

In the next section we see how the schemes of this section can be applied to systems of conservation laws (without source terms).

3. SYSTEMS OF CONSERVATION LAWS

In this section we apply the first and second order schemes of §2 to systems of conservation laws.

Consider the system of hyperbolic conservation laws

$$\underline{w}_t + \underline{f}_x = \underline{0} \qquad (x,t) \in (-\infty,\infty) \times [0,T] \tag{3.1}$$

with initial data

$$\underline{w}(x,0) = \underline{w}_0(x) \tag{3.2}$$

for the function $\underline{w} = \underline{w}(x,t)$ where $\underline{f} = \underline{f}(w)$.

The approximate solution of equations (3.1)-(3.2) is sought by assuming a piecewise constant representation and solving the approximate Riemann problem

$$\underline{w}_t + \tilde{A}(\underline{w}^n_{j-1},\underline{w}^n_j)\underline{w}_x = \underline{0} \ , \qquad (x,t) \in [x_{j-1},x_j] \times (t_n,t_{n+1}) \tag{3.3}$$

where $\tilde{A}(\underline{w}^n_{j-1},\underline{w}^n_j)$ is an approximation to the Jacobian $A(w) = \frac{\partial \underline{f}}{\partial \underline{w}}(w)$, and $\underline{w}^n_{j-1},\underline{w}^n_j$ represent the piecewise constant states at time level n , i.e.

$$\underline{w}(x,t_n) = \begin{cases} \underline{w}^n_{j-1} & x \in (x_{j-1}-\frac{\Delta x}{2},\ x_{j-1}+\frac{\Delta x}{2}) \\ \underline{w}^n_j & x \in (x_j-\frac{\Delta x}{2},\ x_j+\frac{\Delta x}{2}) \ . \end{cases} \tag{3.4}$$

The notation of §2 is assumed. (A specific example of $\tilde{A}$ is given by Roe [1] for the Euler equations of gas dynamics.)

Given the Riemann problem (3.3)-(3.4), we can obtain first and second order schemes for the solution of equations (3.1)-(3.2) as below.

3.1 First order scheme by diagonalisation

Consider the cell $[x_{j-1}, x_j]$ and suppose that the approximate Jacobian $\tilde{A}(\underline{w}^n_{j-1}, \underline{w}^n_j) = \tilde{A}_{j-\frac{1}{2}}$ has n eigenvalues $_i\tilde{\lambda}_{j-\frac{1}{2}}$, i=1,...m with corresponding linearly independent eigenvectors $_i\underline{\tilde{e}}_{j-\frac{1}{2}}$, i=1,...m . If we write

$$\tilde{X}_{j-\frac{1}{2}} = [{}_i\underline{\tilde{e}}_{j-\frac{1}{2}}, \dots {}_m\underline{\tilde{e}}_{j-\frac{1}{2}}] \tag{3.5}$$

as the modal matrix then it is well-known that

$$\tilde{X}^{-1}_{j-\frac{1}{2}} \tilde{A}_{j-\frac{1}{2}} \tilde{X}_{j-\frac{1}{2}} = \tilde{\Lambda}_{j-\frac{1}{2}} \tag{3.6}$$

where

$$\tilde{\Lambda}_{j-\frac{1}{2}} = \text{diag}({}_i\tilde{\lambda}_{j-\frac{1}{2}}, \dots {}_m\tilde{\lambda}_{j-\frac{1}{2}}) \tag{3.7}$$

is a diagonal matrix. Thus, if we define a new dependent variable by

$$\underline{v} = \tilde{X}^{-1}_{j-\frac{1}{2}}\underline{w} \qquad (x,t) \in [x_{j-1}, x_j] \times [t_n, t_{n+1}] \tag{3.8}$$

then equation (3.3) becomes

$$\underline{v}_t + \tilde{\Lambda}_{j-\frac{1}{2}}\underline{v}_x = \underline{0} \tag{3.9}$$

i.e. a set of scalar problems

$$\frac{\partial}{\partial t}({}_iv) + {}_i\tilde{\lambda}_{j-\frac{1}{2}}\frac{\partial}{\partial x}({}_iv) = 0 \qquad i=1, \quad m \tag{3.10}$$

where $\underline{v} = ({}_1v, \dots {}_mv)^T$.

Equations (3.10) can now be solved approximately using the first order upwind scheme given by equations (2.8a-b) where we identify ${}_i\tilde{\lambda}_{j-\frac{1}{2}}$ with the approximation to $a_{j-\frac{1}{2}}$ for each i. Thus the scheme for equations (3.10) written cell-wise is

$$\begin{aligned} {}_iv_{j-1}^{n+1} &= {}_iv_{j-1}^{n} - \frac{\Delta t}{\Delta x}\,{}_i\tilde{\lambda}_{j-\frac{1}{2}}^{-}({}_iv_j^n - {}_iv_{j-1}^n) \\ & \qquad\qquad\qquad\qquad i=1,\ldots m \\ {}_iv_j^{n+1} &= {}_iv_j^n - \frac{\Delta t}{\Delta x}\,{}_i\tilde{\lambda}_{j-\frac{1}{2}}^{+}({}_iv_j^n - {}_iv_{j-1}^n) \end{aligned} \tag{3.11a-b}$$

where ${}_i\tilde{\lambda}_{j-\frac{1}{2}}^{\pm}$ are defined using equations (2.6a-b). Equations (3.11a-b) can be written in system form as

$$\begin{aligned} \underline{v}_{j-1}^{n+1} &= \underline{v}_{j-1}^{n} - \frac{\Delta t}{\Delta x}\tilde{\Lambda}_{j-\frac{1}{2}}^{-}(\underline{v}_j^n - \underline{v}_{j-1}^n) \\ \underline{v}_j^{n+1} &= \underline{v}_j^{n} - \frac{\Delta t}{\Delta x}\tilde{\Lambda}_{j-\frac{1}{2}}^{+}(\underline{v}_j^n - \underline{v}_{j-1}^n) \end{aligned} \tag{3.12a-b}$$

where

$$\tilde{\Lambda}_{j-\frac{1}{2}}^{\pm} = \mathrm{diag}({}_i\tilde{\lambda}_{j-\frac{1}{2}}^{\pm},\ldots {}_m\tilde{\lambda}_{j-\frac{1}{2}}^{\pm}) \quad , \tag{3.13}$$

or, if we transform back using equation (3.8),

$$\begin{aligned} \underline{w}_{j-1}^{n+1} &= \underline{w}_{j-1}^{n} - \frac{\Delta t}{\Delta x}\tilde{A}_{j-\frac{1}{2}}^{-}(\underline{w}_j^n - \underline{w}_{j-1}^n) \\ \underline{w}_j^{n+1} &= \underline{w}_j^{n} - \frac{\Delta t}{\Delta x}\tilde{A}_{j-\frac{1}{2}}^{+}(\underline{w}_j^n - \underline{w}_{j-1}^n) \end{aligned} \tag{3.14a-b}$$

where we have defined

$$\tilde{A}_{j-\frac{1}{2}}^{\pm} = \tilde{X}_{j-\frac{1}{2}}\tilde{\Lambda}_{j-\frac{1}{2}}^{\pm}\tilde{X}_{j-\frac{1}{2}}^{-1} \tag{3.15}$$

as the positive and negative parts of $\tilde{A}_{j-\frac{1}{2}}$.

The scheme given by equations (3.14a-b) can be written point-wise as

$$\underline{w}_j^{n+1} = \underline{w}_j^n - \frac{\Delta t}{\Delta x}\tilde{A}^+_{j-\frac{1}{2}}(\underline{w}_j^n - \underline{w}_{j-1}^n) - \frac{\Delta t}{\Delta x}\tilde{A}^-_{j+\frac{1}{2}}(\underline{w}_{j+1}^n - \underline{w}_j^n) , \tag{3.16}$$

which is an extension of the algorithm given by equation (2.5).

3.2 Flux-Difference Splitting

Consider first an alternative approach to the diagonalisation given in §3.1 which consists of splitting the Jacobian matrix $A = \frac{\partial \underline{f}}{\partial \underline{w}}$, with eigenvalues ${}_i\lambda$ and corresponding linearly independent eigenvectors ${}_i\underline{e}$ into

$$A = A^+ + A^- \tag{3.17}$$

where

$$A^{\pm} = X\, \Lambda^{\pm} X^{-1} \tag{3.18}$$

$$\Lambda^{\pm} = \mathrm{diag}({}_1\lambda^{\pm}, \ldots {}_m\lambda^{\pm}) \tag{3.19}$$

and

$$X = [{}_1\underline{e}, \ldots {}_m\underline{e}] . \tag{3.20}$$

If we expand $\underline{w}(x_j, t_n + \Delta t)$ about (x_j, t_n) as a Taylor series to second order and use equations (3.1) and (3.17) to obtain

$$\underline{w}(x_j,t_n+\Delta t) \simeq \underline{w}(x_j,t_n) + \Delta t\underline{w}_t(x_j,t_n)$$
$$= \underline{w}(x_j,t_n) - \Delta tA\underline{w}_x(x_j,t_n)$$
$$= \underline{w}(x_j,t_n) - \Delta tA^+\underline{w}_x(x_j,t_n) - \Delta tA^-\underline{w}_x(x_j,t_n) \ . \tag{3.21}$$

Thus, defining $\tilde{A}_{j-\frac{1}{2}}$ as an approximation to A at $x_{j-\frac{1}{2}} = \frac{1}{2}(x_{j-1}+x_j)$ and splitting $\tilde{A}_{j-\frac{1}{2}}$ into

$$\tilde{A}_{j-\frac{1}{2}} = \tilde{A}^+_{j-\frac{1}{2}} + \tilde{A}^-_{j-\frac{1}{2}} \tag{3.22}$$

as in §3.1, we get the following first order upwind scheme for equation (3.1) from equation (3.21)

$$\underline{w}_j^{n+1} = \underline{w}_j^n - \Delta t\tilde{A}^+_{j-\frac{1}{2}}\frac{(\underline{w}_j^n-\underline{w}_{j-1}^n)}{\Delta x} - \Delta t\tilde{A}^-_{j+\frac{1}{2}}\frac{(\underline{w}_{j+1}^n-\underline{w}_j^n)}{\Delta x} \ . \tag{3.23}$$

Equation (3.23) is the same as equation (3.16). (N.B. The matrices $A^-,\tilde{A}^-_{j-\frac{1}{2}}$ and $A^+,\tilde{A}^+_{j-\frac{1}{2}}$ are associated with left and right travelling waves, respectively.) To implement this scheme in an upwind manner (by looking at each of the m waves in turn) we proceed as follows.

To implement the algorithm given by equation (3.23) we project $\underline{w}_j^n - \underline{w}_{j-1}^n$ onto the eigenvectors ${}_i\tilde{\underline{e}}_{j-\frac{1}{2}}$ in the form

$$\underline{w}_j^n - \underline{w}_{j-1}^n = \sum_{i=1}^{m} {}_i\tilde{\alpha}_{j-\frac{1}{2}}\ {}_i\tilde{\underline{e}}_{j-\frac{1}{2}} \tag{3.24}$$

so that equation (3.23) becomes

$$\underline{w}_j^{n+1} = \underline{w}_j^{n} - \sum_{i=1}^{m} {}_i\tilde{\nu}^{+}_{j-\frac{1}{2}} \; {}_i\tilde{\alpha}_{j-\frac{1}{2}} \; {}_i\tilde{\underline{e}}_{j-\frac{1}{2}} - \sum_{i=1}^{m} {}_i\tilde{\nu}^{-}_{j+\frac{1}{2}} \; {}_i\tilde{\alpha}_{j+\frac{1}{2}} \; {}_i\tilde{\underline{e}}_{j+\frac{1}{2}} \qquad (3.25)$$

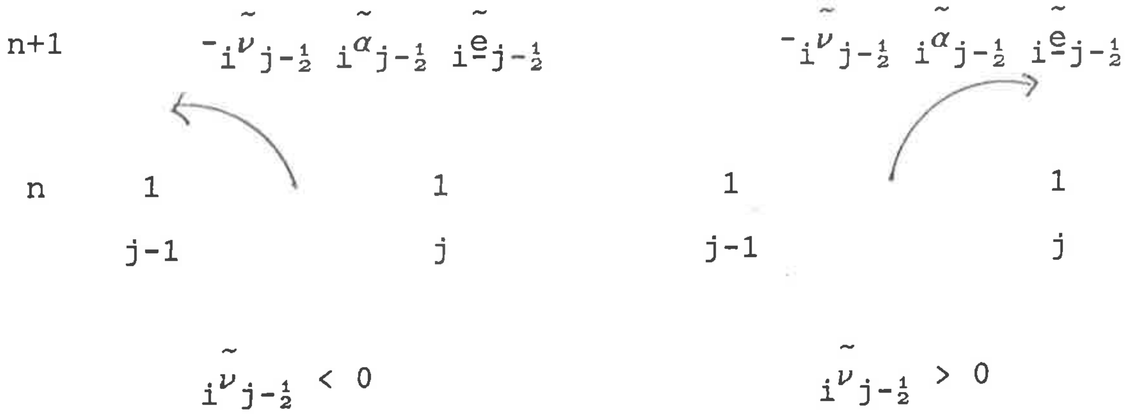
written pointwise, or

$$\underline{w}_{j-1}^{n+1} = \underline{w}_{j-1}^{n} - \sum_{i=1}^{m} {}_i\tilde{\nu}^{-}_{j-\frac{1}{2}} \; {}_i\tilde{\alpha}_{j-\frac{1}{2}} \; {}_i\tilde{\underline{e}}_{j-\frac{1}{2}}$$

$$\underline{w}_{j}^{n+1} = \underline{w}_{j}^{n} - \sum_{i=1}^{m} {}_i\tilde{\nu}^{+}_{j-\frac{1}{2}} \; {}_i\tilde{\alpha}_{j-\frac{1}{2}} \; {}_i\tilde{\underline{e}}_{j-\frac{1}{2}} \qquad (3.26a\text{-}b)$$

written cell-wise, (see figure 3), where we have defined

$${}_i\tilde{\nu}^{\pm}_{j-\frac{1}{2}} = \frac{\Delta t}{\Delta x} \, {}_i\tilde{\lambda}^{\pm}_{j-\frac{1}{2}} \, .$$



$i = 1,\ldots m$

<u>Figure 3</u>

We note that the $\tilde{A}^{\pm}_{j-\frac{1}{2}}$ have eigenvalues ${}_i\tilde{\lambda}^{\pm}_{j-\frac{1}{2}}$ with eigenvectors ${}_i\tilde{\underline{e}}_{j-\frac{1}{2}}$. In addition, equation (3.24) can be written as

$$\underline{w}_j^{n} - \underline{w}_{j-1}^{n} = \tilde{X}_{j-\frac{1}{2}} \, \tilde{\underline{\alpha}}_{j-\frac{1}{2}} \qquad (3.27)$$

where $\tilde{\underline{\alpha}}_{j-\frac{1}{2}} = ({}_1\tilde{\alpha}_{j-\frac{1}{2}}, \ldots {}_m\tilde{\alpha}_{j-\frac{1}{2}})^T$

and using the change of variable given by equation (3.8) we obtain

$$ {}_i\tilde{\alpha}_{j-\frac{1}{2}} = {}_i v_j^n - {}_i v_{j-1}^n , \tag{3.28} $$

i.e. the 'wavestrength' ${}_i\tilde{\alpha}_{j-\frac{1}{2}}$ represent a change in the characteristic variable ${}_i v$ across the cell $[x_{j-1}, x_j]$. Finally, we note that the scheme given by equation (3.25) is the scalar scheme given by equation (2.5) when applied to each of the m-waves.

3.3 Second order scheme

We now derive the Lax-Wendroff second order scheme of §2.2 as applied to the system given by equation (3.1).

Suppose we split the Jacobian matrix $A = \frac{\partial \underline{f}}{\partial \underline{w}}$ as in equations (3.17)-(3.20) and expand $\underline{w}(x_j, t_n+\Delta t)$ about (x_j, t_n) as a Taylor series

$$ \underline{w}(x_j, t_n+\Delta t) \simeq \underline{w}(x_j, t_n) + \Delta t \underline{w}_t(x_j, t_n) + \frac{\Delta t^2}{2} \underline{w}_{tt}(x_j, t_n) . \tag{3.29} $$

Using equation (3.1) we have

$$ \underline{w}_t = - \underline{f}_x = - A\underline{w}_x \tag{3.30} $$

and

$$ \underline{w}_{tt} = - \underline{f}_{xt} = - (\underline{f}_t)_x = - (A\underline{w}_t)_x = (A^2 \underline{w}_x)_x \tag{3.31} $$

so that equation (3.29) becomes

$$ \underline{w}(x_j, t_n + \Delta t) \simeq \underline{w}(x_j, t_n) - \Delta t A \underline{w}_x(x_j, t_n) + \frac{\Delta t^2}{2} (A^2 \underline{w}_x(w_j, t_n))_x . \tag{3.32} $$

We approximate

$$A\underline{w}_x(x_j,t_n) \simeq \tfrac{1}{2}\left[\tilde{A}_{j-\frac{1}{2}}\,\frac{\Delta_-\underline{w}_j^n}{\Delta x} + \tilde{A}_{j+\frac{1}{2}}\,\frac{\Delta_+\underline{w}_j^n}{\Delta x}\right] \tag{3.33}$$

and

$$(A^2\underline{w}_x(x_j,t_n))_x \simeq \frac{1}{\Delta x}\left[\tilde{A}^2_{j+\frac{1}{2}}\frac{\Delta_+\underline{w}_j^n}{\Delta x} - \tilde{A}^2_{j-\frac{1}{2}}\frac{\Delta_-\underline{w}_j^n}{\Delta x}\right] \tag{3.34}$$

where

$$\Delta_+\underline{w}_j^n = \underline{w}_{j+1}^n - \underline{w}_j^n \tag{3.35a}$$

$$\Delta_-\underline{w}_j^n = \underline{w}_j^n - \underline{w}_{j-1}^n \tag{3.35b}$$

and the $\tilde{A}_{j\pm\frac{1}{2}}$ are as before. Combining the expressions given by equations (3.32)-(3.35b) we get the following second order centrally based scheme

$$\begin{aligned}\underline{w}_j^{n+1} = \underline{w}_j^n &- \frac{\Delta t}{2\Delta x}\left[\tilde{A}_{j-\frac{1}{2}}\Delta_-\underline{w}_j^n + \tilde{A}_{j+\frac{1}{2}}\Delta_+\underline{w}_j^n\right] \\ &+ \frac{\Delta t^2}{2\Delta x^2}\left[\tilde{A}^2_{j+\frac{1}{2}}\Delta_+\underline{w}_j^n - \tilde{A}^2_{j+\frac{1}{2}}\Delta_-\underline{w}_j^n\right] .\end{aligned} \tag{3.36}$$

If we split the approximate Jacobians $\tilde{A}_{j\pm\frac{1}{2}}$ as given by equation (3.22), equation (3.36) becomes on rearrangement

$$\begin{aligned}\underline{w}_j^{n+1} = \underline{w}_j^n &- \frac{\Delta t}{\Delta x}\tilde{A}^+_{j-\frac{1}{2}}\Delta_-\underline{w}_j^n - \frac{\Delta t}{\Delta x}\tilde{A}^-_{j+\frac{1}{2}}\Delta_+\underline{w}_j^n \\ &+ \tfrac{1}{2}\frac{\Delta t}{\Delta x}\tilde{A}^+_{j-\frac{1}{2}}\left(I - \frac{\Delta t}{\Delta x}\tilde{A}_{j-\frac{1}{2}}\right)\Delta_-\underline{w}_j^n \\ &- \tfrac{1}{2}\frac{\Delta t}{\Delta x}\tilde{A}^-_{j-\frac{1}{2}}\left(I + \frac{\Delta t}{\Delta x}\tilde{A}_{j-\frac{1}{2}}\right)\Delta_-\underline{w}_j^n \\ &+ \tfrac{1}{2}\frac{\Delta t}{\Delta x}\tilde{A}^-_{j+\frac{1}{2}}\left(I + \frac{\Delta t}{\Delta x}\tilde{A}_{j+\frac{1}{2}}\right)\Delta_+\underline{w}_j^n \\ &- \tfrac{1}{2}\frac{\Delta t}{\Delta x}\tilde{A}^+_{j+\frac{1}{2}}\left(I - \frac{\Delta t}{\Delta x}\tilde{A}_{j+\frac{1}{2}}\right)\Delta_+\underline{w}_j^n .\end{aligned} \tag{3.37}$$

To implement the algorithm given by equation (3.37) we proceed as in §3.2, i.e. we project in the manner

$$\Delta_- \underline{w}_j^n = \sum_{i=1}^{m} {}_i\tilde{\alpha}_{j-\frac{1}{2}} \, {}_i\tilde{\underline{e}}_{j-\frac{1}{2}}$$

and

$$\Delta_+ \underline{w}_j^n = \sum_{i=1}^{m} {}_i\tilde{\alpha}_{j+\frac{1}{2}} \, {}_i\tilde{\underline{e}}_{j+\frac{1}{2}}$$

so that equation (3.37) becomes

$$\begin{aligned} \underline{w}_j^{n+1} = \underline{w}_j^n &- \sum_{i=1}^{m} {}_i\tilde{\nu}^+_{j-\frac{1}{2}} \, {}_i\tilde{\alpha}_{j-\frac{1}{2}} \, {}_i\tilde{\underline{e}}_{j-\frac{1}{2}} \\ &- \sum_{i=1}^{m} {}_i\tilde{\nu}^-_{j+\frac{1}{2}} \, {}_i\tilde{\alpha}_{j+\frac{1}{2}} \, {}_i\tilde{\underline{e}}_{j+\frac{1}{2}} \\ &+ \sum_{i=1}^{m} \tfrac{1}{2} {}_i\tilde{\nu}^+_{j-\frac{1}{2}} (1 - |{}_i\tilde{\nu}_{j-\frac{1}{2}}|) \, {}_i\tilde{\alpha}_{j-\frac{1}{2}} \, {}_i\tilde{\underline{e}}_{j-\frac{1}{2}} \\ &- \sum_{i=1}^{m} \tfrac{1}{2} {}_i\tilde{\nu}^-_{j-\frac{1}{2}} (1 - |{}_i\tilde{\nu}_{j-\frac{1}{2}}|) \, {}_i\tilde{\alpha}_{j-\frac{1}{2}} \, {}_i\tilde{\underline{e}}_{j-\frac{1}{2}} \\ &+ \sum_{i=1}^{m} \tfrac{1}{2} {}_i\tilde{\nu}^-_{j+\frac{1}{2}} (1 - |{}_i\tilde{\nu}_{j+\frac{1}{2}}|) \, {}_i\tilde{\alpha}_{j+\frac{1}{2}} \, {}_i\tilde{\underline{e}}_{j+\frac{1}{2}} \\ &- \sum_{i=1}^{m} \tfrac{1}{2} {}_i\tilde{\nu}^+_{j+\frac{1}{2}} (1 - |{}_i\tilde{\nu}_{j+\frac{1}{2}}|) \, {}_i\tilde{\alpha}_{j+\frac{1}{2}} \, {}_i\tilde{\underline{e}}_{j+\frac{1}{2}} \end{aligned} \tag{3.38}$$

where

$${}_i\tilde{\nu}_{j-\frac{1}{2}} = \frac{\Delta t}{\Delta x} \, {}_i\tilde{\lambda}_{j-\frac{1}{2}} \tag{3.39}$$

and ${}_i\tilde{\nu}^\pm_{j-\frac{1}{2}}$ are as before.

We again note that $\tilde{A}^{\pm}_{j-\frac{1}{2}}$ have eigenvalues ${}_i\tilde{\lambda}^{\pm}_{j-\frac{1}{2}}$ with eigenvectors ${}_i\underline{\tilde{e}}_{j-\frac{1}{2}}$ and that

$$|{}_i\tilde{\nu}_{j-\frac{1}{2}}| = {}_i\tilde{\nu}_{j-\frac{1}{2}} \quad \text{when} \quad {}_i\nu^{+}_{j-\frac{1}{2}} \neq 0 \tag{3.40}$$

$$|{}_i\tilde{\nu}_{j-\frac{1}{2}}| = -\,{}_i\tilde{\nu}_{j-\frac{1}{2}} \quad \text{when} \quad {}_i\nu^{-}_{j-\frac{1}{2}} \neq 0\,. \tag{3.41}$$

The algorithm given by equation (3.38) is written pointwise and is an extension of the first order algorithm given by equation (3.25); moreover it is an extension of the Lax-Wendroff scalar algorithm given by equation (2.23) when applied to each of the m-waves. In addition, we can write the scheme cell-wise in the form

$$\begin{aligned} \underline{w}^{n+1}_{j-1} = \underline{w}^{n}_{j-1} &- \sum_{i=1}^{m} {}_i\tilde{\nu}^{-}_{j-\frac{1}{2}}\ {}_i\tilde{\alpha}_{j-\frac{1}{2}}\ {}_i\underline{\tilde{e}}_{j-\frac{1}{2}} \\ &+ \sum_{i=1}^{m} \tfrac{1}{2}\,{}_i\tilde{\nu}^{-}_{j-\frac{1}{2}}(1 - |{}_i\tilde{\nu}_{j-\frac{1}{2}}|)\,{}_i\tilde{\alpha}_{j-\frac{1}{2}}\ {}_i\underline{\tilde{e}}_{j-\frac{1}{2}} \\ &- \sum_{i=1}^{m} \tfrac{1}{2}\,{}_i\tilde{\nu}^{+}_{j-\frac{1}{2}}(1 - |{}_i\tilde{\nu}_{j-\frac{1}{2}}|)\,{}_i\tilde{\alpha}_{j-\frac{1}{2}}\ {}_i\underline{\tilde{e}}_{j-\frac{1}{2}} \end{aligned} \tag{3.42a}$$

$$\begin{aligned} \underline{w}^{n+1}_{j} = \underline{w}^{n}_{j} &- \sum_{i=1}^{m} {}_i\tilde{\nu}^{+}_{j-\frac{1}{2}}\ {}_i\tilde{\alpha}_{j-\frac{1}{2}}\ {}_i\underline{\tilde{e}}_{j-\frac{1}{2}} \\ &+ \sum_{i=1}^{m} \tfrac{1}{2}\ {}_i\tilde{\nu}^{+}_{j-\frac{1}{2}}(1 - |{}_i\tilde{\nu}_{j-\frac{1}{2}}|)\,{}_i\tilde{\alpha}_{j-\frac{1}{2}}\ {}_i\underline{\tilde{e}}_{j-\frac{1}{2}} \\ &- \sum_{i=1}^{m} \tfrac{1}{2}\ {}_i\tilde{\nu}^{-}_{j-\frac{1}{2}}(1 - |{}_i\tilde{\nu}_{j-\frac{1}{2}}|)\,{}_i\tilde{\alpha}_{j-\frac{1}{2}}\ {}_i\underline{\tilde{e}}_{j-\frac{1}{2}} \end{aligned} \tag{3.42b}$$

which readily yields the scalar algorithm given by equation (2.23) when we employ the transformation

$$\underline{w}_{j-1}^{n} = \tilde{X}_{j-\frac{1}{2}} \underline{v}_{j-1}^{n}$$
$$\underline{w}_{j}^{n} = \tilde{X}_{j-\frac{1}{2}} \underline{v}_{j}^{n}$$

of §3.1 as applied to each component of $\underline{v}$ where

$${}_i v_j^n - {}_i v_{j-1}^n = {}_i\tilde{\alpha}_{j-\frac{1}{2}} .$$

Finally, we can write equations (3.42a-b) in a similar form to equations (2.20a)-(2.22) as

$$\left.\begin{aligned} \underline{w}_{j-1}^{n+1} &= \underline{w}_{j-1}^{n} + {}_i\underline{b}_{j-\frac{1}{2}} \\ \underline{w}_{j}^{n+1} &= \underline{w}_{j}^{n} + {}_i\underline{\Phi}_{j-\frac{1}{2}} - {}_i\underline{b}_{j-\frac{1}{2}} \end{aligned}\right\} \quad {}_i\tilde{\nu}_{j-\frac{1}{2}} > 0 \tag{3.43a}$$

$$\left.\begin{aligned} \underline{w}_{j-1}^{n+1} &= \underline{w}_{j-1}^{n} + {}_i\underline{\Phi}_{j-\frac{1}{2}} - {}_i\underline{b}_{j-\frac{1}{2}} \\ \underline{w}_{j}^{n+1} &= \underline{w}_{j}^{n} + {}_i\underline{b}_{j-\frac{1}{2}} \end{aligned}\right\} \quad {}_i\tilde{\nu}_{j-\frac{1}{2}} < 0 \tag{3.43b}$$

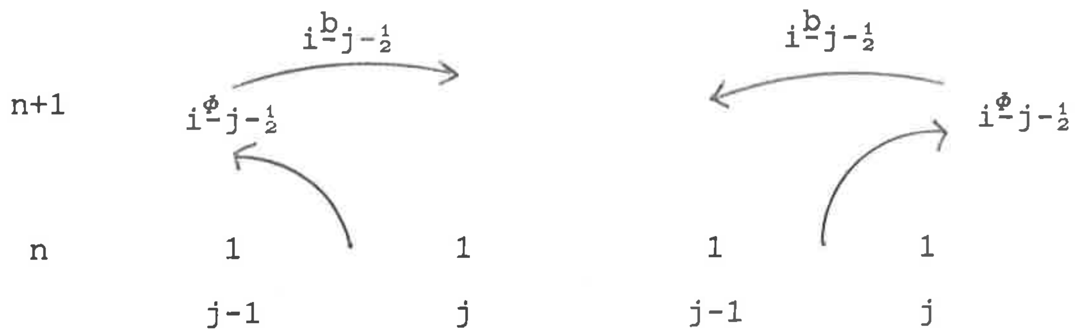
for each $i = 1,\ldots m$, where

$${}_i\underline{\Phi}_{j-\frac{1}{2}} = - {}_i\tilde{\nu}_{j-\frac{1}{2}} \, {}_i\tilde{\alpha}_{j-\frac{1}{2}} \, {}_i\tilde{\underline{e}}_{j-\frac{1}{2}} \tag{3.44}$$

and

$${}_i\underline{b}_{j-\frac{1}{2}} = \tfrac{1}{2}(1 - |{}_i\tilde{\nu}_{j-\frac{1}{2}}|)\,{}_i\underline{\Phi}_{j-\frac{1}{2}} , \tag{3.45}$$

(see Figure 4).



$_i\tilde{\nu}_{j-\frac{1}{2}} < 0$ $\qquad$ $_i\tilde{\nu}_{j-\frac{1}{2}} > 0$

$i = 1,\ldots m$

Figure 4

The second order transfers can be limited in such a way that non-physical oscillations are avoided (see [2], [3]).

In the next section we return to a single conservation law, but with a 'source' term.

4. SCALAR CONSERVATION LAW WITH SOURCE TERM

In this section describe first and second order schemes for a single conservation law with a source term.

Consider the hyperbolic problem

$$u_t + f_x = h(x,u) \; , \quad (x,t) \in (-\infty,\infty) \times [0,T] \tag{4.1}$$

with initial data

$$u(x,0) = u_0(x) \tag{4.2}$$

for the function $u = u(x,t)$, where $f = f(u)$ is a convex flux function and the source term $h(x,u)$ contains no derivatives of u. Equation (4.1) can be written as

$$u_t + a(u)u_x = h(x,u) \tag{4.3}$$

where $a(u) = f'(u)$, so that the characteristics solution of equation (4.1) can be written

$$\frac{du}{dt} = h(x,u) \tag{4.4a}$$

along

$$\frac{dx}{dt} = a(u) \; . \tag{4.4b}$$

If we adopt the notation of §2 and consider $a(u)$ to be constant, say a, in the time interval $[t,t+\Delta t]$ then we can integrate equations (4.4a-b) to give

$$u(x,t+\Delta t) = u(x - a\Delta t,t) + \int_0^{\Delta t} \bar{h}(s)ds \tag{4.5a}$$

where

$$\bar{h}(s) = h(x - a(\Delta t - s) , u(x - a(\Delta t - s),s)) . \qquad (4.5b)$$

4.1 First Order Scheme

We now derive a first order scheme based on integrating equation (4.4a) along the characteristics given by equation (4.4b) in an approximate manner.

Consider the interval $[x_{j-1},x_j]$ of length Δx and suppose $a(u) > 0$ so that equation (4.5a) can be approximated as

$$u_j^{n+1} = u_{Q_{j-\frac{1}{2}}} + \Delta t h(x_{Q_{j-\frac{1}{2}}}, u_{Q_{j-\frac{1}{2}}}) \qquad (4.6)$$

where

$$x_{Q_{j-\frac{1}{2}}} = x_j - a_{j-\frac{1}{2}}\Delta t \qquad (4.7)$$

and $u_{Q_{j-\frac{1}{2}}}$, $a_{j-\frac{1}{2}}$ are approximations to $u(x_{Q_{j-\frac{1}{2}}},t)$, $a(u)$, respectively. We choose $a_{j-\frac{1}{2}}$ as given in equations (2.7a-b) and set

$$\begin{aligned} u_{Q_{j-\frac{1}{2}}} &= u_j^n - \frac{\Delta t}{\Delta x} a_{j-\frac{1}{2}}(u_j^n - u_{j-1}^n) \\ &= u_j^n - \frac{\Delta t}{\Delta x} (f_j^n - f_{j-1}^n) \end{aligned} \qquad (4.8)$$

where $f_j^n = f(u_j^n)$. The approximation given by equation (4.8) is consistent with the first order upwind scheme of §2 in the case $h \equiv 0$.

If we now consider $a(u)$ to take any sign we have the following first order scheme for the solution of equations (4.1)-(4.2) considered cell-wise

$$\left.\begin{aligned} u^{n+1}_{j-1} &= u^n_{j-1} && (4.9a)\\ u^{n+1}_{j} &= u^n_j-\nu_{j-\frac{1}{2}}(u^n_j-u^n_{j-1})+\Delta t h(x_j-a_{j-\frac{1}{2}}\Delta t, u^n_j-\nu_{j-\frac{1}{2}}(u^n_j-u^n_{j-1})) && (4.9b)\end{aligned}\right\}\nu_{j-\frac{1}{2}} > 0$$

$$\left.\begin{aligned} u^{n+1}_{j-1} &= u^n_{j-1}-\nu_{j-\frac{1}{2}}(u^n_j-u^n_{j-1})+\Delta t h(x_{j-1}-a_{j-\frac{1}{2}}\Delta t, u^n_{j-1}\nu_{j-\frac{1}{2}}(u^n_j-u^n_{j-1})) && (4.10a)\\ u^{n+1}_{j} &= u^n_j && (4.10b)\end{aligned}\right\}\nu_{j-\frac{1}{2}} < 0$$

where

$$\nu_{j-\frac{1}{2}} = a_{j-\frac{1}{2}} \frac{\Delta t}{\Delta x} .$$

Alternatively, we can write equations (4.9a)-(4.10b) pointwise, centred on x_j as

$$\begin{aligned} u^{n+1}_j = u^n_j-\nu^+_{j-\frac{1}{2}}(u^n_j-u^n_{j-1}) &+ \Delta t \frac{\nu^+_{j-\frac{1}{2}}}{\nu_{j-\frac{1}{2}}} h(x_{Q^+_{j-\frac{1}{2}}}, u_{Q^+_{j-\frac{1}{2}}})\\ - \nu^-_{j+\frac{1}{2}}(u^n_{j+1}-u^n_j) &+ \Delta t \frac{\nu^-_{j+\frac{1}{2}}}{\nu_{j+\frac{1}{2}}} h(x_{Q^-_{j+\frac{1}{2}}}, u_{Q^-_{j+\frac{1}{2}}})\end{aligned} \qquad (4.11)$$

where

$$u_{Q^\pm_{j-\frac{1}{2}}} = u^n_j - \nu^\pm_{j-\frac{1}{2}}(u^n_j - u^n_{j-1})$$

$$x_{Q^\pm_{j-\frac{1}{2}}} = x_j - a^\pm_{j-\frac{1}{2}}\Delta t$$

and $\nu^{\pm}_{j-\frac{1}{2}}$, $a^{\pm}_{j-\frac{1}{2}}$ are as before. If we expand $h(x_{Q^{\pm}_{j\mp\frac{1}{2}}}, u_{Q^{\pm}_{j\mp\frac{1}{2}}})$ about $(x_{Q^{\pm}_{j\mp\frac{1}{2}}}, u^n_j)$ equation (4.11) gives, to first order,

$$u^{n+1}_j = u^n_j - \nu^+_{j-\frac{1}{2}}(u^n_j - u^n_{j-1}) - \nu^-_{j+\frac{1}{2}}(u^n_{j+1} - u^n_j) + \Delta t \frac{\nu^+_{j-\frac{1}{2}}}{\nu_{j-\frac{1}{2}}} h(x_j - a^+_{j-\frac{1}{2}}\Delta t, u^n_j) + \Delta t \frac{\nu^-_{j+\frac{1}{2}}}{\nu_{j+\frac{1}{2}}} h(x_j - a^-_{j+\frac{1}{2}}\Delta t, u^n_j) . \tag{4.12}$$

Equation (4.12) represents a first order scalar algorithm that can be easily extended to systems.

If we wish to simplify this algorithm further we may expand $h(x_j - a^{\pm}_{j\mp\frac{1}{2}}\Delta t, u^n_j)$ about (x_j, u^n_j) so that, again to first order,

$$u^{n+1}_j = u^n_j - \nu^+_{j-\frac{1}{2}}(u^n_j - u^n_{j-1}) - \nu^-_{j+\frac{1}{2}}(u^n_{j+1} - u^n_j) + \Delta t \frac{\nu^+_{j-\frac{1}{2}}}{\nu_{j-\frac{1}{2}}} h(x_j, u^n_j) + \Delta t \frac{\nu^-_{j+\frac{1}{2}}}{\nu_{j+\frac{1}{2}}} h(x_j, u^n_j) . \tag{4.13}$$

The scheme given by equation (4.13) can be expressed cell-wise as

$$\left.\begin{aligned} u^{n+1}_{j-1} &= u^n_{j-1} \\ u^{n+1}_j &= u^n_j + \Delta t h(x_j, u^n_j) - \nu_{j-\frac{1}{2}}(u^n_j - u^n_{j-1}) \end{aligned}\right\} \nu_{j-\frac{1}{2}} > 0 \tag{4.14a}$$

$$\left.\begin{aligned} u_{j-1}^{n+1} &= u_{j-1}^{n} + \Delta t h(x_{j-1}, u_{j-1}^{n}) - \nu_{j-\frac{1}{2}}(u_j^n - u_{j-1}^n) \\ u_j^{n+1} &= u_j^n \end{aligned}\right\} \nu_{j-\frac{1}{2}} < 0 \tag{4.14b}$$

(see Figure 5).

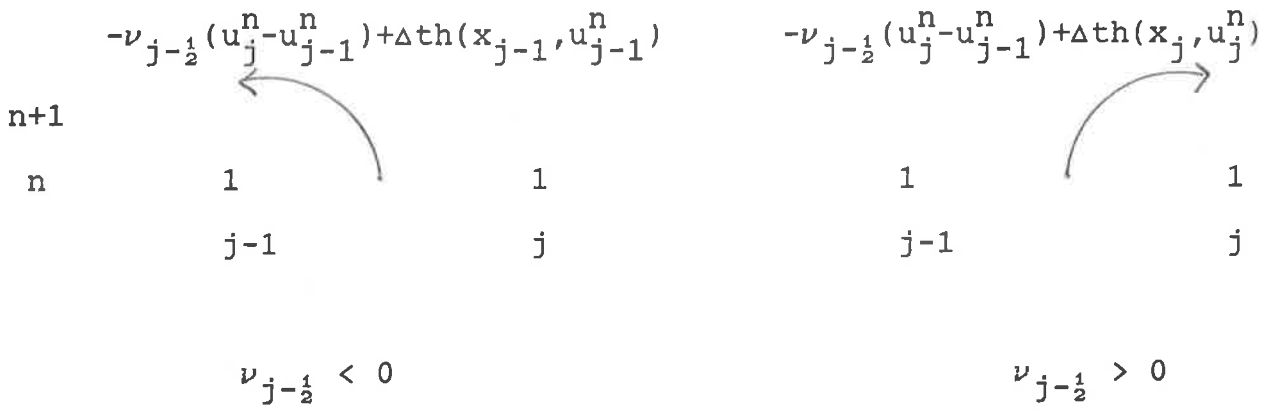


Figure 5

4.2 Second Order Scheme

We now derive a second order scheme for the solution of equations (4.1)-(4.2). Differentiating equation (4.1) with respect to t gives

$$u_{tt} + f_{xt} = h_t = h_u u_t \tag{4.15}$$

where

$$f_{xt} = (f_t)_x = (f' u_t)_x . \tag{4.16}$$

Thus, using equations (4.1) and (4.16), equation (4.15) yields

$$u_{tt} = h_u(h - f_x) - (f'(h - f_x))_x$$

so that expanding as a Taylor series as before

$$u(x_j,t_x+\Delta t) \simeq u + \Delta t(h-f_x) + \frac{\Delta t^2}{2} h_u(h-f_x) - \frac{\Delta t^2}{2} (f'(h-f_x))_x \tag{4.17}$$

where the terms of the right hand side of equation (4.17) are evaluated at (x_j,t_n). Following equation (4.17) we propose the following scheme centred on x_j

$$\begin{aligned} u_j^{n+1} = u_j^n &+ \frac{\Delta t}{2}\left[h_{j+\frac{1}{2}} - \frac{(f_{j+1}^n - f_j^n)}{\Delta x}\right] \\ &+ \frac{\Delta t}{2}\left[h_{j-\frac{1}{2}} - \frac{(f_j^n - f_{j-1}^n)}{\Delta x}\right] \\ &+ \frac{\Delta t^2}{2}\left[\frac{(h_u)_{j+\frac{1}{2}}}{2}\left[h_{j+\frac{1}{2}} - \frac{(f_{j+1}^n - f_j^n)}{\Delta x}\right]\right] \\ &+ \frac{\Delta t^2}{2}\left[\frac{(h_u)_{j-\frac{1}{2}}}{2}\left[h_{j-\frac{1}{2}} - \frac{(f_j^n - f_{j-1}^n)}{\Delta x}\right]\right] \\ &- \frac{\Delta t^2}{2}\left[\frac{f'_{j+\frac{1}{2}}}{} \frac{\left[h_{j+\frac{1}{2}} - \frac{(f_{j+1}^n - f_j^n)}{\Delta x}\right]}{\Delta x} \right. \\ &\left. - \frac{f'_{j-\frac{1}{2}}}{} \frac{\left[h_{j-\frac{1}{2}} - \frac{(f_j^n - f_{j-1}^n)}{\Delta x}\right]}{\Delta x}\right] \end{aligned} \tag{4.18}$$

where $h_{j\pm\frac{1}{2}}$, $(h_u)_{j\pm\frac{1}{2}}$, $f'_{j\pm\frac{1}{2}}$ represent approximations to $h(x,u)$, $h_u(x,u)$, $f'(u)$, respectively, with $x \in [x_{j-1},x_j]$, and $u \in [u^n_{j-1},u^n_j]$, i.e. at time level n . As in §2.2 we set

$$a_{j+\frac{1}{2}} = f'_{j+\frac{1}{2}} = \frac{f^n_{j+1} - f^n_j}{u^n_{j+1} - u^n_j} = \frac{f(u^n_{j+1}) - f(u^n_j)}{u^n_{j+1} - u^n_j} \tag{4.19}$$

and similarly for $a_{j-\frac{1}{2}} = f'_{j-\frac{1}{2}}$. We now wish to choose $h_{j\pm\frac{1}{2}}$, $(h_u)_{j\pm\frac{1}{2}}$ so that equation (4.18) matches equation (4.17) to $O(\Delta^3)$.

Since $(h_u)_{j\pm\frac{1}{2}}$ only appear in the second order terms we set

$$(h_u)_{j\pm\frac{1}{2}} = h_u(x_j,u^n_j) \tag{4.20}$$

and approximate

$$h_{j-\frac{1}{2}} = h(\bar{x},\bar{u}) \tag{4.21}$$

where

$$\bar{x} = p\, x_{j-1} + (1-p)x_j \qquad p \in [0,1] \tag{4.22}$$

$$\bar{u} = q\, u^n_{j-1} + (1-q)u^n_j \qquad q \in [0,1] \tag{4.23}$$

with similar expressions for $h_{j+\frac{1}{2}}$. From equation (4.22) we get

$$\bar{x} = x_j - p(x_j - x_{j-1}) = x_j - p\Delta x \tag{4.24}$$

and from (4.23)

$$\begin{aligned} \bar{u} &= u^n_j - q(u^n_j - u^n_{j-1}) \\ &= u^n_j - q\,\Delta x\, u(x_j,t_n) + O(\Delta x^2) . \end{aligned} \tag{4.25}$$

Thus, using equation (4.24)-(4.25) we rewrite equation (4.21) as

$$h_{j-\frac{1}{2}} = h(x_j - x_-,\ u_j^n - u_-) \tag{4.26}$$

and similarly

$$h_{j+\frac{1}{2}} = h(x_j + x_+,\ u_j^n + u_+) \tag{4.27}$$

where

$$x_{\pm} = 0(\Delta x), \quad u_{\pm} = 0(\Delta x) . \tag{4.28}$$

Substituting the approximations given by equations (4.19)-(4.20) and (4.16)-(4.27) into equation (4.18) we obtain, after expanding all terms about (x_j, u_j^n), the following expression for the right hand side of equation (4.18)

$$\begin{aligned} u &+ \Delta t(h - f_x) + \frac{\Delta t^2}{2}(h - f_x)h_u \\ &+ \frac{\Delta t^2}{2}\left[\left[\frac{f_x^2}{u_x}\right]_x - h\left[\frac{f_x}{u_x}\right]_x\right] \\ &+ \frac{\Delta t}{2}h_x(x_+ - x_-) + \frac{\Delta t}{2}h_u(u_+ - u_-) \\ &+ \frac{\Delta t^2}{2\Delta x}\left[- (x_+ + x_-)\frac{h_x f_x}{u_x} - (u_+ + u_-)\frac{f_x h_u}{u_x}\right] \\ &+ 0(\Delta^3) . \end{aligned} \tag{4.29}$$

All terms in equation (4.29) are evaluated at (x_j, t_n) and we have used the following expressions:

$$\frac{f_{j+1} - f_j}{\Delta x} = f_x - \frac{\Delta x}{2} f_{xx} + 0(\Delta x^2) \tag{4.30}$$

$$\frac{f_j - f_{j-1}}{\Delta x} = f_x + \frac{\Delta x}{2} f_{xx} + 0(\Delta x^2) \tag{4.31}$$

$$(f')_{j\pm\frac{1}{2}} = \frac{f_x}{u_x} \pm \frac{\Delta x}{2}\left(\frac{f_x}{u_x}\right)_x + 0(\Delta x^2) \ . \tag{4.32}$$

In order to make the scheme given by equations (4.18)-(4.19) and (4.26)-(4.28) second order accurate we must choose

$$x_+ - x_- = 0 \tag{4.33}$$

$$x_+ + x_- = \Delta x \tag{4.34}$$

$$u_+ - u_- = 0 \tag{4.35}$$

$$u_+ + u_- = 0 \tag{4.36}$$

so that equation (4.29) yields

$$u + \Delta t(h - f_x) + \frac{\Delta t^2}{2}(h - f_x)h_u + \frac{\Delta t^2}{2}\left[\left(\frac{f_x^2}{u_x}\right)_x - h\left(\frac{f_x}{u_x}\right)_x\right] - \frac{\Delta t^2}{2}\,\frac{h_x f_x}{u_x} + 0(\Delta^3) \ , \tag{4.37}$$

and using $f_x = f'u_x$ equation (4.37) gives

$$u + \Delta t(h - f_x) + \frac{\Delta t^2}{2}(h - f_x)h_u - \frac{\Delta t^2}{2}(f'(h - f_x))_x \ . \tag{4.38}$$

Equation (4.38) is of the correct form as can be seen by comparison with equation (4.17). Thus, solving equations (4.33)-(4.36) we obtain

$$x_{\pm} = \frac{\Delta x}{2}, \qquad u_{\pm} = 0 . \tag{4.39}$$

Therefore the second order algorithm given by equations (4.18)-(4.19), (4.26)-(4.28) and (4.39) becomes

$$\begin{aligned} u_j^{n+1} = u_j^n &+ \tfrac{1}{2}(\Delta t h_{j+\frac{1}{2}} - \nu_{j+\frac{1}{2}}(u_{j+1}^n - u_j^n)) \\ &+ \tfrac{1}{2}(\Delta t h_{j-\frac{1}{2}} - \nu_{j-\frac{1}{2}}(u_j^n - u_{j-1}^n)) \\ &+ \frac{\Delta t (h_u)_{j+\frac{1}{2}}}{4}(\Delta t h_{j+\frac{1}{2}} - \nu_{j+\frac{1}{2}}(u_{j+1}^n - u_j^n)) \\ &+ \frac{\Delta t (h_u)_{j-\frac{1}{2}}}{4}(\Delta t h_{j-\frac{1}{2}} - \nu_{j-\frac{1}{2}}(u_j^n - u_{j-1}^n)) \\ &- \tfrac{1}{2}(\nu_{j+\frac{1}{2}}(\Delta t h_{j+\frac{1}{2}} - \nu_{j+\frac{1}{2}}(u_{j+1}^n - u_j^n)) \\ &\qquad - \nu_{j-\frac{1}{2}}(\Delta t h_{j-\frac{1}{2}} - \nu_{j-\frac{1}{2}}(u_j^n - u_{j-1}^n))) \end{aligned} \tag{4.40a}$$

where
$$\nu_{j\pm\frac{1}{2}} = a_{j\pm\frac{1}{2}} \frac{\Delta t}{\Delta x} = f'_{j\pm\frac{1}{2}} \frac{\Delta t}{\Delta x} \tag{4.40b}$$

$$(h_u)_{j\pm\frac{1}{2}} = h(x_j, u_j^n) \tag{4.40c}$$

and
$$h_{j\pm\frac{1}{2}} = h(x_j \pm \frac{\Delta x}{2}, u_j^n) . \tag{4.40d}$$

Equation (4.40a) can be rewritten as

$$\begin{aligned} u_j^{n+1} = u_j^n &+ (1 + \frac{\Delta t}{2}(h_u)_{j-\frac{1}{2}})(\Delta t h_{j-\frac{1}{2}} - \nu_{j-\frac{1}{2}}(u_j^n - u_{j-1}^n)) \\ &+ \tfrac{1}{2}(1 - \nu_{j+\frac{1}{2}} + \frac{\Delta t}{2}(h_u)_{j+\frac{1}{2}})(\Delta t h_{j+\frac{1}{2}} - \nu_{j+\frac{1}{2}}(u_{j+1}^n - u_j^n)) \\ &- \tfrac{1}{2}(1 - \nu_{j-\frac{1}{2}} + \frac{\Delta t}{2}(h_u)_{j-\frac{1}{2}})(\Delta t h_{j-\frac{1}{2}} - \nu_{j-\frac{1}{2}}(u_j^n - u_{j-1}^n)) \end{aligned} \tag{4.41a}$$

$$\text{or} \quad u_j^{n+1} = u_j^n + (1 + \tfrac{\Delta t}{2}(h_u)_{j+\frac{1}{2}})(\Delta th_{j+\frac{1}{2}} - \nu_{j+\frac{1}{2}}(u_{j+1}^n - u_j^n))$$
$$+ \tfrac{1}{2}(1 + \nu_{j-\frac{1}{2}} + \tfrac{\Delta t}{2}(h_u)_{j-\frac{1}{2}})(\Delta th_{j-\frac{1}{2}} - \nu_{j-\frac{1}{2}}(u_j^n - u_{j-1}^n))$$
$$- \tfrac{1}{2}(1 + \nu_{j+\frac{1}{2}} + \tfrac{\Delta t}{2}(h_u)_{j+\frac{1}{2}})(\Delta th_{j+\frac{1}{2}} - \nu_{j+\frac{1}{2}}(u_{j+1}^n - u_j^n)) .$$

(4.41b)

If we compare equations (4.41a-b) with equations (4.14a)-(4.15b) we can consider the scheme given by equations (4.40a-d) as based on the cell $[x_{j-1}, x_j]$. Moreover, we can consider the scheme as consisting of an increment stage together with a transfer stage:

$$\left.\begin{aligned} u_{j-1}^{n+1} &= u_{j-1}^n + c_{j-\frac{1}{2}} \\ u_j^{n+1} &= u_j^n + \Psi_{j-\frac{1}{2}} - c_{j-\frac{1}{2}} \end{aligned}\right\} \quad \nu_{j-\frac{1}{2}} > 0$$

(4.42a)

$$\left.\begin{aligned} u_{j-1}^{n+1} &= u_{j-1}^n + \Psi_{j-\frac{1}{2}} - c_{j-\frac{1}{2}} \\ u_j^{n+1} &= u_j^n + c_{j-\frac{1}{2}} \end{aligned}\right\} \quad \nu_{j-\frac{1}{2}} < 0$$

(4.42b)

where

$$\Psi_{j-\frac{1}{2}} = (1 + \tfrac{\Delta t}{2}(h_u)_{j-\frac{1}{2}})(\Delta th_{j-\frac{1}{2}} - \nu_{j-\frac{1}{2}}(u_j^n - u_{j-1}^n))$$

(4.43)

$$c_{j-\frac{1}{2}} = - \tfrac{1}{2}(1 - |\nu_{j-\frac{1}{2}}| + \tfrac{\Delta t}{2}(h_u)_{j-\frac{1}{2}})(\Delta th_{j-\frac{1}{2}} - \nu_{j-\frac{1}{2}}(u_j^n - u_{j-1}^n)) .$$

(4.44)

(See Figure 6).

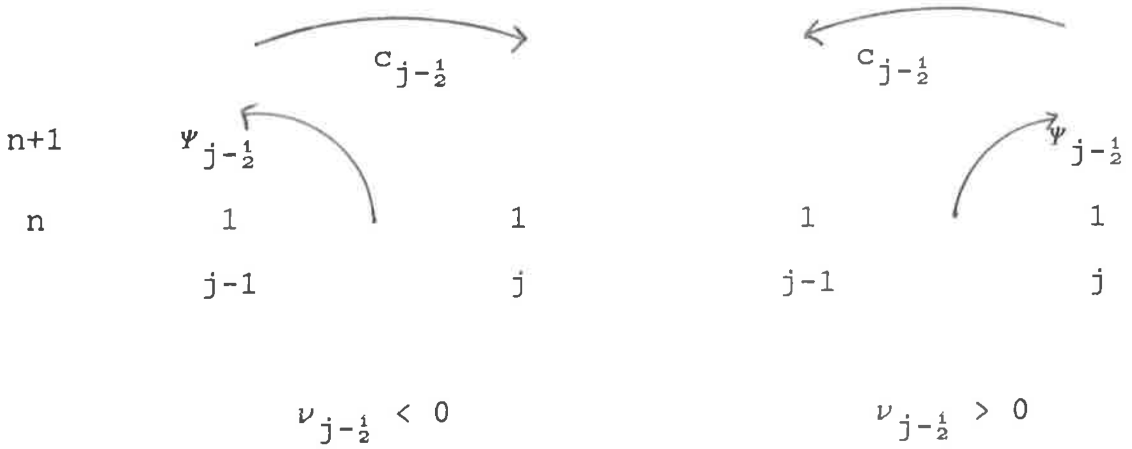


Figure 6

We can also write equations (4.42a)-(4.44) in a way similar to the first order scheme (4.13)

$$
\begin{aligned}
u_j^{n+1} = u_j^n &+ \left[1 + \frac{\Delta t}{2}(h_u)_{j-\frac{1}{2}}\right]\left[\Delta t \frac{\nu^+_{j-\frac{1}{2}}}{\nu_{j-\frac{1}{2}}} h_{j-\frac{1}{2}} - \nu^+_{j-\frac{1}{2}}(u_j^n - u_{j-1}^n)\right] \\
&+ \left[1 + \frac{\Delta t}{2}(h_u)_{j+\frac{1}{2}}\right]\left[\Delta t \frac{\nu^-_{j+\frac{1}{2}}}{\nu_{j+\frac{1}{2}}} h_{j+\frac{1}{2}} - \nu^-_{j+\frac{1}{2}}(u_{j+1}^n - u_j^n)\right] \\
&- \tfrac{1}{2}\left[1 - |\nu_{j-\frac{1}{2}}| + \frac{\Delta t}{2}(h_u)_{j-\frac{1}{2}}\right]\left[\Delta t \frac{\nu^+_{j-\frac{1}{2}}}{\nu_{j-\frac{1}{2}}} h_{j-\frac{1}{2}} - \nu^+_{j-\frac{1}{2}}(u_j^n - u_{j-1}^n)\right] \\
&+ \tfrac{1}{2}\left[1 - |\nu_{j-\frac{1}{2}}| + \frac{\Delta t}{2}(h_u)_{j-\frac{1}{2}}\right]\left[\Delta t \frac{\nu^-_{j-\frac{1}{2}}}{\nu_{j-\frac{1}{2}}} h_{j-\frac{1}{2}} - \nu^-_{j-\frac{1}{2}}(u_j^n - u_{j-1}^n)\right] \\
&+ \tfrac{1}{2}\left[1 - |\nu_{j+\frac{1}{2}}| + \frac{\Delta t}{2}(h_u)_{j+\frac{1}{2}}\right]\left[\Delta t \frac{\nu^+_{j+\frac{1}{2}}}{\nu_{j+\frac{1}{2}}} h_{j+\frac{1}{2}} - \nu^+_{j+\frac{1}{2}}(u_{j+1}^n - u_j^n)\right] \\
&- \tfrac{1}{2}\left[1 - |\nu_{j+\frac{1}{2}}| + \frac{\Delta t}{2}(h_u)_{j+\frac{1}{2}}\right]\left[\Delta t \frac{\nu^-_{j+\frac{1}{2}}}{\nu_{j+\frac{1}{2}}} h_{j+\frac{1}{2}} - \nu^-_{j+\frac{1}{2}}(u_{j+1}^n - u_j^n)\right].
\end{aligned}
\tag{4.45}
$$

Written cell-wise, equation (4.45) becomes

$$u_{j-1}^{n+1} = u_{j-1}^{n} + \left[1 + \frac{\Delta t}{2}(h_u)_{j-\frac{1}{2}}\right]\left[\Delta t\frac{\nu_{j-\frac{1}{2}}^{-}}{\nu_{j-\frac{1}{2}}} h_{j-\frac{1}{2}} - \nu_{j-\frac{1}{2}}^{-}(u_j^n - u_{j-1}^n)\right]$$

$$- \tfrac{1}{2}\left[1 - |\nu_{j-\frac{1}{2}}| + \frac{\Delta t}{2}(h_u)_{j-\frac{1}{2}}\right]\left[\Delta t\frac{\nu_{j-\frac{1}{2}}^{-}}{\nu_{j-\frac{1}{2}}} h_{j-\frac{1}{2}} - \nu_{j-\frac{1}{2}}^{-}(u_j^n - u_{j-1}^n)\right]$$

$$+ \tfrac{1}{2}\left[1 - |\nu_{j-\frac{1}{2}}| + \frac{\Delta t}{2}(h_u)_{j-\frac{1}{2}}\right]\left[\Delta t\frac{\nu_{j-\frac{1}{2}}^{+}}{\nu_{j-\frac{1}{2}}} h_{j-\frac{1}{2}} - \nu_{j-\frac{1}{2}}^{+}(u_j^n - u_{j-1}^n)\right] \tag{4.46a}$$

$$u_{j}^{n+1} = u_{j}^{n} + \left[1 + \frac{\Delta t}{2}(h_u)_{j-\frac{1}{2}}\right]\left[\Delta t\frac{\nu_{j-\frac{1}{2}}^{+}}{\nu_{j-\frac{1}{2}}} h_{j-\frac{1}{2}} - \nu_{j-\frac{1}{2}}^{+}(u_j^n - u_{j-1}^n)\right]$$

$$- \tfrac{1}{2}\left[1 - |\nu_{j-\frac{1}{2}}| + \frac{\Delta t}{2}(h_u)_{j-\frac{1}{2}}\right]\left[\Delta t\frac{\nu_{j-\frac{1}{2}}^{+}}{\nu_{j-\frac{1}{2}}} h_{j-\frac{1}{2}} - \nu_{j-\frac{1}{2}}^{+}(u_j^n - u_{j-1}^n)\right]$$

$$+ \tfrac{1}{2}\left[1 - |\nu_{j-\frac{1}{2}}| + \frac{\Delta t}{2}(h_u)_{j-\frac{1}{2}}\right]\left[\Delta t\frac{\nu_{j-\frac{1}{2}}^{-}}{\nu_{j-\frac{1}{2}}} h_{j-\frac{1}{2}} - \nu_{j-\frac{1}{2}}^{-}(u_j^n - u_{j-1}^n)\right]. \tag{4.46b}$$

In the next section we apply the schemes of this section to systems of conservation laws with source terms.

5. SYSTEMS OF CONSERVATION LAWS WITH SOURCE TERMS

In this section we apply the first and second order schemes of §4 to systems of conservation laws with source terms.

Consider the system of hyperbolic conservation laws

$$\underline{w}_t + \underline{f}_x = \underline{g} \qquad (x,t) \in (-\infty,\infty) \times [0,T] \tag{5.1}$$

for the function $\underline{w} = \underline{w}(x,t)$, where $\underline{f} = \underline{f}(\underline{w})$ and where the source term $\underline{g} = \underline{g}(x,\underline{w})$ contains no derivatives of $\underline{w}$. As in §3, we assume that the approximate solution of equations (5.1) is sought by solving the Riemann problem

$$\underline{w}_t + \tilde{A}(\underline{w}^n_{j-1},\underline{w}^n_j)\underline{w}_x = \underline{g}(x,\underline{w}) , \qquad (x,t) \in [x_{j-1},x_j] \times (t_n,t_{n+1}) \tag{5.2}$$

where $\tilde{A}(\underline{w}^n_{j-1},\underline{w}^n_j)$ is an approximation to the Jacobian $A(\underline{w}) = \frac{\partial \underline{f}}{\partial \underline{w}}(\underline{w})$ and $\underline{w}^n_{j-1},\underline{w}^n_j$ represent piecewise constant states as in equations (3.4). A specific example of $\tilde{A}$ is given by Glaister [5] for compressible flow in a duct of variable cross section. The algorithms of this section are illustrated for this example in §6.

5.1 First order scheme by diagonalisation

Assuming the notation of §3.1, we can 'diagonalise' the system of equations (5.2), using the change of dependent variable given by equation (3.8), to

$$\underline{v}_t + \tilde{\Lambda}_{j-\frac{1}{2}}\underline{v}_x = \underline{h}(x,\underline{v}) \tag{5.3}$$

where

$$\tilde{X}^{-1}_{j-\frac{1}{2}}\, \underline{g}(x,\underline{w}) = \tilde{X}^{-1}_{j-\frac{1}{2}}\, \underline{g}(x,\tilde{X}_{j-\frac{1}{2}}\underline{v}) = \underline{h}(x,\underline{v})\ . \tag{5.4}$$

Equation (5.3) represents the sequence of scalar problems

$$\frac{\partial}{\partial t}({}_i v) + {}_i\tilde{\lambda}_{j-\frac{1}{2}} \frac{\partial}{\partial x}({}_i v) = h_i(x, {}_1 v, \ldots {}_m v)\ , \quad i = 1,\ldots m\ . \tag{5.5}$$

Equations (5.5) can now be solved using the first order upwind scheme given by equations (4.13), where we identify ${}_i\tilde{\lambda}_{j-\frac{1}{2}}$ with the approximation to $a_{j-\frac{1}{2}}$ for each i. Thus the scheme for equations (5.5) cell-wise is

$$\begin{aligned} {}_i v^{n+1}_{j-1} &= {}_i v^n_{j-1} - \frac{\Delta t}{\Delta x}\, {}_i\tilde{\lambda}^-_{j-\frac{1}{2}}({}_i v^n_j - {}_i v^n_{j-1}) \\ &\quad + \Delta t \frac{{}_i\tilde{\lambda}^-_{j-\frac{1}{2}}}{{}_i\tilde{\lambda}_{j-\frac{1}{2}}}\, h_i(x_{j-1}, {}_1 v^n_{j-1}, \ldots {}_m v^n_{j-1}) \end{aligned} \tag{5.6a}$$

$$i = 1,\ldots m$$

$$\begin{aligned} {}_i v^{n+1}_{j} &= {}_i v^n_{j} - \frac{\Delta t}{\Delta x}\, {}_i\tilde{\lambda}^+_{j-\frac{1}{2}}({}_i v^n_j - {}_i v^n_{j-1}) \\ &\quad + \Delta t \frac{{}_i\tilde{\lambda}^+_{j-\frac{1}{2}}}{{}_i\tilde{\lambda}_{j-\frac{1}{2}}}\, h_i(x_{j}, {}_1 v^n_{j}, \ldots {}_m v^n_{j}) \end{aligned} \tag{5.6b}$$

$$i = 1,\ldots m$$

where ${}_i\tilde{\lambda}^{\pm}_{j-\frac{1}{2}}$ are as before. Equations (5.6a-b) can now be written in system form as

$$\begin{aligned} \underline{v}^{n+1}_{j-1} &= \underline{v}^n_{j-1} - \frac{\Delta t}{\Delta x}\, \tilde{\Lambda}^-_{j-\frac{1}{2}}(\underline{v}^n_j - \underline{v}^n_{j-1}) \\ &\quad + \Delta t \tilde{\Lambda}^{-1}_{j-\frac{1}{2}} \tilde{\Lambda}^-_{j-\frac{1}{2}}\, \underline{h}(x_{j-1}, \underline{v}^n_{j-1}) \end{aligned} \tag{5.7a}$$

$$\underline{v}_j^{n+1} = \underline{v}_j^n - \frac{\Delta t}{\Delta x} \tilde{\Lambda}^+_{j-\frac{1}{2}}(\underline{v}_j^n - \underline{v}_{j-1}^n) + \Delta t \tilde{\Lambda}^{-1}_{j-\frac{1}{2}} \tilde{\Lambda}^+_{j-\frac{1}{2}} \underline{h}(x_j, \underline{v}_j^n) \tag{5.7b}$$

where

$$\tilde{\Lambda}^{\pm}_{j-\frac{1}{2}} = \text{diag}({}_1\tilde{\lambda}^{\pm}_{j-\frac{1}{2}}, \ldots {}_m\tilde{\lambda}^{\pm}_{j-\frac{1}{2}}) \ . \tag{5.8}$$

If we transform back using equations (3.8) and (5.4) equations (5.7a-b) become

$$\underline{w}_{j-1}^{n+1} = \underline{w}_{j-1}^n - \frac{\Delta t}{\Delta x} \tilde{A}^-_{j-\frac{1}{2}}(\underline{w}_j^n - \underline{w}_{j-1}^n) + \Delta t \tilde{A}^{-1}_{j-\frac{1}{2}} \tilde{A}^-_{j-\frac{1}{2}} \underline{g}(x_{j-1}, \underline{w}_{j-1}^n) \tag{5.9a}$$

$$\underline{w}_j^{n+1} = \underline{w}_j^n - \frac{\Delta t}{\Delta x} \tilde{A}^+_{j-\frac{1}{2}}(\underline{w}_j^n - \underline{w}_{j-1}^n) + \Delta t \ \tilde{A}^{-1}_{j-\frac{1}{2}} \ \tilde{A}^+_{j-\frac{1}{2}} \ \underline{g}(x_j, \underline{w}_j^n) \tag{5.9b}$$

where $\tilde{A}^{\pm}_{j-\frac{1}{2}}$ are given by equation (3.15) and have eigenvalues ${}_i\tilde{\lambda}^{\pm}_{j-\frac{1}{2}}$, $i = 1, \ldots m$. The scheme given by equations (5.9a-b) can be written pointwise as

$$\underline{w}_j^{n+1} = \underline{w}_j^n - \frac{\Delta t}{\Delta x}\tilde{A}^+_{j-\frac{1}{2}}(\underline{w}_j^n - \underline{w}_{j-1}^n) - \frac{\Delta t}{\Delta x} \tilde{A}^-_{j+\frac{1}{2}}(\underline{w}_{j+1}^n - \underline{w}_j^n) + \Delta t \tilde{A}^{-1}_{j-\frac{1}{2}} \tilde{A}^+_{j-\frac{1}{2}} \underline{g}(x_j, \underline{w}_j^n) + \Delta t \tilde{A}^{-1}_{j+\frac{1}{2}} \tilde{A}^-_{j+\frac{1}{2}} \underline{g}(x_j, \underline{w}_j^n) \tag{5.10}$$

which is an extension of the algorithm given by equation (4.13).

To implement this scheme in an upwind manner by looking at each of the m waves with wavespeeds ${}_i\tilde{\lambda}_{j-\frac{1}{2}}$ $i = 1,\ldots m$ we proceed as follows.

5.2 Flux-Difference Splitting

Following the approach of §3.2 we split the Jacobian $A = \frac{\partial \underline{f}}{\partial \underline{w}}$ into

$$A = A^{+} + A^{-} \tag{5.11}$$

and $\underline{g}$ similarly as

$$\underline{g} = \underline{g}^{+} + \underline{g}^{-} \tag{5.12}$$

where

$$\underline{g}^{\pm} = A^{-1}A^{\pm}\,\underline{g} \,. \tag{5.13}$$

The matrices $A^{\pm}$ are defined by equations (3.18)-(3.20). Expanding $\underline{w}(x_j,t_n + \Delta t)$ about (x_j,t_n) as a Taylor series and using equations (5.1) and (5.11)-(4.13) we obtain

$$\begin{aligned}\underline{w}(x_j,t_n + \Delta t) &\simeq \underline{w}(x_j,t_n) + \Delta t\underline{w}_t(x_j,t_n)\\ &= \underline{w}(x_j,t_n) + \Delta t(\underline{g}(x_j,t_n) - A\underline{w}_x(x_j,t_n))\\ &= \underline{w}(x_j,t_n) + \Delta t(\underline{g}^{+}(x_j,t_n) - A^{+}\underline{w}_x(x_j,t_n))\\ &\qquad + \Delta t(\underline{g}^{-}(x_j,t_n) - A^{-}\underline{w}_x(x_j,t_n)) \,.\end{aligned} \tag{5.14}$$

Thus, defining $\tilde{A}_{j-\frac{1}{2}}, \underline{g}_{j-\frac{1}{2}}$ as approximations to $A, \underline{g}$ at $x_{j-\frac{1}{2}} = \frac{1}{2}(x_j + x_{j-1})$ and time level n, and splitting $\tilde{A}_{j-\frac{1}{2}}, \underline{g}_{j-\frac{1}{2}}$ into

$$\tilde{A}_{j-\frac{1}{2}} = \tilde{A}^{+}_{j-\frac{1}{2}} + \tilde{A}^{-}_{j-\frac{1}{2}} \tag{5.15}$$

$$\underline{g}_{j-\frac{1}{2}} = \underline{g}^{+}_{j-\frac{1}{2}} + \underline{g}^{-}_{j-\frac{1}{2}} \tag{5.16}$$

where

$$\underline{g}^{\pm}_{j-\frac{1}{2}} = \tilde{A}^{-1}_{j-\frac{1}{2}} \tilde{A}^{\pm}_{j-\frac{1}{2}} \underline{g}_{j-\frac{1}{2}} \tag{5.17}$$

we get the following first order upwind scheme for equation (5.1) from equation (5.14):

$$\begin{aligned} \underline{w}_j^{n+1} = \underline{w}_j^n &+ \Delta t\left[\underline{g}^{+}_{j-\frac{1}{2}} - \tilde{A}^{+}_{j-\frac{1}{2}}\frac{(\underline{w}_j^n - \underline{w}_{j-1}^n)}{\Delta x}\right] \\ &+ \Delta t\left[\underline{g}^{-}_{j+\frac{1}{2}} - \tilde{A}^{-}_{j+\frac{1}{2}}\frac{(\underline{w}_{j+1}^n - \underline{w}_j^n)}{\Delta x}\right] . \end{aligned} \tag{5.18}$$

(N.B. $\tilde{A}^{\pm}_{j-\frac{1}{2}}$ are as in §5.1 and are associated with right(+) and left(-) travelling waves.)

Comparing equations (5.18), (5.10) and (5.17) we see that we can take the approximations $\underline{g}_{j\pm\frac{1}{2}}$ used for updating $\underline{w}_j^n$ to be $\underline{g}_{j\pm\frac{1}{2}} = \underline{g}(x_j, \underline{w}_j^n)$.

To implement the algorithm given by equation (5.10) written cell-wise in equations (5.9a-b) we project as before:

$$\underline{w}_j^n - \underline{w}_{j-1}^n = \sum_{i=1}^{m} {}_i\tilde{\alpha}_{j-\frac{1}{2}} \; {}_i\tilde{\underline{e}}_{j-\frac{1}{2}} \tag{5.19}$$

$$\underline{g}(x_{j-1}, \underline{w}_{j-1}^n) = -\frac{1}{\Delta x} \sum_{i=1}^{m} {}_i\tilde{\lambda}_{j-\frac{1}{2}} \; {}_i\tilde{\beta}_{j-\frac{1}{2}} \; {}_i\tilde{\underline{e}}_{j-\frac{1}{2}} \tag{5.20}$$

and

$$\underline{g}(x_j, \underline{w}_j^n) = -\frac{1}{\Delta x} \sum_{i=1}^{m} {}_i\tilde{\lambda}_{j-\frac{1}{2}} \; {}_i\tilde{\gamma}_{j-\frac{1}{2}} \; {}_i\tilde{\underline{e}}_{j-\frac{1}{2}} \tag{5.21}$$

to give expressions for ${}_i\tilde{\alpha}_{j-\frac{1}{2}}, {}_i\tilde{\beta}_{j-\frac{1}{2}}$ and ${}_i\tilde{\gamma}_{j-\frac{1}{2}}$; i.e. project $\underline{g}^+_{j-\frac{1}{2}}, \underline{g}^-_{j+\frac{1}{2}}$ in equation (5.18) onto ${}_i\tilde{\underline{e}}_{j-\frac{1}{2}}, {}_i\tilde{\underline{e}}_{j+\frac{1}{2}}$, respectively. Equations (5.9a-b) and (5.19)-(5.21) now give

$$\underline{w}_{j-1}^{n+1} = \underline{w}_{j-1}^n - \frac{\Delta t}{\Delta x} \sum_{i=1}^{m} {}_i\tilde{\lambda}^-_{j-\frac{1}{2}} \; {}_i\tilde{\delta}_{j-\frac{1}{2}} \; {}_i\tilde{\underline{e}}_{j-\frac{1}{2}} \tag{5.22a}$$

and

$$\underline{w}_{j}^{n+1} = \underline{w}_{j}^n - \frac{\Delta t}{\Delta x} \sum_{i=1}^{m} {}_i\tilde{\lambda}^+_{j-\frac{1}{2}} \; {}_i\tilde{\epsilon}_{j-\frac{1}{2}} \; {}_i\tilde{\underline{e}}_{j-\frac{1}{2}} \, , \tag{5.22b}$$

where

$${}_i\tilde{\delta}_{j-\frac{1}{2}} = {}_i\tilde{\alpha}_{j-\frac{1}{2}} + {}_i\tilde{\beta}_{j-\frac{1}{2}} \tag{5.23}$$

and

$$ {}_i\tilde{\varepsilon}_{j-\frac{1}{2}} = {}_i\tilde{\alpha}_{j-\frac{1}{2}} + {}_i\tilde{\gamma}_{j-\frac{1}{2}} \; . \qquad (5.24)$$

(N.B. $\tilde{A}_{j-\frac{1}{2}}, \tilde{A}^{\pm}_{j-\frac{1}{2}}$ have eigenvalues ${}_i\tilde{\lambda}_{j-\frac{1}{2}}, {}_i\tilde{\lambda}^{\pm}_{j-\frac{1}{2}}$, respectively.) The schematic representation of the scheme given by equations (5.22a)-(5.24) can be seen in Figure 7.

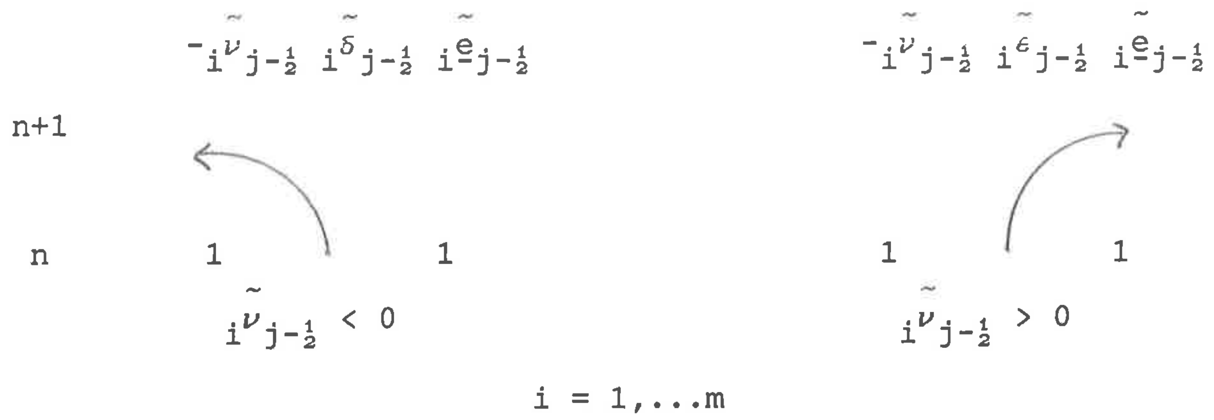


Figure 7

Written pointwise, equations (5.22a-b) become

$$ \underline{w}_j^{n+1} = \underline{w}_j^n - \sum_{i=1}^{m} {}_i\tilde{\nu}^{+}_{j-\frac{1}{2}}\; {}_i\tilde{\varepsilon}_{j-\frac{1}{2}}\; {}_i\tilde{\underline{e}}_{j-\frac{1}{2}} - \sum_{i=1}^{m} {}_i\tilde{\nu}^{-}_{j+\frac{1}{2}}\; {}_i\tilde{\delta}_{j+\frac{1}{2}}\; {}_i\tilde{\underline{e}}_{j+\frac{1}{2}} \qquad (5.25)$$

where ${}_i\tilde{\nu}^{\pm}_{j-\frac{1}{2}} = \frac{\Delta t}{\Delta x}\,{}_i\tilde{\lambda}^{\pm}_{j-\frac{1}{2}}$. Equation (5.25) could have been derived from equation (5.10) by projecting $\underline{w}^n_j - \underline{w}^n_{j-1}$, $\underline{w}^n_{j+1} - \underline{w}^n_j$ onto the local eigenvectors ${}_i\tilde{\underline{e}}_{j-\frac{1}{2}}$, ${}_i\tilde{\underline{e}}_{j+\frac{1}{2}}$, respectively, as given by equation (5.19), and projecting the term $\underline{g}(x_j,\underline{w}^n_j)$ occurring as $\tilde{A}^{-1}_{j-\frac{1}{2}}\,\tilde{A}^{+}_{j-\frac{1}{2}}\,\underline{g}(x_j,\underline{w}^n_j)$, $\tilde{A}^{-1}_{j+\frac{1}{2}}\,\tilde{A}^{-}_{j+\frac{1}{2}}\,\underline{g}(x_j,\underline{w}^n_j)$ onto the local eignvectors ${}_i\tilde{\underline{e}}_{j-\frac{1}{2}}$, ${}_i\tilde{\underline{e}}_{j+\frac{1}{2}}$, respectively, as given by equations (5.20)-(5.21).

As in §2, we can use equations (3.8) and (5.19) to represent ${}_i\tilde{\alpha}_{j-\frac{1}{2}}$ as

$${}_i\tilde{\alpha}_{j-\frac{1}{2}} = {}_iv^n_j - {}_iv^n_{j-1} = \left\{\tilde{X}^{-1}_{j-\frac{1}{2}}(\underline{w}^n_j - \underline{w}^n_{j-1})\right\}_i \tag{5.26}$$

in addition, we can use equations (5.4) (5.20) and (5.21) to represent the 'additional wavestrengths' ${}_i\tilde{\beta}_{j-\frac{1}{2}}$, ${}_i\tilde{\gamma}_{j-\frac{1}{2}}$ as

$${}_i\tilde{\beta}_{j-\frac{1}{2}} = -\left[\frac{\tilde{\Lambda}^{-1}_{j-\frac{1}{2}}\tilde{X}^{-1}_{j-\frac{1}{2}}\underline{g}(x_{j-1},\underline{w}^n_{j-1})}{\Delta x}\right]_i = -\frac{\Delta x}{{}_i\tilde{\lambda}_{j-\frac{1}{2}}}\,h_i(x_{j-1},\underline{v}^n_{j-1}) \tag{5.27}$$

and

$${}_i\tilde{\gamma}_{j-\frac{1}{2}} = -\left[\frac{\tilde{\Lambda}^{-1}_{j-\frac{1}{2}}\tilde{X}^{-1}_{j-\frac{1}{2}}\underline{g}(x_j,\underline{w}^n_j)}{\Delta x}\right]_i = -\frac{\Delta x}{{}_i\tilde{\lambda}_{j-\frac{1}{2}}}\,h_i(x_j,\underline{v}^n_j)\,. \tag{5.28}$$

Finally, we note that the scheme given by equation (5.25) is the scalar scheme given by equation (4.13) when applied to each ofthe m-waves.

5.3 Second order scheme

We now derive a second order scheme for the solution of equation (5.1) using the scalar scheme of §4.2.

Suppose we split the Jacobian matrix $A = \frac{\partial \underline{f}}{\partial \underline{w}}$, and the source term $\underline{g}$ as in equations (5.11)-(5.13) and expand $\underline{w}(x_j, t_n + \Delta t)$ about (x_j, t_n) as the truncated Taylor series

$$\underline{w}(x_j, t_n + \Delta t) \simeq \underline{w}(x_j, t_n) + \Delta t \underline{w}_t(w_j, t_n) + \frac{\Delta t^2}{2} \underline{w}_{tt}(x_j, t_n) . \tag{5.29}$$

Using equation (5.1) we have

$$\underline{w}_t = \underline{g} - \underline{f}_x = \underline{g} - A\underline{w}_x \tag{5.30}$$

and

$$\begin{aligned} \underline{w}_{tt} &= \underline{g}_t - \underline{f}_{xt} = \underline{g}_{\underline{w}} \underline{w}_t - (\underline{f}_t)_x \\ &= \underline{g}_{\underline{w}} \underline{w}_t - (A\underline{w}_t)_x = \underline{g}_{\underline{w}}(\underline{g} - A\underline{w}_x) - (A(\underline{g} - A\underline{w}_x))_x \end{aligned} \tag{5.31}$$

so that equation (5.29) becomes

$$\begin{aligned} \underline{w}(x_j, t_n + \Delta t) \simeq{} & \underline{w} + \Delta t(\underline{g} - A\underline{w}_x) \\ & + \frac{\Delta t^2}{2} \underline{g}_{\underline{w}}(\underline{g} - A\underline{w}_x) - \frac{\Delta t^2}{2}(A(\underline{g} - A\underline{w}_x))_x , \end{aligned} \tag{5.32}$$

where the terms on the right hand side are evaluated at (x_j, t_n).

We approximate

$$(\underline{g} - A\underline{w}_x)\Big|_{(x_j,t_n)} \simeq \tfrac{1}{2}\Big[\underline{g}_{j-\frac{1}{2}} - \tilde{A}_{j-\frac{1}{2}}\frac{(\underline{w}_j^n - \underline{w}_{j-1}^n)}{\Delta x} + \underline{g}_{j+\frac{1}{2}} - \tilde{A}_{j+\frac{1}{2}}\frac{(\underline{w}_{j+1}^n - \underline{w}_j^n)}{\Delta x}\Big] \tag{5.33a}$$

$$\underline{g}_{\underline{w}}(\underline{g} - A\underline{w}_x)\Big|_{(x_j,t_n)} \simeq \tfrac{1}{2}\Big[(\underline{g}_{\underline{w}})_{j-\frac{1}{2}}\Big[\underline{g}_{j-\frac{1}{2}} - \tilde{A}_{j-\frac{1}{2}}\frac{(\underline{w}_j^n - \underline{w}_{j-1}^n)}{\Delta x}\Big] + (\underline{g}_{\underline{w}})_{j+\frac{1}{2}}\Big[\underline{g}_{j+\frac{1}{2}} - \tilde{A}_{j+\frac{1}{2}}\frac{(\underline{w}_{j+1}^n - \underline{w}_j^n))}{\Delta x}\Big]\Big] \tag{5.33b}$$

and

$$(A(\underline{g} - A\underline{w}_x))_x \simeq \frac{1}{\Delta x}\Big[\tilde{A}_{j+\frac{1}{2}}\Big[\underline{g}_{j+\frac{1}{2}} - \tilde{A}_{j+\frac{1}{2}}\frac{(\underline{w}_{j+1}^n - \underline{w}_j^n)}{\Delta x}\Big] - \tilde{A}_{j-\frac{1}{2}}\Big[\underline{g}_{j-\frac{1}{2}} - \tilde{A}_{j-\frac{1}{2}}\frac{(\underline{w}_j^n - \underline{w}_{j-1}^n)}{\Delta x}\Big]\Big] \tag{5.34}$$

where the approximations $\underline{g}_{j-\frac{1}{2}}, (\underline{g}_{\underline{w}})_{j-\frac{1}{2}}$ to $\underline{g}, \underline{g}_{\underline{w}}$ at (x, t_n), $x \in [x_{j-1}, x_j]$, will be determined by the scalar algorithm of §4.2. Substituting the expressions given by equations (5.33a-b)-(5.34) into equation (5.32) and using equations (5.15)-(5.17) gives the following scheme centred on x_j

$$
\begin{aligned}
\underline{w}_j^{n+1} = \underline{w}_j^n &+ \frac{\Delta t}{2}\Big[\underline{g}^+_{j-\frac{1}{2}} - \tilde{A}^+_{j-\frac{1}{2}}\frac{(\underline{w}_j^n - \underline{w}_{j-1}^n)}{\Delta x} \\
&\qquad + \underline{g}^-_{j-\frac{1}{2}} - \tilde{A}^-_{j-\frac{1}{2}}(\underline{w}_j^n - \underline{w}_{j-1}^n)\Big] \\
&+ \frac{\Delta t}{2}\Big[\underline{g}^+_{j+\frac{1}{2}} - \tilde{A}^+_{j+\frac{1}{2}}\frac{(\underline{w}_{j+1}^n - \underline{w}_j^n)}{\Delta x} \\
&\qquad + \underline{g}^-_{j+\frac{1}{2}} - \tilde{A}^-_{j+\frac{1}{2}}\frac{(\underline{w}_{j+1}^n - \underline{w}_j^n)}{\Delta x}\Big] \\
&+ \frac{\Delta t^2}{2}\Big[(\underline{g}_{\underline{w}})_{j-\frac{1}{2}}\Big\{\underline{g}^+_{j-\frac{1}{2}} - \tilde{A}^+_{j-\frac{1}{2}}\frac{(\underline{w}_j^n - \underline{w}_{j-1}^n)}{\Delta x} \\
&\qquad + \underline{g}^-_{j-\frac{1}{2}} - \tilde{A}^-_{j-\frac{1}{2}}\frac{(\underline{w}_j^n - \underline{w}_{j-1}^n)}{\Delta x}\Big\}\Big] \\
&+ \frac{\Delta t^2}{2}\Big[(\underline{g}_{\underline{w}})_{j+\frac{1}{2}}\Big\{\underline{g}^+_{j+\frac{1}{2}} - \tilde{A}^+_{j+\frac{1}{2}}\frac{(\underline{w}_{j+1}^n - \underline{w}_j^n)}{\Delta x} \\
&\qquad + \underline{g}^-_{j+\frac{1}{2}} - \tilde{A}^-_{j+\frac{1}{2}}\frac{(\underline{w}_{j+1}^n - \underline{w}_j^n)}{\Delta x}\Big\}\Big] \\
&- \frac{\Delta t^2}{2\Delta x}\Big[\tilde{A}_{j+\frac{1}{2}}\Big\{\underline{g}^+_{j+\frac{1}{2}} - \tilde{A}^+_{j+\frac{1}{2}}\frac{(\underline{w}_{j+1}^n - \underline{w}_j^n)}{\Delta x} \\
&\qquad + \underline{g}^-_{j+\frac{1}{2}} - \tilde{A}^-_{j+\frac{1}{2}}\frac{(\underline{w}_{j+1}^n - \underline{w}_j^n)}{\Delta x}\Big\}
\end{aligned}
$$

$$- \tilde{A}_{j-\frac{1}{2}}\left[\underline{g}^{+}_{j-\frac{1}{2}} - \tilde{A}^{+}_{j-\frac{1}{2}}\frac{(\underline{w}^{n}_{j} - \underline{w}^{n}_{j-1})}{\Delta x}\right.$$

$$\left. + \underline{g}^{-}_{j-\frac{1}{2}} - \tilde{A}^{-}_{j-\frac{1}{2}}\frac{(\underline{w}^{n}_{j} - \underline{w}^{n}_{j-1})}{\Delta x}\right]\Bigg] . \tag{5.35}$$

To express this scheme as an increment stage and a transfer stage as in §3.3 we rearrange equation (5.35) as

$$\begin{aligned}
\underline{w}^{n+1}_{j} = \underline{w}^{n}_{j} &+ \left[I + \frac{\Delta t}{2}(\underline{g}_{\underline{w}})_{j-\frac{1}{2}}\right]\left[\Delta t\underline{g}^{+}_{j-\frac{1}{2}} - \frac{\Delta t}{\Delta x}\tilde{A}^{+}_{j-\frac{1}{2}}(\underline{w}^{n}_{j} - \underline{w}^{n}_{j-1})\right] \\
&+ \left[I + \frac{\Delta t}{2}(\underline{g}_{\underline{w}})_{j+\frac{1}{2}}\right]\left[\Delta t\underline{g}^{-}_{j+\frac{1}{2}} - \frac{\Delta t}{\Delta x}\tilde{A}^{-}_{j+\frac{1}{2}}(\underline{w}^{n}_{j+1} - \underline{w}^{n}_{j})\right] \\
&- \tfrac{1}{2}\left[I - \frac{\Delta t}{\Delta x}\tilde{A}_{j-\frac{1}{2}} + \frac{\Delta t}{2}(\underline{g}_{\underline{w}})_{j-\frac{1}{2}}\right]\left[\Delta t\underline{g}^{+}_{j-\frac{1}{2}} - \frac{\Delta t}{\Delta x}\tilde{A}^{+}_{j-\frac{1}{2}}(\underline{w}^{n}_{j} - \underline{w}^{n}_{j-1})\right] \\
&- \tfrac{1}{2}\left[I + \frac{\Delta t}{\Delta x}\tilde{A}_{j+\frac{1}{2}} + \frac{\Delta t}{2}(\underline{g}_{\underline{w}})_{j+\frac{1}{2}}\right]\left[\Delta t\underline{g}^{-}_{j+\frac{1}{2}} - \frac{\Delta t}{\Delta x}\tilde{A}^{-}_{j+\frac{1}{2}}(\underline{w}^{n}_{j+1} - \underline{w}^{n}_{j})\right] \\
&+ \tfrac{1}{2}\left[I + \frac{\Delta t}{\Delta x}\tilde{A}_{j-\frac{1}{2}} + \frac{\Delta t}{2}(\underline{g}_{\underline{w}})_{j-\frac{1}{2}}\right]\left[\Delta t\underline{g}^{-}_{j-\frac{1}{2}} - \frac{\Delta t}{\Delta x}\tilde{A}^{-}_{j-\frac{1}{2}}(\underline{w}^{n}_{j} - \underline{w}^{n}_{j-1})\right] \\
&+ \tfrac{1}{2}\left[I - \frac{\Delta t}{\Delta x}\tilde{A}_{j+\frac{1}{2}} + \frac{\Delta t}{2}(\underline{g}_{\underline{w}})_{j+\frac{1}{2}}\right]\left[\Delta t\underline{g}^{+}_{j+\frac{1}{2}} - \frac{\Delta t}{\Delta x}\tilde{A}^{+}_{j+\frac{1}{2}}(\underline{w}^{n}_{j} - \underline{w}^{n}_{j-1})\right] .
\end{aligned} \tag{5.36}$$

We project

$$\underline{w}^{n}_{j} - \underline{w}^{n}_{j-1} = \sum_{i=1}^{m} {}_{i}\tilde{\alpha}_{j-\frac{1}{2}}\, {}_{i}\tilde{\underline{e}}_{j-\frac{1}{2}} \tag{5.37}$$

$$g_{j+\frac{1}{2}} = -\frac{1}{\Delta x}\sum_{i=1}^{m} {}_i\tilde{\lambda}_{j+\frac{1}{2}}\ {}_i\tilde{\beta}_{j+\frac{1}{2}}\ {}_i\tilde{e}_{j+\frac{1}{2}} \tag{5.38}$$

$$g_{j-\frac{1}{2}} = -\frac{1}{\Delta x}\sum_{i=1}^{m} {}_i\tilde{\lambda}_{j-\frac{1}{2}}\ {}_i\tilde{\gamma}_{j-\frac{1}{2}}\ {}_i\tilde{\underline{e}}_{j-\frac{1}{2}} \tag{5.39}$$

so that we can apply the scalar algorithm of §4.2 to each of the m-waves. Also,

$$\frac{\Delta t}{2}(\underline{g}_{\underline{w}})_{j-\frac{1}{2}}\ {}_i\tilde{\underline{e}}_{j-\frac{1}{2}} = \sum_{k=1}^{m} {}_{ki}\tilde{\omega}_{j-\frac{1}{2}}\ {}_k\tilde{\underline{e}}_{j-\frac{1}{2}} \tag{5.40}$$

and

$$\frac{\Delta t}{2}(\underline{g}_{\underline{w}})_{j+\frac{1}{2}}\ {}_i\tilde{\underline{e}}_{j+\frac{1}{2}} = \sum_{k=1}^{m} {}_{ki}\tilde{\theta}_{j+\frac{1}{2}}\ {}_k\tilde{\underline{e}}_{j+\frac{1}{2}} \tag{5.41}$$

Thus comparing equations (5.36)-(5.41) with the scheme of §4.2, we see that ${}_i\tilde{\beta}_{j+\frac{1}{2}}$ and ${}_i\tilde{\gamma}_{j-\frac{1}{2}}$ should be evaluated at $x_j + \frac{\Delta x}{2}, \underline{w}_j^n$ and $x_j - \frac{\Delta x}{2}, \underline{w}_j^n$. Similarly, ${}_{ki}\tilde{\omega}_{j-\frac{1}{2}}$, ${}_{ki}\tilde{\theta}_{j+\frac{1}{2}}$ should be evaluated at $x_j, \underline{w}_j^n$. This means that $\underline{g}_{j-\frac{1}{2}} = \underline{g}(x_j - \frac{\Delta x}{2}, \underline{w}_j^n)$, $\underline{g}_{j+\frac{1}{2}} = \underline{g}(x_j + \frac{\Delta x}{2}, \underline{w}_j^n)$, $(\underline{g}_{\underline{w}})_{j\pm\frac{1}{2}} = \underline{g}_{\underline{w}}(w_j, \underline{w}_j^n)$. Thus we project

$$\underline{g}(x_j + \frac{\Delta x}{2}, \underline{w}_j^n) = -\frac{1}{\Delta x}\sum_{i=1}^{m} {}_i\tilde{\lambda}_{j+\frac{1}{2}}\ {}_i\tilde{\beta}_{j+\frac{1}{2}}\ {}_i\tilde{\underline{e}}_{j+\frac{1}{2}} \tag{5.42}$$

so that

$$ {}_i\tilde{\beta}_{j+\frac{1}{2}} = -\frac{\left[\tilde{\Lambda}^{-1}_{j+\frac{1}{2}}\tilde{X}^{-1}_{j+\frac{1}{2}}\underline{g}(x_j + \frac{\Delta x}{2},\underline{w}^n_j)\right]_i}{\Delta x} = -\frac{\Delta x}{{}_i\tilde{\lambda}_{j+\frac{1}{2}}} h_i(x_j + \frac{\Delta x}{2},\underline{v}^n_j) \tag{5.43} $$

and

$$ \underline{g}(x_j - \frac{\Delta x}{2},\underline{w}^n_j) = -\frac{1}{\Delta x}\sum_{i=1}^{m} {}_i\tilde{\lambda}_{j-\frac{1}{2}}\ {}_i\tilde{\gamma}_{j-\frac{1}{2}}\ {}_i\tilde{\underline{e}}_{j-\frac{1}{2}} \ . \tag{5.44} $$

Thus

$$ {}_i\tilde{\gamma}_{j-\frac{1}{2}} = -\frac{\left[\tilde{\Lambda}^{-1}_{j-\frac{1}{2}}\tilde{X}^{-1}_{j-\frac{1}{2}}\underline{g}(x_j - \frac{\Delta x}{2},\underline{w}^n_j)\right]_i}{\Delta x} = -\frac{\Delta x}{{}_i\tilde{\lambda}_{j-\frac{1}{2}}} h_i(x_j - \frac{\Delta x}{2},\underline{v}^n_j) \ . \tag{5.45} $$

Moreover,

$$ \frac{\Delta t}{2}(\underline{g}_{\underline{w}})_{j-\frac{1}{2}}\tilde{X}_{j-\frac{1}{2}} = \tilde{X}_{j-\frac{1}{2}}\tilde{\Omega}_{j-\frac{1}{2}} \tag{5.46} $$

$$ \frac{\Delta t}{2}(\underline{g}_{\underline{w}})_{j+\frac{1}{2}}\tilde{X}_{j+\frac{1}{2}} = \tilde{X}_{j+\frac{1}{2}}\tilde{\theta}_{j+\frac{1}{2}} \tag{5.47} $$

where $\tilde{\Omega}_{j-\frac{1}{2}} = \{{}_{ki}\tilde{\omega}_{j-\frac{1}{2}}\}$, $\tilde{\theta}_{j+\frac{1}{2}} = \{{}_{ki}\tilde{\theta}_{j+\frac{1}{2}}\}$, and hence

$$ \begin{aligned} \tilde{\Omega}_{j-\frac{1}{2}} &= \frac{\Delta t}{2}\tilde{X}^{-1}_{j-\frac{1}{2}}(\underline{g}_{\underline{w}})_{j-\frac{1}{2}}\tilde{X}_{j-\frac{1}{2}} \\ &= \frac{\Delta t}{2}\tilde{X}^{-1}_{j-\frac{1}{2}}\ \underline{g}_{\underline{w}}(x_j,\underline{w}^n_j)\tilde{X}_{j-\frac{1}{2}} \\ &= \frac{\Delta t}{2}\ \underline{h}_{\underline{v}}(x_j,\underline{v}^n_j) \end{aligned} \tag{5.48} $$

$$\tilde{\theta}_{j+\frac{1}{2}} = \frac{\Delta t}{2}\tilde{X}^{-1}_{j+\frac{1}{2}}(g_{\underline{w}})_{j+\frac{1}{2}}\tilde{X}_{j+\frac{1}{2}}$$
$$= \frac{\Delta t}{2}\tilde{X}^{-1}_{j+\frac{1}{2}}\, g_{\underline{w}}(x_j,\underline{w}^n_j)\tilde{X}_{j+\frac{1}{2}}$$
$$= \frac{\Delta t}{2}\, \underline{h}_{\underline{v}}(x_j,\underline{v}^n_j) \ . \tag{5.49}$$

We note that the expressions given for ${}_i\tilde{\beta}_{j+\frac{1}{2}}$, ${}_i\tilde{\gamma}_{j-\frac{1}{2}}$, ${}_{ki}\tilde{\omega}_{j-\frac{1}{2}}$, ${}_{ki}\tilde{\beta}_{j+\frac{1}{2}}$ by equations (5.42)-(5.49) are consistent with the algorithm of §4.2. Using equations (5.16)-(5.17) and (5.37)-(5.39) we can write

$$\Delta t\ \underline{g}^{\pm}_{j-\frac{1}{2}} - \frac{\Delta t}{\Delta x}\tilde{A}^{\pm}_{j-\frac{1}{2}}(\underline{w}^n_j - \underline{w}^n_{j-1}) = -\frac{\Delta t}{\Delta x}\sum_{i=1}^{m} {}_i\tilde{\lambda}^{\pm}_{j-\frac{1}{2}}\ {}_i\tilde{\epsilon}_{j-\frac{1}{2}}\ {}_i\tilde{\underline{e}}_{j-\frac{1}{2}} \tag{5.50}$$

$$\Delta t\ \underline{g}^{\pm}_{j+\frac{1}{2}} - \frac{\Delta t}{\Delta x}\tilde{A}^{\pm}_{j+\frac{1}{2}}(\underline{w}^n_{j+1} - \underline{w}^n_{j}) = -\frac{\Delta t}{\Delta x}\sum_{i=1}^{m} {}_i\tilde{\lambda}^{\pm}_{j+\frac{1}{2}}\ {}_i\tilde{\delta}_{j+\frac{1}{2}}\ {}_i\tilde{\underline{e}}_{j+\frac{1}{2}} \tag{5.51}$$

where

$${}_i\tilde{\delta}_{j+\frac{1}{2}} = {}_i\tilde{\alpha}_{j+\frac{1}{2}} + {}_i\tilde{\beta}_{j+\frac{1}{2}} \tag{5.52}$$

and

$${}_i\tilde{\epsilon}_{j-\frac{1}{2}} = {}_i\tilde{\alpha}_{j-\frac{1}{2}} + {}_i\tilde{\gamma}_{j-\frac{1}{2}} \ . \tag{5.53}$$

Finally, we need to consider expressions like

$D = \left[I + \frac{\Delta t \tilde{A}}{\Delta x}_{j+\frac{1}{2}} + \frac{\Delta t}{2}(g_{\underline{w}})_{j+\frac{1}{2}}\right]\left[\Delta t \bar{g}_{j+\frac{1}{2}} - \frac{\Delta t}{\Delta x}(\underline{w}^n_{j+1} - \underline{w}^n_j)\right]$. Now, using equation (5.51) we can write

$$D = -\frac{\Delta t}{\Delta x}\left[I + \frac{\Delta t}{\Delta x}\tilde{A}_{j+\frac{1}{2}} + \frac{\Delta t}{2}(g_{\underline{w}})_{j+\frac{1}{2}}\right]\left[\sum_{i=1}^{m} {}_i\tilde{\bar{\lambda}}_{j+\frac{1}{2}}\ {}_i\tilde{\delta}_{j+\frac{1}{2}}\ {}_i\tilde{\underline{e}}_{j+\frac{1}{2}}\right]$$

$$= -\frac{\Delta t}{\Delta x}\sum_{i=1}^{m}\left[I + \frac{\Delta t}{\Delta x}\tilde{A}_{j+\frac{1}{2}} + \frac{\Delta t}{2}(g_{\underline{w}})_{j+\frac{1}{2}}\right]{}_i\tilde{\bar{\lambda}}_{j+\frac{1}{2}}\ {}_i\tilde{\delta}_{j+\frac{1}{2}}\ {}_i\tilde{\underline{e}}_{j+\frac{1}{2}}\ .$$

(5.54)

Also, since $\tilde{A}_{j+\frac{1}{2}}$ has eigenvalues ${}_i\tilde{\lambda}_{j+\frac{1}{2}}$ with eigenvectors ${}_i\tilde{\underline{e}}_{j+\frac{1}{2}}$ we can use equation (5.41) to rewrite equation (5.54) as

$$D = -\frac{\Delta t}{\Delta x}\sum_{i=1}^{m} {}_i\tilde{\bar{\lambda}}_{j+\frac{1}{2}}\ {}_i\tilde{\delta}_{j+\frac{1}{2}}\left[{}_i\tilde{\underline{e}}_{j+\frac{1}{2}} + \frac{\Delta t}{\Delta x}\ {}_i\tilde{\lambda}_{j+\frac{1}{2}}\ {}_i\tilde{\underline{e}}_{j+\frac{1}{2}} + \sum_{k=1}^{m} {}_{ki}\tilde{\theta}_{j+\frac{1}{2}}\ {}_k\tilde{\underline{e}}_{j+\frac{1}{2}}\right]$$

$$= -\sum_{i=1}^{m} {}_i\tilde{\bar{\nu}}_{j+\frac{1}{2}}\ {}_i\tilde{\delta}_{j+\frac{1}{2}}\left[(1 + {}_i\tilde{\nu}_{j+\frac{1}{2}})\,{}_i\tilde{\underline{e}}_{j+\frac{1}{2}} + {}_i\tilde{\underline{\xi}}_{j+\frac{1}{2}}\right]$$

(5.55)

where

$${}_i\tilde{\underline{\xi}}_{j+\frac{1}{2}} = \sum_{k=1}^{m} {}_{ki}\tilde{\theta}_{j+\frac{1}{2}}\ {}_k\tilde{\underline{e}}_{j+\frac{1}{2}}$$

(5.56)

and

$${}_i\tilde{\nu}_{j+\frac{1}{2}} = \frac{\Delta t}{\Delta x}\ {}_i\tilde{\lambda}_{j+\frac{1}{2}},\quad {}_i\tilde{\nu}^{\pm}_{j+\frac{1}{2}} = \frac{\Delta t}{\Delta x}\ {}_i\tilde{\lambda}^{\pm}_{j+\frac{1}{2}}\ .$$

(5.57)

Similarly, defining

$$
{}_i\tilde{\underline{\zeta}}_{j-\frac{1}{2}} = \sum_{k=1}^{m} {}_{ki}\tilde{\omega}_{j-\frac{1}{2}} \; {}_k\tilde{\underline{e}}_{j-\frac{1}{2}}
\tag{5.58}
$$

we can write equation (5.37) as

$$
\begin{aligned}
\underline{w}_j^{n+1} = \underline{w}_j^{n} &- \sum_{i=1}^{m} {}_i\tilde{\nu}^{+}_{j-\frac{1}{2}} \; {}_i\tilde{\epsilon}_{j-\frac{1}{2}} ({}_i\tilde{\underline{e}}_{j-\frac{1}{2}} + {}_i\tilde{\underline{\zeta}}_{j-\frac{1}{2}}) \\
&- \sum_{i=1}^{m} {}_i\tilde{\nu}^{-}_{j+\frac{1}{2}} \; {}_i\tilde{\delta}_{j+\frac{1}{2}} ({}_i\tilde{\underline{e}}_{j+\frac{1}{2}} + {}_i\tilde{\underline{\zeta}}_{j+\frac{1}{2}}) \\
&+ \sum_{i=1}^{m} \tfrac{1}{2} {}_i\tilde{\nu}^{+}_{j-\frac{1}{2}} \; {}_i\tilde{\epsilon}_{j-\frac{1}{2}} \left[(1 - |{}_i\tilde{\nu}_{j-\frac{1}{2}}|) {}_i\tilde{\underline{e}}_{j-\frac{1}{2}} + {}_i\tilde{\underline{\zeta}}_{j-\frac{1}{2}} \right] \\
&+ \sum_{i=1}^{m} \tfrac{1}{2} {}_i\tilde{\nu}^{-}_{j+\frac{1}{2}} \; {}_i\tilde{\delta}_{j+\frac{1}{2}} \left[(1 - |{}_i\tilde{\nu}_{j+\frac{1}{2}}|) {}_i\tilde{\underline{e}}_{j+\frac{1}{2}} + {}_i\tilde{\underline{\zeta}}_{j+\frac{1}{2}} \right] \\
&- \sum_{i=1}^{m} \tfrac{1}{2} {}_i\tilde{\nu}^{+}_{j-\frac{1}{2}} \; {}_i\tilde{\epsilon}_{j-\frac{1}{2}} \left[(1 - |{}_i\tilde{\nu}_{j-\frac{1}{2}}|) {}_i\tilde{\underline{e}}_{j-\frac{1}{2}} + {}_i\tilde{\underline{\zeta}}_{j-\frac{1}{2}} \right] \\
&- \sum_{i=1}^{m} \tfrac{1}{2} {}_i\tilde{\nu}^{+}_{j+\frac{1}{2}} \; {}_i\tilde{\delta}_{j+\frac{1}{2}} \left[(1 - |{}_i\tilde{\nu}_{j+\frac{1}{2}}|) {}_i\tilde{\underline{e}}_{j+\frac{1}{2}} + {}_i\tilde{\underline{\zeta}}_{j+\frac{1}{2}} \right] .
\end{aligned}
\tag{5.59}
$$

Finally, we can summarise this scheme by writing it in the cell-wise fashion

$$\underline{w}_{j-1}^{n+1} = \underline{w}_{j-1}^{n} - \sum_{i=1}^{m} {}_i\tilde{\nu}^{-}_{j-\frac{1}{2}}\ {}_i\tilde{\delta}_{j-\frac{1}{2}}\left({}_i\tilde{\underline{e}}_{j-\frac{1}{2}} + {}_i\tilde{\underline{\xi}}_{j-\frac{1}{2}}\right)$$

$$+ \sum_{i=1}^{m} \tfrac{1}{2}\ {}_i\tilde{\nu}^{-}_{j-\frac{1}{2}}\ {}_i\tilde{\delta}_{j-\frac{1}{2}}\left[(1 - |{}_i\tilde{\nu}_{j-\frac{1}{2}}|) + {}_i\tilde{\underline{\xi}}_{j-\frac{1}{2}}\right]$$

$$- \sum_{i=1}^{m} \tfrac{1}{2}\ {}_i\tilde{\nu}^{+}_{j-\frac{1}{2}}\ {}_i\tilde{\delta}_{j-\frac{1}{2}}\left[(1 - |{}_i\tilde{\nu}_{j-\frac{1}{2}}|)\,{}_i\tilde{\underline{e}}_{j-\frac{1}{2}} + {}_i\tilde{\underline{\xi}}_{j-\frac{1}{2}}\right] \qquad (5.60a)$$

$$\underline{w}_{j}^{n+1} = \underline{w}_{j}^{n} - \sum_{i=1}^{m} {}_i\tilde{\nu}^{+}_{j-\frac{1}{2}}\ {}_i\tilde{\epsilon}_{j-\frac{1}{2}}\left({}_i\tilde{\underline{e}}_{j-\frac{1}{2}} + {}_i\tilde{\underline{\xi}}_{j-\frac{1}{2}}\right)$$

$$+ \sum_{i=1}^{m} \tfrac{1}{2}\ {}_i\tilde{\nu}^{+}_{j-\frac{1}{2}}\ {}_i\tilde{\epsilon}_{j-\frac{1}{2}}\left[(1 - |{}_i\tilde{\nu}_{j-\frac{1}{2}}|)\,{}_i\tilde{\underline{e}}_{j-\frac{1}{2}} + {}_i\tilde{\underline{\xi}}_{j-\frac{1}{2}}\right]$$

$$- \sum_{i=1}^{m} \tfrac{1}{2}\ {}_i\tilde{\nu}^{-}_{j-\frac{1}{2}}\ {}_i\tilde{\epsilon}_{j-\frac{1}{2}}\left[(1 - |{}_i\tilde{\nu}_{j-\frac{1}{2}}|)\,{}_i\tilde{\underline{e}}_{j-\frac{1}{2}} + {}_i\tilde{\underline{\xi}}_{j-\frac{1}{2}}\right] \qquad (5.60b)$$

where $_i\tilde{\underline{e}}_{j\pm\frac{1}{2}}$, $_i\tilde{\nu}_{j\pm\frac{1}{2}}$, $_i\tilde{\nu}^+_{j\pm\frac{1}{2}}$, $_i\tilde{\nu}^-_{j\pm\frac{1}{2}}$ are as before,

$$_i\tilde{\delta}_{j-\frac{1}{2}} = {_i\tilde{\alpha}_{j-\frac{1}{2}}} + {_i\tilde{\beta}_{j-\frac{1}{2}}} \tag{5.61}$$

$$_i\tilde{\varepsilon}_{j-\frac{1}{2}} = {_i\tilde{\alpha}_{j-\frac{1}{2}}} + {_i\tilde{\gamma}_{j-\frac{1}{2}}} \tag{5.62}$$

and $_i\tilde{\beta}_{j-\frac{1}{2}}$, $_i\tilde{\gamma}_{j-\frac{1}{2}}$ represent projections onto the local eigenvectors $_i\tilde{\underline{e}}_{j-\frac{1}{2}}$ of $\underline{g}(x_j - \frac{\Delta x}{2}, \underline{w}^n_{j-1})$, $\underline{g}(x_j - \frac{\Delta x}{2}, \underline{w}^n_j)$, respectively.

In addition, following equations (5.40)-(5.41), (5.48)-(5.49), (5.56) and (5.58), $_i\tilde{\xi}_{j-\frac{1}{2}}$, $_i\tilde{\zeta}_{j-\frac{1}{2}}$ represent $\frac{\Delta t}{2}\frac{\partial \underline{g}}{\partial \underline{w}}(x_{j-1}, \underline{w}^n_{j-1})$, $\frac{\Delta t}{2}\frac{\partial \underline{g}}{\partial \underline{w}}(x_j, \underline{w}^n_j)$ applied to $_i\tilde{\underline{e}}_{j-\frac{1}{2}}$. Finally, we can write equations (5.60a-b) in a similar form to equations (3.43a)-(3.43b), i.e. increment and transfer form, as

$$\left.\begin{aligned} \underline{w}^{n+1}_{j-1} &= \underline{w}^n_{j-1} + {_i\underline{c}_{j-\frac{1}{2}}} \\ \underline{w}^{n+1}_{j} &= \underline{w}^n_{j} + {_i\underline{\Phi}_{j-\frac{1}{2}}} - {_i\underline{b}_{j-\frac{1}{2}}} \end{aligned}\right\} {_i\tilde{\lambda}_{j-\frac{1}{2}}} > 0 \tag{5.63a}$$

$$\left.\begin{aligned} \underline{w}^{n+1}_{j-1} &= \underline{w}^n_{j-1} + {_i\underline{\Psi}_{j-\frac{1}{2}}} - {_i\underline{c}_{j-\frac{1}{2}}} \\ \underline{w}^{n+1}_{j} &= \underline{w}^n_{j} + {_i\underline{b}_{j-\frac{1}{2}}} \end{aligned}\right\} {_i\tilde{\lambda}_{j-\frac{1}{2}}} < 0 \tag{5.63b}$$

for each $i = 1,\ldots m$, where

$$ {}_i\underline{\Phi}_{j-\frac12} = -\,{}_i\tilde{\nu}_{j-\frac12}\;{}_i\tilde{\varepsilon}_{j-\frac12}\left({}_i\tilde{\underline{e}}_{j-\frac12} + {}_i\tilde{\underline{\xi}}_{j-\frac12}\right) \tag{5.64} $$

$$ {}_i\underline{\Psi}_{j-\frac12} = -\,{}_i\tilde{\nu}_{j-\frac12}\;{}_i\tilde{\delta}_{j-\frac12}\left({}_i\tilde{\underline{e}}_{j-\frac12} + {}_i\tilde{\underline{\xi}}_{j-\frac12}\right) \tag{5.65} $$

$$ {}_i\underline{b}_{j-\frac12} = -\,\tfrac12\,{}_i\tilde{\nu}_{j-\frac12}\;{}_i\tilde{\varepsilon}_{j-\frac12}\left[(1 - |{}_i\tilde{\nu}_{j-\frac12}|)\,{}_i\tilde{\underline{e}}_{j-\frac12} + {}_i\tilde{\underline{\xi}}_{j-\frac12}\right] \tag{5.66} $$

$$ {}_i\underline{c}_{j-\frac12} = -\,\tfrac12\,{}_i\tilde{\nu}_{j-\frac12}\;{}_i\tilde{\delta}_{j-\frac12}\left[(1 - |{}_i\tilde{\nu}_{j-\frac12}|)\,{}_i\tilde{\underline{e}}_{j-\frac12} + {}_i\tilde{\underline{\xi}}_{j-\frac12}\right] . \tag{5.67} $$

This scheme is represented schematically in Figure 8. As a special case, equations (5.63a)-(5.67) reduce to equations (3.43a)-(3.45) in the case $\underline{g} = \underline{0}$.

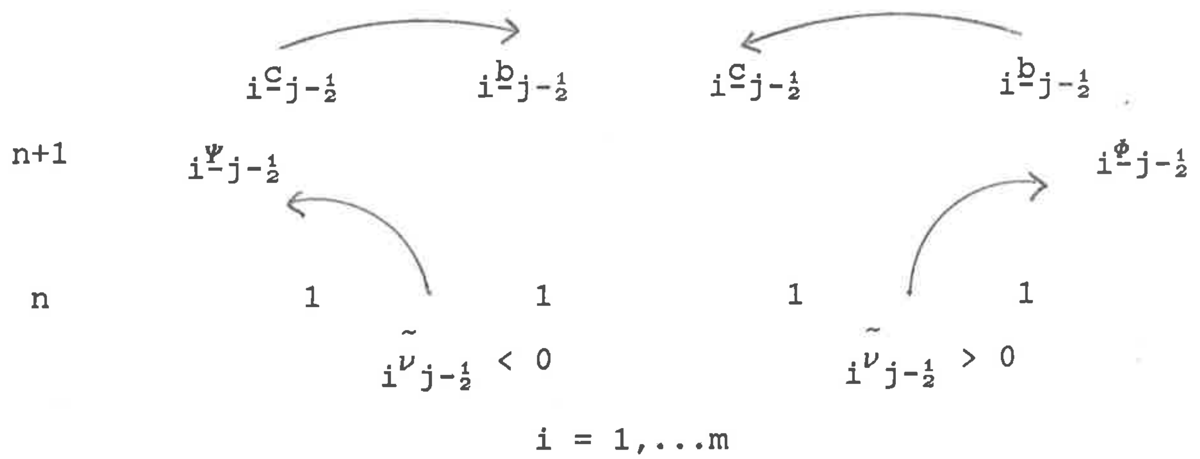


Figure 8

In the next section we discuss the special cases of compressible flow in a duct of variable cross section, and incompressible flow in a channel.

6. EXAMPLES

In this section we discuss the application of the algorithm of §5.3 to particular systems of conservations laws. Firstly, we consider the Euler equations of gas dynamics with source terms arising from flow in a duct of smoothly varying cross-section. Secondly, we consider the non-linear shallow water equations with source terms arising from flow in a channel whose lower surface is smoothly varying.

6.1 Euler equations

The Euler equations for the compressible flow of an ideal gas in a duct of cross-section $S(x)$ can be written as

$$\underline{w}_t + \underline{f}_x = \underline{g} \tag{6.1}$$

where

$$\underline{w} = S(x)(\rho,\rho u,e)^T \tag{6.2}$$

$$\underline{f}(\underline{w}) = S(x)(\rho u,p+\rho u^2,u(e+p))^T \tag{6.3}$$

$$\underline{g}(x,\underline{w}) = (0,pS'(x),0)^T \tag{6.4}$$

and

$$e = \frac{p}{\gamma-1} + \tfrac{1}{2}\rho u^2 \tag{6.5}$$

(see Glaister [5]). The quantities $\rho = \rho(x,t)$, $u = u(x,t)$, $p = p(x,t)$, $e = e(x,t)$ and γ represent the density, velocity, pressure, total energy and the ratio of specific heat capacities of the fluid, respectively, at a general point x and at time t. The special cases $S = 1,x,x^2$ refer to flows with slab, cylindrical or spherical symmetry, respectively. Following Glaister, we define new variables $\mathfrak{R} = S(x)\rho$, $U = u$, $P = S(x)p$, $E = S(x)e$ so that

$$\underline{w} = (\mathfrak{R},\mathfrak{R}U,E)^T \tag{6.6}$$

$$\underline{f} = (\mathfrak{R}U,P+\mathfrak{R}U^2,U(E+P))^T \tag{6.7}$$

$$\underline{g} = (0,PS'(x)/S(x),0)^T \tag{6.8}$$

and

$$E = \frac{P}{\gamma-1} + \tfrac{1}{2}\mathfrak{R}U^2 . \tag{6.9}$$

For the algorithm of §5.3 we devise an approximate Riemann problem as given by equation (5.2). In the specific example of this section we use the approximate Riemann problem proposed by Glaister [5] based on the work of Roe [1].

The approximate Jacobian $\tilde{A}(\underline{w}^n_{j-1},\underline{w}^n_j)$, to $A = \frac{\partial \underline{f}}{\partial \underline{w}}$ in the cell $[x_{j-1},x_j]$ at time level n is

$$\tilde{A} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{(\gamma-3)\tilde{U}^2}{2} & (3-\gamma)\tilde{U} & \gamma-1 \\ \frac{(\gamma-1)\tilde{U}^3}{2} - \tilde{H}\tilde{U} & \tilde{H} - (\gamma-1)\tilde{U}^2 & \gamma\tilde{U} \end{bmatrix} ; \tag{6.10}$$

where the averages of U and the enthalpy $H = (E+P)/\mathfrak{R}$ are given by

$$\tilde{U} = \frac{\sqrt{\mathfrak{R}_{j-1}}\, U_{j-1} + \sqrt{\mathfrak{R}_j}\, U_j}{\sqrt{\mathfrak{R}_{j-1}} + \sqrt{\mathfrak{R}_j}} \tag{6.11}$$

$$\tilde{H} = \frac{\sqrt{\mathfrak{R}_{j-1}}\, H_{j-1} + \sqrt{\mathfrak{R}_j}\, H_j}{\sqrt{\mathfrak{R}_{j-1}} + \sqrt{\mathfrak{R}_j}} . \tag{6.12}$$

The eigenvalues of $\tilde{A}$ are

$$_{1,2,3}\tilde{\lambda}_{j-\frac{1}{2}} = \tilde{U} + \tilde{a},\ \tilde{U} - \tilde{a},\ \tilde{U} \tag{6.13a-c}$$

where

$$\tilde{a} = \sqrt{(\gamma-1)(\tilde{H}-\tfrac{1}{2}\tilde{U}^2)} , \tag{6.14}$$

with corresponding eigenvectors

$$ {}_{1,2,3}\underline{\tilde{e}}_{j-\frac{1}{2}} = \begin{bmatrix} 1 \\ \tilde{U}+\tilde{a} \\ \tilde{H}+\tilde{U}\tilde{a} \end{bmatrix} , \begin{bmatrix} 1 \\ \tilde{U}-\tilde{a} \\ \tilde{H}-\tilde{U}\tilde{a} \end{bmatrix} , \begin{bmatrix} 1 \\ \tilde{U} \\ \frac{1}{2}\tilde{U}^2 \end{bmatrix} . \tag{6.15a-c} $$

Thus the modal matrix $\tilde{X}_{j-\frac{1}{2}} = [{}_1\underline{\tilde{e}}_{j-\frac{1}{2}}, {}_2\underline{\tilde{e}}_{j-\frac{1}{2}}, {}_3\underline{\tilde{e}}_{j-\frac{1}{2}}]$ has the inverse

$$ \tilde{X}^{-1}_{j-\frac{1}{2}} = \frac{1}{2\tilde{a}^2} \begin{bmatrix} \frac{(\gamma-1)\tilde{U}^2}{2} - \tilde{U}\tilde{a} & \tilde{a}-(\gamma-1)\tilde{U} & \gamma - 1 \\ \frac{(\gamma-1)\tilde{U}^2}{2} + \tilde{U}\tilde{a} & -\tilde{a}-(\gamma-1)\tilde{U} & \gamma - 1 \\ 2\tilde{a}^2 - (\gamma-1)\tilde{U}^2 & 2\tilde{U}(\gamma-1) & -2(\gamma-1) \end{bmatrix} . \tag{6.16} $$

To apply the scheme of §5.3 given by equations (5.60a-b) we need to calculate the quantities ${}_i\tilde{\alpha}_{j-\frac{1}{2}}$, ${}_i\tilde{\beta}_{j-\frac{1}{2}}$, ${}_i\tilde{\gamma}_{j-\frac{1}{2}}$, ${}_i\underline{\tilde{\xi}}_{j-\frac{1}{2}}$, ${}_i\underline{\tilde{\zeta}}_{j-\frac{1}{2}}$, $i = 1,2,3$. Denoting

$$ \Delta Y = Y_j - Y_{j-1} , \tag{6.17} $$

equations (5.26), (6.6) and (6.16) yield

$$ {}_1\tilde{\alpha}_{j-\frac{1}{2}} = \frac{1}{2\tilde{a}^2}\left[\Delta P + \tilde{a}(\Delta(\mathfrak{R}U) - \tilde{U}\Delta\mathfrak{R})\right] \tag{6.18a} $$

$$ {}_2\tilde{\alpha}_{j-\frac{1}{2}} = \frac{1}{2\tilde{a}^2}\left[\Delta P - \tilde{a}(\Delta(\mathfrak{R}U) - \tilde{U}\Delta\mathfrak{R})\right] \tag{6.18b} $$

$$ {}_3\tilde{\alpha}_{j-\frac{1}{2}} = \Delta\mathfrak{R} - \frac{\Delta P}{\tilde{a}^2} \tag{6.18c} $$

where we have used equation (6.9) and the property of $\tilde{U}$ that

$$\Delta(\mathfrak{R}U^2) - 2\tilde{U}\Delta(\mathfrak{R}U) + \tilde{U}^2\Delta\mathfrak{R} = 0 . \tag{6.19}$$

If we define the average of $\mathfrak{R}_{j-1}, \mathfrak{R}_j$

$$\tilde{\mathfrak{R}} = \sqrt{\mathfrak{R}_{j-1}\mathfrak{R}_j} \tag{6.20}$$

then

$$\Delta(\mathfrak{R}U) - \tilde{U}\Delta\mathfrak{R} = \tilde{\mathfrak{R}}\Delta U \tag{6.21}$$

so that the expressions given in equations (6.18a-b) simplify to

$$_{1,2}\tilde{\alpha}_{j-\frac{1}{2}} = \frac{1}{2\tilde{a}^2}(\Delta P \pm \tilde{\mathfrak{R}}\tilde{a}\Delta U) . \tag{6.22a-b}$$

In addition, using equations (5.43), (5.45), (6.8), (6.13a-c) and (6.16) we obtain

$$_{1}\tilde{\beta}_{j-\frac{1}{2}} = \kappa \frac{\left[(\gamma-1)\tilde{U} - \tilde{a}\right]}{\tilde{U} + \tilde{a}} P_{j-1} \tag{6.23a}$$

$$_{2}\tilde{\beta}_{j-\frac{1}{2}} = \kappa \frac{\left[(\gamma-1)\tilde{U} - \tilde{a}\right]}{\tilde{U} - \tilde{a}} P_{j-1} \tag{6.23b}$$

$$_{3}\tilde{\beta}_{j-\frac{1}{2}} = - 2\kappa(\gamma-1)P_{j-1} \tag{6.23c}$$

and

$$ {}_1\tilde{\gamma}_{j-\frac{1}{2}} = \kappa \frac{\left[(\gamma-1)\tilde{U} - \tilde{a}\right]}{\tilde{U} + \tilde{a}} P_j \tag{6.24a}$$

$$ {}_2\tilde{\gamma}_{j-\frac{1}{2}} = \kappa \frac{\left[(\gamma-1)\tilde{U} + \tilde{a}\right]}{\tilde{U} - \tilde{a}} P_j \tag{6.24b}$$

$$ {}_3\tilde{\gamma}_{j-\frac{1}{2}} = - 2\kappa(\gamma-1)P_j \tag{6.24c}$$

where

$$\kappa = \frac{S'(x_j - \frac{\Delta x}{2})}{2\tilde{a}^2\Delta x \; S(x_j - \frac{\Delta x}{2})} . \tag{6.25}$$

Finally, using equations (6.8) and (6.9)

$$\underline{g} = \left[0, \frac{S'(x)}{S(x)}(\gamma-1)(E - \frac{1}{2}\frac{M^2}{\mathfrak{R}}), 0\right]^T \tag{6.26}$$

where $M = \mathfrak{R}U$ so that from equation (6.6)

$$\underline{w} = (\mathfrak{R}, M, E)^T \tag{6.27}$$

and hence

$$\underline{g}_{\underline{w}} = (\gamma-1)\frac{S'(x)}{S(x)} \begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{2}U^2 & -U & 1 \\ 0 & 0 & 0 \end{bmatrix} . \tag{6.28}$$

Therefore the expressions $_i\tilde{\xi}_{j-\frac{1}{2}}$, $_i\tilde{\xi}_{j-\frac{1}{2}}$ of §5.3 become

$$_1\tilde{\xi}_{j-\frac{1}{2}} = \left[0, \tau_{j-1}\left\{\frac{\tilde{a}^2}{\gamma-1} + \tfrac{1}{2}(\tilde{U}-U_{j-1})^2 + \tilde{a}(\tilde{U}-U_{j-1})\right\}, 0\right]^T \quad (6.29a)$$

$$_2\tilde{\xi}_{j-\frac{1}{2}} = \left[0, \tau_{j-1}\left\{\frac{\tilde{a}^2}{\gamma-1} + \tfrac{1}{2}(\tilde{U}-U_{j-1})^2 - \tilde{a}(\tilde{U}-U_{j-1})\right\}, 0\right]^T \quad (6.29b)$$

$$_3\tilde{\xi}_{j-\frac{1}{2}} = (0, \tau_{j-1}\tfrac{1}{2}(\tilde{U}-U_{j-1})^2, 0)^T \quad (6.29c)$$

$$_1\tilde{\xi}_{j-\frac{1}{2}} = \left[0, \tau_j\left\{\frac{\tilde{a}^2}{\gamma-1} + \tfrac{1}{2}(\tilde{U}-U_j)^2 + \tilde{a}(\tilde{U}-U_j)\right\}, 0\right]^T \quad (6.30a)$$

$$_2\tilde{\xi}_{j-\frac{1}{2}} = \left[0, \tau_j\left\{\frac{\tilde{a}^2}{\gamma-1} + \tfrac{1}{2}(\tilde{U}-U_j)^2 - \tilde{a}(\tilde{U}-U_j)\right\}, 0\right]^T \quad (6.30b)$$

$$_3\tilde{\xi}_{j-\frac{1}{2}} = (0, \tau_j\tfrac{1}{2}(\tilde{U}-U_j)^2, 0)^T \quad (6.30c)$$

where

$$\tau_j = \frac{\Delta t(\gamma-1)S'(x_j)}{2S(x_j)} . \quad (6.31)$$

We observe, however, that the $_i\tilde{\xi}_{j-\frac{1}{2}}$, $_i\tilde{\xi}_{j-\frac{1}{2}}$ occur in the second order terms of equation (5.35) and since

$$\tilde{U} - U_j = -\frac{\Delta U_j}{1 + \sqrt{\mathcal{R}_j/\mathcal{R}_{j-1}}} \tag{6.32}$$

$$\tilde{U} - U_{j-1} = \frac{\Delta U_j}{1 + \sqrt{\mathcal{R}_{j-1}/\mathcal{R}_j}} \tag{6.33}$$

we could approximate the expressions in equations (6.29a)-(6.30c) by

$$_1\tilde{\xi}_{j-\frac{1}{2}} = {}_2\tilde{\xi}_{j-\frac{1}{2}} = \frac{\tilde{a}^2\Delta t S'(x_{j-1})}{2S(x_{j-1})}(0,1,0)^T, \quad {}_3\tilde{\xi}_{j-\frac{1}{2}} = \underline{0} \, , \tag{6.34a-c}$$

$$_1\tilde{\xi}_{j-\frac{1}{2}} = {}_2\tilde{\xi}_{j-\frac{1}{2}} = \frac{\tilde{a}^2\Delta t S'(x_j)}{2S(x_j)}(0,1,0)^T, \quad {}_3\tilde{\xi}_{j-\frac{1}{2}} = \underline{0} \, . \tag{6.35a-c}$$

6.2 Non-linear shallow water equations

The shallow water equations for the flow of an incompressible fluid in a channel of rectangular cross-section can be written as

$$\underline{w}_t + \underline{f}_x = \underline{g} \tag{6.36}$$

where

$$\underline{w} = (g(\eta+h), \quad g(\eta+h)u)^T$$

(6.37)

$$\underline{f}(\underline{w}) = (g(\eta+h)u, \quad g(\eta+h)u^2 + \tfrac{1}{2}g^2(\eta+h)^2)^T$$

(6.38)

and

$$\underline{g}(x,w) = (0, g^2(\eta+h)h'(x))^T$$

(6.39)

(see Glaister [6]). The quantities $\eta = \eta(x,t)$, $u = u(x,t)$ and $h(x)$ represent the free surface elevation, velocity and an undisturbed depth of the fluid, respectively, at a general point x and at time t . The acceleration of gravity is represented by g . Following Glaister, we define $\Phi = g(\eta+h)$ so that

$$\underline{w} = (\Phi, \Phi u)^T$$

(6.40)

$$\underline{f} = (\Phi u, \quad \Phi u^2 + \tfrac{1}{2}\Phi^2)^T$$

(6.41)

and

$$\underline{g} = (0, \quad g\Phi h'(x))^T .$$

(6.42)

For the algorithm of §5.3 we devise an approximate 'Riemann problem' as given by equation (5.2). In the specific example of this section we use the approximate Riemann problem proposed by Glaister [6].

The approximate Jacobian $\tilde{A}(\underline{w}^n_{j-1}, \underline{w}^n_j)$, to $A = \frac{\partial \underline{f}}{\partial \underline{w}}$ in the cell $[x_{j-1}, x_j]$ at time level n is

$$\tilde{A} = \begin{bmatrix} 0 & 1 \\ \bar{\Phi} - \tilde{u}^2 & 2\tilde{u} \end{bmatrix} , \tag{6.43}$$

where the averages of u and Φ are given by

$$\tilde{u} = \frac{\sqrt{\Phi_{j-1}}\, u_{j-1} + \sqrt{\Phi_j}\, u_j}{\sqrt{\Phi_{j-1}} + \sqrt{\Phi_j}} \tag{6.44}$$

$$\bar{\Phi} = \tfrac{1}{2}(\Phi_{j-1} + \Phi_j) \quad . \tag{6.45}$$

The eigenvalues of $\tilde{A}$ are

$$_{1,2}\tilde{\lambda}_{j-\frac{1}{2}} = \tilde{u} + \tilde{\Psi} , \quad \tilde{u} - \tilde{\Psi} , \tag{6.46a-b}$$

with corresponding eigenvectors

$$_{1,2}\tilde{\underline{e}}_{j-\frac{1}{2}} \quad \begin{bmatrix} 1 \\ \tilde{u} + \tilde{\Psi} \end{bmatrix} , \quad \begin{bmatrix} 1 \\ \tilde{u} - \tilde{\Psi} \end{bmatrix} , \tag{6.47a-b}$$

where

$$\tilde{\Psi} = \sqrt{\bar{\Phi}} \quad . \tag{6.48}$$

Thus the modal matrix $\tilde{X}_{j-\frac{1}{2}} = [{}_1\tilde{e}_{j-\frac{1}{2}}, {}_2\tilde{e}_{j-\frac{1}{2}}]$ has the inverse

$$\tilde{X}^{-1}_{j-\frac{1}{2}} = \frac{1}{2\Psi}\begin{bmatrix} \tilde{\Psi} - \tilde{u} & 1 \\ \tilde{u} + \tilde{\Psi} & -1 \end{bmatrix} . \tag{6.49}$$

To apply the scheme of §5.3 given by equations (5.60a-b) we need to calculate the quantities ${}_i\tilde{\alpha}_{j-\frac{1}{2}}$, ${}_i\tilde{\beta}_{j-\frac{1}{2}}$, ${}_i\tilde{\gamma}_{j-\frac{1}{2}}$, ${}_i\tilde{\xi}_{j-\frac{1}{2}}$, ${}_i\tilde{\xi}_{j-\frac{1}{2}}$, $i = 1,2$. Denoting

$$\Delta Y = Y_j - Y_{j-1} , \tag{6.50}$$

equations (5.26), (6.40) and (6.49) yield

$${}_1\tilde{\alpha}_{j-\frac{1}{2}} = \frac{1}{2\Psi}\left[\tilde{\Psi}\Delta\Phi + (\Delta(\Phi u) - \tilde{u}\Delta\Phi)\right] \tag{6.51a}$$

$${}_2\tilde{\alpha}_{j-\frac{1}{2}} = \frac{1}{2\Psi}\left[\tilde{\Psi}\Delta\Phi - (\Delta(\Phi u) - \tilde{u}\Delta\Phi)\right] . \tag{6.51b}$$

If we define the average of Φ_{j-1}, Φ_j

$$\tilde{\Phi} = \sqrt{\Phi_{j-1}\Phi_j} \tag{6.52}$$

then

$$\Delta(\Phi u) - \tilde{u}\Delta\Phi = \tilde{\Phi}\Delta u \tag{6.53}$$

so that the expressions given in equations (6.51a-b) simplify to

$$ {}_{1,2}\tilde{\alpha}_{j-\frac{1}{2}} = \tfrac{1}{2}\Delta\Phi \pm \tfrac{1}{2}\,\frac{\tilde{\Phi}}{\tilde{\Psi}}\,\Delta u \; . \tag{6.54} $$

In addition, using equations (5.43), (5.45), (6.42), (6.46a-b) and (6.49) we obtain

$$ {}_{1}\tilde{\beta}_{j-\frac{1}{2}} = -\,\frac{\kappa\;\Phi_{j-1}}{\tilde{u} + \tilde{\Psi}} \tag{6.55a} $$

$$ {}_{2}\tilde{\beta}_{j-\frac{1}{2}} = \frac{\kappa\;\Phi_{j-1}}{\tilde{u} - \tilde{\Psi}} \tag{6.55b} $$

and

$$ {}_{1}\tilde{\gamma}_{j-\frac{1}{2}} = -\,\frac{\kappa\;\Phi_{j}}{\tilde{u} + \tilde{\Psi}} \tag{6.56a} $$

$$ {}_{2}\tilde{\gamma}_{j-\frac{1}{2}} = \frac{\kappa\;\Phi_{j}}{\tilde{u} - \tilde{\Psi}} \tag{6.56b} $$

where

$$ \kappa = \frac{gh'(x_j - \frac{\Delta x}{2})}{2\tilde{\Psi}} \; . \tag{6.57} $$

Finally, using equations (6.40) and (6.42)

$$ \underline{g}_{\underline{w}} = gh'(x)\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} . \tag{6.58} $$

Therefore the expressions of §5.3 become

$$ {}_1\tilde{\underline{\xi}}_{j-\frac{1}{2}} = (0, \tau_{j-1}) \tag{6.59a} $$

$$ {}_2\tilde{\underline{\xi}}_{j-\frac{1}{2}} = (0, \tau_{j-1}) \tag{6.59b} $$

$$ {}_1\tilde{\underline{\xi}}_{j-\frac{1}{2}} = (0, \tau_{j}) \tag{6.60a} $$

$$ {}_2\tilde{\underline{\xi}}_{j-\frac{1}{2}} = (0, \tau_{j}) \tag{6.60b} $$

where

$$ \tau_j = \frac{\Delta t g h'(x_j)}{2} . \tag{6.61} $$

7. CONCLUSION

We have presented first and second order schemes for scalar conservation laws with source terms and have applied them to systems of conservation laws with source terms.

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