FINITE ELEMENT METHOD
WITH CLEBSCH REPRESENTATION

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1. INTRODUCTION

The aim of this paper is to describe a Finite Element Method (FEM) for solving ideal compressible inviscid fluid flows. Developing any such method involves three parts:

a) choosing a variational principle from which flow equations can be derived;

b) choosing the space of trial functions to approximate the solution;

c) adopting a method of solving the resulting non-linear equations.

These three parts are not entirely independent since the choice of variationa principle to some extent predetermines the choice of finite elements and the method of solving the non-linear equations thus obtained.

A non-linear set of equations may have discontinuous solutions (weak solutions), and in general these weak solutions are not unique but depend on the particular formulation of the equations. The solutions on either side of a discontinuity have to be joined by Rankine-Hugoniot conditions which are obtained by integrating the conservation laws over a slim volume containing the discontinuity. It is clear that one particular form of equations, namely conservation form, is most convenient. Of course, in regions where solutions are smooth any equivalent form of equations will produce a unique solution. But since the conservative form of equations has to be used near discontinuities it may as well be used in the whole region. This is the reason why most numerical schemes for solving fluid flows use Euler's equations in conservative form. The finite difference methods take as a starting point the differential equations and discretise them in an appropriate manner. Hence it is natural to discretise the equations in the conservative form. The finite element methods differ from finite difference methods in that they require some form of variational principle as a starting point. Now we have a choice of variational principles: one method is to seek weak solutions by the Petrov-Galerkin method, with as many variational principles as there are differential

equations. The disadvantage of this method is that the choice of a test space for non-self-adjoint or non-linear equations is not clear.

A more satisfactory approach when it is possible is as follows: a physical problem, like perfect fluid flow, has a physical variational principle and all the equations of motion are derivable from the one principle. These equations are not necessarily in conservation form, but in addition to the equations of motion the conservation laws are also derivable from the variational principle with the aid of Noether's theorem. Thus the equations of motion can be used in regions of smooth flow while Rankine-Hugoniot conditions obtained from conservation laws can be used for "joining" discontinuities. The disadvantage of this method is that discontinuities have to be dealt with separately from the rest of the solution. But, on the other hand, the equations of motion are in a simple canonical form which should be easier to solve than the conservative form.

In order to demonstrate the method an FEM for one-dimensional steady flow through a pipe with varying diameter is described in this paper. The main feature of the method is the use of the same equations for both supersonic and subsonic flow without addition of any numerical viscosity.

2. VARIATIONAL PRINCIPLE

It has been shown, Detyna (1981), that the steady flow of a perfect fluid without spin can be described by the variational principle

$$\delta A = \delta \int_{\Omega} d_3 \times L = 0 \tag{2.1}$$

where the Lagrangian L is

$$L = \rho[H - \tfrac{1}{2}(\nabla\phi + S\nabla\lambda + H\nabla\beta)^2 - E(\rho,S)] : \tag{2.2}$$

where H is the total enthalpy and $E(\rho, S)$ is the internal energy; ρ and S are mass density and entropy respectively. The functions ϕ, λ and β are potentials without any particular physical meaning, but they define the velocity

$$\underline{v} = -(\nabla\phi + S\nabla\lambda + H\nabla\beta). \tag{2.3}$$

The equations of motion follow from (2.1) by taking variations with respect to each variable in turn:

$$\delta p : \quad \nabla\cdot(\rho\underline{v}) = 0, \tag{2.4}$$

$$\delta\lambda : \quad \nabla\cdot(\rho S\underline{v}) = 0, \tag{2.5}$$

$$\delta\beta : \quad \nabla\cdot(\rho H\underline{v}) = 0, \tag{2.6}$$

$$\delta\rho : \quad \tfrac{1}{2}\underline{v}^2 + h = H, \tag{2.7}$$

$$\delta S : \quad \underline{v}\cdot\nabla\lambda = T, \tag{2.8}$$

$$\delta S : \quad \underline{v}\cdot\nabla\beta = 0, \tag{2.9}$$

where $h = \partial(\rho E)/\partial\rho$ and $T = \partial E/\partial S$ are specific enthalpy and temperature respectively. The first three equations describe the conservation of mass, entropy and energy.

Since the Lagrangian (2.2) is invariant with respect to translation of the co-ordinates, the conservation of momentum is also induced, Detyna (1981):

$$\nabla\cdot(\rho\underline{v}\underline{v} + \underline{\underline{I}}p) = 0 \tag{2.10}$$

where $p \equiv L$ is pressure.

The equations (2.4) to (2.9) are used in the regions of smooth flow while conservation laws (2.4), (2.6) and (2.10) determine conditions at discontinuities, where entropy S is no longer conserved; hence equation (2.5) is replaced by a physical requirement of non-reversibility

$$\Delta S \geq 0$$

across the discontinuity. (It should be noted that there are only five independent physical variables ρ, S and $\underline{v}$ and hence the six conservation equations including (2.5) are overdetermined).

In order to show how the FEM can be applied to this problem, one-dimensional flow through a pipe with varying diameter is studied. Let us suppose the flow is in the x-direction: then the integration in (2.1)

over y- and z-co-ordinates can be performed since no functions depend on y or z. Furthermore, the one-dimensional flow is necessarily homenergic $H = H_0$ and the variable β can be transformed out by replacing ϕ with $\phi - H_0\beta$.

Thus the variational principle (2.1) now reads

$$\delta A = \delta\int dx\ [H_0 - \tfrac{1}{2}(\phi_x + S\lambda_x)^2 - E(\rho,S)]\rho R = 0 \qquad (2.11$$

where $R = R(x)$ is the cross-section of the pipe at point x.

This variational principle differs from the previous one in that it is explicitly dependent on the x-coordinate through the function $R(x)$. Consequently, the momentum is no longer conserved and we have instead an equation

$$(\rho R v^2 + Rp)_x = pR_x \qquad (2.12$$

which describes the rate of change of momentum with pdR/dx as its source.

The equations of motion can be obtained from the variational principle (2.11) in a standard way and are not given here.

3. FINITE ELEMENTS

The Lagrangian in (2.11) depends on functions ρ and S and first derivatives of ϕ and λ: hence the simplest trial functions are piecewise constant for ρ and S and piecewise linear for ϕ and λ. Let us divide the computational region (x_0,x_N) into N elements with nodal values at $x_0,x_1,\ldots,x_N$. The nodal values of ϕ and λ are denoted by ϕ_i and λ_i, $i=0,1,\ldots N$; the constant values of ρ and S on the element i defined as (x_{i-1},x_i) are denoted ρ_i, S_i, $i=1,2,\ldots,N$.

With such trial functions the action A in (2.11) is

$$A = \sum_{i=1}^{N} \rho_i R_i\left[H_0 - \tfrac{1}{2}\left(\frac{\phi_i-\phi_{i-1}+S_i(\lambda_i-\lambda_{i-1})}{\Delta_i}\right)^2 - E_i\right]\Delta_i \qquad (3.1)$$

where $\Delta_i = x_i - x_{i-1}$ and R_i is the integral

$$R_i = \Delta_i^{-1}\int_{x_{i-1}}^{x_i} R(x)\,dx$$

The approximate equations are now obtained by minimizing action A with respect to all nodal values ϕ_i, λ_i and constants ρ_i, S_i.

$$\frac{\partial A}{\partial \rho_i} = [H_0 - \tfrac{1}{2}v_i^2 - h_i]R_i\Delta_i = 0 \tag{3.2}$$

$$\frac{\partial A}{\partial S_i} = [v_i(\lambda_i - \lambda_{i-1}) - T_i\Delta_i]R_i\rho_i = 0 \tag{3.3}$$

$$\frac{\partial A}{\partial \phi_i} = \rho_i R_i v_i - \rho_{i+1}R_{i+1}v_{i+1} = 0 \tag{3.4}$$

$$\frac{\partial A}{\partial \lambda_i} = \rho_i R_i S_i v_i - \rho_{i+1}R_{i+1}S_{i+1}v_{i+1} = 0 \tag{3.5}$$

where $v_i = (\phi_i - \phi_{i-1} + S_i(\lambda_i - \lambda_{i-1}))/\Delta_i$ is the velocity of the fluid in element i.

The equations (3.2) to (3.5) are non-linear and we propose to solve them by iterations.

It has been noted, Buneman (1981), Detyna (1981, 1982), that the variables ρ and ϕ, ρS and λ are conjugate in the Hamiltonian sense. Therefore it is natural to solve the above equations in appropriate pairs - (3.2) with (3.4) and (3.3) with (3.5). Let us deal with the first pair of equations. The equation (3.2) depends on a single value ρ_i (through h_i) and two values of ϕ, ϕ_{i-1} and ϕ_i (through v_i). On the other hand, the equation (3.4) depends on three values of ϕ, ϕ_{i-1}, ϕ_i and ϕ_{i+1}, and two values of density, ρ_i and ρ_{i+1}.

Therefore, in the subsonic region, we can regard equation (3.2) as a definition of ρ_i, while equation (3.4) is

$$-\frac{\rho_i R_i}{\Delta_i}\phi^{i-1} + \left(\frac{\rho_i R_i}{\Delta_i} + \frac{\rho_{i+1}R_{i+1}}{\Delta_{i+1}}\right)\phi^i - \frac{\rho_{i+1}R_{i+1}}{\Delta_{i+1}}\phi^{i+1} = -\frac{\rho_i R_i T_i S_i}{v_i} + \frac{\rho_{i+1}R_{i+1}T_{i+1}S_{i+1}}{v_{i+1}} \tag{3.6}$$

an elliptic "three-point" formula for ϕ^i, where a superscript denotes a new iterative value.

In the supersonic region, the equations (3.2) and (3.4) are regarded as equations for ϕ and ρ respectively:

$$\delta\phi^i = \delta\phi^{i-1} + (\tfrac{1}{2}v_i^2 + h_i - H_0)\,\Delta_i/v_i \tag{3.7}$$

$$\rho^{i+1} = \rho^i(R_i v_i/R_{i+1}v_{i+1}), \tag{3.8}$$

where equation (3.7) was obtained by assuming a new approximation for ϕ_i to be $\phi^i = \phi_i + \delta\phi_i$ and linearising (3.2) with respect to $\delta\phi_i$. The equations (3.7) and (3.8) are "marching algorithms". The two methods of solution of equations (3.2) and (3.4) are well suited for the boundary conditions: for subsonic flow equation (3.6) requires Dirichlet or Neumann conditions while equations (3.7) and (3.8) for supersonic flow require Cauchy conditions.

The second pair of equations, (3.3) and (3.5), are different in that the only boundary conditions that can be given are the Cauchy ones, say λ_0 and S_1. This is because it is physically inappropriate to prescribe values of entropy at both ends of the flow. Hence the equations are solved by the marching algorithm in an obvious manner. In the case of transonic flow, equation (3.5) is replaced at the point of the shock by a formula giving the increase of entropy from the conservation laws.

Let us suppose the inflow is supersonic while outflow is subsonic with full boundary conditions (ρ_1, v_1 and S_1) at the inflow and one other boundary condition (say v_N or p_N) at the outflow. These boundary conditions overdetermine the flow if all the equations (3.2) to (3.5) are used. But a careful study of the above described method of solution shows that (3.8) for $i = j$, where x_j is the shock point, cannot be used since the element $j + 1$ is already in the subsonic region and ρ_{j+1} should be calculated from equation (3.2) for $i = j + 1$. That means the mass is not conserved across the shock unless it is moved to a new position - a procedure equivalent to a calculation

of shock speed in time dependent problems. A linear approximation to a new nodal position x^j which preserves mass gives

$$x^j = x_j + s\delta\tau \tag{3.9}$$

where

$$\delta\tau = \begin{cases} \Delta_j/v_j & \text{for } s > 0 \\ \Delta_{j+1}/v_{j+1} & \text{for } s < 0 \end{cases}$$

and $$s = (\rho_{j+1}v_{j+1}R_{j+1} - \rho_j v_j R_j)/(\rho_{j+1}R_{j+1} - \rho_j R_j)$$

It is seen that the equation (3.9) can be interpreted as the "movement" of the shock with speed s in time $\delta\tau$. This interpretation is indeed justifiable since, say equation (3.7), can be derived from time dependent upwind equations with variable time step $\Delta t_i = \Delta_i/v_i$ as in (3.9). Notwithstanding the interpretation, the equation (3.9) can be used either for calculation of a new nodal position $x = x^j$ of the shock (shock fitting) or for indicating the element in which the shock occurs without recalculating the new nodal values (shock capture).

4. NUMERICAL RESULTS

In order to test the method, flow in various pipes with different boundary conditions was studied. As an example we show in detail the solution of the flow in the pipe shown in Fig. 1 with inlet conditions $v_1 = 1.4$, $\rho_1 = 1.0$ and $S_1 = 0.0$; at the outlet the one condition is $v_N = 0.4$ or $P_N = 1.7$. (The above values are in the standard units normalised to the sound velocity at the inlet). According to the exact solution, the condition at the outlet results from a shock at $x_s = 3.8619$. We note that the method requires a boundary condition for ϕ (or velocity) at the outlet rather than pressure, but these two values are connected through the formula

$$p = \rho(H_0 - \tfrac{1}{2}v^2 - E)$$

stating that the Lagrangian is the pressure. In order to start the iterating

procedure the following distributions of density, velocity and entropy were chosen:

$$\rho_i = \begin{cases} 1.0 & \text{for } x_i < 7.0 \\ 1.8 & \text{for } x_i > 7.0 \end{cases}$$

$$v_i = \begin{cases} 1.4/\rho_i R_i & \text{for } i < N \\ 0.4 & \text{for } i = N \end{cases}$$

$$S_i = 0.0$$

as shown in Fig. 2 together with the exact solution. Such a trial solution ensures the shock position at the beginning of iterations is far removed from its exact location.

The results for $N = 5, 9, 17, 33, 65$ are shown in Figs. 3.1 to 3.5. It is seen that even with only 5 elements the solution is well approximated with shock position calculated with accuracy better than 8% of the tube length. The error in the shock position $\delta x = |x_s - x_s^E|$/tube length is shown in the following table:

N	5	9	17	33	65
δx	.078	.012	.0093	.0068	.0053

The iterative procedure was carried out until the average change δv in velocity

$$\delta v = \sqrt{\sum_i |v^i - v_i|/N}$$

was less than 10^{-8}. The number of iterations required was of the order of 20 regardless of the number N of elements. No significant change in solutions is noted if iterations are terminated when $\delta v \leq 10^{-6}$ but their number decreases by a quarter.

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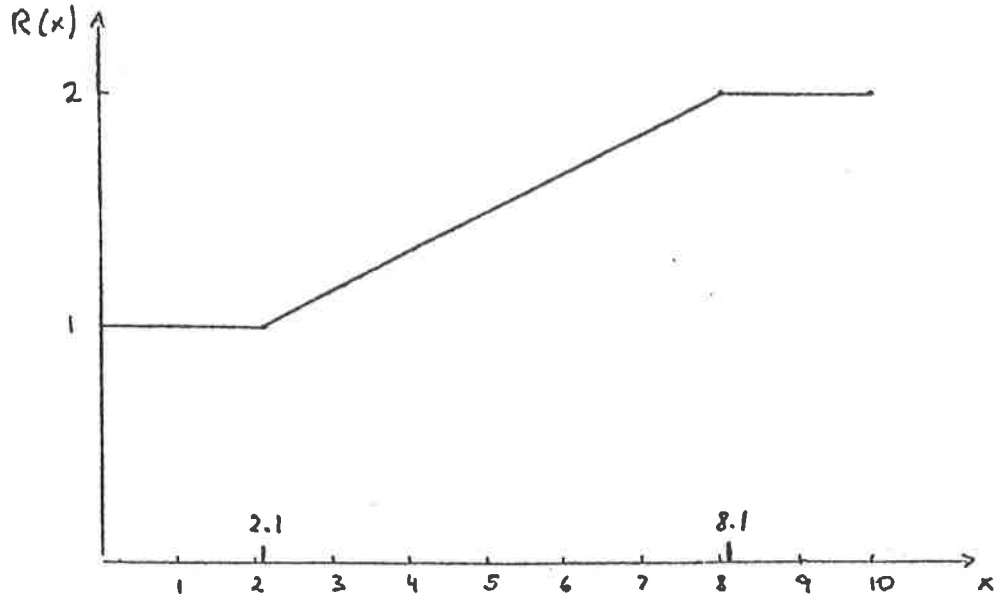

FIGURE 1 : Cross-section R(x) of the pipe studied numerically.

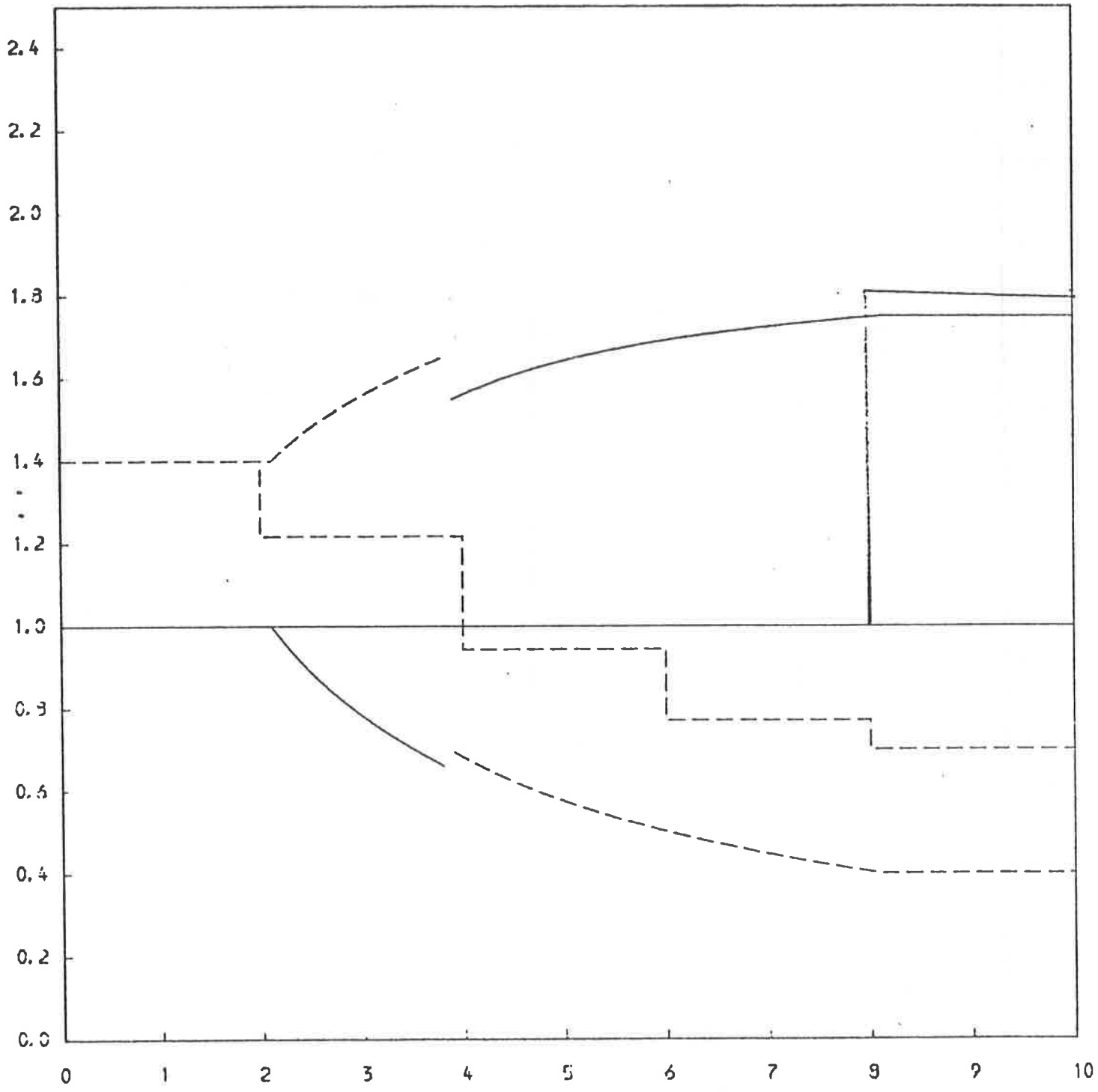

FIGURE 2 : The trial and exact solutions for density ρ (continuous line) and velocity v (broken line). The trial solution is the step function.

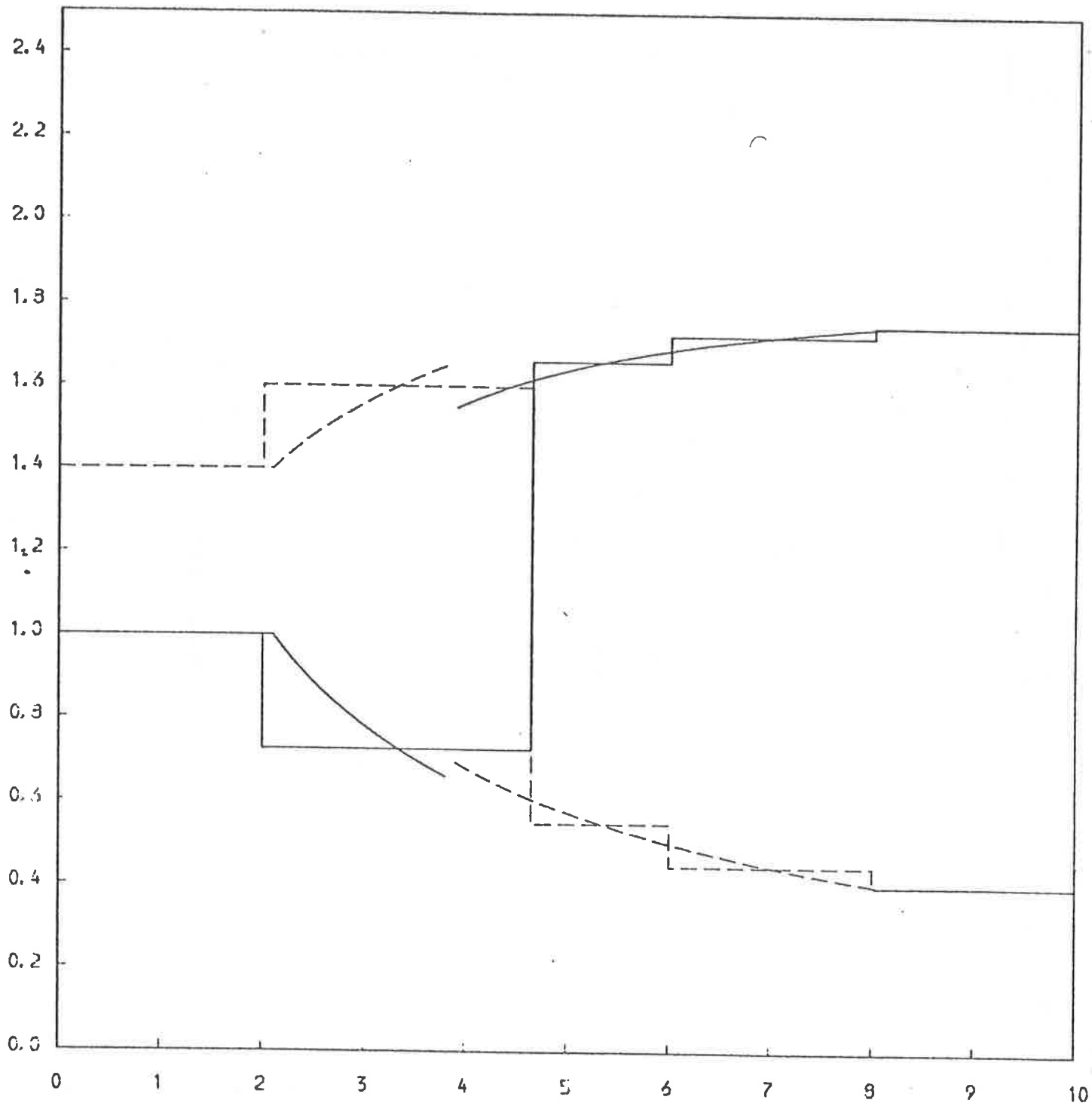

FIGURE 3.1 N = 5

FIGURE 3 : The approximate and exact solutions for density ρ (continuous line) and velocity v (broken line). The approximate solution is the step function. The number N of elements used is shown on each figure.

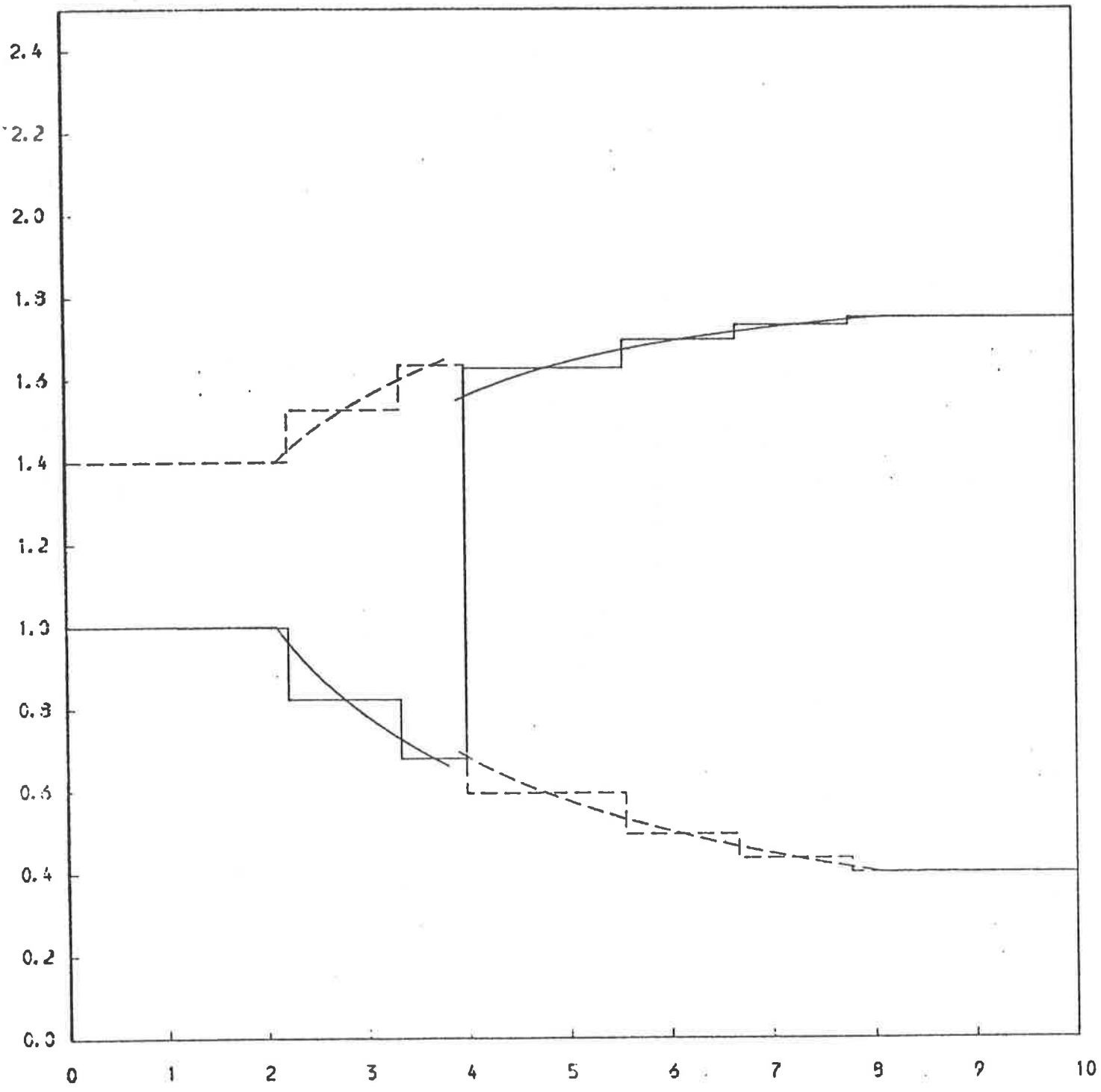

FIGURE 3.2 N = 9

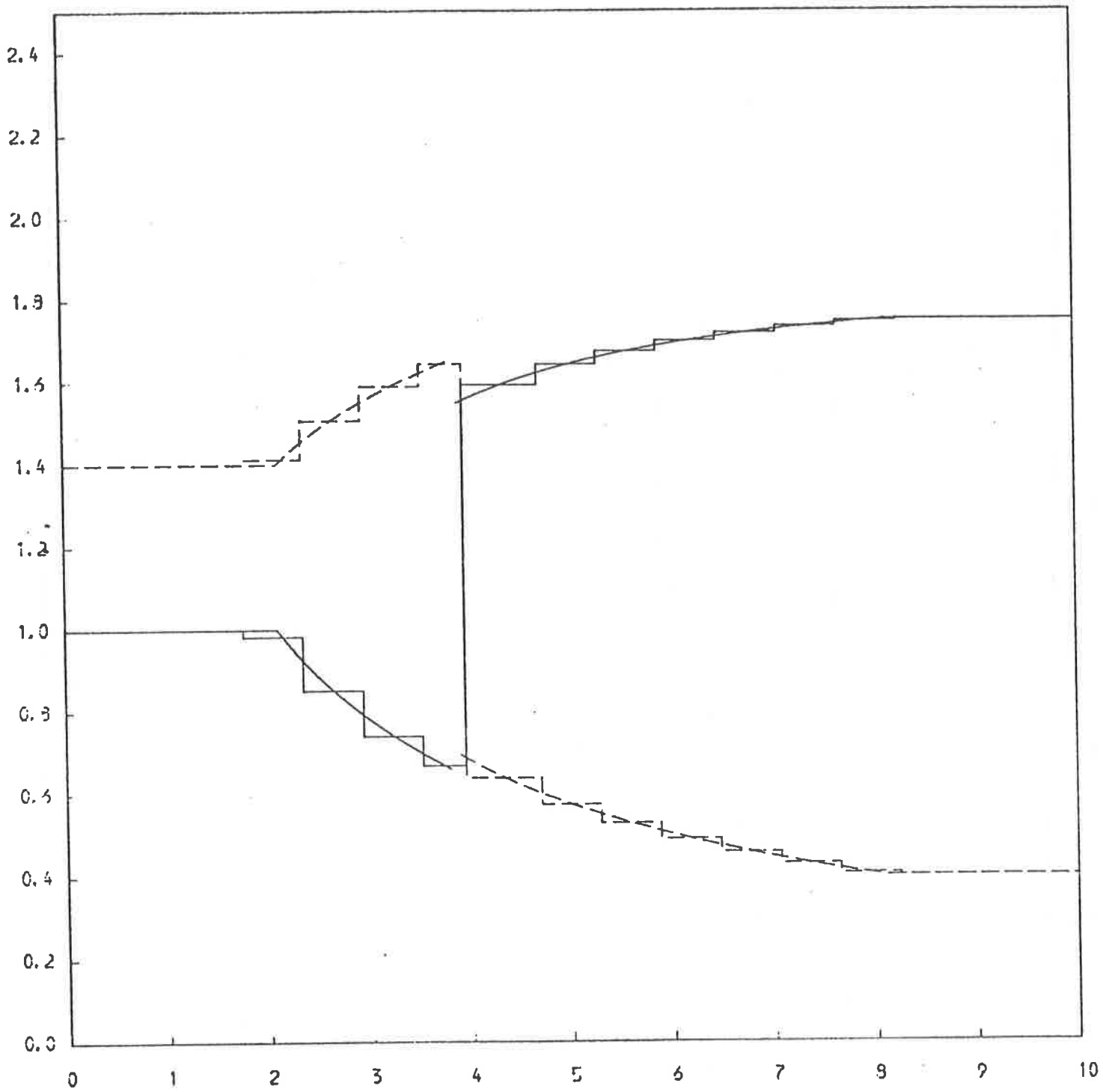

FIGURE 3.3 N = 17

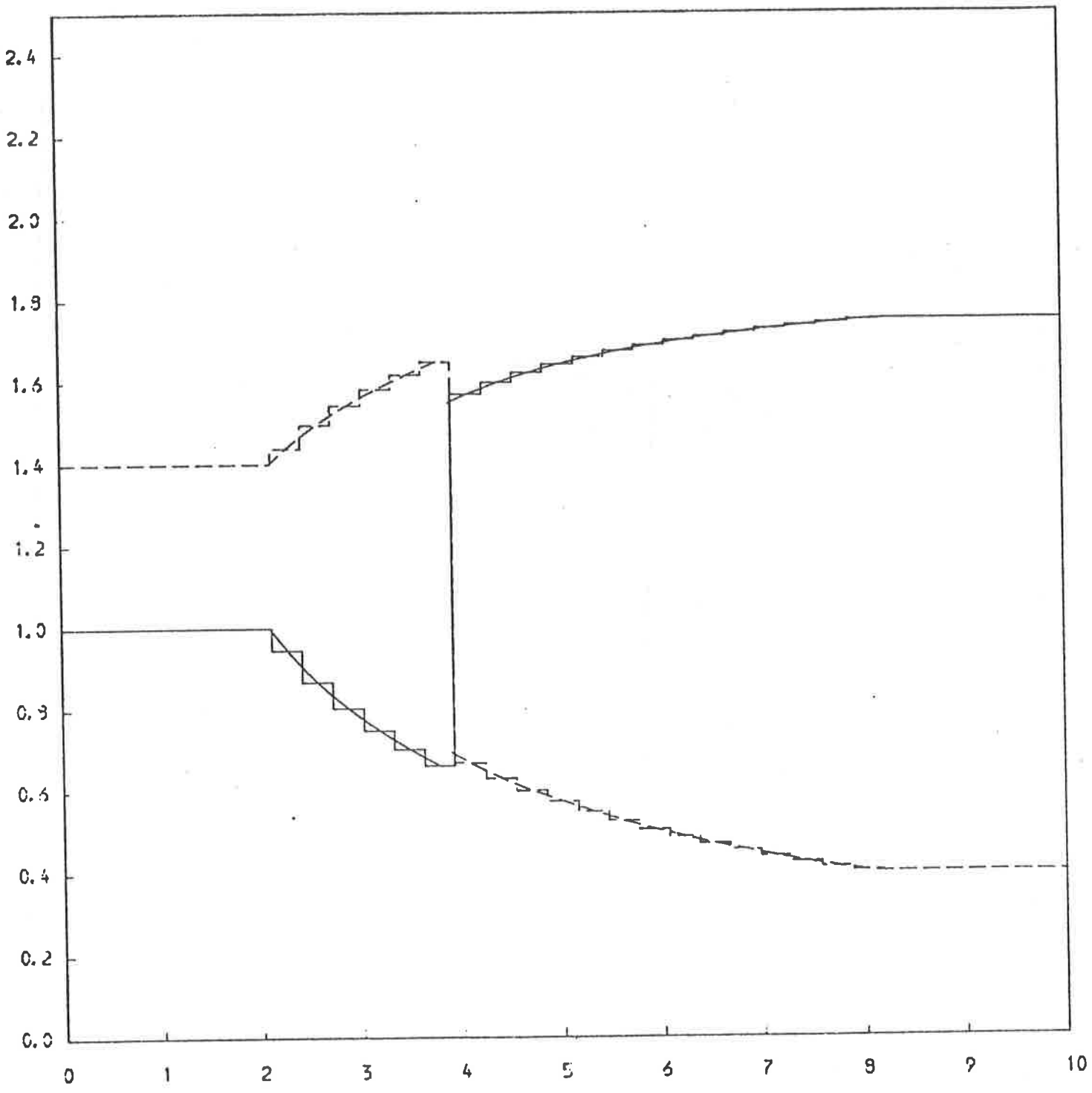

FIGURE 3.4 N = 33

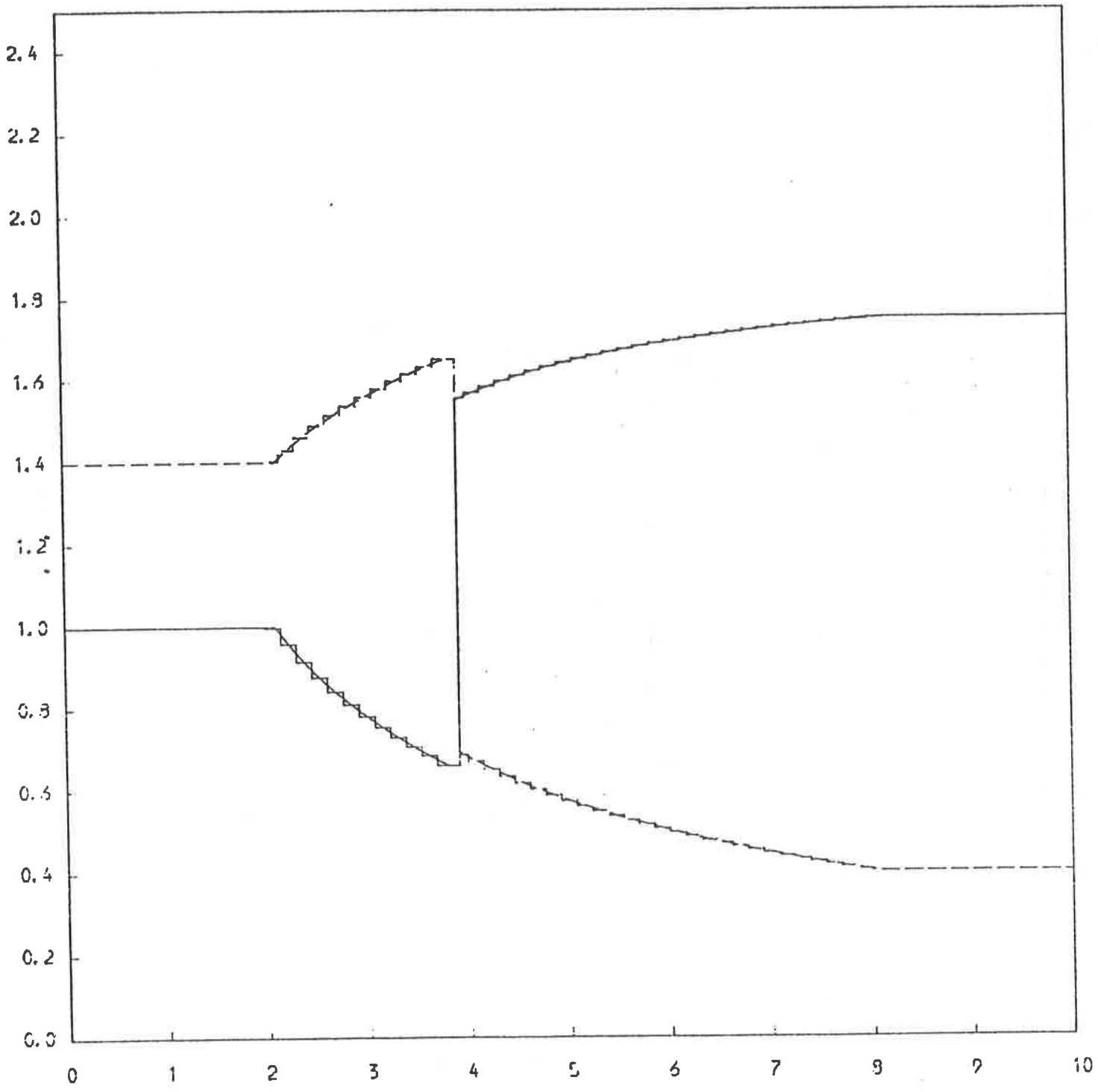

FIGURE 3.5 N = 65