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by

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A First Principles Derivation of a Simple Non–linear Aerodynamic Model*

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Abstract

Based on the principle of a square law for the resistance of an object to fluid flow, a aerodynamic model of a two dimensional plate is derived. The resulting equations of motion are highly non-linear and yet allow for closed-form potential and kinetic energy expression. The model also lends itself quite easily to linearisation, which provides insight into local behavior of the system. In all, it provides an excellent "test–bed" upon which to evaluate the efficacy of various linear feedback controllers applied to a non-linear plant.

Keywords Nonlinear dynamics, Linearisation, Aerodynamics.

1 Introduction

A standard model of the resistance of a particle moving through a liquid stipulates that the force acting on the body is proportional to the velocity of the fluid squared [1, 3]. More specifically, if a body with cross–sectional area A and coefficient of drag κ moves at velocity w through a fluid with density ρ, the force acting upon that particle can be written as

$$F = \rho\kappa A w^2. \tag{1}$$

This model can be viewed in the following way. Suppose a body moves through a fluid consisting of atoms with high mean-free-path. Suppose, too, that a fraction κ of the particles collide inelastically with this body. Define further the mass rate of the fluid to be Φ, and the rate of change in the momentum of the fluid that impinges upon the body as Δ_M. Then the force acting upon the body will be proportional to the change in momentum of the fluid with respect to time. Since the momentum change is proportional to the mass-rate, fluid velocity, and particle cross–section, which in this case may be written as the product of the cross-sectional area A times the capture ratio κ, the expression for the force acting on the body may be written as: 13 July 95

$$\begin{aligned} F &= d\Delta_M/dt \\ &= \kappa\Phi A w \\ &= \kappa\rho|w|w. \end{aligned} \tag{2}$$

It should be noted that this model is not necessarily aerodynamic in nature; however, despite the fact that this model is limited theoretically to the forces acting on bodies in an ethereal fluid, *e.g.,* space vehicles at high altitude, solar sails, etc., its simplicity provides a method of deriving nonlinear aerodynamic models with which to test schemes by which the dynamics of nonlinear systems are modified by feedback.

2 Flate Plate Model

In this section we apply the paradigm of the first section to a specific example; a two–dimensional plate of length ℓ and mass m pivoting about a point at a distance d from its center of mass, as depicted in Figure 1.

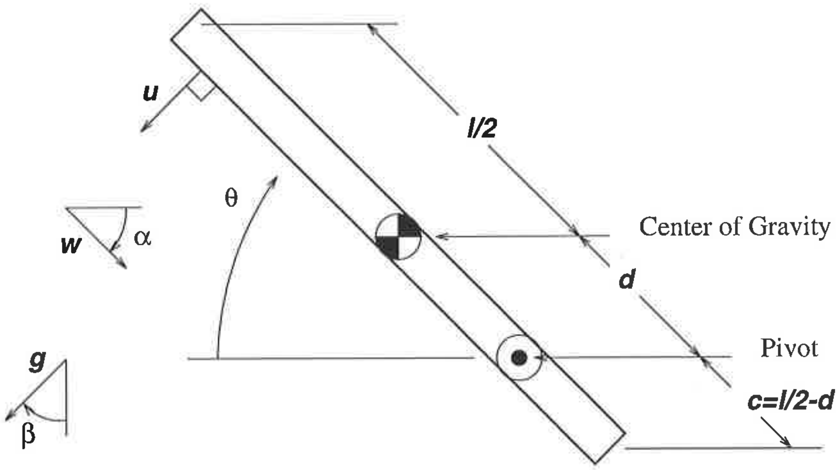

Figure 1: A Schematic of the Flying Plate

In this example, the fluid velocity is w and the acceleration due to gravity is g. These are defined to be horizontal and vertical, but in general may take the form:

$$w = \begin{bmatrix} w_0 \cos\alpha \\ w_0 \sin\alpha \end{bmatrix} , \quad g = \begin{bmatrix} -g_0 \sin\beta \\ -g_0 \cos\beta \end{bmatrix} \tag{3}$$

For simplicity, we consider only the case with $\alpha = 0$ and $\beta = -\frac{\pi}{2}$. The solution to this problem requires the solution of an integral. If v is defined to be the velocity of the fluid relative to an infinitesimal section of the plate dr, then as the plate rotates around its pivot with rotational speed $\dot{\theta}$,

$$v = w + \dot{\theta} r n, \tag{4}$$

where n is the unit norm to the surface of the plate, *i.e.*,

$$n = \begin{bmatrix} -\sin\theta \\ -\cos\theta \end{bmatrix} \tag{5}$$

and r is the distance from the pivot.

Using the paradigm of Section 1, the cross-sectional area which the fluid "sees" at the length of the plate dr is the inner product of unit vector in the fluid direction and the unit vector normal to the plate. Thus, the infinitesimal force F_f of the fluid acting on the infinitesimal section of the plate dr is

$$\begin{aligned} df_r &= \kappa\rho A\|v\|vdr \\ &= \kappa\rho\left(\frac{1}{\|v\|}|v^T n|\right)\|v\|vdr \\ &= \kappa\rho|v^T n|vdr \end{aligned} \tag{6}$$

Given a force f acting on a point displaced from a pivot by vector g, the resultant torque τ about the pivot may be written as

$$\tau = f \times g, \tag{7}$$

where $\times$ denotes the cross product. In two dimensions, however, torque is no longer a vector but a scalar, and thus (7) is reduced to

$$\tau = f^T g_\perp \tag{8}$$

where $g_\perp$ is perpendicular to g in the right hand sense.

Returning to the problem at hand, we note that $g_\perp = nr$. Thus, the infinitesimal force df_r acting at point r along the plate produced about the pivot the infinitesimal torque

$$\begin{aligned} d\tau_f &= rn^T df_r \\ &= \kappa\rho|v^T n|v^T nrdr \end{aligned} \tag{9}$$

One of the more subtle points in the integration of the torque along the length of the plate is the fact that the term $|v^T n|$ changes sign at the point where the relative fluid flow along the plate is zero. The integral which describes the total torque must then be split into two if $v^T n$ disappears as a function of r in the region $[-c, \ell - c]$. Since

$$\begin{aligned} v^T n &= (w + \dot\theta rn)^T n \\ &= w^T n + \dot\theta rn^T n \\ &= w^T n + \dot\theta r \end{aligned} \tag{10}$$

the point where $v^T n$ is zero is $\bar{\xi} = \frac{w^T n}{\dot{\theta}}$.

Thus, the torque that results from the fluid flow is

$$\tau_f = \sigma \int_{-c}^{\xi} \kappa\rho(n^T v)^2 r dr - \sigma \int_{\xi}^{\ell-c} \kappa\rho(n^T v)^2 r dr, \tag{11}$$

where $c = \ell/2 - d$ and where

$$\xi = \begin{cases} \frac{w^T n}{\dot{\theta}} & , \quad -c \leq \xi \leq \ell - c \\ \ell - c & , \quad \xi > \ell - c \\ -c & , \quad -c > \xi \end{cases} \tag{12}$$

The variable $\sigma = \pm 1$ in front of the integrals in (11) depends on the relative direction of the fluid relative to the point ξ on the plate, which is determined by the sign of the inner product $v^T n$. Since this function is monotonic as a function of r, the value of σ may be determined by the difference of the relative motions of the end-points of the plate, *i.e.*,

$$\begin{aligned} \sigma &= \operatorname{sign}\left(\left(w^T n + \dot{\theta}(\ell - c)\right) - \left(w^T n + \dot{\theta}(-c)\right)\right) \\ &= \operatorname{sign}\left(\dot{\theta}\right) \end{aligned} \tag{13}$$

In the case where $\dot{\theta} = 0$, the gradient of the wind velocity along the plate is zero, requiring a different approach for determining σ. The value of σ is determined in this case by the sign of the inner product $w^T n$, *i.e.*,

$$\sigma = \operatorname{sign}\left(w^T n\right). \tag{14}$$

Substituting expression (10) and (13) in (12), we have

$$\begin{aligned} \tau_f &= \sigma\kappa\rho\left(\int_{-c}^{\xi}\left(w^T n + \dot{\theta}r\right)^2 r dr - \int_{\xi}^{\ell-c}\left(w^T n + \dot{\theta}r\right)^2 r dr\right) \\ &= \sigma\kappa\rho\left(\int_{-c}^{\xi}\left((w^T n)^2 r + 2w^T n\dot{\theta}r^2 + \dot{\theta}^2 r^3\right) dr \right. \\ &\qquad \left. - \int_{\xi}^{\ell-c}\left((w^T n)^2 r + 2w^T n\dot{\theta}r^2 + \dot{\theta}^2 r^3 dr\right)\right) \\ &= \sigma\kappa\rho\left(\left[\tfrac{1}{2}(w^T n)^2 r^2 + \tfrac{2}{3}w^T n\dot{\theta}r^3 + \tfrac{1}{4}\dot{\theta}^2 r^4\right]_{-c}^{\xi} \right. \\ &\qquad \left. - \left[\tfrac{1}{2}(w^T n)^2 r^2 + \tfrac{2}{3}w^T n\dot{\theta}r^3 + \tfrac{1}{4}\dot{\theta}^2 r^4\right]_{\xi}^{\ell-c}\right) \\ &= \sigma\kappa\rho\left(\tfrac{1}{2}\ell_2(w^T n)^2 + \tfrac{2}{3}\ell_3 w^T n\dot{\theta} + \tfrac{1}{4}\ell_4\dot{\theta}^2\right) \end{aligned} \tag{15}$$

where

$$\begin{aligned}
\ell_2 &= 2\xi^2 - c^2 - (\ell - c)^2 \\
\ell_3 &= 2\xi^3 + c^3 - (\ell - c)^3 \\
\ell_4 &= 2\xi^4 - c^4 - (\ell - c)^4
\end{aligned} \tag{16}$$

with ξ defined as in (12).

In addition to the torque produced by the fluid flow, the other torques which affect the dynamics of the plate are the torque due to the angular acceleration T_a, the torque due to gravity T_g, and the applied torque T_i. The torque due to angular acceleration may be written as

$$T_a = -I\ddot{\theta} \tag{17}$$

where I is the moment of inertia of the plate about the pivot, with the negative sign reflecting that it is a reactive force. For a plate made from a material of uniform density $\delta = m/l$, the moment of inertia may be written

$$\begin{aligned}
I &= \textstyle\int_{-c}^{\ell-c} \delta r^2 dr \\
&= \tfrac{m}{\ell} \left[\tfrac{1}{3} r^3\right]_{-c}^{\ell-c} \\
&= \frac{m\hat{\ell}_3}{3\ell}
\end{aligned} \tag{18}$$

where $\hat{\ell}_3 = (\ell - c)^3 + c^3$. The torque due to gravity may be simply written as

$$T_g = mdg^T n \tag{19}$$

and the applied torque as

$$T_i = \tau. \tag{20}$$

The dynamical principle that states that the sum of all torque of a free body is zero implies that

$$I\ddot{\theta} - \kappa\rho \left(\frac{1}{2}\ell_2(w^T n)^2 + \frac{2}{3}\ell_3\dot{\theta}w^T n + \frac{1}{4}\ell_4\dot{\theta}^2\right) + mdg^T n + \tau = 0 \tag{21}$$

which gives the dynamics of the plate.

3 System Energy

As a means of determining the passivity of the system, the energy of the system is determined. In particular, the system energy may be written as

$$E(\theta, \dot{\theta}) = V(\theta) + K(\dot{\theta}) + \Delta_E(\theta, \dot{\theta}) \tag{22}$$

where $V(\theta)$ is the potential energy, $K(\dot{\theta})$ is the kinetic energy, and Δ_E is the energy loss (gain). In accordance with the law of conservation of energy, we insist that

$$E(\theta, \dot{\theta}) = 0; \tag{23}$$

therefore, since $V(\theta)$ and $K(\dot{\theta})$ are defined to be positive functions, the rate of change of the sign of $\Delta_E(\theta, \dot{\theta})$ will indicate whether the system is passive.

The potential energy function is defined at the integral of the torque produced by the wind as a function of θ with $\dot{\theta} = 0$. By convention it has minimum energy at zero. Thus, for $d > 0$ and $\theta \in [0, \pi]$

$$\begin{aligned} V(\theta) &= \int_0^\theta \tau_f(\phi, \dot{\phi})\ d\phi\ |_{\dot{\phi}=0} \\ &= \int_0^\theta \kappa\rho((l-c)^2 - c^2)(w^T n)^2\ d\phi \\ &= \tfrac{1}{2}\kappa\rho((l-c)^2 - c^2) w_0^2 \int_0^\theta \sin^2\phi\ d\phi \\ &= \tfrac{1}{4}\kappa\rho((l-c)^2 - c^2) w_0^2 \int_0^\theta 1 - \cos 2\phi\ d\phi \\ &= \tfrac{1}{4}\kappa\rho((l-c)^2 - c^2) w_0^2 \left[\phi - \tfrac{1}{2}\sin 2\phi\right]_0^\theta \\ &= \tfrac{1}{4}\kappa\rho((l-c)^2 - c^2) w_0^2 \left(\theta - \tfrac{1}{2}\sin 2\theta\right) \end{aligned} \tag{24}$$

Taking into account the symmetries of the problem, the potential energy function may be written as

$$V(\theta) = \begin{cases} \tfrac{1}{4}\kappa\rho\hat{\ell}_2 w_0^2 \left(|\theta| - \tfrac{1}{2}\sin 2|\theta|\right) & ;\ w_0 d > 0\ \ ,\ \ \theta \in [-\pi, \pi] \\ \tfrac{1}{4}\kappa\rho\hat{\ell}_2 w_0^2 \left(|\theta - \pi| - \tfrac{1}{2}\sin 2|\theta - \pi|\right) & ;\ w_0 d < 0\ \ ,\ \ \theta \in [0, 2\pi] \end{cases} \tag{25}$$

where $\hat{\ell}_2 = ((l-c)^2 - c^2)$.

The kinetic energy is more easily written; it is simply

$$K(\dot{\theta}) = \frac{1}{2} J \dot{\theta}^2. \tag{26}$$

4 Linearisation

Suppose that we have a differential equation which describes the evolution of a system described by the differential equation

$$\dot{z} = f(z, v) \tag{27}$$

where the function f, the state vector, and the input vector are respectively

$$f(z,v) = \begin{bmatrix} f_1(z,v) \\ f_2(z,v) \\ \vdots \\ f_N(z,v) \end{bmatrix}, \quad z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_N \end{bmatrix}, \quad v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_M \end{bmatrix}. \tag{28}$$

Probably the most common form of linearisation [2] is a Taylor's expansion of the dynamical equations about a fixed point (z_0, v_0). This produces the equations

$$\dot{x} = Ax + Bu \tag{29}$$

where

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial z_1} & \frac{\partial f_1}{\partial z_2} \\ \frac{\partial f_2}{\partial z_1} & \frac{\partial f_2}{\partial z_2} \end{bmatrix}, \quad B = \begin{bmatrix} \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial v} \end{bmatrix} \tag{30}$$

$$x = z - z_0, \quad u = v - v_0.$$

The flying plate of the previous section may be written in the format of (27), *i.e.*,

$$\begin{aligned} f_1(z(\theta,\dot{\theta})) &= \dot{\theta} \\ f_2(z(\theta,\dot{\theta})) &= \tfrac{\ell\sigma\kappa\rho}{4m\ell_3}\left(6\ell_2 w^2 \sin^2\theta + 8\ell_3 w\dot{\theta}\sin\theta + 3\ell_4\dot{\theta}^2\right) \end{aligned} \tag{31}$$

with

$$z = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}, \quad v = \tau. \tag{32}$$

For these equations, all equilibria points must have the property that $\dot{\theta} = 0$, otherwise the linearised system would be time-varying. This greatly simplifies the linearisation as the terms containing $\dot{\theta}$ in A and B may be set to zero after the linearisation. It also implies that ξ in (12) is either $-c$ or $\ell - c$. With the assumption that $g_0 = 0$, $\alpha = 0$, and $\beta = 0$, (30) may be written as

$$\begin{aligned} A &= \begin{bmatrix} 0 & 1 \\ \frac{3\kappa\rho\sigma\ell\ell_2}{ml_3}(w^T n w^T \frac{\partial n}{\partial \theta}) & \frac{2\kappa\rho\sigma\ell}{m} w^T u \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ \frac{3\kappa\rho\sigma\ell\ell_2 w_0^2 \sin\theta\cos\theta}{ml_3} & \frac{2\kappa\rho\sigma\ell w_0 \sin\theta}{m} \end{bmatrix} \\ B &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{aligned} \tag{33}$$

Here σ is the sign of ξ in (12), which must be defined in the limit as $\dot{\theta}$ goes to zero.

Furthermore, we note that the torque $v_0 = \tau = -\tau_f$ under equilibrium conditions, *i.e.*, with $\dot{\theta} = 0$ may be written

$$\begin{aligned} v_0 &= -\tfrac{1}{2}\sigma\kappa\rho\ell_2(w^T n)^2 \\ &= -\tfrac{1}{2}\sigma\kappa\rho\ell_2 w_0^2 \sin^2\theta. \end{aligned} \tag{34}$$

The simplicity of this formulation is conducive to inversion, thus for any given τ within its appropriate range, the equilibrium state z_0 is

$$z_0 = \begin{bmatrix} \arcsin\left(\sqrt{\frac{2\tau}{\sigma\kappa\rho\ell_2 w_0^2}}\right) \\ 0 \end{bmatrix} \tag{35}$$

The linearisation of this system provides some insight about the system's behavior; in particular, the local behavior about a point θ may characterised by the position of the eigenvalues of the state matrix $A(\theta)$.

For $\theta \in [0, \pi]$, the eigenvalues of the system may be directly computed, and are

$$\begin{aligned} \Lambda\left(A(\theta)\right) &= \tfrac{1}{m}\ell\kappa\rho\sigma w_0 \sin\theta \pm \\ & \tfrac{w_0}{\ell_3 m}\sqrt{\ell\ell_3\kappa\rho\sin\theta\left(3\sigma\ell_2 m\cos(\theta) + \ell\ell_3\kappa\rho\sin\theta\right)}. \end{aligned} \tag{36}$$

The linearisation also provides a means of characterising the behavior of the system further. In particular, the damping factor ζ [2] provides a qualitative measure of the dampedness of the system. This quantity may be written as

$$\zeta = \sqrt{\frac{-\kappa\rho\sigma\ell\ell_3 \sin\theta}{3\ell_2 m \cos\theta}} \tag{37}$$

Perhaps a more instructive measure of dampedness is the angle ϕ between the radial line from the stable eigenvalues to the origin and the negative real axis. This measure is related to ζ, namely $\phi = \arccos(\zeta)$.

5 Numerical Experiments

In the first experiment, we examine the time response of the state variables θ and $\dot{\theta}$ from their equilibrium points with a slight positive perturbation the variable $\dot{\theta}$. In Figure 5, we plot the time response with $\theta = 0$, $\dot{\theta} = \frac{\pi}{10}$, $w_0 = 4$, $\ell = 4$, $d = 1$, $m = 10$, $g = 0$, $\rho = 1$, $\kappa = 1$, and $\tau = 0$. In Figure 5, the constants remain the same with the exception that $d = -1$. The first case corresponds to a system that is unstable about the equilibrium point $\theta = 0$, while the second is stable about this points. In both plots we note that the total stored energy, *i.e.*, the potential and the kinetic energy, is continuously decreasing, indicating that the system is passive.

The behavior of the system that corresponds to the data represented in Figure 5 may be reasonable well accounted for by the insights gleaned from the linearisation of the system. In particular, the eigenvalues of the system as they vary with the plate angle θ, as shown in Figure 5, reflect the nature of the response of the system as shown in Figure 5. This plot indicates that initially (about $\theta = \pi/2$) the system is unstable, as the real part of one of the eigenvalues goes positive. At $\theta = 3\pi/4$, the system is again stable, but as the plate angle approaches $\theta = \pi$, the eigenvalues become increasingly undamped. This can be inferred from the fact that ζ is tending to zero. We plot $\arccos(\zeta)$ as defined in (37) and corresponding with the data represented in figure 5 versus the plate angle in Figure 5. The increasingly undamped nature of the system about $\theta = \pi$, with its concomitant increase the period of the oscillation is evident in Figure 5, as the period between the zero crossings of $\dot{\theta}$ grows longer.

6 Concluding Remarks

In this note we have modeled a two-dimensional plate based on first principles. Since the model assumes a fluid with infinite mean-free-path, it is not aero-

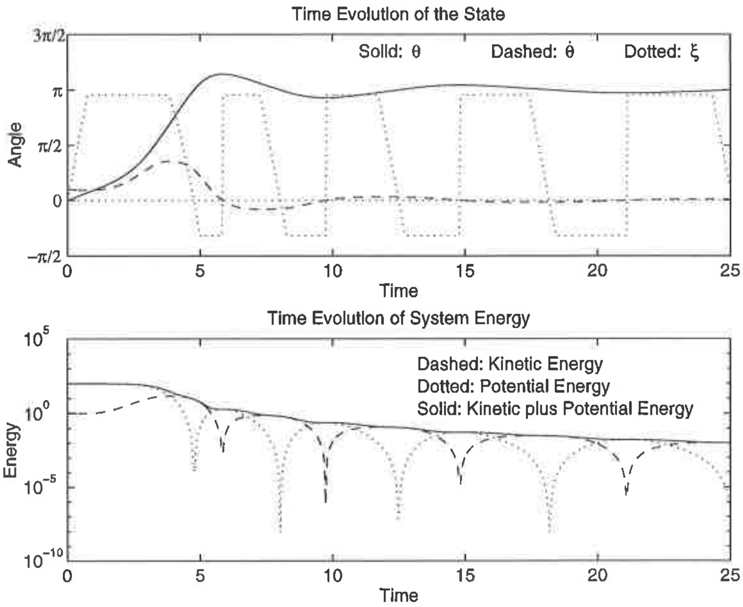

Figure 2: Time Evolution of the State Variables

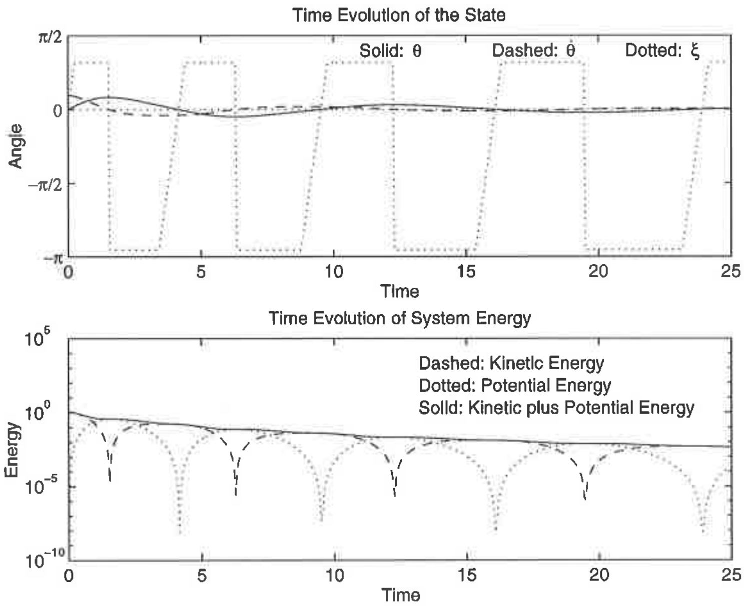

Figure 3: Time Evolution of the State Variables

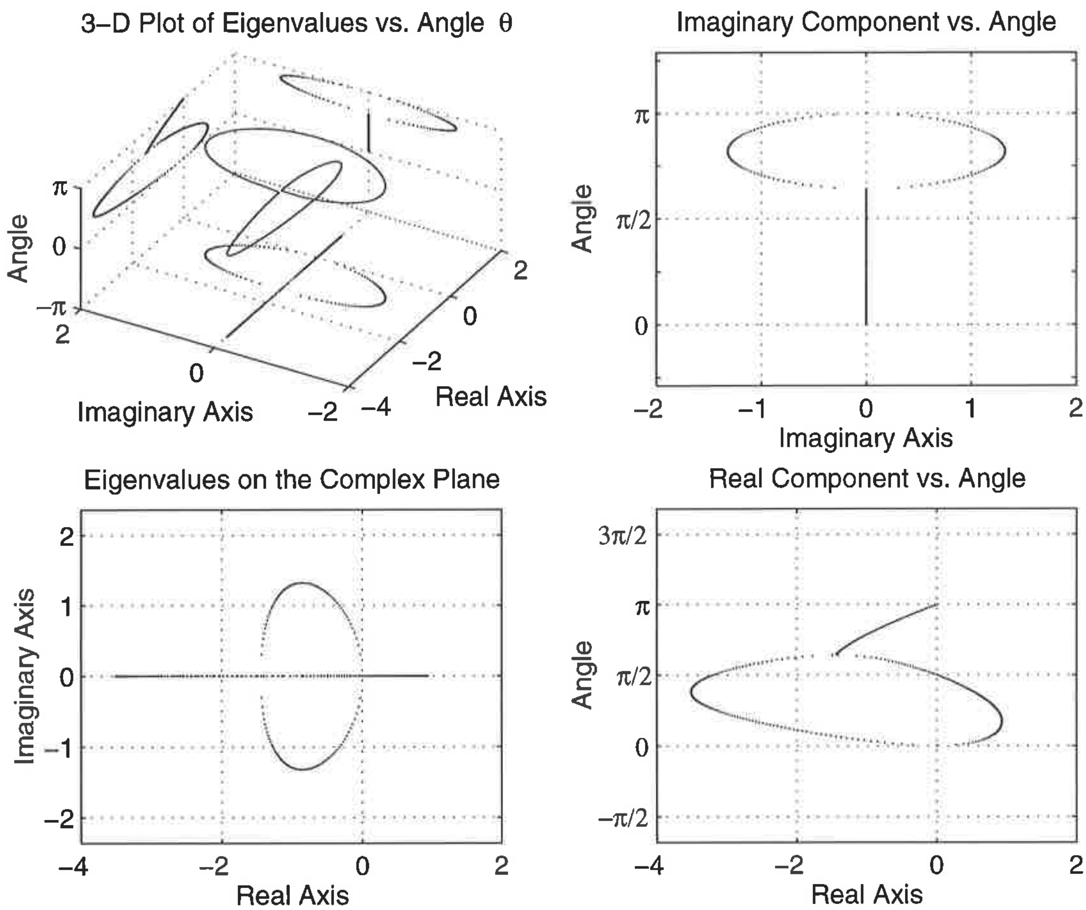

Figure 4: Eigenvalues of Linearised Equations

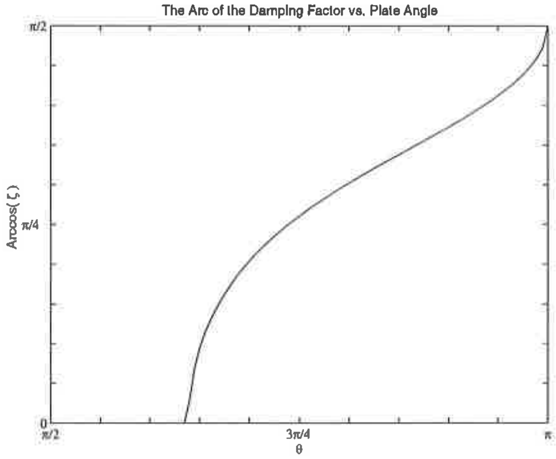

Figure 5: Relative Damping Angle versus Plate Angle

dynamic in the strictest sense; however, it does seem to produce reasonable behavior. Furthermore, since it can be written down exactly and is highly non-linear, it can provide a reasonable means of testing the efficacy of linear controllers applied to non-linear plants.

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