CONVERGENCE OF ROE'S SCHEME
FOR THE GENERAL NON-LINEAR SCALAR WAVE EQUATION

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Abstract

Convergence is proved for the approximation generated by the second order scheme of P. Roe to a weak solution of the non-linear scalar wave equation for variable wave speeds.

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CONTENTS

1. INTRODUCTION

In a recent paper Le Roux [2] proved the convergence of a quasi-second order scheme for the non-linear scalar wave equation

$$u_t + f'(u)u_x = 0$$

The scheme is second order accurate except in certain special regions where it is only first order accurate. In an earlier report [1] we proved similar convergence for the scheme of P. Roe [3] in the case where the non-linear term $f(u)$ is monotonic, i.e. when the wave speed is one-signed. In the present report we extend this proof to more general $f(u)$, showing that Roe's scheme converges to a weak solution of the Cauchy problem for non-monotonic $f(u)$. We have taken the opportunity to co-ordinate the results and simplify the notation of the earlier report : as a result the present report is self-contained.

Section 2 contains a description of the problem and the difference scheme. Some results preliminary to the main theorem are contained in Section 3 while the main convergence theorem itself is proved in Section 4. Conclusions and remarks are in Section 5.

2. THE PROBLEM AND THE DIFFERENCE SCHEME

(a) The Problem.

We consider the equation

$$u_t + [f(u)]_x = 0 \tag{2.1}$$

for (x,t) in $\mathbb{R} \times (0,T)$, $T>0$ and f in $C^1(\mathbb{R})$, with

$$u(x,0) = u_0(x) \tag{2.2}$$

for x in $\mathbb{R}$ and u_0 in $L^\infty(\mathbb{R})$, assumed to be of locally bounded variation on $\mathbb{R}$ and therefore satisfying, for all real δ,

$$\forall R \geq 0 \quad , \quad \int_{|x|<R} |u_0(x+\delta) - u_0(x)| \, dx \leq C(R)|\delta| \tag{2.3}$$

where C is an increasing function on $[0, \infty)$, independent of δ.

The Cauchy problem associated with (2.1) and (2.2) is to find a bounded function u which satisfies (2.1), (2.2). A weak solution to the Cauchy problem is a function u in $L^\infty(\mathbb{R} \times (0,T))$ which satisfies an integral form of (2.1), namely,

$$\iint_{\mathbb{R} \times (0,T)} \left(u \frac{\partial \psi}{\partial t} + f(u) \frac{\partial \psi}{\partial x} \right) dx\, dt + \int_{\mathbb{R}} u_0(x)\psi(x,0)dx = 0 \tag{2.4}$$

for all test functions ψ in $C^2(\mathbb{R} \times [0,T])$ of compact support in $\mathbb{R} \times [0,T]$

We consider the approximations generated by the finite difference scheme of Roe [3] and discuss their convergence to such a weak solution of the Cauchy problem.

Let h be the spatial grid size, with $0<h<h_0$, and Δt be the time grid size, related to h by the fixed positive number q through

$$q = \frac{\Delta t}{h} \tag{2.5}$$

In a neighbourhood of the gridpoint $(kh, n\Delta t)$ define the rectangle

$$I_k \times J_n = [(k-\tfrac{1}{2})h, (k+\tfrac{1}{2})h) \times [(n-\tfrac{1}{2})qh, (n+\tfrac{1}{2})qh) \tag{2.6}$$

for $k \in \mathbb{Z}$, $n \in \mathbb{N}$ and $n \leq N = [T/qh] + 1$, where $[y]$ denotes the integer part of y.

We approach a weak solution of (2.1), (2.2) in the sense of (2.4) by a piecewise constant function u_h defined on $\mathbb{R} \times (0,T)$ by

$$u_h(x,t) = u_k^n \quad \text{for} \quad (x,t) \in I_k \times J_n , \tag{2.7}$$

where the initial condition (2.2) is projected onto the space of piecewise constant functions by the restriction

$$u_k^o = \frac{1}{h} \int_{I_k} u_o(x)\, dx \tag{2.8}$$

(b) The Difference Scheme.

The values u_k^n are calculated as follows (see [3]). For brevity we write

$$u_k = u_k^n \quad , \; u^k = u_k^{n+1} \tag{2.9}$$

whenever there is no danger of confusion.

Let $\nu_{k-\frac{1}{2}}$ be the approximation

$$\nu_{k-\frac{1}{2}} = q \frac{\delta f_{k-\frac{1}{2}}}{\delta u_{k-\frac{1}{2}}} \tag{2.10}$$

to the CFL number in $I_{k-\frac{1}{2}}$ where $f_{k-\frac{1}{2}} = f(u_{k-\frac{1}{2}})$ and $\delta f_{k-\frac{1}{2}} = f_k - f_{k-1}$

Let also

$$s_{k-\frac{1}{2}} = \mathrm{sgn}(\nu_{k-\frac{1}{2}}) = \pm 1 \tag{2.11}$$

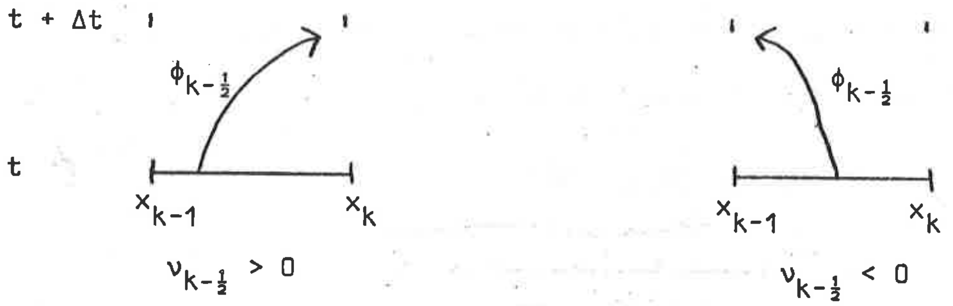
be the sign of $\nu_{k-\frac{1}{2}}$, and define

$$\phi_{k-\frac{1}{2}} = -q\delta f_{k-\frac{1}{2}} = -\nu_{k-\frac{1}{2}} \delta u_{k-\frac{1}{2}} \tag{2.12}$$

to be the flux increment or <u>fluctuation</u> in the cell $I_{k-\frac{1}{2}}$. The direction indicated by (2.11) will be called the downwind direction.

We obtain a first order accurate scheme when the quantity $\phi_{k-\frac{1}{2}}$ is added to the value of u at the downwind end of the cell over the time step Δt. (If $\nu_{k-\frac{1}{2}} = 0$, then $\phi_{k-\frac{1}{2}} = 0$ so that no ambiguity arises). This is Godunov's first order upwinded scheme, which can be represented graphically as in Fig. 1.

Fig. 1 First Order Scheme



Now let $k' = k - s_{k-\frac{1}{2}}$, $\alpha_{k-\frac{1}{2}} = \frac{1}{2}(1 - |\nu_{k-\frac{1}{2}}|)$ (2.13)

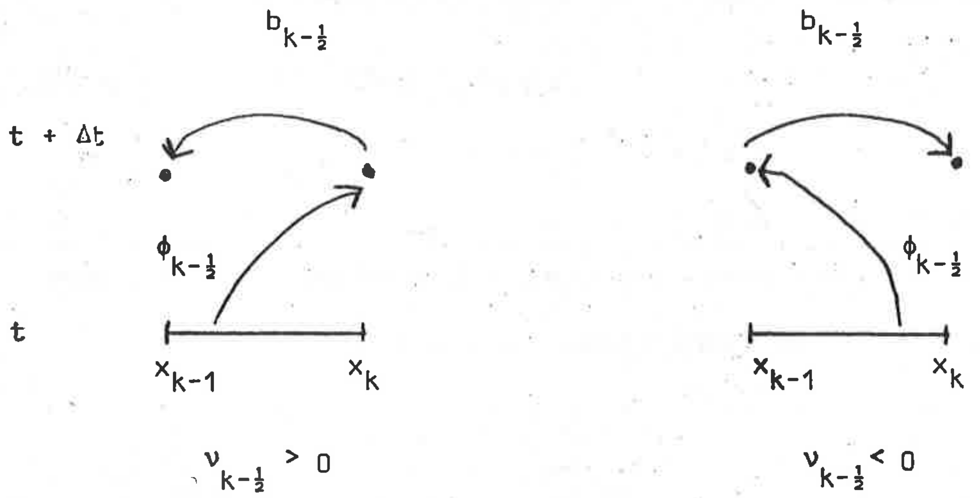
and define the quantity $b_{k-\frac{1}{2}}$ by

$$|b_{k-\frac{1}{2}}| = \tfrac{1}{2}|s_{k-\frac{1}{2}} + s_{k'-\frac{1}{2}}| \min\{|\alpha_{k-\frac{1}{2}}\phi_{k-\frac{1}{2}}|, |\alpha_{k'-\frac{1}{2}}\phi_{k'-\frac{1}{2}}|\} \quad (2.14)$$

where $\frac{1}{2}|s_{k'-\frac{1}{2}} + s_{k+\frac{1}{2}}|$ is 1 except in the special case of expansion waves where it is zero (see § 5) and with the sign of $b_{k-\frac{1}{2}}$ equal to that of $\phi_{k-\frac{1}{2}}$.

If $b_{k-\frac{1}{2}}$ is now transferred across the cell <u>against</u> the stream we generate a scheme which is second order except at isolated points, where the minimum selection in (2.14) changes. This scheme may a.e. be identified as either the Lax-Wendroff scheme or the second order upwinded scheme of Warming & Beam, depending on the choice in (2.14). The transfer of $b_{k-\frac{1}{2}}$ can be regarded as an antidiffusion step (see [2]) and the whole process can be represented graphically as in Fig. 2.

Fig. 2 Second Order Scheme



This is Roe's scheme which has been very successful in eliminating oscillations in approximations to the solution of the problem (2.1), (2.2). It may be written in the form

$$\left.\begin{aligned} u^k &= u_k + \phi_{k-\frac{1}{2}} + b_{k+\frac{1}{2}} - b_{k-\frac{1}{2}} \qquad && \nu_{k-\frac{1}{2}},\ \nu_{k+\frac{1}{2}} \geq 0 \\ u^k &= u_k + \phi_{k+\frac{1}{2}} - b_{k+\frac{1}{2}} + b_{k-\frac{1}{2}} \qquad && \nu_{k-\frac{1}{2}},\ \nu_{k+\frac{1}{2}} \leq 0 \end{aligned}\right\} \qquad (2.15)$$

and

$$\left.\begin{aligned} u^k &= u_k && \nu_{k-\frac{1}{2}} < 0,\ \nu_{k+\frac{1}{2}} > 0 \\ u^k &= u_k + \phi_{k-\frac{1}{2}} + \phi_{k+\frac{1}{2}} && \\ &\qquad - b_{k-\frac{1}{2}} - b_{k+\frac{1}{2}} && \nu_{k-\frac{1}{2}} > 0,\ \nu_{k+\frac{1}{2}} < 0 \end{aligned}\right\} \qquad (2.16)$$

In subsequent work it will be convenient to define two numbers $\beta_{k-\frac{1}{2}}$ and $\beta'_{k-\frac{1}{2}}$, namely,

$$\beta_{k-\frac{1}{2}} = \frac{b_{k-\frac{1}{2}}}{\alpha_{k-\frac{1}{2}}\ \phi_{k-\frac{1}{2}}} \qquad\qquad \beta'_{k-\frac{1}{2}} = \frac{b_{k-\frac{1}{2}}}{\alpha_{k'-\frac{1}{2}}\ \phi_{k'-\frac{1}{2}}}, \tag{2.17}$$

which, from the definition (2.14), have the properties

$$0 \le \beta_{k-\frac{1}{2}} \le 1 \qquad\qquad |\beta'_{k-\frac{1}{2}}| \le 1 \tag{2.18}$$

3. PRELIMINARIES

We now prove two lemmas and quote a theorem which will be used in the proof of the main convergence theorem in §4.

Lemma 1

A difference scheme in the form

$$u^k = u_k + \xi_{k+\frac{1}{2}}\ \phi_{k+\frac{1}{2}} + \zeta_{k-\frac{1}{2}}\ \phi_{k-\frac{1}{2}}, \tag{3.1}$$

where $\phi_{k\pm\frac{1}{2}}$ is defined by (2.12) and $\xi_{k+\frac{1}{2}}$, $\zeta_{k-\frac{1}{2}}$ may be data dependent, conserves local stability in the sense

$$\inf\ \{u_{k-1}, u_k, u_{k+1}\} \le u^k \le \sup\ \{u_{k-1}, u_k, u_{k+1}\} \tag{3.2}$$

if the following inequalities are satisfied:

$$\left.\begin{aligned} 0 &\le \zeta_{k-\frac{1}{2}}\ \nu_{k-\frac{1}{2}} \\ 0 &\le -\xi_{k+\frac{1}{2}}\ \nu_{k+\frac{1}{2}} \\ 0 &\le -\xi_{k+\frac{1}{2}}\ \nu_{k+\frac{1}{2}} + \zeta_{k-\frac{1}{2}}\ \nu_{k-\frac{1}{2}} \le 1 \end{aligned}\right\} \tag{3.3}$$

Proof

$$\begin{aligned} u^k &= u_k + \xi_{k+\frac{1}{2}}\ \phi_{k+\frac{1}{2}} + \zeta_{k-\frac{1}{2}}\ \phi_{k-\frac{1}{2}} \\ &= u_{k-1}\ \{\zeta_{k-\frac{1}{2}}\ \nu_{k-\frac{1}{2}}\} \\ &+ u_k\ \{1 + \xi_{k+\frac{1}{2}}\ \nu_{k+\frac{1}{2}} - \zeta_{k-\frac{1}{2}}\ \nu_{k-\frac{1}{2}}\} \\ &+ u_{k+1}\ \{-\xi_{k+\frac{1}{2}}\ \nu_{k+\frac{1}{2}}\} \end{aligned} \tag{3.4}$$

If the inequalities (3.3) are satisfied the coefficients of the u's are non-negative and we obtain

$$u^k \le \{\zeta_{k-\frac{1}{2}}\, \nu_{k-\frac{1}{2}}\}\, u_{max} + \{1 + \xi_{k+\frac{1}{2}}\, \nu_{k+\frac{1}{2}} - \zeta_{k-\frac{1}{2}}\, \nu_{k-\frac{1}{2}}\}\, u_{max}$$

$$+ \{-\xi_{k+\frac{1}{2}}\, \nu_{k+\frac{1}{2}}\}\, u_{max} \quad ,$$

from which $u^k \le u_{max}$, where

$$u_{max} = \sup\, \{u_{k+1}\; u_k,\; u_{k+1}\}$$

Similarly, $u^k \ge u_{min}$, where

$$u_{min} = \inf\, \{u_{k-1},\; u_k,\; u_{k+1}\}$$

This completes the proof.

<u>Lemma 2</u>

A difference scheme in the form (3.1) conserves local bounded variation in the sense

$$\sum_{|k|\le K} |u^{n+1}_{k+1} - u^{n+1}_k| \le \sum_{|k|\le K+n} |u^o_{k+1} - u^o_k| \qquad (3.5)$$

for all $K>0$ if the following inequalities are satisfied.

$$\left.\begin{array}{l} 0 \le -\,\xi_{k+\frac{1}{2}}\, \nu_{k+\frac{1}{2}} \\ 0 \le \zeta_{k+\frac{1}{2}}\, \nu_{k+\frac{1}{2}} \\ 0 \le (\zeta_{k+\frac{1}{2}} - \xi_{k+\frac{1}{2}})\nu_{k+\frac{1}{2}} \le 1 \end{array}\right\} \qquad (3.6)$$

<u>Proof</u>

$$u^{k+1} - u^k = u_{k+1} - u_k + \xi_{k+3/2}\, \phi_{k+3/2} + (\zeta_{k+\frac{1}{2}} - \xi_{k+\frac{1}{2}})\phi_{k+\frac{1}{2}}$$

$$-\zeta_{k-\frac{1}{2}}\, \phi_{k-\frac{1}{2}} \qquad (3.7)$$

$$= \{-\xi_{k+3/2}\ \nu_{k+3/2}\}\delta u_{k+3/2}$$

$$+ \{1 - (\zeta_{k+\frac{1}{2}} - \xi_{k+\frac{1}{2}})\nu_{k+\frac{1}{2}}\}\delta u_{k+\frac{1}{2}}$$

$$+\{\zeta_{k-\frac{1}{2}}\ \nu_{k-\frac{1}{2}}\}\delta u_{k-\frac{1}{2}} \tag{3.8}$$

Taking absolute values and summing over $|k|\leq K$, we obtain

$$\sum_{|k|\leq K} |u^{k+1} - u^{k}| \leq \sum_{|k|\leq K} \{| -\xi_{k+\frac{1}{2}}\ \nu_{k+\frac{1}{2}}| + | 1 -(\zeta_{k+\frac{1}{2}} - \xi_{k+\frac{1}{2}})\ \nu_{k+\frac{1}{2}}|$$

$$+ |\zeta_{k+\frac{1}{2}}\ \nu_{k+\frac{1}{2}}|\}|u_{k+1} - u_{k}|$$

$$+ |-\xi_{K+3/2}\ \nu_{K+3/2}|\ |u_{K+2/3} - u_{K+\frac{1}{2}}|$$

$$+ |\zeta_{-K-\frac{1}{2}}\ \nu_{-K-\frac{1}{2}}|\ |u_{-K-\frac{1}{2}} - u_{-K-3/2}|, \tag{3.9}$$

using summation by parts. If the inequalities (3.6) hold we may remove the modulus signs in the coefficients, obtaining

$$\sum_{|k|\leq K} |u^{n+1}_{k+1} - u^{n+1}_{k}| \leq \sum_{|k|\leq K} |u^{n}_{k+1} - u^{n}_{k}| \tag{3.10}$$

Repeated application gives (3.5) as required.

This completes the proof.

Helly's Theorem

We now quote Helly's Theorem (see [4] pp. 29-30), which will be required in §4.

Let the sequence of functions $\{\lambda_n(x)\}_0^\infty$ be of uniformly bounded variation in $a \leq x \leq b$ and such that

$$|\lambda_n(a)| < A \qquad (n = 0, 1, 2, \ldots\ldots)$$

for some constant A. Then there exists a set of integers

$$n_0 < n_1 < n_2 < \ldots\ldots.$$

and a function $\lambda(x)$ of bounded variation in $a \leq x \leq b$ such that

$$\lim_{i\to\infty} \lambda_{n_i}(x) = \lambda(x) \qquad (a \leq x \leq b)$$

That is, given a sequence of functions which are uniformly bounded and of uniformly bounded variation on an interval, then it is possible to extract a subsequence which converges to a function of bounded variation.

4. CONVERGENCE OF THE DIFFERENCE SCHEME

We can now state and prove our main theorem.

Theorem

Suppose that u_o lies in $L^\infty(\mathbb{R}) \cap BV_{loc}(\mathbb{R})$ and that the condition

$$\sup_k |\nu_k| \leq \tfrac{1}{2} \tag{4.1}$$

is satisfied. Then the family of approximations $\{u_h\}$ generated by Roe's difference scheme (2.15), (2.16) from initial data (2.8) contains a subsequence $\{u_{h_m}\}$ which converges in $L^1_{loc}(\mathbb{R} \times (0,T))$ towards a weak solution of (2.1), (2.2) as $h_m \to 0$.

Proof

The proof is in three main parts. First we show that the piecewise constant function (2.7) generated by Roe's scheme is uniformly bounded and of uniformly bounded variation in space and time. Then we show that from the family of such functions we can extract a sequence convergent in $L^1_{loc}(\mathbb{R} \times (0,T))$. Finally, we show that the limit function is in fact a weak solution of the problem.

Roe's scheme may be written in the form

$$u^k = u_k + \xi_{k+\frac{1}{2}}\,\phi_{k+\frac{1}{2}} + \zeta_{k-\frac{1}{2}}\,\phi_{k-\frac{1}{2}} \tag{4.2}$$

where $\xi_{k+\frac{1}{2}}$, $\zeta_{k-\frac{1}{2}}$ are given by

$$\zeta_{k-\frac{1}{2}} = \begin{cases} 1 + (b_{k+\frac{1}{2}} - b_{k-\frac{1}{2}})/\phi_{k-\frac{1}{2}} & \nu_{k\pm\frac{1}{2}} > 0 \\ 0 & \nu_{k-\frac{1}{2}} < 0 \\ 1 - b_{k-\frac{1}{2}}/\phi_{k-\frac{1}{2}} & \nu_{k+\frac{1}{2}} < 0,\ \nu_{k-\frac{1}{2}} > 0 \end{cases} \quad (4.3)$$

$$\xi_{k+\frac{1}{2}} = \begin{cases} 0 & \nu_{k+\frac{1}{2}} > 0 \\ 1 - (b_{k+\frac{1}{2}} - b_{k-\frac{1}{2}})/\phi_{k+\frac{1}{2}} & \nu_{k\pm\frac{1}{2}} < 0 \\ 1 - b_{k+\frac{1}{2}}/\phi_{k+\frac{1}{2}} & \nu_{k+\frac{1}{2}} < 0,\ \nu_{k-\frac{1}{2}} > 0 \end{cases} \quad (4.4)$$

Using the definitions of α, β, β' in (2.13) and (2.17), these become

$$\zeta_{k-\frac{1}{2}} = \begin{cases} 1 + (\beta'_{k+\frac{1}{2}} - \beta_{k-\frac{1}{2}})\alpha_{k-\frac{1}{2}} & \nu_{k\pm\frac{1}{2}} > 0 \\ 0 & \nu_{k-\frac{1}{2}} < 0 \\ 1 - \beta_{k-\frac{1}{2}}\,\alpha_{k-\frac{1}{2}} & \nu_{k+\frac{1}{2}} < 0,\ \nu_{k-\frac{1}{2}} > 0 \end{cases} \quad (4.5)$$

$$\xi_{k+\frac{1}{2}} = \begin{cases} 0 & \nu_{k+\frac{1}{2}} > 0 \\ 1 + (\beta'_{k-\frac{1}{2}} - \beta_{k+\frac{1}{2}})\alpha_{k+\frac{1}{2}} & \nu_{k\pm\frac{1}{2}} < 0 \\ 1 - \beta_{k+\frac{1}{2}}\,\alpha_{k+\frac{1}{2}} & \nu_{k+\frac{1}{2}} < 0\ \ \nu_{k-\frac{1}{2}} > 0 \end{cases} \quad (4.6)$$

We have from (2.18) and (2.13) the inequalities

$$-2 \leq (\beta'_{k\pm\frac{1}{2}} - \beta_{k\mp\frac{1}{2}}) \leq 1 \quad (4.7)$$

and

$$0 \leq \tilde{\alpha}_{k\pm\frac{1}{2}} \leq \tfrac{1}{2} \quad (4.8)$$

from which we deduce from (4.5) and (4.6) that

$$\left.\begin{array}{l} 0 \leq -\xi_{k+\frac{1}{2}}\,\nu_{k+\frac{1}{2}} \\ 0 \leq \zeta_{k-\frac{1}{2}}\,\nu_{k-\frac{1}{2}} \end{array}\right\} \qquad (4.9)$$

Consider now the expression

$$-\xi_{k+\frac{1}{2}}\,\nu_{k+\frac{1}{2}} + \zeta_{k-\frac{1}{2}}\,\nu_{k-\frac{1}{2}} \qquad (4.10)$$

If $\nu_{k\pm\frac{1}{2}}$ are of the same sign we have (taking the positive sign as example)

$$\begin{aligned} -\xi_{k+\frac{1}{2}}\,\nu_{k+\frac{1}{2}} + \zeta_{k-\frac{1}{2}}\,\nu_{k-\frac{1}{2}} &= (1 + (\beta_{k+\frac{1}{2}} - \beta_{k-\frac{1}{2}})\alpha_{k-\frac{1}{2}})\nu_{k-\frac{1}{2}} \\ &\leq (1 + \alpha_{k-\frac{1}{2}})\nu_{k-\frac{1}{2}} \\ \text{from (4.7)} \quad &= (3/2 - |\nu_{k-\frac{1}{2}}|)\nu_{k-\frac{1}{2}} \\ &\leq 1 \qquad (4.11) \end{aligned}$$

by condition (4.1).

If $\nu_{k+\frac{1}{2}}$, $\nu_{k-\frac{1}{2}}$ are of opposite sign then there are two cases to consider.

For an expansion wave, i.e. $\nu_{k+\frac{1}{2}} > 0$, $\nu_{k-\frac{1}{2}} < 0$, then $\xi_{k+\frac{1}{2}} = \zeta_{k-\frac{1}{2}} = 0$ so that trivially

$$-\xi_{k+\frac{1}{2}}\,\nu_{k+\frac{1}{2}} + \zeta_{k-\frac{1}{2}}\,\nu_{k-\frac{1}{2}} \leq 1 \qquad (4.12)$$

while for a compression wave (shock), i.e. $\nu_{k+\frac{1}{2}} < 0$, $\nu_{k-\frac{1}{2}} > 0$, (4.10) is

$$\begin{aligned} -\xi_{k+\frac{1}{2}}\,\nu_{k+\frac{1}{2}} + \zeta_{k-\frac{1}{2}}\,\nu_{k-\frac{1}{2}} &= -(1-\beta_{k+\frac{1}{2}}\,\alpha_{k+\frac{1}{2}})\nu_{k+\frac{1}{2}} + (1-\beta_{k-\frac{1}{2}}\,\alpha_{k-\frac{1}{2}})\nu_{k-\frac{1}{2}} \\ &\leq \nu_{k-\frac{1}{2}} - \nu_{k+\frac{1}{2}} \\ \text{by condition (4.1)} \quad &\leq 1 \qquad (4.13) \end{aligned}$$

Consider next the expression

$$(\zeta_{k+\frac{1}{2}} - \xi_{k+\frac{1}{2}})\nu_{k+\frac{1}{2}} \qquad (4.14)$$

From (4.5), (4.6)

$$\left.\begin{array}{l} \zeta_{k+\frac{1}{2}} = 0 \quad \text{if} \quad \nu_{k+\frac{1}{2}} < 0 \\ \xi_{k+\frac{1}{2}} = 0 \quad \text{if} \quad \nu_{k+\frac{1}{2}} > 0 \end{array}\right\} \qquad (4.15)$$

so that using (4.11), we have

$$(\zeta_{k+\frac{1}{2}} - \xi_{k+\frac{1}{2}})\nu_{k+\frac{1}{2}} \leq 1 \tag{4.16}$$

Now from (4.9), (4.11), (4.12), (4.13) we see that the conditions of Lemma 1 of § 3. are satisfied, and hence

$$\inf \{u_{k-1}, u_k, u_{k+1}\} \leq u^k \leq \sup \{u_{k-1}, u_k, u_{k+1}\} \tag{4.17}$$

By induction we may readily deduce that

$$\| u_k\|_{L^\infty(\mathbb{R} \times [0,T))} \leq \|u_0\|_{L^\infty(\mathbb{R})} \tag{4.18}$$

Also from (4.9) and (4.16) the conditions of Lemma 2 of §3 are met and hence

$$h\sum_{|k|\leq K} | u_{k+1}^{n+1} - u_k^{n+1} | \leq h \sum_{|k|\leq K+n} | u_{k+1}^0 - u_k^0 | \tag{4.19}$$

for all $K > 0$.

Choose $R > 0$ and set $K = [R/h]$: then, using (2.3), (4.19) becomes

$$\begin{aligned} h \sum_{|k|\leq K} |u_{k+1}^{n+1} - u_k^n| &\leq \int_{|x|\leq R+t/q} | u_0(x + h,t) - u_0(x,t)| \, dx \\ &\leq C(R + T/q)h \\ &\leq C(R + T/q)h_0 \end{aligned} \tag{4.20}$$

where $C(R + \frac{T}{q})$ is a constant depending only on the region Ω_R defined by

$$\Omega_R = (-R,R) \times (0,T) \tag{4.21}$$

Summarizing, we have shown that Roe's scheme generates the family of functions $\{u_h(x,t)\}$ (see 2.7)) with the following properties:-

a) $u_h(x,t) = u_k^n$ in the rectangle$((k-\frac{1}{2})h, (k+\frac{1}{2})h)\times((n-\frac{1}{2})qh, n+\frac{1}{2})qh)$.

b) $u_h(x,t)$ is uniformly bounded by $||u_0||_{L^\infty(\mathbb{R})}$, from (4.18)

c) $u_h(x,t)$ is of uniformly bounded variation in the x co-ordinate, from (4.20)

d) $u(x,t)$ is of uniformly bounded variation in the time co-ordinate, since, from (2.14), (2.15) and (2.16),

$$|u_k^{n+1} - u_k^n| \le \max \{|u_{k+1}^n - u_k^n|, |u_k^n - u_{k-1}^n|\}, \qquad (4.22)$$

so that, from (b) there is also a bound on the time variation of $u_h(x,t)$.

Now, following Oleinik [5], let $t=t_m$ $(m=1,2,\ldots.)$ be a countable everywhere dense set on the segment $[0,T]$ in Ω_R. By Helly's theorem, (see § 3), on any straight line t = constant > 0 we can extract from $\{u_h\}$ a subsequence, converging at every point of this straight line for $h \to 0$.

Hence on the line $t = t_1$ we extract a sequence $\{u_{h_1}\}$ from $\{u_h\}$, then on the line $t = t_2$ we extract from $\{u_{h_1}\}$ a subsequence $\{u_{h_2}\}$ and so on. Then, by means of the diagonal process (see [6] pp. 301), which consists of taking the i'th element of the i'th sequence we can extract a sequence $\{u_h^i\} = \{u_{h_i}^i\}$ $(i\to\infty,\ h\to 0)$ which converges at every point of the family of straight lines $t = t_m$ $(m = 1,2,\ldots)$ for $i \to \infty$.

We now show that $\{u_h^i\}$ is Cauchy in $L^1(\Omega_R)$ for any $t \in (0,T)$

i.e. $$\int_{\Omega_R} |u_h^i(x,t) - u_h^j(x,t)|\, dx\, dt \to 0 \text{ as } i,\ j \to \infty,\ h \to 0, \forall x,t \qquad (4.23)$$

Since u_h is constant on $((k-\frac{1}{2})h,\ (k+\frac{1}{2})h) \times ((n-\frac{1}{2})qh,\ (n+\frac{1}{2})qh)$ we have $u_h(x,t) = u_h(x,\ nqh)$, where $n = [t/qh + \frac{1}{2}]$ and $[y]$ again denotes the integer part of y.

Since the set $t = t_m(m= 1,2,\ldots.)$ is everywhere dense we can choose from it a sequence $\{t_{m_s}\}$ converging to t for $m_s \to \infty$.

Setting $n_s = \left[\frac{t_{m_s}}{qh} + \frac{1}{2}\right]$, we have

$$\int_{\Omega_R} |u_h^i(x,t) - u_h^j(x,t)|\, dx\, dt \le \int_{\Omega_R} |u_h^i(x,nqh) - u_h^i(x,n_s qh)|\,dx\, dt$$

$$+ \int_{\Omega_R} |u_h^j(x,nqh) - u_h^j(x,n_s qh)|\, dx\, dt + \int_{\Omega_R} |u_h^i(x,n_s qh) - u_h^j(x,n_s qh)|\,dx\, dt \qquad (4.24)$$

The first term on the right hand side of (4.24) is bounded by

$$\int_0^T h \sum_{|k|\leq K} |u_k^n - u_k^{n_s}| dt$$

since $K = \left[\frac{R}{h}\right]$, which in turn is bounded by

$$\int_0^T h \sum_{|k|\leq K} \sum_{n=n_1}^{n_2-1} |u_k^{n+1} - u_k^n| dt \tag{4.25}$$

writing n_1, n_2 for the minimum and maximum of n, n_s respectively.

Now from (4.22)

$$h \sum_{|k|\leq K} \sum_{n_1}^{n_2-1} | u_k^{n+1} - u_k^n| \leq 2h \sum_{n_1}^{n_2-1} \sum_{|k|\leq K} |u_{k+1}^n - u_k^n|$$

$$\leq 2\sum_{n_1}^{n_2-1} C(R + T/q)h$$

from (4.20)

$$\leq 2(n_2 - n_1) C(R + T/q)h$$

$$= (2/q) |t-t_{m_s}|C(R + T/q)$$

$$\to 0 \text{ as } t_{m_s} \to t \tag{4.26}$$

Thus the first term on the right hand side of (4.24) tends to 0 as $t_{m_s} \to t$ and the same is true for the second term.

Since the sequence t_{m_s} has been chosen from the set $t = t_m$ $(m = 1, 2, \ldots\ldots)$ and since the sequence $\{u_h^i\}$ is convergent on each line $t = t_m$ it is also convergent on $t = t_{m_s}$ and thus is Cauchy on $t = t_{m_s}$. Hence the last term in (4.24) $\to 0$ as $i \to \infty$ $j \to \infty$. Thus we have proved (4.23) and shown that the sequence $\{u_h^i\}$ converges to a function $u(x,t)$ in $L^1(\Omega_R)$.

So we have obtained a sequence $\{u_{h_R}^i\}$ from $\{u_h\}$ converging in $L^1(\Omega_R)$, and similarly we may obtain from $\{u_{h_R}^i\}$ a sequence $\{u_{h_{R+1}}^i\}$ converging in $L^1(\Omega_{R+1})$ and so on. Then by the diagonal process (see above) we may obtain a sequence $\{u_{h_{R+m}}^m\}$ extracted from $\{u_h\}$ which is convergent in $L^1_{loc}(\mathbb{R} \times (0,T))$ to $u(x,t)$. It is evident that $u(x,t) \in L^\infty(\mathbb{R} \times (0,T))$

It remains to show that u is a weak solution of (2.1), (2.2), i.e. to show that it satisfies (2.4). We introduce a test function $\psi(x,t) \in C^2(\mathbb{R} \times [0,T])$ with compact support. The L^2 projection of ψ onto the space of functions constant on each set $I_k \times J_n$ is given by

$$\psi_h(x,t) = \psi_k^n = \frac{1}{qh^2} \iint_{I_k \times J_n} \psi(x,t)\, dx\, dt, \tag{4.27}$$

where $(x,t) \in I_k \times J_n$.

We now write Roe's scheme (2.15), (2.16) in the form

$$u^k = u_k + \lambda_{k-\frac{1}{2}} \phi_{k-\frac{1}{2}} + (1 - \lambda_{k+\frac{1}{2}})\phi_{k+\frac{1}{2}} \pm (b_{k+\frac{1}{2}} - b_{k-\frac{1}{2}}) \tag{4.28}$$

where

$$\lambda_{k-\frac{1}{2}} = \tfrac{1}{2}(1 + s_{k-\frac{1}{2}}) = \begin{cases} 0 & \nu_{k-\frac{1}{2}} < 0 \\ 1 & \nu_{k-\frac{1}{2}} > 0 \end{cases} \tag{4.29}$$

Multiplying (4.28) by ψ_k^n, taking absolute values and summing by parts gives

$$\Big| \sum_k \sum_n [u_k^n(\psi_k^n - \psi_k^{n-1}) + qf(u_k^n)\{\lambda_{k+\frac{1}{2}}^n \psi_{k+1}^n - \lambda_{k-\frac{1}{2}}^n \psi_k^n + (1-\lambda_{k+\frac{1}{2}}^n)\psi_k^n$$

$$-(1-\lambda_{k-\frac{1}{2}}^n)\psi_{k-1}^n\}]h + \sum_k u_k^o \psi_k^o h \Big| \le \sum_k \sum_n |b_{k+\frac{1}{2}}^n| \, |\psi_{k+1}^n - \psi_k^n| h \tag{4.30}$$

From (2.14) and (4.1) we have

$$|b_{k+\frac{1}{2}}| \le (1/8) \max_k |u_{k+1} - u_k| \tag{4.31}$$

and using the Mean Value Theorem and (4.18) we obtain

$$\sum_k \sum_n |b_{k+\frac{1}{2}}| \, |\psi_{k+1}^n - \psi_k^n| h \le \frac{T}{8q} \left\|\frac{d\psi}{dx}\right\|_{L^\infty(\mathbb{R} \times (0,T))} \sum_{|k| \le K+n} |u_{k+1}^o - u_k^o| h \tag{4.32}$$

which $\to 0$ as $h \to 0$.

Consider now the expression

$$\frac{1}{h}\{\psi_k^n - \psi_{k-1}^n + \lambda_{k+\frac{1}{2}}^n(\psi_{k+1}^n - \psi_k^n) - \lambda_{k-\frac{1}{2}}^n(\psi_k^n - \psi_{k-1}^n)\}, \tag{4.33}$$

As $h \to 0$ we obtain

$$(1 + \lambda^+ - \lambda^-)\phi_x \tag{4.34}$$

where λ^+, λ^- are limits taken from the right, left, respectively, and are equal except on a set of measure zero (i.e. at isolated points).

Hence as $h \to 0$ the inequality (4.30) becomes the equality (2.4) and u is a weak solution of the problem (2.1), (2.2).

The proof of the theorem is complete.

5. CONCLUSIONS

We have proved that the approximation generated by Roe's scheme (2.15),(2.16) converges to a weak solution of the problem in § 2.

Note that equation (4.17) demonstrates the important property of monotonicity preservation in which monotone data remains monotone after a time step. It is this property of Roe's scheme which has been found particularly valuable in eliminating unwanted oscillations in shock problems, and it is here proved generally for the first time for variable $\nu_{k-\frac{1}{2}}$ (see also [7]).

Two further remarks need to be made concerning minor variations in the implementation of Roe's scheme.

First, although the restriction $\sup_k |\nu_{k-\frac{1}{2}}| \le \frac{1}{2}$ in (4.1) is required to obtain the inequality (4.13), it is needed only for that isolated case. Otherwise the theorem in § 4 would be true for

$$\sup_k |\nu_{k-\frac{1}{2}}| \le 1 \tag{5.1}$$

We have not yet been able to prove the theorem under the less restrictive condition (5.1) but it is known that, in practice, Roe's scheme is highly successful for a great variety of data so long as (5.1) holds.

Secondly, there is a minor difference between the scheme as stated in § 2 and as used elsewhere. In the original scheme the factor $\frac{1}{2}|s_{k-\frac{1}{2}} + s_{k'-\frac{1}{2}}|$ in (2.14) is absent but in the version treated in this report we have found it necessary to include it in order to obtain the inequality (4.12) for an expansion wave in the presence of certain data. The factor has the effect of making $b_{k-\frac{1}{2}} = 0$ but only in the special cases where $s_{k-\frac{1}{2}} = -s_{k'-\frac{1}{2}}$, i.e. in the two cells adjacent to an expansion. We have not been able to prove the convergence theorem in § 4 without this change and indeed it is known that Roe's original scheme fails to treat expansion fans correctly under certain conditions and that modifications are needed in this area. The present modification however simply reduces the order of accuracy to one at isolated points and does not resolve the difficulty.

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