DIFFERENCE SCHEMES FOR THE SHALLOW WATER EQUATIONS

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ABSTRACT

A finite difference scheme is presented for the solution of the non-linear shallow water equations based on flux difference splitting. The scheme is applied to two standard test problems in one-dimension.

1. INTRODUCTION

The work of Glaister [1],[2] presents an approximate Riemann solver for the Euler equations of gas dynamics for a general equation of state. In the present report we develop a similar type of approximate Riemann solver for the non-linear shallow water equations.

In §2 we derive the two-dimensional shallow water equations and consider the Jacobians of the corresponding flux functions, while in §3 we define the Riemann problem for the shallow water equations and derive approximate solutions for the one-dimensional case. Then in §4 we discuss the two-dimensional case incorporating operator splitting and in §5 describe a technique for dealing with source terms. In §6 we describe two one-dimensional problems with exact solutions, and finally in §7 give the numerical results for the problems of §6 using the scheme of §3.

2. SHALLOW WATER EQUATIONS

In this section we derive the shallow water approximation in the case of incompressible, irrotational flow and give the eigenvalues and eigenvectors of the Jacobians of the corresponding flux functions.

2.1 Equations of flow

We begin by considering a typical physical situation with water occupying the region $(x,z) \in R$, $-h(x,z) \leq y \leq \eta(x,z,t)$, (see figure 1). The depth of the undisturbed water is given by $h(x,z)$ (so that $y = -h(x,z)$ represents the bottom of the sea, say) and $\eta = \eta(x,z,t)$ represents the free surface elevation of the water. The undisturbed free surface of the water is taken as lying in the x-z plane and the y-axis is taken vertically upwards (see figure 1).

The equations of flow of an incompressible fluid in three dimensions are

$$u_x + v_y + w_z = 0 \quad \text{(continuity)} \tag{2.1}$$

$$\frac{Du}{Dt} = u_t + uu_x + vu_y + wu_z = -\frac{1}{\rho} p_x \tag{2.2a}$$

$$\frac{Dv}{Dt} = v_t + uv_x + vv_y + wv_z = -\frac{1}{\rho} p_y - g \tag{2.2b}$$

$$\frac{Dw}{Dt} = w_t + uw_x + vw_y + ww_z = -\frac{1}{\rho} p_z \tag{2.2c}$$

(equations of motion)

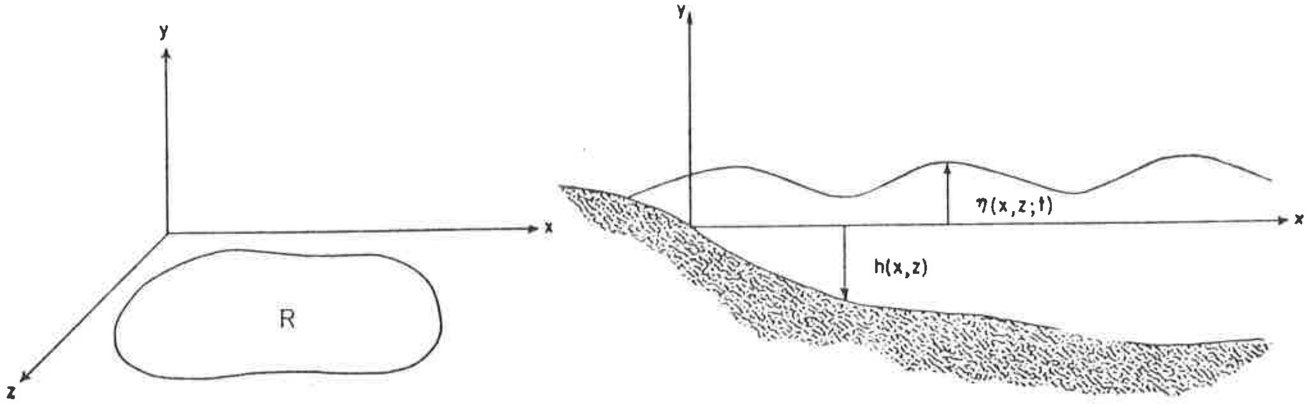


Figure 1

where $u = u(x,y,z,t)$, $v = v(x,y,z,t)$, $w = w(x,y,z,t)$ and $p = p(x,y,z,t)$ represent the velocity in each of the co-ordinate directions x,y,z and the pressure of the fluid, respectively. The material derivative is given by

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} \qquad (2.3)$$

and ρ, g represent the density of the fluid and the acceleration of gravity, respectively. In addition to equations (2.1)-(2.2c) we have the following boundary conditions

$$\frac{D}{Dt}(\eta - y) = 0 \qquad \text{(kinematic condition at the free surface } y = \eta) \qquad (2.4)$$

$$p = 0 \qquad \text{(dynamical condition at the free surface } y = \eta) \tag{2.5}$$

and

$$\frac{D}{Dt}(y+h) = 0 \qquad \text{(zero normal velocity at the lower boundary } y = -h). \tag{2.6}$$

Using equation (2.3) the boundary conditions given by equations (2.4) and (2.6) become

$$(\eta_t + u\eta_x + w\eta_z - v)\Big|_{y=\eta} = 0 \tag{2.7}$$

and

$$(uh_x + wh_z + v)\Big|_{y=-h} = 0 \ . \tag{2.8}$$

The two-dimensional shallow water equations are a consequence of equations (2.1)-(2.2c),(2.5) and (2.7)-(2.8) together with the shallow water approximation on the y-component of the acceleration (see below). We proceed as follows.

Integration of equation (2.1) with respect to y yields

$$\int_{-h}^{\eta}(u_x)dy + \int_{-h}^{\eta}(w_z)dy + v\Big|_{-h}^{\eta} = 0 \ , \tag{2.9}$$

and using the conditions given by equations (2.7)-(2.8) this becomes

$$\int_{-h}^{\eta} u_x dy + \int_{-h}^{\eta} w_z dy + \eta_t + u\Big|_{y=\eta} .\eta_x + w\Big|_{y=\eta} .\eta_z + u\Big|_{y=-h} .h_x + w\Big|_{y=-h} .h_z = 0 \ . \tag{2.10}$$

Introducing the relations

$$\frac{\partial}{\partial x}\int_{-h(x,z)}^{\eta(x,z,t)} u\,dy = \int_{-h}^{\eta} u_x\,dy + u\Big|_{y=\eta} .\eta_x + u\Big|_{y=-h} .h_x \tag{2.11}$$

and

$$\frac{\partial}{\partial z}\int_{-h(x,z)}^{\eta(x,z,t)} w\,dy = \int_{-h}^{\eta} w_z\,dy + w\Big|_{y=-\eta} .\eta_z + w\Big|_{y=-h} .h_z \tag{2.12}$$

equation (2.10) can be written as

$$\frac{\partial}{\partial x}\int_{-h}^{\eta} u\,dy + \frac{\partial}{\partial z}\int_{-h}^{\eta} w\,dy = -\eta_t \quad . \tag{2.13}$$

2.2 Shallow water approximation

We now introduce the shallow water approximation, namely, the assumption that the y-component of the acceleration of the water particles has a negligible effect on the pressure p. Formally, if we set the y-component of the acceleration to zero, i.e.

$$\frac{Dv}{Dt} = 0 \tag{2.14}$$

then integrating equation (2.2b) and using the surface condition on the pressure given by equation (2.5) yields the hydrostatic pressure relation

$$p = \rho g(\eta - y) \tag{2.15}$$

Following equation (2.15) we obtain

$$p_x = \rho g\eta_x \tag{2.16a}$$

$$p_z = \rho g\eta_z \tag{2.16b}$$

so that p_x and p_z are independent of y. It follows from equations (2.2a) and (2.2c) that the x and z components of the acceleration of the water particles are independent of y. Hence the x and z components of the velocity, i.e. u and w, are also independent of y, for all t if they were at any given time, say at $t = 0$. We shall assume this to be true in all cases so that $u = u(x,z,t)$ and $w = w(x,z,t)$ depend on x,z and t only. In view of equations (2.16a-b) and the preceding remarks the equations of motion (2.2a) and (2.2c) become

$$u_t + uu_x + wu_z = - g\eta_x \tag{2.17}$$

$$w_t + uw_x + ww_t = - g\eta_z \quad . \tag{2.18}$$

Finally, because of the assumptions made above we have

$$\int_{-h}^{\eta} u \, dy = u(\eta + h) \tag{2.19}$$

and

$$\int_{-h}^{\eta} w \, dy = w(\eta + h) \quad , \tag{2.20}$$

so that equation (2.13) becomes

$$\eta_t + \{u(\eta+h)\}_x + \{w(\eta+h)\}_z = 0 \quad . \tag{2.21}$$

Equations (2.17)-(2.18) and (2.21) are the governing equations that we shall work with in this report.

2.3 Conservation Form

Before investigating the structure of the equations we write them in standard conservation form.

Firstly, equation (2.21) can be written as

$$(\eta+h)_t + \{u(\eta+h)\}_x + \{w(\eta+h)\}_z = 0 \ , \tag{2.22}$$

since $h_t = 0$. Secondly, multiplying equation (2.22) by u and adding to equation (2.17) multiplied by $\eta + h$ yields

$$\{(\eta+h)u\}_t + \{(\eta+h)u^2\}_x + \{(\eta+h)uw\}_z = - g\eta_x(\eta+h) \ . \tag{2.23}$$

Similarly, equations (2.22) and (2.17) yield

$$\{(\eta+h)w\}_t + \{(\eta+h)uw\}_x + \{(\eta+h)w^2\}_z = - g\eta_z(\eta+h) \ . \tag{2.24}$$

If we now write the terms on the right hand side of equations (2.23) and (2.24) as $-\frac{g}{2}\{(\eta+h)^2\}_x + g(\eta+h)h_x$ and $-\frac{g}{2}\{(\eta+h)^2\}_z + g(\eta+h)h_z$ respectively, then we can rewrite the governing equations (2.22)-(2.24) in conservation form as

$$\underline{w}_t + \underline{F}_x + \underline{G}_z = \underline{f} + \underline{g} \tag{2.25}$$

where

$$\underline{w} = (\phi,\ \phi u,\ \phi w)^T \tag{2.26}$$

$$\underline{F}(\underline{w}) = (\phi u,\ \phi u^2 + \frac{\phi^2}{2},\ \phi uw)^T \tag{2.27}$$

$$\underline{G}(\underline{w}) = (\phi w,\ \phi uw,\ \phi w^2 + \frac{\phi^2}{2})^T \tag{2.28}$$

$$\underline{f} = (0,\ g\phi h_x,\ 0)^T \tag{2.29}$$

$$\underline{g} = (0,\ 0,\ g\phi h_z)^T \tag{2.30}$$

and

$$\phi = g(\eta + h) \quad . \tag{2.31}$$

Equation (2.25) has been written so that the right hand side does not contain any derivatives of flow variables. However, the vectors $\underline{f}$ and $\underline{g}$ are associated with derivatives in the x and z directions, respectively, as a consequence of the terms h_x and h_z. Moreover, equations (2.25)-(2.31) represent a system of hyperbolic equations for the variables $\phi = \phi(x,z,t)$, $m = m(x,z,t) = \phi u$ and $n = n(x,z,t) = \phi w$.

2.4 Jacobians

We now construct the Jacobian A, of the flux function $\underline{F}(\underline{w})$, given by

$$A = \frac{\partial \underline{F}}{\partial \underline{w}} , \tag{2.27}$$

and find its eigenvalues and (right) eigenvectors, since this information, together with a similar analysis for the Jacobian of $\underline{G}$, will form the basis for an approximate 'Riemann' solver.

With the definitions of m and n above we have

$$\underline{w} = (\Phi, m, n)^T \tag{2.23}$$

and

$$\underline{F}(\underline{w}) = \left[m, \frac{m^2}{\Phi} + \frac{\Phi^2}{2}, \frac{mn}{\Phi}\right]^T . \tag{2.24}$$

From equations (2.23) and (2.24) we then have the following expression for the Jacobian

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -u^2 + \Phi & 2u & 0 \\ -uw & w & u \end{bmatrix} . \tag{2.25}$$

(2.5) Eigenvalues and Eigenvectors

The eigenvalues λ_i and corresponding eigenvectors $\underline{e}_i$ of A are then found to be

$$\lambda_1 = u + \sqrt{\Phi} , \quad \underline{e}_1 = \begin{bmatrix} 1 \\ u + \sqrt{\Phi} \\ w \end{bmatrix} \tag{2.26a}$$

$$\lambda_2 = u - \sqrt{\Phi} , \quad \underline{e}_2 = \begin{bmatrix} 1 \\ u - \sqrt{\Phi} \\ w \end{bmatrix} \tag{2.26b}$$

$$\lambda_3 = u , \quad \underline{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} . \tag{2.26c}$$

A similar analysis can be carried out for the Jacobian $\frac{\partial \underline{G}}{\partial \underline{w}}$.

In the next section we develop an approximate 'Riemann' solver in the one-dimensional case using the results of this section.

3. ONE-DIMENSIONAL CASE

In this section we define, for the one-dimensional shallow water equations, an analagous problem to the Riemann problem of one-dimensional gas dynamics, and derive approximate solutions to this problem.

3.1 Equations of flow

The governing equations we shall look at in this section can be written as

$$\underline{w}_t + \underline{H}_x = \underline{h} \qquad (3.1)$$

where

$$\underline{w} = (\phi, \phi u)^T \qquad (3.2a)$$

$$\underline{H}(\underline{w}) = (\phi u, \phi u^2 + \frac{\phi^2}{2})^T \qquad (3.2b)$$

and

$$\underline{h} = (0, g\phi h'(x))^T . \qquad (3.3)$$

We have here assumed slab symmetry so that the flow variables are independent of z, i.e. $\phi = \phi(x,t) = g(\eta+h)$ and $u = u(x,t)$ where $\eta = \eta(x,t)$ and $h = h(x)$. Using the results of §2 we find that the eigenvalues and eigenvectors of the Jacobian $C = \frac{\partial \underline{H}}{\partial \underline{w}}$ given by

$$C = \begin{bmatrix} 0 & 1 \\ \phi - u^2 & 2u \end{bmatrix} \quad (3.4)$$

are

$$\lambda_1 = u + \sqrt{\phi}\ , \qquad \underset{=}{e}_1 = \begin{bmatrix} 1 \\ u + \sqrt{\phi} \end{bmatrix} \quad (3.5a)$$

$$\lambda_2 = u - \sqrt{\phi}\ , \qquad \underset{=}{e}_2 = \begin{bmatrix} 1 \\ u - \sqrt{\phi} \end{bmatrix} \quad (3.5b)$$

3.2 Riemann problem for the shallow water equations (RPSW)

The Riemann problem of gas dynamics can be defined for any system of hyperbolic conservation laws and in the present context can be interpreted as follows.

We begin by considering a one-dimensional flow where a membrane is placed at $x = 0$. In addition we consider an initial state in which the fluid at the right-hand side, $x > 0$, is in a constant state (r) given by ϕ_r, u_r and the fluid at the left-hand side, $x < 0$, is in a constant state (l) given by ϕ_l, u_l. The problem is then that of finding the resulting fluid flow when the membrane is removed.

3.3 Approximate Riemann problem

For the remainder of this section we deal with the special case when the undisturbed depth is constant, so that equations (3.1)-(3.3) become

$$\underset{-}{w}_t + \underset{-}{H}_x = \underset{-}{0} \quad (3.6)$$

where

$$\underline{w} = (\phi, \phi u)^T \tag{3.7}$$

and

$$\underline{H}(\underline{w}) = (\phi u, \phi u^2 + \frac{\phi^2}{2})^T . \tag{3.8}$$

The approximate solution of equations (3.6)-(3.8) is sought by assuming a piecewise constant approximation and solving the approximate Riemann problem

$$\underline{w}_t + \tilde{C}(\underline{w}_L, \underline{w}_R)\underline{w}_x = \underline{0}, (x,t) \in [x_L, x_R] \times (t_n, t_{n+1}) \tag{3.9}$$

where $\tilde{C}(\underline{w}_L, \underline{w}_R)$ is an approximation to the Jacobian $C(\underline{w}) = \frac{\partial \underline{H}}{\partial \underline{w}}$, and $\underline{w}_L, \underline{w}_R$ represent the piecewise constant states at time t_n, i.e.

$$\underline{w}(x, t_n) = \begin{cases} \underline{w}_L & x \in (x_L - \frac{\Delta x}{2}, x_L + \frac{\Delta x}{2}) \\ \underline{w}_R & x \in (x_R - \frac{\Delta x}{2}, x_R + \frac{\Delta x}{2}) \end{cases} . \tag{3.10}$$

We assume a constant mesh spacing, Δx in the x-direction. To solve this approximate Riemann problem we begin by determining the approximation $\tilde{C}(\underline{w}_L, \underline{w}_R)$ to the Jacobian C in a similar way to that of Glaister [1] for the Euler equations with a general equation of state.

3.4 Wavespeeds for nearby states

Consider two adjacent states, $\underline{w}_L, \underline{w}_R$ (left and right) close to an average state $\underline{w}$, at points L and R on an x-coordinate line. We seek constants α_1, α_2 such that

$$\Delta \underline{w} = \sum_{j=1}^{2} \alpha_j \underline{e}_j \qquad (3.11)$$

to within $O(\Delta^2)$, where $\Delta(\cdot) = (\cdot)_R - (\cdot)_L$. Writing equations (3.11) in full we have

$$\Delta\Phi = \alpha_1 + \alpha_2 \qquad (3.12a)$$

and

$$\Delta(\Phi u) = \alpha_1(u + \sqrt{\Phi}) + \alpha_2(u - \sqrt{\Phi}) \; . \qquad (3.12b)$$

From equations (3.12a-b) we then have the following expressions for α_1 and α_2

$$\alpha_1 = \tfrac{1}{2}\Delta\Phi + \frac{1}{2\sqrt{\Phi}}(\Delta(\Phi u) - u\Delta\Phi) \qquad (3.13a)$$

$$\alpha_2 = \tfrac{1}{2}\Delta\Phi - \frac{1}{2\sqrt{\Phi}}(\Delta(\Phi u) - u\Delta\Phi) \; . \qquad (3.13b)$$

A routine calculation verifies that

$$\Delta \underline{H} = \sum_{j=1}^{2} \lambda_j \alpha_j \underline{e}_j \qquad (3.14)$$

to $O(\Delta^2)$. We are now in a position to construct the approximate Riemann solver for general discontinuities.

3.5 Decomposition for general $\underline{w}_L, \underline{w}_R$

Consider the algebraic problem of finding average eigenvalues $\tilde{\lambda}_1, \tilde{\lambda}_2$ and corresponding average eigenvectors $\tilde{\underline{e}}_1, \tilde{\underline{e}}_2$ such that relations (3.11) and (3.14) hold exactly for arbitrary states $\underline{w}_L, \underline{w}_R$, not necessarily close. Specifically, we seek averages $\tilde{u}$ and $\tilde{\Psi}$ in terms of two adjacent states $\underline{w}_L, \underline{w}_R$ (on an x-coordinate line) such that

$$\Delta\underline{w} = \sum_{j=1}^{2} \tilde{\alpha}_j \tilde{\underline{e}}_j \qquad (3.15)$$

and

$$\Delta\underline{H} = \sum_{j=1}^{2} \tilde{\lambda}_j \tilde{\alpha}_j \tilde{\underline{e}}_j \ , \qquad (3.16)$$

where

$$\Delta(\cdot) = (\cdot)_R - (\cdot)_L \qquad (3.17)$$

$$\underline{w} = (\phi, \phi u)^T \qquad (3.18)$$

$$\underline{H}(\underline{w}) = (\phi u, \phi u^2 + \frac{\phi^2}{2})^T \qquad (3.19)$$

$$\tilde{\lambda}_{1,2} = \tilde{u} \pm \tilde{\Psi} \qquad (3.20a\text{-}b)$$

$$\tilde{\underline{e}}_{1,2} = \begin{bmatrix} 1 \\ \tilde{u} \pm \tilde{\Psi} \end{bmatrix} \qquad (3.21a\text{-}b)$$

and

$$\tilde{\alpha}_{1,2} = \tfrac{1}{2}\Delta\phi \pm \tfrac{1}{2}\frac{(\Delta(\phi u) - \tilde{u}\Delta\phi)}{\tilde{\Psi}} \ . \qquad (3.22a\text{-}b)$$

The problem of finding averages $\tilde{u}$ and $\tilde{\Psi}$ subject to equations (3.15)-(3.21b) will subsequently be denoted by (*). (N.B. The quantity $\tilde{\Psi}$ represents an average for $\sqrt{\phi}$.) We note that problem (*) is equivalent to seeking an approximation to the Jacobian C, namely $\tilde{C}$, with eigenvalues $\tilde{\lambda}_i$ and eigenvectors $\tilde{\underline{e}}_i$ such that

$$\tilde{C}\Delta\underline{w} = \Delta\underline{H} \ . \qquad (3.23)$$

The property given by equation (3.23) ensures that we will obtain a conservative algorithm. Moreover equation (3.23) guarantees that any discontinuity, e.g. a bore, will move at the correct speed.

The first step in the analysis of problem (*) is to write out equations (3.15) and (3.16) explicitly, namely,

$$\Delta\Phi = \tilde{\alpha}_1 + \tilde{\alpha}_2 \qquad (3.24a)$$

$$\Delta(\Phi u) = \tilde{\alpha}_1(\tilde{u}+\tilde{\Psi}) + \tilde{\alpha}_2(\tilde{u}-\tilde{\Psi}) \qquad (3.24b)$$

$$\Delta(\Phi u) = \tilde{\alpha}_1(\tilde{u}+\tilde{\Psi}) + \tilde{\alpha}_2(\tilde{u}-\tilde{\Psi}) \qquad (3.24c)$$

and

$$\Delta(\Phi u^2+\frac{\Phi^2}{2}) = \tilde{\alpha}_1(\tilde{u}+\tilde{\Psi})^2 + \tilde{\alpha}_2(\tilde{u}-\tilde{\Psi})^2 . \qquad (3.24d)$$

Equations (3.24b) and (3.24c) are the same, and are automatically satisfied by any average: similarly, equation (3.24a) is automatically satisfied. Thus it remains to determine $\tilde{u}$ and $\tilde{\Psi}$ using equation (3.24d).

Equation (3.24d) can be rewritten using the expressions for $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ from equations (3.22a-b) as

$$\begin{aligned}\Delta(\Phi u^2) + \Delta(\frac{\Phi^2}{2}) &= (\tilde{u}^2+\tilde{\Psi}^2)(\tilde{\alpha}_1+\tilde{\alpha}_2) + 2\tilde{u}\tilde{\Psi}(\tilde{\alpha}_1-\tilde{\alpha}_2) \\ &= (\tilde{u}^2+\tilde{\Psi}^2)\Delta\Phi + 2\tilde{u}(\Delta(\Phi u)-\tilde{u}\Delta\Phi), \qquad (3.25)\end{aligned}$$

and on rearrangement becomes

$$\tilde{u}^2\Delta\Phi - 2\tilde{u}\Delta(\Phi u) + \Delta(\Phi u^2) = \tilde{\Psi}^2\Delta\Phi - \Delta(\frac{\Phi^2}{2}) . \qquad (3.26)$$

Firstly, if we set

$$\tilde{u}^2 - 2\tilde{u}\Delta(\phi u) + \Delta(\phi u^2) = 0 \tag{3.27}$$

then from equation (3.26) we have

$$\tilde{\Psi}^2\Delta\phi - \Delta(\frac{\phi^2}{2}) = 0 \tag{3.28}$$

which yields

$$\tilde{\Psi}^2 = \frac{\Delta\left[\frac{\phi^2}{2}\right]}{\Delta\phi} = \frac{\frac{1}{2}(\phi_R^2 - \phi_L^2)}{(\phi_R - \phi_L)} = \frac{1}{2}(\phi_R + \phi_L) \ . \tag{3.29}$$

This gives the following average $\tilde{\Psi}$ for $\sqrt{\phi}$,

$$\tilde{\Psi} = \sqrt{\tfrac{1}{2}(\phi_R+\phi_L)} \ , \tag{3.30}$$

i.e. the square root of the arithmetic mean of ϕ_R and ϕ_L . Only one solution of the quadratic equation (3.27) for $\tilde{u}$ is productive, namely

$$\tilde{u} = \frac{\Delta(\phi u) - \sqrt{(\Delta(\phi u)^2 - \Delta\phi\Delta(\phi u^2)}}{\Delta\phi}$$

$$= \frac{\sqrt{\phi_R}\,u_R + \sqrt{\phi_L}\,u_L}{\sqrt{\phi_R} + \sqrt{\phi_L}} \ . \tag{3.31}$$

Thus the averages $\tilde{u}$ and $\tilde{\Psi}$ given by equations (3.30)-(3.31) represent a solution of equation (3.24d). In addition, using equation (3.31), we have

$$\Delta(\phi u) - \tilde{u}\Delta\phi = \sqrt{\phi_R\phi_L}\,\Delta u \ . \tag{3.32}$$

Hence, if we define an average of ϕ by

$$\tilde{\phi} = \sqrt{\phi_R\phi_L} \tag{3.33}$$

equations (3.22a-b) simplify to

$$\tilde{\alpha}_{1,2} = \tfrac{1}{2}\Delta\Phi \pm \tfrac{1}{2}\frac{\tilde{\Phi}}{\tilde{\Psi}}\Delta u \quad . \tag{3.34a-b}$$

Alternatively, noting the results that have just been found, we could begin to solve equation (3.26) by defining

$$\hat{\Phi} = \sqrt{\Phi_R\Phi_L} \tag{3.35}$$

$$\bar{\Phi} = \tfrac{1}{2}(\Phi_R+\Phi_L) \tag{3.36}$$

and

$$\hat{u} = \frac{\sqrt{\Phi_R}\, u_R + \sqrt{\Phi_L}\, u_L}{\sqrt{\Phi_R} + \sqrt{\Phi_L}} \quad . \tag{3.37}$$

Then, using equations (3.35)-(3.37) we can show that

$$\Delta(\Phi u^2) = \hat{u}^2\Delta\Phi + 2\hat{u}\hat{\Phi}\Delta u \tag{3.38}$$

$$\Delta(\Phi u) = \hat{u}\Delta\Phi + \hat{\Phi}\Delta u \tag{3.39}$$

and

$$\Delta\left[\frac{\Phi^2}{2}\right] = \bar{\Phi}\Delta\Phi \tag{3.40}$$

Substituting the expressions given by equations (3.38)-(3.40) into equation (3.26) and rearranging yields

$$\left((\hat{u}-\tilde{u})^2 + \bar{\Phi} - \tilde{\Psi}^2\right)\Delta\Phi + 2\hat{\Phi}(\hat{u}-\tilde{u})\Delta u = 0 \ . \tag{3.41}$$

We require equation (3.24d) to be satisfied for all variations $\Delta\Phi$ and Δu, i.e. setting the coefficients of $\Delta\Phi$ and Δu in equation (3.41) to zero yields

$$\hat{\Phi}(\hat{u} - \tilde{u}) = 0 \tag{3.42a}$$

and

$$(\hat{u}-\tilde{u})^2 + \bar{\Phi} - \tilde{\Psi}^2 = 0 \quad . \tag{3.42b}$$

The only physical solution of equation (3.42a) is

$$\tilde{u} = \hat{u} = \frac{\sqrt{\Phi_R}\, u_R + \sqrt{\Phi_L}\, u_L}{\sqrt{\Phi_R} + \sqrt{\Phi_L}} \tag{3.43}$$

and hence from equation (3.42b)

$$\tilde{\Psi} = \sqrt{\bar{\Phi}} = \sqrt{\tfrac{1}{2}(\Phi_R+\Phi_L)} \; . \tag{3.44}$$

In addition, following equations (3.43), (3.32) and (3.33) we arrive at the simplified expressions for $\tilde{\alpha}_{1,2}$ given by equations (3.34a-b).

Summarising, we now have a one-dimensional Riemann solver for the shallow water equations, with a constant undisturbed depth, and can apply it using a first order upwind scalar algorithm as follows.

If at time level n we have data $\underline{w}_L, \underline{w}_R$ given at either end of the cell (x_L, x_R), then update $\underline{w}$ to time level $n+1$ in an upwind manner as follows. Schematically, we increment $\underline{w}$ as in Figure 2.

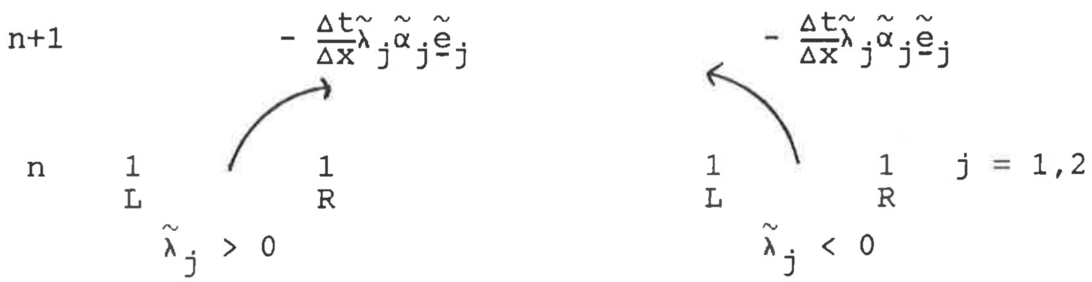


Figure 2

where $\Delta x = x_R - x_L$ and Δt is the time interval from level n to $n+1$. Specifically, we

$$\text{add} \quad -\frac{\Delta t}{\Delta x}\tilde{\lambda}_j\tilde{\alpha}_j\tilde{\underline{e}}_j \quad \text{to} \quad \underline{w}_R \quad \text{if} \quad \tilde{\lambda}_j > 0$$

or

$$\text{add} \quad -\frac{\Delta t}{\Delta x}\tilde{\lambda}_j\tilde{\alpha}_j\tilde{\underline{e}}_j \quad \text{to} \quad \underline{w}_L \quad \text{if} \quad \tilde{\lambda}_j < 0 \, ,$$

where $\tilde{\lambda}_j$, $\tilde{\alpha}_j$, $\tilde{\underline{e}}_j$ are given by

$$\tilde{\lambda}_{1,2} = \tilde{u} \pm \tilde{\Psi} \tag{3.45a-b}$$

$$\tilde{\alpha}_{1,2} = \tfrac{1}{2}\Delta\Phi \pm \tfrac{1}{2}\frac{\tilde{\Phi}}{\tilde{\Psi}}\Delta u \tag{3.46a-b}$$

$$\tilde{\underline{e}}_{1,2} = \begin{bmatrix} 1 \\ \tilde{u} \pm \tilde{\Psi} \end{bmatrix} \tag{3.47a-b}$$

with

$$\tilde{u} = \frac{\sqrt{\Phi_R}\, u_R + \sqrt{\Phi_L}\, u_L}{\sqrt{\Phi_R} + \sqrt{\Phi_L}} \tag{3.48a}$$

$$\tilde{\Phi} = \sqrt{\Phi_R \Phi_L} \tag{3.48b}$$

$$\tilde{\Psi} = \sqrt{\tfrac{1}{2}(\Phi_R + \Phi_L)} \tag{3.48c}$$

and

$$\Delta(\cdot) = (\cdot)_R - (\cdot)_L \; .$$

In addition, we can use the idea of flux limiters [3] to create a second order algorithm which is oscillation free.

The approximate Riemann solver that has been constructed in this section takes the form of a conservative algorithm, and has the property that discontinuities in the solution will move at the correct speed as guaranteed by equations (3.15) and (3.16).

Finally, we note that the approximate Jacobian $\tilde{C}$ satisfying equation (3.23) can be written as

$$\tilde{C}(\underline{w}_L, \underline{w}_R) = \begin{bmatrix} 0 & 1 \\ \tilde{\Psi}^2 - \tilde{u}^2 & 2\tilde{u} \end{bmatrix} . \qquad (3.49)$$

In the next section we extend the scheme of this section to the two-dimensional case incorporating the technique of operator splitting.

4. TWO-DIMENSIONAL CASE

In this section we develop approximate solutions of the two-dimensional shallow water equations along the same lines as those of §3 incorporating the technique of operator splitting.

4.1 Equations of flow

The governing equations considered in this section are the two-dimensional equations of §2.3 when the undisturbed depth h(x,z) = constant, so that equation (2.25) becomes

$$\underline{w}_t + \underline{F}_x + \underline{G}_z = \underline{0} \qquad (4.1)$$

where

$$\underline{w} = (\phi,\phi u,\phi w)^T \qquad (4.2a)$$

$$\underline{F}(w) = (\phi u,\phi u^2 + \frac{\phi^2}{2},\phi uw)^T \qquad (4.2b)$$

and

$$\underline{G}(\underline{w}) = (\phi w,\phi uw,\phi w^2 + \frac{\phi^2}{2})^T . \qquad (4.2c)$$

We wish to solve equations (4.1)-(4.2c) approximately using an extension of the Riemann solver of §3 and the technique of operator splitting, i.e. we solve successively

$$\tfrac{1}{2}\underline{w}_t + \underline{F}_x = \underline{0} \qquad (4.3a)$$

and

$$\tfrac{1}{2}\underline{w}_t + \underline{G}_z = \underline{0} \qquad (4.3b)$$

along x- and z-coordinate lines, respectively. We shall discuss the solution of equation (4.3a) and the solution of equation (4.3b) will then follow by symmetry. Using the results of §2 we find that the eigenvalues and eigenvectors of the Jacobian $A = \frac{\partial \underline{F}}{\partial \underline{w}}$ given by

$$A = \begin{bmatrix} 0 & 1 & 0 \\ \Phi - u^2 & 2u & 0 \\ -uw & w & u \end{bmatrix} \tag{4.4}$$

are

$$\lambda_{1,2} = u \pm \sqrt{\Phi}, \quad \lambda_3 = u \tag{4.5a-c}$$

$$\underline{e}_{1,2} = \begin{bmatrix} 1 \\ u \pm \sqrt{\Phi} \\ w \end{bmatrix}, \quad \underline{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{4.6a-c}$$

4.2 Wavespeeds for nearby states

Consider two adjacent states $\underline{w}_L, \underline{w}_R$ (left and right) close to an average state $\underline{w}$, at points L and R on an x-coordinate line. As in §3.4 we seek constants $\alpha_1, \alpha_2, \alpha_3$ such that

$$\Delta\underline{w} = \sum_{j=1}^{3} \alpha_j \underline{e}_j \tag{4.7}$$

to within $O(\Delta^2)$, where $\Delta(\cdot) = (\cdot)_R - (\cdot)_L$. Writing equation (4.7) in full we have

$$\Delta\Phi = \alpha_1 + \alpha_2 \tag{4.8a}$$

$$\Delta(\Phi u) = \alpha_1(u + \sqrt{\Phi}) + \alpha_2(u - \sqrt{\Phi}) \tag{4.8b}$$

and

$$\Delta(\Phi w) = \alpha_1 w + \alpha_2 w + \alpha_3 \ . \tag{4.8c}$$

From equations (4.8a-c) we have the following expressions for α_1, α_2 and α_3

$$\alpha_1 = \tfrac{1}{2}\Delta\Phi + \frac{1}{2\sqrt{\Phi}}(\Delta(\Phi u) - u\Delta\Phi) \tag{4.9a}$$

$$\alpha_2 = \tfrac{1}{2}\Delta\Phi - \frac{1}{2\sqrt{\Phi}}(\Delta(\Phi u) - u\Delta\Phi) \tag{4.9b}$$

and

$$\alpha_3 = \Delta(\Phi w) - w\Delta\Phi \; . \tag{4.9c}$$

A routine calculation verifies that

$$\Delta \underline{F} = \sum_{j=1}^{3} \lambda_j \alpha_j \underline{e}_j \tag{4.10}$$

to within $O(\Delta^2)$. We are now in a position to construct the approximate Riemann solver in the x-direction.

4.3 Decomposition for general $\underline{w}_L, \underline{w}_R$

Consider the algebraic problem of finding average eigenvalues $\tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\lambda}_3$ and corresponding average eigenvectors $\tilde{\underline{e}}_1, \tilde{\underline{e}}_2, \tilde{\underline{e}}_3$ such that relations (4.7) and (4.10) hold for arbitrary states $\underline{w}_L, \underline{w}_R$ not necessarily close. Specifically, we seek averages $\tilde{u}, \tilde{w}$ and $\tilde{\Psi}$ in terms of two adjacent states $\underline{w}_L, \underline{w}_R$ (on an x-coordinate line) such that

$$\Delta \underline{w} = \sum_{j=1}^{3} \tilde{\alpha}_j \tilde{\underline{e}}_j \tag{4.11}$$

and

$$\Delta \underline{F} = \sum_{j=1}^{3} \tilde{\lambda}_j \tilde{\alpha}_j \tilde{\underline{e}}_j \; , \tag{4.12}$$

where

$$\Delta(\cdot) = (\cdot)_R - (\cdot)_L \tag{4.13}$$

$$\underline{w} = (\phi, \phi u, \phi w)^T \tag{4.14}$$

$$\underline{F}(\underline{w}) = (\phi u, \phi u^2 + \frac{\phi^2}{2}, \phi uw)^T \tag{4.15}$$

$$\tilde{\lambda}_{1,2} = \tilde{u} \pm \tilde{\Psi}\ , \quad \tilde{\lambda}_3 = \tilde{u} \tag{4.16a-c}$$

$$\tilde{\underline{e}}_{1,2} = \begin{bmatrix} 1 \\ \tilde{u} \pm \tilde{\Psi} \\ \tilde{w} \end{bmatrix}, \quad \tilde{\underline{e}}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{4.17a-c}$$

and

$$\tilde{\alpha}_{1,2} = \tfrac{1}{2}\Delta\phi \pm \tfrac{1}{2}\frac{(\Delta(\phi u) - \tilde{u}\Delta\phi)}{\tilde{\Psi}}\ , \quad \tilde{\alpha}_3 = \Delta(\phi w) - \tilde{w}\Delta\phi\ . \tag{4.18a-c}$$

The problem of finding averages $\tilde{u}, \tilde{w}$ and $\tilde{\Psi}$ subject to equations (4.11)-(4.17c) will subsequently be denoted by (**) which is equivalent to seeking an approximation to the Jacobian A, namely $\tilde{A}$ with eigenvalues $\tilde{\lambda}_i$ and eigenvectors $\tilde{\underline{e}}_i$ such that

$$\tilde{A}\Delta\tilde{\underline{w}} = \Delta\underline{F}\ . \tag{4.18}$$

The first step in the analysis of problem (*) is to write out equations (4.11) and (4.12) explicitly, namely,

$$\Delta\phi = \tilde{\alpha}_1 + \tilde{\alpha}_2 \tag{4.19a}$$

$$\Delta(\phi u) = \tilde{\alpha}_1(\tilde{u}+\tilde{\Psi}) + \tilde{\alpha}_2(\tilde{u}-\tilde{\Psi}) \tag{4.19b}$$

$$\Delta(\phi w) = \tilde{\alpha}_1\tilde{w} + \tilde{\alpha}_2\tilde{w} + \tilde{\alpha}_3 \tag{4.19c}$$

$$\Delta(\phi u) = \tilde{\alpha}_1(\tilde{u}+\tilde{\Psi}) + \tilde{\alpha}_2(\tilde{u}-\tilde{\Psi}) \tag{4.19d}$$

$$\Delta(\phi u^2+\frac{\phi^2}{2}) = \tilde{\alpha}_1(\tilde{u}+\tilde{\Psi})^2 + \tilde{\alpha}_2(\tilde{u}-\tilde{\Psi})^2 \tag{4.19e}$$

and

$$\Delta(\phi uw) = \tilde{\alpha}_1(\tilde{u}+\tilde{\Psi})\tilde{w} + \tilde{\alpha}_2(\tilde{u}-\tilde{\Psi})\tilde{w} + \tilde{\alpha}_3\tilde{u}\ . \tag{4.19f}$$

Equations (4.19b) and (4.19d) are the same, and are automatically satisfied by any average: similarly equations (4.19a) and (4.19c) are automatically satisfied. Thus it remains to determine $\tilde{u}, \tilde{w}$ and $\tilde{\Psi}$ using equations (4.19e) and (4.19f).

If we define

$$\hat{\Phi} = \sqrt{\Phi_R \Phi_L} \tag{4.20}$$

$$\bar{\Phi} = \tfrac{1}{2}(\Phi_R + \Phi_L) \tag{4.21}$$

and

$$\hat{u} = \frac{\sqrt{\Phi_R}\, u_R + \sqrt{\Phi_L}\, u_L}{\sqrt{\Phi_R} + \sqrt{\Phi_L}} \tag{4.22}$$

as in §3.5, and use the difference properties

$$\Delta(\Phi u^2) = \hat{u}^2 \Delta\Phi + 2\hat{u}\hat{\Phi}\Delta u \tag{4.23}$$

$$\Delta(\Phi u) = \hat{u}\Delta\Phi + \hat{\Phi}\Delta u \tag{4.24}$$

$$\Delta\left(\frac{\Phi^2}{2}\right) = \bar{\Phi}\Delta\Phi \tag{4.25}$$

then equation (4.19e) becomes

$$((\hat{u}-\tilde{u})^2 + \bar{\Phi} - \tilde{\Psi}^2)\Delta\Phi + 2\hat{\Phi}(\hat{u}-\tilde{u})\Delta u = 0 \; . \tag{4.26}$$

Therefore, if equation (4.19e) is to be satisfied for all variations $\Delta\Phi$ and Δu then from equation (4.26) we have

$$\hat{\Phi}(\hat{u} - \tilde{u}) = 0 \tag{4.27a}$$

and

$$(\hat{u}-\tilde{u})^2 + \bar{\Phi} - \tilde{\Psi}^2 = 0 \; . \tag{4.27b}$$

The only physical solution of equation (4.27a) is

$$\tilde{u} = \hat{u} = \frac{\sqrt{\Phi_R}\, u_R + \sqrt{\Phi_L}\, u_L}{\sqrt{\Phi_R} + \sqrt{\Phi_L}} \tag{4.28}$$

and hence from equation (4.27a)

$$\tilde{\Psi} = \sqrt{\bar{\Phi}} = \sqrt{\tfrac{1}{2}(\Phi_R+\Phi_L)} \; . \tag{4.29}$$

In addition, since

$$\Delta(\Phi u) - \tilde{u}\Delta\Phi = \sqrt{\Phi_R \Phi_L}\,\Delta u = \tilde{\Phi}\Delta u \qquad (4.30)$$

where

$$\tilde{\Phi} = \sqrt{\Phi_R \Phi_L} \qquad (4.31)$$

equations (4.18a-b) simplify to

$$\tilde{\alpha}_{1,2} = \tfrac{1}{2}\Delta\Phi + \tfrac{1}{2}\frac{\tilde{\Phi}}{\tilde{\Psi}}\,\Delta u \,. \qquad (4.32)$$

Finally, using equations (4.18c) and (4.19b) equation (4.19f) can be rewritten as

$$\Delta(\Phi uw) - \tilde{u}\Delta(\Phi w) = \tilde{w}(\Delta(\Phi u) - \tilde{u}\Delta\Phi) \,. \qquad (4.33)$$

However, using equation (4.28) the left hand side of equation (4.33) becomes

$$\Delta(\Phi uw) - \tilde{u}\Delta(\Phi w) = \tilde{\Phi}\frac{(\sqrt{\Phi_R}\,w_R + \sqrt{\Phi_L}\,w_L)}{\sqrt{\Phi_R} + \sqrt{\Phi_L}}\Delta u \qquad (4.34)$$

and from equation (4.30) the right hand side of equation (4.33) becomes

$$\tilde{w}(\Delta(\Phi u) - \tilde{u}\Delta\Phi) = \tilde{w}\tilde{\Phi}\Delta u \,. \qquad (4.35)$$

Thus, combining equations (4.34)-(4.35) equation (4.33) yields

$$\tilde{\Phi}\,\frac{(\sqrt{\Phi_R}\,w_R + \sqrt{\Phi_L}\,w_L)}{\sqrt{\Phi_R} + \sqrt{\Phi_L}}\,\Delta u = \tilde{w}\tilde{\Phi}\Delta u \qquad (4.36)$$

so that

$$\tilde{w} = \frac{\sqrt{\Phi_R}\,w_R + \sqrt{\Phi_L}\,w_L}{\sqrt{\Phi_R} + \sqrt{\Phi_L}} \,. \qquad (4.37)$$

Equation (4.37) now gives

$$\Delta(\Phi w) - \tilde{w}\Delta\Phi = \tilde{\Phi}\Delta w \qquad (4.38)$$

so that the expression for $\tilde{\alpha}_3$ in equation (4.18c) simplifies to

$$\tilde{\alpha}_3 = \tilde{\Phi}\Delta w \,. \qquad (4.39)$$

By symmetry, similar results hold for the Jacobian $\frac{\partial \underline{G}}{\partial \underline{w}}$.

We can now apply the results of this section to find approximate solutions to the two dimensional shallow water equations with an undisturbed depth $h(x,z)$ = constant. In particular, we shall use first order upwind differencing as in §3.5 together with the technique of operator splitting. Incorporating the results found here, together with a one-dimensional scalar algorithm, we perform a sequence of one-dimensional calculations along computational grid lines in the x- and z-directions in turn. The algorithm along a line z = constant can be described as follows. Supposing at time level n we have data $\underline{w}_L, \underline{w}_R$ given at either end of a cell (x_L, x_R) (on a line $z = z_0$), then we update $\underline{w}$ to time level $n+1$ in an upwind manner as shown below. Schematically, we increment $\underline{w}$ as in Figure 3

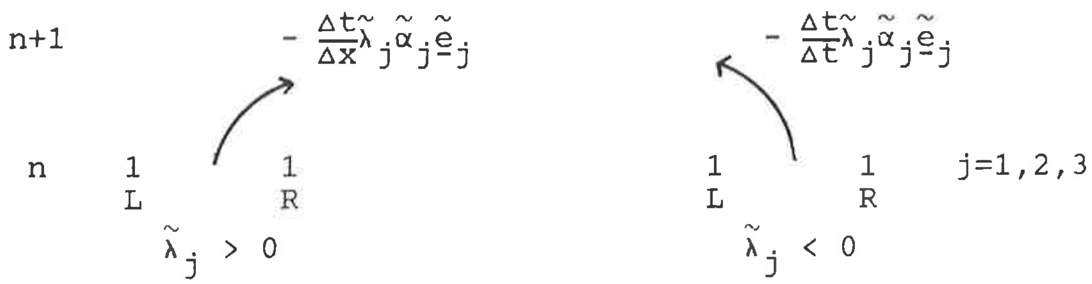


Figure 3

where $\Delta x = x_R - x_L$ and Δt is the time interval from level n to n+1. Specifically, we

$$\text{add} \quad -\frac{\Delta t}{\Delta x}\tilde{\lambda}_j\tilde{\alpha}_j\tilde{\underline{e}}_j \quad \text{to} \quad \underline{w}_R \quad \text{if} \quad \tilde{\lambda}_j > 0$$

or

$$\text{add} \quad -\frac{\Delta t}{\Delta x}\tilde{\lambda}_j\tilde{\alpha}_j\tilde{\underline{e}}_j \quad \text{to} \quad \underline{w}_L \quad \text{if} \quad \tilde{\lambda}_j < 0 ,$$

where $\tilde{\lambda}_j, \tilde{\alpha}_j, \tilde{\underline{e}}_j$ are given by

$$\tilde{\lambda}_{1,2,3} = \tilde{u} + \tilde{\Psi}, \quad \tilde{u} - \tilde{\Psi}, \quad \tilde{u} \tag{4.40a-c}$$

$$\tilde{\alpha}_{1,2,3} = \tfrac{1}{2}\Delta\Phi + \tfrac{1}{2}\frac{\tilde{\Phi}}{\tilde{\Psi}}\Delta u, \quad \tfrac{1}{2}\Delta\Phi - \tfrac{1}{2}\frac{\tilde{\Phi}}{\tilde{\Psi}}\Delta u, \quad \tilde{\alpha}_3 = \tilde{\Phi}\Delta w \tag{4.41a-c}$$

$$\tilde{\underline{e}}_{1,2,3} = \begin{bmatrix} 1 \\ \tilde{u} + \tilde{\Psi} \\ \tilde{w} \end{bmatrix}, \quad \begin{bmatrix} 1 \\ \tilde{u} - \tilde{\Psi} \\ \tilde{w} \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{4.42a-c}$$

with

$$\tilde{u} = \frac{\sqrt{\Phi_R}\, u_R + \sqrt{\Phi_L}\, u_L}{\sqrt{\Phi_R} + \sqrt{\Phi_L}} \tag{4.43a}$$

$$\tilde{w} = \frac{\sqrt{\Phi_R}\, w_R + \sqrt{\Phi_L}\, w_L}{\sqrt{\Phi_R} + \sqrt{\Phi_L}} \tag{4.43b}$$

$$\tilde{\Phi} = \sqrt{\Phi_R \Phi_L} \tag{4.43c}$$

$$\tilde{\Psi} = \sqrt{\tfrac{1}{2}(\Phi_R + \Phi_L)} \tag{4.43d}$$

and

$$\Delta(\cdot) = (\cdot)_R - (\cdot)_L \quad . \tag{4.43e}$$

Similar results apply for updating in the z-direction. As in §3.5 we can use the idea of flux limiters [3] to create a second order algorithm which is oscillation free.

The approximate Riemann solver we have constructed in this section is a conservative algorithm when incorporated with operator splitting and has the one dimensional property of capturing discontinuities guaranteed by equations (4.11), (4.12).

Finally, we note that the approximate Jacobian $\tilde{A}$ satisfying equation (4.18) can be written as

$$\tilde{A}(\underline{w}_L,\underline{w}_R) = \begin{bmatrix} 0 & 1 & 0 \\ \tilde{\Psi}^2-\tilde{u}^2 & 2\tilde{u} & 0 \\ -\tilde{u}\tilde{w} & \tilde{w} & \tilde{u} \end{bmatrix} . \qquad (4.44)$$

In the next section we extend the schemes of this section and §3 to include the case when the undisturbed depth $h(x,z)$ is not constant.

5. SOURCE TERMS

In this section we extend the algorithms of §3 and §4 to include the case when the undisturbed depth is not constant.

We shall use a technique suggested by Roe [4] in the context of gasdynamics and used by Glaister [5] which treats the source terms arising from a variable undisturbed depth. The source terms are projected onto the local eigenvectors of the Jacobian and then upwinded using the previously described first order algorithm.

5.1 One-Dimensional Case

The governing equations in the one-dimensional case are now

$$\underline{w}_t + \underline{H}_x = \underline{h} \qquad (5.1)$$

where

$$\underline{w} = (\phi, \phi u)^T \qquad (5.2)$$

$$\underline{H}(\underline{w}) = (\phi u, \phi u^2 + \frac{\phi^2}{2})^T \qquad (5.3)$$

and

$$\underline{h}(\underline{w}) = (0, g\phi h'(x))^T. \qquad (5.4)$$

Consider the interval $[x_L, x_R]$ and denote by $\underline{w}_L, \underline{w}_R$ the approximations to $\underline{w}$ at x_L, x_R, respectively. We rewrite equation (5.1) as

$$\underline{w}_t + \frac{\partial \underline{H}}{\partial \underline{w}} \underline{w}_x = \underline{h}(\underline{w}) \qquad (5.5)$$

and now solve approximately the associated Riemann problem

$$\underline{w}_t + C\underline{w}_x = \underline{h}(\underline{w}) \qquad (5.6)$$

with data $\underline{w}_L, \underline{w}_R$ either side of the point $\frac{1}{2}(x_L + x_R)$, linearising by considering C as a constant matrix. We shall use the approximate form

$$\frac{\underline{w}_P^{n+1} - \underline{w}_P^n}{\Delta t} + \tilde{C}\frac{(\underline{w}_R - \underline{w}_L)}{\Delta x} = \tilde{\underline{h}}(\underline{w}^n) \qquad (5.7)$$

where $\tilde{C}$ is the matrix of §3.5, $\tilde{\underline{h}}$ is an approximation to $\underline{h}(\underline{w})$ (see below) and P may be L or R. The matrix $\tilde{C}$ is given by equation (3.49) and its eigenvalues and eigenvectors are given by equations (3.45a-b), (3.47a-b), (3.48a) and (3.48c). A suitable approximation for the source term $\underline{h} = (0, g\phi h'(x))^T$ is

$$\tilde{\underline{h}} = (0, g\tilde{\phi}\frac{\Delta h}{\Delta x})^T \qquad (5.8)$$

where $\tilde{\phi}$ is given by equation (3.48b), (see Glaister [5]).

We now rewrite equation (5.7) as

$$\underline{w}_P^{n+1} = \underline{w}_P^n + \Delta t\tilde{\underline{h}} - \frac{\Delta t}{\Delta x}\tilde{C}(\underline{w}_R - \underline{w}_L), \qquad (5.9)$$

project $\underline{w}_R - \underline{w}_L$ and $\tilde{\underline{h}}$ onto the local eigenvectors given by equations (3.47a-b) and update $\underline{w}^n$ to $\underline{w}^{n+1}$ as follows. If

$$\underline{w}_R - \underline{w}_L = \sum_{j=1}^{2} \tilde{\alpha}_j \tilde{\underline{e}}_j \qquad (5.10)$$

then

$$\tilde{C}(\underline{w}_R - \underline{w}_L) = \sum_{j=1}^{2} \tilde{\lambda}_j \tilde{\alpha}_j \tilde{\underline{e}}_j \qquad (5.11)$$

since $\tilde{C}$ has eigenvalues $\tilde{\lambda}_i$ with eigenvectors $\tilde{\underline{e}}_i$ given by equations (3.45a-b) and (3.47a-b), respectively. We note that the results of §3.5 hold for variable $h(x)$ provided we evaluate terms like ϕ_R as $\phi_R = g(\eta_R + h(x_R))$. Projecting the source term $\tilde{\underline{h}}(\underline{w}^n)$ given by equation (5.8) as

$$\tilde{\underline{h}}(\underline{w}^n) = -\frac{1}{\Delta x}\sum_{j=1}^{2}\tilde{\beta}_j\tilde{\underline{e}}_j \tag{5.12}$$

enables equation (5.9) to be written as

$$\underline{w}_P^{n+1} = \underline{w}_P^n + \frac{\Delta t}{\Delta x}\sum_{j=1}^{2}\tilde{\lambda}_j\tilde{\gamma}_j\tilde{\underline{e}}_j \tag{5.13}$$

where

$$\tilde{\gamma}_j = \tilde{\alpha}_j + \tilde{\beta}_j/\tilde{\lambda}_j \tag{5.14}$$

and P may be L or R. To update $\underline{w}^n$ to $\underline{w}^{n+1}$ we use the method of upwind differencing as in §3, i.e. for each cell $[x_L, x_R]$ we

add $-\frac{\Delta t}{\Delta x}\tilde{\lambda}_j\tilde{\gamma}_j\tilde{\underline{e}}_j$ to $\underline{w}_R$ when $\tilde{\lambda}_j > 0$

or

add $-\frac{\Delta t}{\Delta x}\tilde{\lambda}_j\tilde{\gamma}_j\tilde{\underline{e}}_j$ to $\underline{w}_L$ when $\tilde{\lambda}_j < 0$

as shown in Figure 4.

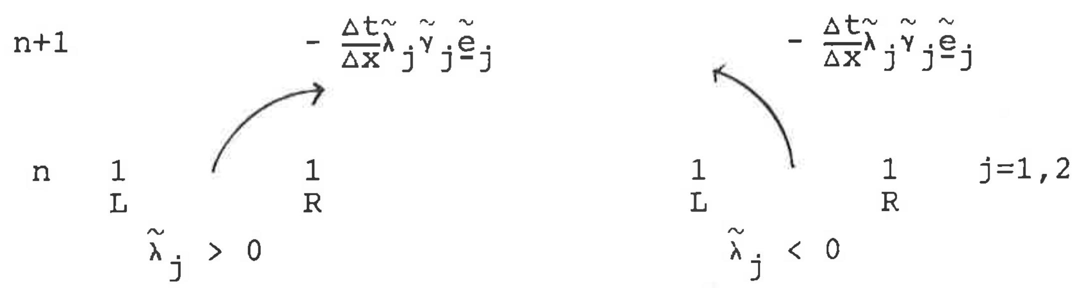


Figure 4

Following the algebra through, equation (5.10) gives

$$\tilde{\alpha}_{1,2} = \tfrac{1}{2}\Delta\Phi \pm \tfrac{1}{2}\frac{\tilde{\Phi}}{\tilde{\Psi}}\Delta u \tag{5.15a-b}$$

as in equations (3.46a-b) and (3.48b-c), and equations (5.8) and (5.12) yield

$$\tilde{\beta}_{1,2} = \mp \tfrac{1}{2} g \frac{\tilde{\Phi}}{\tilde{\Psi}} \Delta h \quad . \tag{5.16a-b}$$

(Note that the quantity Δh is independent of time and therefore has to be worked out only once.)

We turn now to the two-dimensional case.

5.2 Two-Dimensional Case

The governing equations in the two-dimensional case are

$$\underline{w}_t + \underline{F}_x + \underline{G}_z = \underline{f} + \underline{g} \tag{5.17}$$

where

$$\underline{w} = (\phi,\phi u,\phi w)^T \tag{5.18}$$

$$\underline{F}(\underline{w}) = (\phi u,\phi u^2 + \frac{\phi^2}{2},\phi uw)^T \tag{5.19}$$

$$\underline{G}(\underline{w}) = (\phi w,\phi uw,\phi w^2 + \frac{\phi^2}{2})^T \tag{5.20}$$

$$\underline{f}(\underline{w}) = (0,g\phi h_x,0)^T \tag{5.21}$$

and

$$\underline{g}(\underline{w}) = (0,0,g\phi h_z)^T \; . \tag{5.22}$$

We seek to solve equations (5.17)-(5.22) approximately using the technique of operator splitting, i.e. we solve successively

$$\tfrac{1}{2}\underline{w}_t + \underline{F}_x = \underline{f} \tag{5.23a}$$

and

$$\tfrac{1}{2}\underline{w}_t + \underline{G}_z = \underline{g} \tag{5.23b}$$

along x- and z-coordinate lines, respectively. The vectors $\underline{f}$ and $\underline{g}$ are associated with the x- and z-directions, respectively: this is a consequence of the terms h_x and h_z representing changes in the x- and z-directions, respectively. We shall discuss the solution of equation (5.23a), and the solution of equation (5.23b) will then follow by symmetry. The technique we adopt is an extension of §5.1.

Consider the interval $[x_L, x_R]$ on an x-coordinate line, $z = z_0$ and denote by $\underline{w}_L, \underline{w}_R$ the approximations to $\underline{w}$ at x_L, x_R, respectively. We now rewrite equation (5.23a) as

$$\underline{w}_t + \frac{\partial \underline{F}}{\partial \underline{w}} \underline{w}_x = \underline{f}(\underline{w}) \tag{5.24}$$

and solve approximately the associated Riemann problem

$$\underline{w}_t + A\underline{w}_x = \underline{f}(\underline{w}) \tag{5.25}$$

with data $\underline{w}_L, \underline{w}_R$ either side of the point $\frac{1}{2}(x_L + x_R)$, linearising by considering A as a constant matrix. We use the approximate form

$$\frac{\underline{w}_P^{n+1} - \underline{w}_P^n}{\Delta t} + \tilde{A}\frac{(\underline{w}_R - \underline{w}_L)}{\Delta x} = \tilde{\underline{f}}(\underline{w}^n) \tag{5.26}$$

where $\tilde{A}$ is the matrix of §4.3, $\tilde{\underline{f}}$ is an approximation to $\underline{f}(\underline{w})$ and P may be L or R. The matrix $\tilde{A}$ is given by equation (4.44) and its eigenvalues and eigenvectors are given by equations (4.40a-c), (4.42a-c), (4.43a-b) and (4.43d). A suitable approximation for the source term $\underline{f} = (0, g\phi h_x, 0)^T$ is

$$\tilde{\underline{f}} = (0, g\tilde{\phi}\frac{\Delta h}{\Delta x}, 0)^T \tag{5.27}$$

where $\tilde{\phi}$ is given by equation (4.43c) and $\Delta h, \Delta x$ are given by

$$\Delta h = h(x_R, z_0) - h(x_L, z_0) \tag{5.28}$$

$$\Delta x = x_R - x_L \tag{5.29}$$

where $z = z_0$ is the x-coordinate line being considered.

We now rewrite equation (5.26) as

$$\underline{w}_P^{n+1} = \underline{w}_P^n + \Delta t\tilde{\underline{f}} - \frac{\Delta t}{\Delta x}\tilde{A}(\underline{w}_R - \underline{w}_L), \tag{5.30}$$

project $\underline{w}_R - \underline{w}_L$ and $\tilde{\underline{f}}$ onto the local eigenvectors given by equations (4.42a-c) and update $\underline{w}^n$ to $\underline{w}^{n+1}$ as follows. If

$$\underline{w}_R - \underline{w}_L = \sum_{j=1}^{3} \tilde{\alpha}_j \tilde{\underline{e}}_j \tag{5.31}$$

then

$$\tilde{A}(\underline{w}_R - \underline{w}_L) = \sum_{j=1}^{3} \tilde{\lambda}_j \tilde{\alpha}_j \tilde{\underline{e}}_j \tag{5.32}$$

since $\tilde{A}$ has eigenvalues $\tilde{\lambda}_i$ with eigenvectors $\tilde{\underline{e}}_i$ given by equations (4.40a-c) and (4.42a-c), respectively. As in §5.1, we note that the results of §4.3 hold for variable $h(x,z)$ provided we evaluate terms like Φ_R as $\Phi_R = g(\eta_R + h(x_R, z_0))$. Projecting the source term $\tilde{\underline{f}}(\underline{w}^n)$, given by equations (5.27)-(5.29), as

$$\tilde{\underline{f}}(\underline{w}^n) = -\frac{1}{\Delta x} \sum_{j=1}^{3} \tilde{\beta}_j \tilde{\underline{e}}_j \tag{5.33}$$

enables equation (5.30) to be written as

$$\underline{w}_P^{n+1} = \underline{w}_P^n + \frac{\Delta t}{\Delta x} \sum_{j=1}^{3} \tilde{\lambda}_j \tilde{\gamma}_j \tilde{\underline{e}}_j \tag{5.34}$$

where

$$\tilde{\gamma}_j = \tilde{\alpha}_j + \tilde{\beta}_j / \tilde{\lambda}_j \tag{5.35}$$

and P may be L or R. To update $\underline{w}^n$ to $\underline{w}^{n+1}$ we use the method of upwind differencing as in §4, i.e. for each cell $[x_L, x_R]$ we

add $-\frac{\Delta t}{\Delta x} \tilde{\lambda}_j \tilde{\gamma}_j \tilde{\underline{e}}_j$ to $\underline{w}_R$ when $\tilde{\lambda}_j > 0$

or

add $-\frac{\Delta t}{\Delta x} \tilde{\lambda}_j \tilde{\gamma}_j \tilde{\underline{e}}_j$ to $\underline{w}_L$ when $\tilde{\lambda}_j < 0$

as shown in Figure 5.

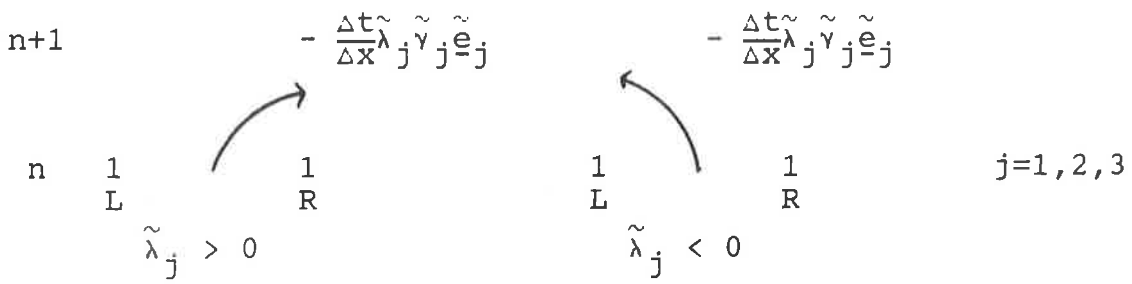


Figure 5

Following the algebra through, equation (5.31) gives

$$\tilde{\alpha}_{1,2} = \tfrac{1}{2}\Delta\Phi \pm \tfrac{1}{2}\frac{\tilde{\Phi}}{\tilde{\Psi}}\Delta u, \quad \tilde{\alpha}_3 = \tilde{\Phi}\Delta w \qquad (5.36a\text{-}c)$$

as in equations (4.41a-c) and (4.43c-d), and equations (5.27) and (5.33) yield

$$\tilde{\beta}_{1,2} = \mp \tfrac{1}{2}g\frac{\tilde{\Phi}}{\tilde{\Psi}}\Delta h, \quad \tilde{\beta}_3 = 0 \ . \qquad (5.37a\text{-}c)$$

(As in §5.1 the quantity Δh is independent of time and therefore has to be worked out only once.)

Similar results apply for updating in the z-direction.

In the next section we describe two one-dimensional problems with exact solutions.

6. TEST PROBLEMS

In this section we describe two test problems for the one-dimensional shallow water equations which have exact solutions.

(i) Bore reflection

This test problem is concerned with the reflection by a wall of a fluid governed by the one-dimensional, constant depth shallow water equations. We consider a region $0 \leq x \leq 1$ with initial conditions at $t = 0$,

$$\left.\begin{aligned} \Phi &= \Phi_0 = gh_0 \\ u &= -u_0 \end{aligned}\right\} \qquad (6.1)$$

where the constant undisturbed fluid depth is h_0. This represents a fluid of zero constant elevation moving towards $x = 0$. The boundary $x = 0$ is a rigid wall and the exact solution describes the reflection of a bore from the wall. The fluid is brought to rest at $x = 0$ and, denoting initial values by (0), values behind the bore by (+) and values ahead of the bore by (-), we can postulate an exact solution of the form

$$\Phi = \Phi_+, \quad u = u_+ = 0 \quad \text{for} \quad \frac{x}{t_0} < S \qquad (6.2a)$$

$$\Phi_0 = \Phi_- = \Phi_0, \quad u = u_- = -u_0 \quad \text{for} \quad \frac{x}{t_0} > S, \qquad (6.2b)$$

at a time $t_0 > 0$ (see Figures 6 and 7).

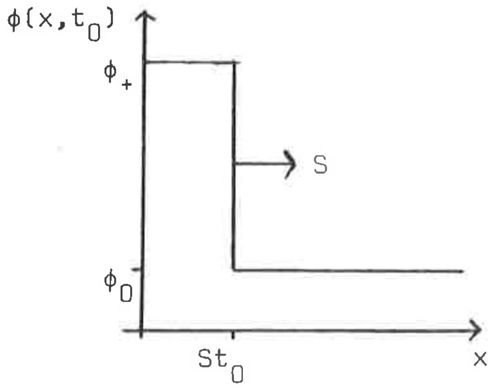


Figure 6

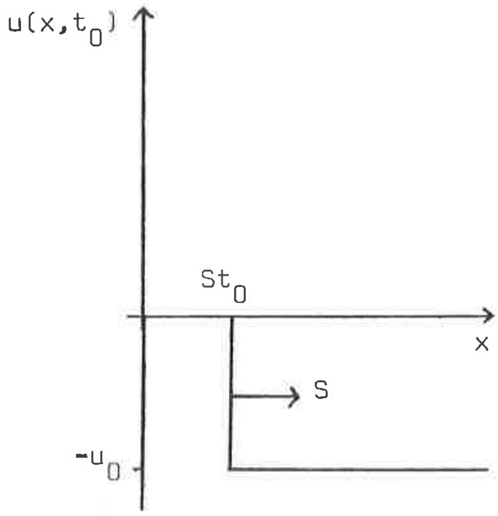


Figure 7

The bore moves out from the origin with speed S, and S, ϕ_+ are given by the following conditions at the discontinuity

$$S = \frac{[\phi u]}{[\phi\ ]} = \frac{[\phi u^2 + \frac{\phi^2}{2}]}{[\phi u]} \qquad (6.3)$$

where $[v] = v_+ - v_-$ denotes the jump in v across the bore (see [6]). We therefore need to solve equations (6.3) for S, ϕ_+ subject to the initial conditions given by equation (6.1).

If we write out equations (6.3), using equations (6.1)-(6.2b), we obtain

$$S = \frac{\phi_0 u_0}{\phi_+ - \phi_0} \qquad (6.4a)$$

$$S = \frac{\frac{1}{2}\phi_+^2 - \frac{1}{2}\phi_0^2 - \phi_0 u_0^2}{\phi_0 u_0} . \qquad (6.4b)$$

From equation (6.4a) we find

$$\phi_+ = \phi_0\left[1 + \frac{u_0}{S}\right] \qquad (6.5)$$

and substituting for ϕ_+ from equation (6.5) into equation (6.4b) yields the following cubic equation for S

$$S^3 + u_0 S^2 - \phi_0 S - \tfrac{1}{2}\phi_0 u_0 = 0 . \qquad (6.6)$$

By inspection, the sum of the roots of equation (6.6) is negative while their product is positive. Therefore the only positive real root of equation (6.6) gives the required speed of the bore, and hence ϕ_+ is determined by equation (6.5). The elevation of the fluid at rest behind the bore is given by $\eta_+ = \phi_+/g - h_0 = (\phi_+ - \phi_0)/g$.

(ii) A breaking dam

This test problem arises when there is water of constant depth either side of a dam and the dam breaks. Consider a horizontal tank of constant cross section extending to infinity in both directions with a thin partition at the section $x = 0$. For $x > 0$ the water has the depth h_0 while for $x < 0$ the depth is h_1, with $h_0 < h_1$. In addition the water is assumed to be at rest initially. Thus we can write the initial conditions as

$$\left.\begin{array}{l} \phi = \phi_0 = gh_0 \\ u = 0 \end{array}\right\} x > 0 \qquad (6.7a)$$

$$\left.\begin{array}{l} \phi = \phi_1 = g(\eta_1 + h_0) = gh_1 \\ u = 0 \end{array}\right\} x < 0 \qquad (6.7b)$$

where the initial elevations are η_1 and $\eta_0 = 0$ for $x < 0$ and $x > 0$, respectively. When the dam is destroyed at $t = 0$ the initial discontinuity breaks up into a bore travelling to the right and a depression wave travelling to the left. The derivation of the exact solution can be found in Stoker [6] and as indicated in Figures 8 and 9, consists of four different regions marked (1), (3), (2), (0) at any time $t_0 > 0$.

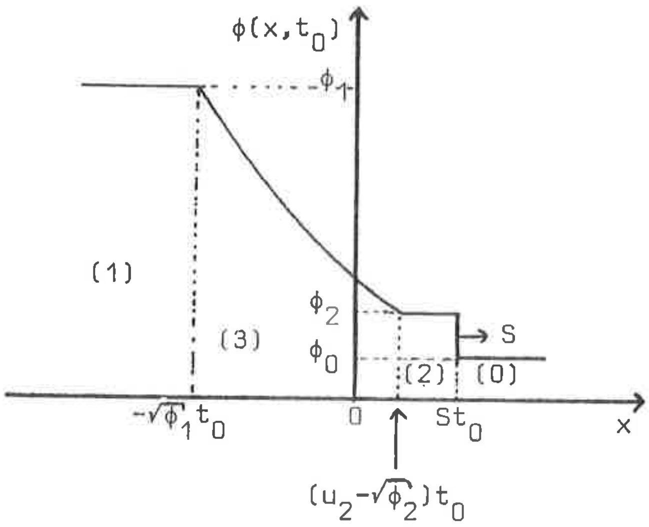


Figure 8

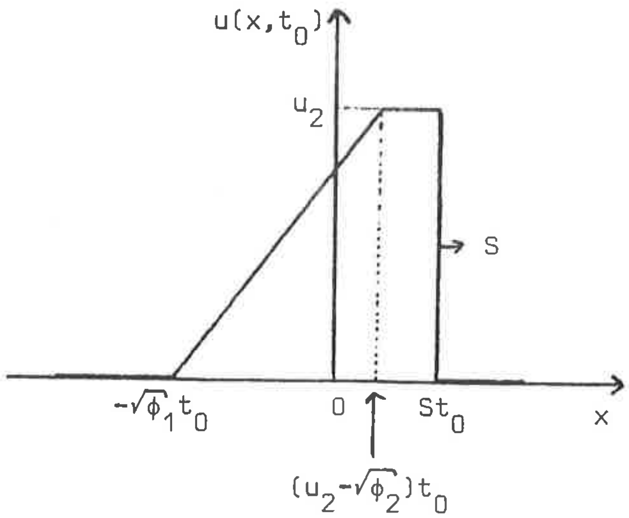


Figure 9

Thus the exact solution can be written as

Region 1 $\qquad -\infty < x \le -\sqrt{\Phi_1}\, t_0$

$$\left.\begin{aligned} \Phi &= \Phi_1 \\ u &= 0 \end{aligned}\right\} \qquad (6.8a)$$

Region 2 $\qquad -\sqrt{\Phi_1}\, t_0 < x \le (u_2 - \sqrt{\Phi_2})t_0$

$$\left.\begin{aligned} \Phi &= \tfrac{1}{9}(2\sqrt{\Phi_1} - \frac{x}{t_0})^2 \\ u &= \tfrac{2}{3}(\sqrt{\Phi_1} + \frac{x}{t_0}) \end{aligned}\right\} \qquad (6.8b)$$

Region 3 $\qquad (u_2 - \sqrt{\Phi_2})t_0 < x < St_0$

$$\left.\begin{aligned} \Phi &= \Phi_2 \\ u &= u_2 \end{aligned}\right\} \qquad (6.8c)$$

Region 4 $\qquad St_0 < x < \infty$

$$\left.\begin{aligned} \Phi &= \Phi_0 \\ u &= 0 \end{aligned}\right\} \qquad (6.8d)$$

where the quantities Φ_2 and u_2 are given in terms of the bore speed S by

$$\Phi_2 = \tfrac{1}{2}\left[\sqrt{1 + 8\frac{S^2}{\Phi_0}} - 1\right]\Phi_0 \qquad (6.9)$$

$$u_2 = S - \frac{\Phi_0}{4S}\left[1 + \sqrt{1 + 8\frac{S^2}{\Phi_0}}\right] \qquad (6.10)$$

and the bore speed S is the real positive root of

$$u_2 + 2\sqrt{\Phi_2} - 2\sqrt{\Phi_1} = 0 \ . \qquad (6.11)$$

(N.B. Substituting for Φ_2 and u_2 from equations

(6.9)-(6.10) into equation (6.11) yields the required equation.)

In the next section we display the numerical results obtained for the test problems of this section using the scheme of §3.5.

7. NUMERICAL RESULTS

In this section we give the numerical results achieved for the two one-dimensional problems of §6 using the scheme presented in §3.

Problem 1 Bore reflection

For this problem we apply a reflection condition at $x = 0$, i.e. we consider an image cell at the boundary and impose equal elevation and equal and opposite velocity at either end of the cell. This results in no net movement in the cell.

Figures 10 and 11 refer to the problem of §6.1 using 50 mesh points and the 'Superbee' limiter (see [3]). The initial incoming velocity is $u_0 = -1$, and the initial incoming elevation Φ_0 chosen so that the jump in the discontinuity, Φ_+/Φ_0 takes the values 2 and 3 in Figure 10 and 11, respectively.

Problem 2 A bursting dam

For this problem we consider a fixed region in space, $0 \leq x \leq 1$ and the initial discontinuity is at $x = 0.5$. At the boundaries we allow only outgoing waves and no incoming waves.

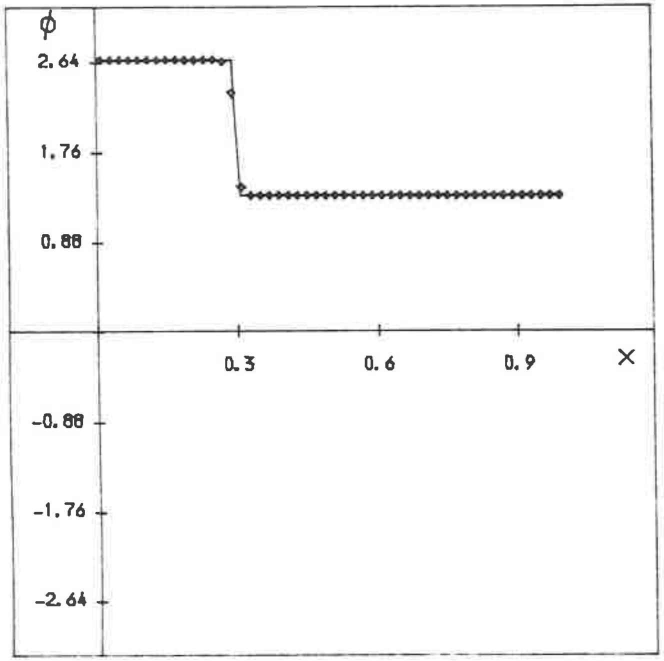
Figures 12, 13, 14 and refer to the problem of §6.2 using 50 mesh points and the Superbee limiter. In each case we take the initial elevation to the left of the discontinuity as

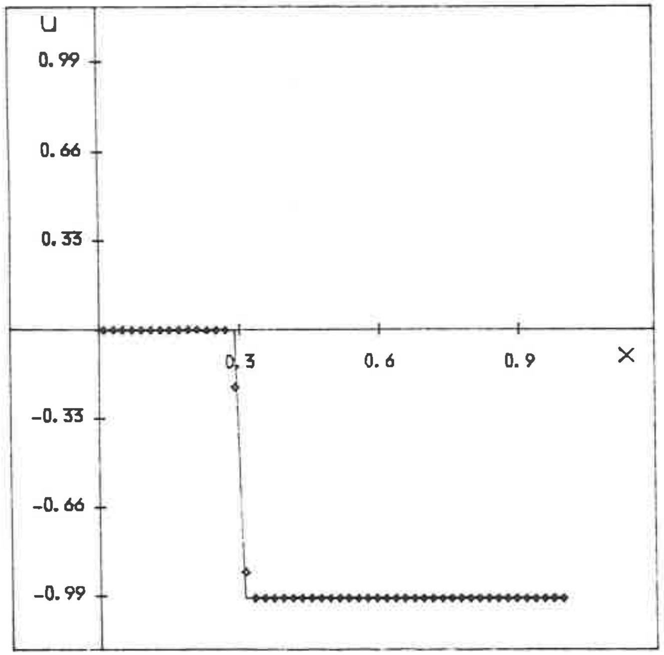
$\phi_1 = 1$, and to the right of the discontinuity we take $\phi_0 = 0.1, 0.2$ and 0.5, corresponding respectively to Figures 12, 13 and 14. We use 25 time steps with $\Delta t = 0.01$ for each case.

For both problems we display ϕ, related to the elevation η, and the fluid speed u. In all cases we observe a good representation of the solution and correct propagation speeds of the discontinuities.

Finally, we give the c.p.u. time to compute the results obtained for problem 2. Using an Amdahl V7 takes 0.0048 c.p.u. seconds to compute one time step and a total of 0.12 c.p.u. seconds to reach a real time of 0.25 seconds using 25 time steps.

SOLUTION OF THE SHALLOW WATER EQUATIONS - Bore Reflection





at time t = 0.296

KEY

ϕ - Elevation

u - Velocity

—— Exact solution

◇◇◇◇◇◇ Approximate solution

PARAMETERS

50 Mesh points

65 Time steps

$\Delta x = 0.02$

$\Delta t = 0.0046$

Elevation ratio = 2

'Superbee' limiter used

INITIAL CONDITIONS

$\phi = 1.332$

$u = -1.000$

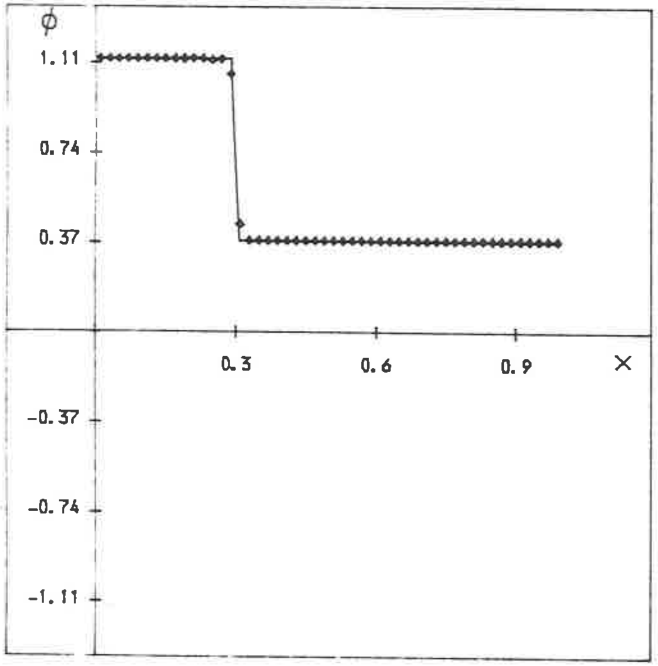
0 1

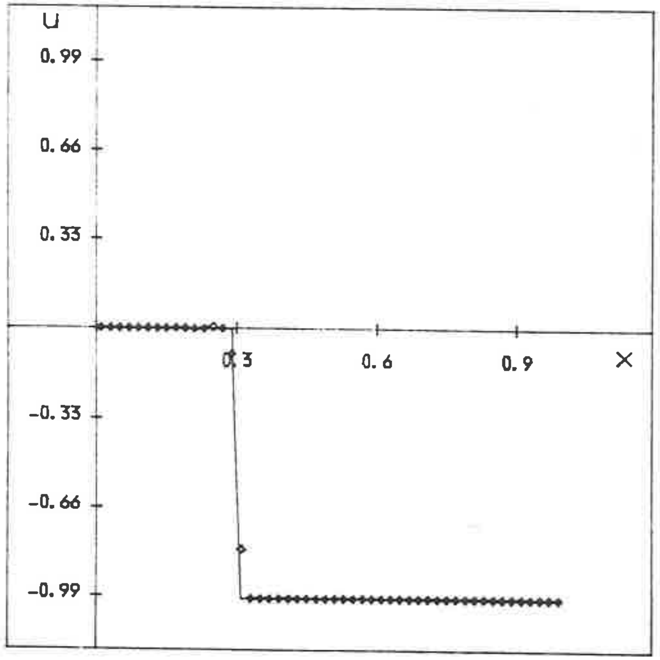
Reflected Boundary Conditions

at x = 0

Figure 10

SOLUTION OF THE SHALLOW WATER EQUATIONS - Bore Reflection





at time t = 0.600

KEY

ϕ - Elevation

u - Velocity

—— Exact solution

◇◇◇◇◇◇ Approximate solution

PARAMETERS

50 Mesh points

103 Time steps

$\Delta x = 0.02$

$\Delta t = 0.0058$

Elevation ratio = 3

'Superbee' limiter used

INITIAL CONDITIONS

$\phi = 0.375$

$u = -1.000$

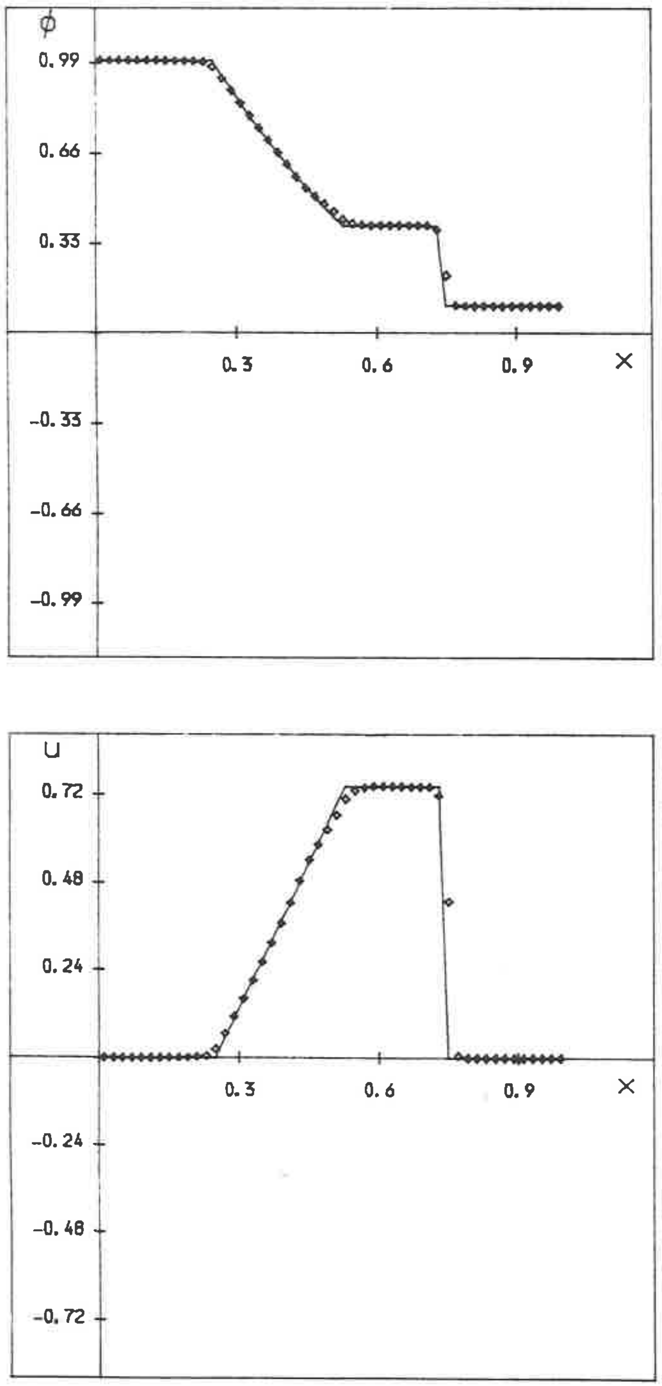
0 1

Reflected Boundary Conditions

at $x = 0$

Figure 11

SOLUTION OF THE SHALLOW WATER EQUATIONS - A Bursting Dam



at time t = 0.250

KEY

ϕ - Elevation

u - Velocity

—— Exact solution

◇◇◇◇◇◇ Approximate solution

PARAMETERS

50 Mesh points

25 Time steps

$\Delta x = 0.02$

$\Delta t = 0.0100$

Elevation ratio = 10

'Superbee' limiter used

INITIAL CONDITIONS

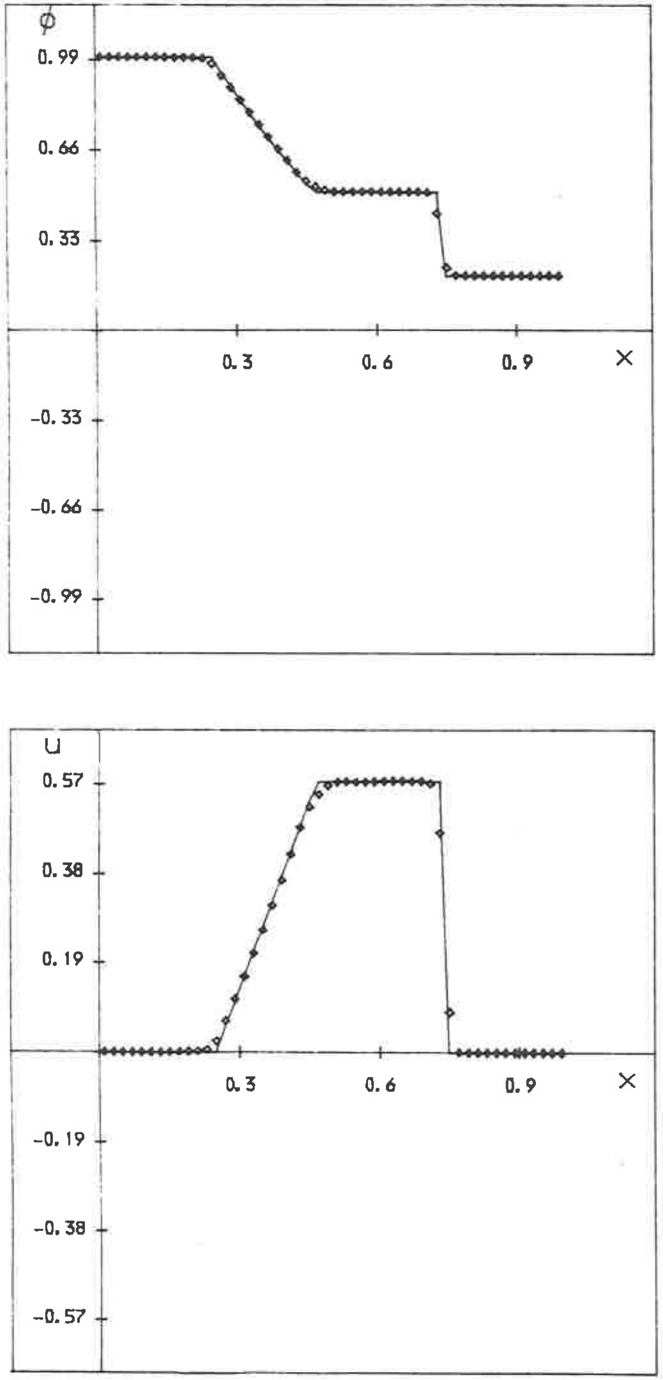
$0 \le x < 0.5$	$0.5 < x \le 1$
$\phi = 1.000$	$\phi = 0.100$
$u = 0.000$	$u = 0.000$

0 0.5 1

Transparent Boundary Conditions
at x = 0 and x = 1

Figure 12

SOLUTION OF THE SHALLOW WATER EQUATIONS - A Bursting Dam



at time t = 0.250

KEY

φ - Elevation

u - Velocity

—— Exact solution

◇◇◇◇◇◇ Approximate solution

PARAMETERS

50 Mesh points

25 Time steps

$\Delta x = 0.02$

$\Delta t = 0.0100$

Elevation ratio = 5

'Superbee' limiter used

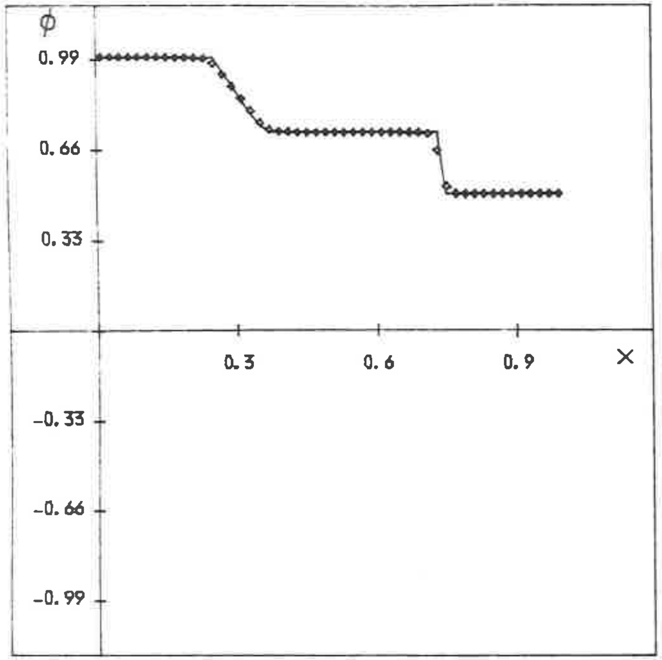
INITIAL CONDITIONS

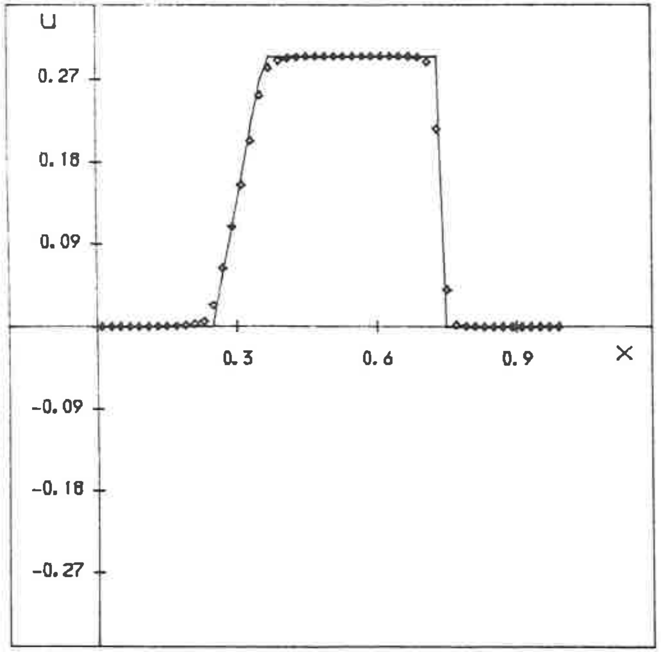
0 – 0.5	0.5 – 1
$\phi = 1.000$	$\phi = 0.200$
$u = 0.000$	$u = 0.000$

Transparent Boundary Conditions at x = 0 and x = 1

Figure 13

SOLUTION OF THE SHALLOW WATER EQUATIONS - A Bursting Dam





at time t = 0.250

KEY

ϕ - Elevation

u - Velocity

—— Exact solution

◇◇◇◇◇ Approximate solution

PARAMETERS

50 Mesh points

25 Time steps

$\Delta x = 0.02$

$\Delta t = 0.0100$

Elevation ratio = 2

'Superbee' limiter used

INITIAL CONDITIONS

$\phi = 1.000$	$\phi = 0.500$
$u = 0.000$	$u = 0.000$

0 0.5 1

Transparent Boundary Conditions

at $x = 0$ and $x = 1$

Figure 14

8. CONCLUSIONS

We have presented a technique for obtaining approximate solutions to the shallow water equations using a Riemann problem. In the two-dimensional case the scheme incorporates the technique of operator splitting. The scheme achieves satisfactory results for two one-dimensional test problems involving discontinuities.

ACKNOWLEDGEMENTS

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