ROBUST MULTIPLE EIGENVALUE ASSIGNMENT BY STATE FEEDBACK IN LINEAR SYSTEMS

by

J. Kautsky* and N.K. Nichols†

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* School of Mathematical Sciences,
Flinders University,
Bedford Park,
S.A. 5042,
Australia.

† Department of Mathematics,
University of Reading,
Reading, Berkshire,
RG6 2AX,
Great Britain.

1. INTRODUCTION

For the problem of eigenvalue assignment by feedback in a multi-input linear control system, a solution is robust, or well-conditioned, if the assigned eigenvalues are as insensitive as possible to perturbations in the coefficient matrices of the closed loop system [1]. It is known that eigenvalue sensitivity is inversely proportional to the cosine of the smallest "canonical angle" between the right and left invariant subspaces corresponding to distinct eigenvalues [3], [4]. The sensitivity, or conditioning, of the eigenvalues is, therefore, determined by the choice of the corresponding eigenvectors of the closed loop system.

Recently we have developed a reliable numerical method for constructing robust solutions to the pole assignment problem [1] [2]. In this procedure the feedback is obtained by selecting from known subspaces linearly independent eigenvectors, corresponding to the assigned poles of the closed loop system matrix, such that the modal matrix of eigenvectors is as well-conditioned as possible in the Frobenius norm. In the case of simple eigenvalues, the eigenvectors determining each invariant subspace are unique, up to scaling, and we can show that the square of the Frobenius condition number of the modal matrix is equal to a weighted sum of the squares of the condition numbers of the eigenvalues. Therefore, minimizing the Frobenius conditioning of the modal matrix guarantees that the assigned poles are as insensitive to perturbations in the system and gain matrices as is feasible. It can be shown, using norm equivalences, that the resulting feedback gains and corresponding transient response of the closed loop system are also guaranteed, then, to be as reasonably bounded as may be expected and that a lower bound on the stability margin is maximized [1].

In the case of multiple eigenvalues the bases of the invariant subspaces are not uniquely defined. We show here that, provided the eigenvectors spanning each subspace are chosen to be orthonormal, then the Frobenius condition of the modal matrix remains invariant under changes of basis. Explicitly, we show that, under this assumption, the Frobenius condition number is equal to the sum of the inverse squares of the cosines of all the canonical angles between the right and left invariant subspaces corresponding to distinct eigenvalues and that for some scaling of the eigenvectors, this measure is exactly equal to a weighted sum of the squares of the condition numbers of the eigenvalues. Minimizing the Frobenius conditioning of the modal matrix, therefore, also guarantees good conditioning of the assigned poles in the multiple eigenvalue case and again leads to other desirable properties of the closed loop system.

A technique for selecting complete orthonormal bases for the invariant subspaces corresponding to the assigned eigenvalues, such as to minimize the Frobenius conditioning of the modal matrix, is also described here. This procedure is a modification of the method we have previously developed for solving the robust pole placement problem.

In the next section we examine conditioning, or robustness measures, and in Section 3 the numerical algorithm for determining a robust solution is described. Results and conclusions are given in Sections 4 and 5.

2. MEASURES OF ROBUSTNESS

We define a closed loop, linear, dynamic system with $n \times n$ coefficient matrix M to be robust if its eigenvalues, or poles, are as insensitive to perturbations in M as possible. For non-defective systems, the sensitivity, or condition number, of a distinct eigenvalue, λ_j, of multiplicity p_j, is given by the inverse of the cosine of the smallest "canonical angle" between its right and left invariant subspaces, $\mathcal{X}_j$ and $\mathcal{Y}_j$. We let $\hat{X}_j$, $\hat{Y}_j$ give orthonormal bases for $\mathcal{X}_j$, $\mathcal{Y}_j$ such that

$$\hat{Y}_j^T \hat{X}_j = \Sigma_j \equiv \mathrm{diag}\{\sigma_{j1}, \sigma_{j2}, \ldots \sigma_{jp_j}\}, \tag{2.1}$$

where $1 \geqq \sigma_{j1} \geqq \sigma_{j2} \geqq .. \geqq \sigma_{jp_j} > 0$, $\forall_j$. (To construct $\hat{X}_j$, $\hat{Y}_j$, we take any orthonormal bases, given by X_j, Y_j, of the invariant subspaces, find the singular value decomposition (SVD) given by $Y_j^T X_j = U_j \Sigma_j V_j^*$, and choose $\hat{X}_j = X_j V_j$ and $\hat{Y}_j = Y_j U_j$.) Then σ_{jk}, $k = 1,2,..p_j$ are the cosines of the canonical angles associated with the subspaces $\mathcal{X}_j$, $\mathcal{Y}_j$ and are independent of the choice of bases.

If M is non-defective and a perturbation $O(\varepsilon)$ is made in the coefficients of the matrix M, then the corresponding first order perturbation in the eigenvalue λ_j of M is of the order of $\varepsilon n c_j$, where the sensitivity, or condition number, c_j, is given by

$$c_j \equiv \sigma_{jp_j}^{-1} \geqq 1, \tag{2.2}$$

that is, the inverse of the cosine of the smallest corresponding canonical angle. If M is defective, then the corresponding perturbation in some eigenvalue is at least an order of magnitude worse in ε, and, therefore, defective system matrices are necessarily less robust than those which are non-defective.

We note that in the case λ_j is a simple eigenvalue, ($p_j = 1$), then c_j may be written directly as

$$c_j \equiv \| \underline{y}_j \|_2 \, \| \underline{x}_j \|_2 \, / \, |y_j^T x_j| , \tag{2.3}$$

where $\underline{x}_j$, $\underline{y}_j$ are right and left eigenvectors corresponding to λ_j.

We now assume, without loss of generality, that $X = [X_1, X_2, \ldots X_q]$, and $Y = [Y_1, Y_2, \ldots Y_q]$ are the modal matrices of right and left eigenvectors of (non-defective) matrix M, respectively, where X_j, Y_j give full bases for the right and left invariant subspace corresponding to eigenvalue λ_j of multiplicity p_j, $\sum_{j=1}^{q} p_j = n$, and X, Y are scaled such that all the columns $\underline{x}_k$ of X have unit length ($\| \underline{x}_k \|_2 = 1$) and $Y^T X = I$. Different scalings of the eigenvectors are then given by XD^{-1} and DY^T, respectively, where D is a block diagonal matrix given by

$$D = \mathrm{diag}\{d_1 I_{p_1}, d_2 I_{p_2}, \ldots d_q I_{p_q}\}. \tag{2.4}$$

We consider now three measures of the robustness of M. The first is

$$\nu_1 = \max_j c_j \equiv \max_j \sigma_{jp_j}^{-1} , \tag{2.5}$$

the maximum of the condition numbers of the eigenvalues. Alternatively, we have as a measure of robustness

$$\nu_2(D) = \kappa_2(XD^{-1}) \equiv \| XD^{-1} \|_2 \, \| DX^{-1} \|_2 , \tag{2.6}$$

the ℓ_2 condition number of the scaled modal matrix. It can be shown that

$$1 \leq \nu_1 \leq \nu_2(D), \tag{2.7}$$

so $\nu_2(D)$ gives an upper bound on ν_1, and that both measures attain their (common) minimal value simultaneously, when the eigenvalues of M are perfectly conditioned $(c_j = 1, \quad \forall_j)$.

The third measure is proportional to the Frobenius condition number, discussed in the introduction, and is given by

$$\nu_3(D) = \kappa_F(XD^{-1})/\kappa_F(D) \equiv \| XD^{-1} \|_F \| DX^{-1} \|_F / \| D \|_F \| D^{-1} \|_F . \tag{2.8}$$

Under the assumptions,

$$\| XD^{-1} \|_F = \| D^{-1} \|_F = \Big(\sum_{j=1}^{q} p_j d_j^{-2}\Big)^{\frac{1}{2}}, \quad \| DX^{-1} \|_F = \| DY^T \|_F , \tag{2.9}$$

and, hence,

$$\nu_3(D) = \| DY^T \|_F / \| D \|_F \equiv \Big(\sum_{j=1}^{q} d_j^2 \| Y_j^T \|_F^2\Big)^{\frac{1}{2}} \Big/ \Big(\sum_{j=1}^{q} p_j d_j^2\Big)^{\frac{1}{2}} . \tag{2.10}$$

We remark that the first two measures are of interest theoretically [1], but it is the third measure which is used in practice.

We now establish the relationship between the measure $\nu_3(D)$ and the condition numbers c_j of the eigenvalues. We make the assumption that the bases, given by X_j, of the invariant subspaces $\mathcal{X}_j$ are orthonormal. Then, letting $\hat{X}_j$, $\hat{Y}_j$ denote the particular orthonormal bases satisfying (2.1), we may write $X_j = \hat{X}_j Z_j$, where Z_j is unitary. By the assumptions, $Y_j^T X_j = I$ and, therefore, $Y_j = \hat{Y}_j \Sigma_j^{-1} Z_j^{-T}$. It follows that

$$\| Y_j^T \|_F^2 = \| Z_j^{-1} \Sigma_j^{-1} \hat{Y}_j^T \|_F^2 = \sum_{k=1}^{p_j} \sigma_{jk}^{-2} \tag{2.11}$$

and, therefore,

$$\| DY^T \|_F^2 = \sum_{j=1}^{q} \Big(d_j^2 \sum_{k=1}^{p_j} \sigma_{jk}^{-2}\Big) . \tag{2.12}$$

We have also

$$c_j^2 \equiv \sigma_{jp_j}^{-2} < \sum_{k=1}^{p_j} \sigma_{jk}^{-2} \leqq p_j\, \sigma_{jp_j}^{-2} \equiv p_j\, c_j^2, \tag{2.13}$$

and we may, thus, write

$$\| Y_j^T \|_F \equiv \Big(\sum_{k=1}^{p_j} \sigma_{jk}^{-2}\Big)^{\frac{1}{2}} = \theta_j c_j, \tag{2.14}$$

where

$$1 < \theta_j \leqq \sqrt{p_j}. \tag{2.15}$$

From (2.11)-(2.12) we then find

$$\sum_{j=1}^{q} d_j^2 c_j^2 \leqq \| DY^T \|_F^2 = \sum_{j=1}^{q} d_j^2 \theta_j^2 c_j^2 \leqq \sum_{j=1}^{q} d_j^2 p_j c_j^2. \tag{2.16}$$

This proves the following theorem.

Theorem 1 Let M be a non-degenerate matrix with eigenvalues λ_j of multiplicity p_j and complete orthonormal bases, given by X_j, for the corresponding invariant subspaces, $j = 1,2,..q$, and let $X = [X_1, X_2, ..X_q]$, $D = \mathrm{diag}\{d_1 I_{p_1}, d_2 I_{p_2} .. d_j I_{p_j}\}$. Then

$$\| DX^{-1} \|_F^2 = \sum_{j=1}^{q} \hat{d}_j^2 c_j^2, \tag{2.17}$$

where

$$d_j < \hat{d}_j \leqq \sqrt{p_j}\, d_j. \tag{2.18}$$

□

From (2.12) we obtain directly

$$\nu_3(D) = \Big(\sum_{j=1}^{q} d_j^2 \sum_{k=1}^{p_j} \sigma_{jk}^{-2}\Big)^{\frac{1}{2}} \Big/ \Big(\sum_{j=1}^{q} p_j d_j^2\Big)^{\frac{1}{2}}, \tag{2.19}$$

and we observe that $\nu_3(D)$ takes its minimal value, unity, if and only if $c_j \equiv \sigma_{jp_j}^{-1} = 1$, $\forall_j$, or, equivalently, X is unitary. We have, furthermore, from (2.16) that

$$\nu_3(D) \leq \Big(\sum_{j=1}^{q} p_j d_j^2 c_j^2\Big)^{\frac{1}{2}} / \Big(\sum_{j=1}^{q} p_j d_j^2\Big)^{\frac{1}{2}} \leq \max_j c_j \equiv \nu_1 \qquad (2.20)$$

Using (2.8) and norm equivalences, we also find

$$\nu_3(D) \geq \kappa_2(XD^{-1})/\kappa_F(D) = \nu_2(D)/\kappa_F(D). \qquad (2.21)$$

We conclude then that

$$1 \leq \nu_3(D) \leq \nu_1 \leq \nu_2(D) \leq \kappa_F(D)\nu_3(D), \qquad (2.22)$$

and, therefore, the measures ν_1, $\nu_2(D)$ and $\nu_3(D)$ are mathematically equivalent and take their minimal values simultaneously, when the closed loop system is perfectly robust.

From (2.16) it also follows that

$$\nu_3(D) \geq \Big(\sum_{j=1}^{q} d_j^2 c_j^2\Big)^{\frac{1}{2}} / \|D\|_F, \qquad (2.23)$$

and, thus, for a particular choice of the weights d_j minimizing any of the three robustness measures minimizes an upper bound on the correspondingly weighted sum of squares of the condition numbers.
In the next section we describe a procedure for constructing the modal matrix X of eigenvectors such as to minimize $\nu_3(D)$.

3. ROBUST POLE ASSIGNMENT

We now consider the time invariant linear multivariable system described by the matrix pair (A, B), where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and B is of full rank. The robust pole assignment problem is defined as follows [1].

Problem 1 Given matrix pair (A, B), $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and set $\mathcal{L} = \{\lambda_j \in \mathbb{C},\ j = 1,2,..n\}$ where $\lambda_j \in \mathcal{L} \Leftrightarrow \overline{\lambda}_j \in \mathcal{L}$, find matrix $F \in \mathbb{R}^{m\times n}$ and non-singular matrix $X \in \mathbb{C}^{n\times n}$, satisfying

$$(A + BF)X = X\Lambda, \tag{3.1}$$

where $\Lambda = \mathrm{diag}\{\lambda_j\}$, such that some measure ν of the conditioning of the eigenproblem is optimized.

□

We remark that the measure ν could be chosen to be any of the three measures defined in §2, but here we are mainly interested in the measure $\nu_3(D)$, (as given by (2.10)).

We remark also that no assumption on the controllability of (A, B) is made. Although the uncontrollable modes of the system cannot be affected by the feedback F, as long as these modes are included in the set $\mathcal{L}$ to be assigned, a solution to the feedback problem may exist [5] and eigenvectors corresponding to these modes can be modified. Therefore the conditioning of uncontrollable modes can be improved by an appropriate choice of F.

We remark finally that in the robust pole placement problem (Problem 1), the choice of eigenvectors which may be assigned is restricted such that the closed loop system matrix $M \equiv A + BF$ is non-defective. This restriction implies a simple limitation on the multiplicity of the poles which may be assigned.

Conditions under which a given non-singular matrix X of eigen-vectors may be assigned are given by the following.

Theorem 2 Given Λ and non-singular X, then there exists F, a solution satisfying (3.1) if and only if

$$U_1^T(AX - X\Lambda) = 0, \tag{3.2}$$

where

$$B = [U_0, U_1]\begin{bmatrix} Z \\ 0 \end{bmatrix}, \tag{3.3}$$

with $U = [U_0, U_1]$ orthogonal and Z non-singular. Then F is given explicitly by

$$F = Z^{-1}U_0^T(X\Lambda X^{-1} - A) \tag{3.4}$$

□

The proof is given elsewhere [1].

An immediate consequence of Theorem 2 is the following

Corollary 2.1 The eigenvector $\underline{x}_j$ of $M \equiv A + BF$ corresponding to the assigned eigenvalue $\lambda_j \in \mathcal{L}$ must belong to the space

$$\mathcal{S}_j = \mathcal{N}\{U_1^T(A - \lambda_j I)\}, \tag{3.5}$$

where the dimension of $\mathcal{S}_j$ is given by

$$\dim(\mathcal{S}_j) = m + \dim(\mathcal{N}\{[B|A - \lambda_j I]^T\}). \tag{3.6}$$

□

(Here $\mathcal{N}\{\cdot\}$ denotes right null space). The proof is again given in [1].

The robust pole assignment problem now reduces to the problem of selecting independent vectors $\underline{x}_j \in \mathcal{S}_j$, $j = 1, 2, \ldots, n$, such that the closed loop system matrix is as robust as possible. From the corollary we deduce that any mode can be assigned arbitrarily with multiplicity at most m. (In the case of a controllable mode, $\dim(\mathcal{S}_j) = m$, and, therefore, m is the maximum number of independent eigenvectors which can be chosen to correspond to the pole. For an uncontrollable mode of multiplicity k, $\dim(\mathcal{S}_j) = m + k$, and essentially the same result holds [1].

Theorem 2 has a number of further consequences. From the theorem it can be shown that minimizing the conditioning of the modal matrix X leads to other desirable properties in the closed loop system. In particular, it can be shown that the feedback matrix F, the transient response of the closed loop system and the maximum stability margin can all be bounded in terms of the robustness measure $\nu_2(D) \equiv \kappa_2(XD^{-1})$ and the given data of the problem [1]. From the equivalence of the measures, as derived in §2, it follows that minimizing any of the measures ν_i, $i = 1, 2, 3$, then minimizes upper bounds on the gain matrix and transient response, and maximizes a lower bound on the stability margin. We remark that the optimal robustness which can be achieved is limited, however, and a lower bound on the attainable conditioning can be given in terms of the poles to be assigned [1].

We now present a procedure for constructing a solution to the robust pole assignment problem (Problem 1) which minimizes the robustness measure $\nu_3(D)$ under the assumptions of §2. Three steps are required.

Step A: Determine the decomposition of matrix B, given by (3.3), and construct orthonormal bases, given by S_j, for the spaces $\mathcal{S}_j$, corresponding to distinct eigenvalues $\lambda_j \in \mathcal{L}$ $j = 1, 2, \ldots, q$.

Step X: Select submatrices $X_j = S_j W_j \subset \mathcal{S}_j$ such that $X_j^* X_j = I$ and $X = [X_1, X_2, \ldots, X_q]$ is well-conditioned, in the sense of the Frobenius measure $\nu_3(D)$.

Step F: Determine the matrix M by solving $MX = X\Lambda$ and find F explicitly from (3.4).

The first and third steps, Step A and Step F are easily accomplished using QR or SVD (Householder or Singular Value) decompositions of matrices and standard techniques for the solution of linear equations. The key step, Step X, is accomplished by an iterative process in which each choice of basis, given by X_j, is updated in turn, for $j = 1, 2, \ldots, q$, in such a way that the measure $\nu_3(D)$ is minimized by each update. The procedure is a modification of Method 1, described in [1], in which a rank-one up-date to matrix X is made at each step of the iteration. Here rank-p_j updates to matrix X are made and at each step a non-linearly constrained least square problem must be solved. We show here that this problem can be solved explicitly. The iteration may be initialized using any set of independent bases $X_j \subset \mathcal{S}_j$ such that $X_j^* X_j = I$. The process is stopped when the reduction in the measure $\nu_3(D)$ after a full sweep ($j = 1, 2, \ldots, q$) is less than a given tolerance.

The technique for determining the update is described here for the case $D = I$ and λ_j real, $j = 1, 2, \ldots, q$. (A detailed description of the complete method is given in [2].) The problem is to find W_j with $W_j^* W_j = I$ to minimize $\|X^{-1}\|_F$ where $X_j = S_j W_j$ and $X_- = [X_1, X_2, \ldots X_{j-1}, X_{j+1}, \ldots, X_q]$ is assumed known. We may write

$$\|X^{-1}\|_F = \|[X_-, S_j W_j]^{-1}\|_F = \|[Y_-, Y_j]^T\|_F = \|Y^T\|_F. \tag{3.7}$$

By QR decomposition we obtain

$$X_- = [Q_1, Q_2]\begin{bmatrix} R_1 \\ 0 \end{bmatrix}, \tag{3.8}$$

and then $Y^T X = I$ implies

$$\| Y^T \|_F = \left\| \begin{bmatrix} R_1^{-1} & -R_1^{-1}R_2R_3^{-1} \\ 0 & R_3^{-1} \end{bmatrix} \right\|_F, \tag{3.9}$$

where

$$R_2 = Q_1^T S_j W_j, \qquad R_3 = Q_2^T S_j W_j. \tag{3.10}$$

To minimize $\nu_3(I)$ it is thus necessary to minimize

$$\| R_1^{-1}R_2R_3^{-1} \|_F + \| R_3^{-1} \|_F. \tag{3.11}$$

We now determine, by a further QR decomposition, a unitary matrix $V = [V_1, V_2]$ such that

$$Q_2^T S_j = [R_4, 0][V_1, V_2]^* = R_4 V_1^*, \tag{3.12}$$

and let

$$U \equiv \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = V^* W_j \equiv \begin{bmatrix} V_1^* W_j \\ V_2^* W_j \end{bmatrix}, \qquad U_3 = U_2 U_1^{-1} \tag{3.13}$$

Then $U^*U = I$, since $W_j^* W_j = I$, and we may complete U such that $[U, U^\perp]$ is unitary and $U^*[U, U^\perp] = [I, 0]$. From (3.10), (3.12) and (3.13) it then follows that $R_3 = R_4 U_1$, and we have

$$\| R_3^{-1} \|_F = \| U_1^{-1} R_4^{-1} \|_F = \| [U, U^\perp]^* U U_1^{-1} R_4^{-1} \|_F = \left\| \begin{bmatrix} I \\ U_3 \end{bmatrix} R_4^{-1} \right\|_F. \tag{3.14}$$

We also have

$$R_1^{-1}R_2R_3^{-1} = R_1^{-1}Q_1^T S_j V U U_1^{-1} R_4^{-1} = R_1^{-1} Q_1^T S_j V \begin{bmatrix} I \\ U_3 \end{bmatrix} R_4^{-1}, \tag{3.15}$$

and denoting $\tilde{W} = U_3 R_4^{-1}$, it follows that to minimize (3.11) it is necessary to minimize

$$\left\| \begin{bmatrix} R_1^{-1} Q_1^T S_j V \\ I \end{bmatrix} \begin{bmatrix} I \\ \tilde{W} \end{bmatrix} \right\|_F \equiv \left\| \begin{bmatrix} R_1^{-1} Q_1^T S_j \\ V^* \end{bmatrix} [V_1 + V_2 \tilde{W}] \right\|_F, \tag{3.16}$$

which takes the form of a standard least square problem for $\tilde{W}$. The solution $\tilde{W}$ is determined by a further QR decomposition and the required W_j is obtained from

$$W_j = (V_1 + V_2\tilde{W}R_4)Z \, , \tag{3.17}$$

where Z is constructed by a Cholesky (or Schur) decomposition using

$$Z^*Z = I + R_4^*\tilde{W}^*\tilde{W}R_4^* \, . \tag{3.18}$$

The up-date minimizing $\nu_3(D)$ with respect to the choice of X_j is thus obtained <u>explicitly</u> using three QR and one Cholesky decompositions, which can all be computed efficiently and stably using standard library software.

4. RESULTS

To illustrate the form of the robust solutions determined by the method described in §3 we give here the results obtained for a simple test problem.

Test Example $n = 3 \quad m = 2$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

The eigenvalues of A are $\{1.0, 2.0, 3.0\}$. We assign the stable eigenvalue set $\mathcal{L} = \{-0.2, -0.2, -10.0\}$. The assigned eigenvectors are selected to be such that

$$X = \begin{bmatrix} 0.84284 & -0.35924 & 0.93741 \\ 0.49283 & 0.21617 & -0.29576 \\ 0.21617 & 0.90786 & -0.18381 \end{bmatrix},$$

and the feedback F is calculated to be

$$F = \begin{bmatrix} -6.7866 & 12.855 & -5.9053 \\ 2.0781 & -4.5713 & 0.86316 \end{bmatrix}.$$

We observe that the first two columns of X form an orthonormal basis for the two-dimensional invariant subspace corresponding to the assigned eigenvalue $\lambda = -0.2$, of multiplicity two, and the third column gives the single eigenvector corresponding to the assigned simple eigenvalue $\lambda = -10.0$, selected such as to minimize the robustness measure $\nu_3(I)$. The solution has robustness $\nu_3(I) = 2.7209/\sqrt{3}$.

With a different set of initial vectors, the solution

$$X = \begin{bmatrix} -0.91621 & 0.00000 & 0.93741 \\ -0.36861 & 0.39210 & -0.29576 \\ 0.15711 & 0.91992 & -0.18381 \end{bmatrix}$$

is obtained. Here the first two vectors of X form a different orthonormal basis for the (same) invariant subspace corresponding to the multiple eigenvalue, and the third vector, selected to minimize the conditioning of the simple eigenvalue with respect to this subspace, is the same as that chosen previously. The robustness measure takes the same value $\nu_3(I) = 2.7209/\sqrt{3}$ and the same feedback F is determined.

To demonstrate the effects of perturbations in the system coefficients, we round the feedback matrix F to three significant figures and calculate the eigenvalues of the resulting closed loop system matrix. Rounding the feedback matrix here corresponds to introducing maximum absolute errors of about ± 0.05 into the system matrix. For robust solutions such perturbations should only cause errors of the same order of magnitude in the poles of the closed loop system. For this test example the absolute errors in the assigned eigenvalues due to these perturbations are $\{0.00284, 0.01269, 0.0225\}$, respectively. A maximum relative error of about 6% is thus obtained in the assigned poles, well within the predicted perturbation for a robust system.

5. CONCLUSIONS

A closed loop system design for pole placement is robust if the assigned eigenvalues are as insensitive as possible to perturbations in the system and feedback matrices. We show here that in the case of multiple eigenvalue assignment a robust design can be achieved by selecting the corresponding invariant subspaces such as to minimize the Frobenius condition of the modal matrix of eigenvectors spanning the subspaces, subject to the subspace bases being orthonormal. A reliable numerical technique for determining a feedback which minimizes this measure of robustness is described, and an illustration is presented. The results derived are extensions of earlier work [1] applicable to simple eigenvalue assignment. Generalizations of this approach to robust pole assignment for problems of feedback in degenerate (descriptor) systems and for output feedback problems are now being developed.

6. REFERENCES

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