Re-iterative Methods For

Integral Equations

Paul Antony Burton

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Department of Mathematics.

University of Reading.

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Abstract

A review is given of the Galerkin, iterated Galerkin and re-iterated Galerkin methods for finding approximate solutions to integral equations of the second kind. The process of re-iteration, first described by Porter & Stirling, is applied to the Kantorovich method to produce two new re-iteration methods. The first, the modified iterated Kantorovich method, is a straightforward application of the re-iterated Galerkin method to a regularized Kantorovich equation. The second method differs by applying the Kantorovich regularization at each iteration. This second method is tested on some example integral equations and its convergence compared with that of the re-iterated Galerkin method. Finally it is shown how the presence of an eigenvalue can affect the convergence of these methods.

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Contents

Table of Notation

$k(x,t)$	kernel
K	integral operator
S, SK	convergence operators
μ_1^+, μ_1^-	eigenvalues
$\tilde{\mu}_1$	approximate eigenvalue
(ϕ, ψ)	inner product
$\|.\|$	norm
H	Hilbert space
$L_2(0,1)$	space of square integrable functions
$R_K(\phi)$	Rayleigh quotient
I	identity operator
P_n	projection
$\rho(T)$	spectral radius of T
$\kappa_0, \alpha, \beta, \lambda$	parameters
$\phi(x)$	particular function
E_n	n-dimensional subspace
$\{\chi_1, ..., \chi_n\}$	basis functions
sup	supremum

Chapter 1

Introduction

1.1 Integral Equations

In certain problems it is not uncommon to encounter equations in which the unknown function appears under an integral sign. Such equations are called integral equations, and are often very difficult, if not impossible, to solve analytically.

Numerical techniques for integral equations, however, can prove to be especially powerful, and since many initial and boundary value problems for differential equations can be expressed as single integral equations, without the need for extra conditions, these techniques are also extremely versatile.

There are two main types of linear integral equation: a) the integral equation of the first kind,

$$f(x) = \int_a^b k(x,t)\phi(t)dt, \qquad (a \leq x \leq b), \tag{1.1}$$

in which the unknown function $\phi(t)$ appears solely under the integral sign, and b) the integral equation of the second kind,

$$\phi(x) = f(x) + \lambda \int_a^b k(x,t)\phi(t)dt, \qquad (a \leq x \leq b), \tag{1.2}$$

in which the unknown function $\phi(x)$ also appears outside the integral. In both cases the functions f(x) and k(x,t) are presumed known.

The function k(x,t) is called the *kernel* of the equation, and f(x) is called the *free term*.

Certain forms of equations (1.1) and (1.2) are known by special names. If the limits of integration a and b are constants, then the equations are called *Fredholm equation's*, and are said to be *homogeneous* if $f(x)$=0, whilst if a=0 and b=x, (ie. variable limit of integration) then the equations are known as *Volterra equation's* (again homogeneous if $f(x) = 0$).

Here we will be discussing techniques for finding approximate solutions to Fredholm equations of the second kind. An example of such an equation is

$$\phi(x) = 1 + \lambda \int_0^1 2max(x,t)\phi(t)dt, \qquad (0 \leq x \leq 1),$$

where the freeterm $f(x)$=1, and the kernel $k(x,t)$=2max(x,t).

Porter & Stirling [3] show how equation (1.2) can be solved using the equation

$$\phi = f + \lambda K \phi$$

or

$$A\phi = f,$$

in $L_2(0,1)$, where $A = (I - \lambda K)$, and K is a compact operator on this space, denoted by

$$(K\phi)(x) = \int_a^b k(x,t)\phi(t)dt,$$

where I is the identity map.

A is assumed to be invertible throughout.

1.2 Projection methods

We wish to find approximations to the solution of integral equations of the form

$$\phi = f + \lambda K\phi \tag{1.3}$$

in some Hilbert space H, such as $L_2(0,1)$.

To do this we look for an approximation p_n to ϕ in some finite dimensional subspace E_n of H, by solving the equation

$$p_n = P_n f + P_n \lambda K \phi_n \tag{1.4}$$

in E_n, where P_n is an orthogonal projection of H onto E_n.

If the sequence of subspaces (E_n) has the property that $\forall \phi \in H$, $P_n\phi \to \phi$ as $n \to \infty$, then since $\phi = A^{-1}f$, we have that $AP_n\phi \to f$ as $n \to \infty$.

We would like p_n to be in some sense the best approximation to ϕ in E_n. One way of achieving this is to choose p_n such that it minimises $\|Ap_n - f\|$.

Hence,

$$\|Ap_n - f\| \leq \|AP_n\phi - f\| \to 0 \quad as \;\; n \to \infty\,,$$

$$\Rightarrow \quad AP_n \to f \qquad \text{as} \quad n \to \infty,$$

$$\Rightarrow \quad p_n \to A^{-1}f = \phi \quad \text{as} \quad n \to \infty,$$

which leads to what is called the Petrov-Galerkin method. Another method is to construct p_n to be that part of the solution ϕ which lies in the subspace E_n, so that the closer E_n is to H, the closer the approximation p_n is to ϕ. If E_n is spanned by an orthonormal basis $\{\chi_1, \ldots, \chi_n\}$ then p_n will be of the form

$$p_n = \sum_{i=1}^{n} \alpha_i \chi_i, \tag{1.5}$$

for some scalars α_i. This process is known as the Galerkin method, which we shall now discuss in more detail.

Chapter 2

Galerkin Methods

2.1 The Galerkin Method

If our approximation p_n is chosen such that $Ap_n - f$ is orthogonal to the subspace E_n, then we have

$$(Ap_n - f, \chi_j) = 0, \quad j = 1, \ldots, n$$

$$\Rightarrow \quad (Ap_n, \chi_j) = (f, \chi_j), \quad j = 1, \ldots, n$$

where the χ_j's are the orthonormal basis functions of E_n.

Using our definition of p_n from equation (1.5) and $A = I - K$ (taking $\lambda = 1$ in equation (1.3) for convenience) we arrive at

$$\sum_{i=1}^{n} \alpha_i \{(\chi_i, \chi_j) - (K\chi_i, \chi_j)\} = (f, \chi_j), \quad j = 1, \ldots, n \tag{2.1}$$

This system of n equations can be solved simultaneously, provided the determinant of coefficients is non-zero, to obtain the α_i's which, via equation (1.5), provide us with our approximation p_n to the solution ϕ of our integral equation.

In terms of the projection P_n of H onto E_n,

$$P_n p_n = P_n f + P_n K p_n$$

$$\Rightarrow \quad p_n = (I - P_n K)^{-1} P_n f,$$

since $P_n p_n = p_n$.

The process described above is called *Galerkin's method*, and is a well-established technique for finding approximate solutions to integral equations. The accuracy of the approximation, determined by the norm of the difference between the exact solution and the approximation, ie. $\|p_n - \phi\|$, is dependent upon the size of the subspace chosen, since $p_n \to \phi$ as $n \to \infty$. From a computational point of view, an increase in the dimension of the subspace results in both an increase in the number of inner-products to be evaluated, and in the number of simultaneous equations to be solved. These considerations have to be balanced against the extra degree of accuracy obtained.

2.2 Degenerate Kernels

Integral equations with degenerate kernels are such that they can be reduced to a set of linear equations. Thus, the integral equation $\phi = f + K\phi$, where the kernel of K is not degenerate, can be approximated by

$$\phi_N = f + K_N \phi_N \tag{2.2}$$

where K_N is a degenerate kernel of dimension N (ie. of the form $\sum_{i=1}^{N} a_i(x) b_i(t)$) and ϕ_N is the corresponding approximation to ϕ.

In 1975, Sloan, Burn and Datyner [8] published a paper on degenerate kernel methods for integrals. Whereas previous methods chose K_N to be a good approximation to K over the whole space H, their approach was to construct a degenerate kernel which was a good approximation in the sense in which K

appears in the equation, ie. as an operator on ϕ.

That is, $K_N\phi$ will be a good approximation to $K\phi$ if

$$K_N\chi_n = K\chi_n \quad n = 1, \ldots, N$$

where $\{\chi_1, \ldots, \chi_N\}$ forms a suitable basis.

Using P_n, the orthogonal projection onto the subspace E_n, so that $P_n\phi$ is the part of ϕ that lies in E_n, we have

$$K_N P_n = K P_n$$

$$\Rightarrow \quad (K - K_N)\phi = (K - K_N)(\phi - P_n\phi).$$

Conventional methods yield

$$\|(K - K_N)\phi\| \leq \|K - K_N\|\|\phi\| = constant,$$

whereas now we have

$$\|(K - K_N)\phi\| \leq \|K - K_N\|\|\phi - P_n\phi\| \to 0 \ \ as\ n \to \infty.$$

This method allows our approximate integral equation

$$\phi \approx \phi_N = f + K_N\phi_N$$

to be reduced to

$$\phi_N = f + \sum_{n=1}^{N} \alpha_n K\chi_n, \tag{2.3}$$

where the α's satisfy

$$\sum_{j=1}^{N} \alpha_n\{(\chi_i, \chi_j) - (\chi_i, K\chi_j)\} = (\chi_i, f) \quad i = 1, \ldots, N. \tag{2.4}$$

Notice that this equation is identical to the expression for the α's in Galerkin's method, so the α's are the same for both methods. However, $\phi_N \neq p_n$. In fact

$$\phi_N = f + K\sum_{n=1}^{N} \alpha_n \chi_n = f + Kp_n \tag{2.5}$$

The approximations are related by a single iteration of the integral equation.

2.3 The Iterated Galerkin Method

If we take our Galerkin approximation $p_n = (I - P_nK)^{-1}P_nf$, and iterate it (writing ϕ_N as $\hat{p}$), we observe that

$$\begin{aligned}
\hat{p_n} &= f + Kp_n \\
&= f + K(I - P_nK)^{-1}P_nf \\
&= f + KP_n(I - KP_n)^{-1}f \\
&= [(I - KP_n) + KP_n](I - KP_n)^{-1}f \\
&= (I - KP_n)^{-1}f.
\end{aligned}$$

Now

$$\begin{aligned}
\hat{p_n} - \phi &= (I - KP_n)^{-1}f - (I - K)^{-1}f \\
&= (I - KP_n)^{-1}[(I - K) - (I - KP_n)](I - K)^{-1}f \\
&= -(I - KP_n)^{-1}K(I - P_n)\phi \\
&= (I - KP_n)^{-1}(K - KP_n)(p_n - \phi)
\end{aligned}$$

Hence $\|\hat{p_n} - \phi\| \leq \|S\|\|p_n - \phi\|$, where $S = (I - KP_n)^{-1}(K - KP_n)$. So iteration provides a more accurate approximation provided $\|S\| < 1$, but since $\|S\| \leq \|(I - KP_n)^{-1}\|\|K(I - P_n)\|$, we can ensure that $\|S\| < 1$ by choosing the subspace E_n to be suitably large.

This process is non-repeatable however. If we try to iterate our iterated approximation we obtain

$$\hat{\hat{p_n}} = f + K\hat{p_n},$$

but now

$$\hat{\hat{p_n}} - \phi = K(\hat{p_n} - \phi)$$

which only gives an improvement if $\|K\| < 1$, which is not something we have any control over.

Despite this, iteration provides us with a means of improving our approximation without directly having to increase the size of the subspace E_n. (For more details see Sloan [7]).

2.4 The Re-iterated Galerkin Method

The iterated Galerkin method provides us with an approximation $\hat{p_0} = f + Kp_0$ to the exact solution ϕ, which implies that

$$\phi = \hat{p_0} + \phi_1$$

where ϕ_1 is a correction term. By substituting this into the integral equation $\phi = f + K\phi$ we obtain

$$\phi_1 = \hat{r_0} + K\phi_1 \tag{2.6}$$

where $\hat{r_0} = f - (I - K)\hat{p_0}$, the residual error incurred in approximating ϕ by $\hat{p_0}$. This leads to the method of re-iteration discovered by Porter & Stirling [4].

Notice that we now have an integral equation to solve for ϕ_1. We proceed as before, by seeking an approximation p_1 to ϕ_1 in the subspace E_n, using Galerkin's method.

When finding our Galerkin approximation to $\phi = f + K\phi$ we used

$$p_0 = \sum_{i=1}^{N} \alpha_i \chi_i$$

where

$$\sum_{i=1}^{N} \alpha_i \{(\chi_i, \chi_j) - (K\chi_i, \chi_j)\} = (f, \chi_j) \quad j = 1, \ldots, N$$

and where $\{\chi_1, \ldots, \chi_N\}$ forms an orthonormal basis to E_n.

Now we wish to find an approximation to $\phi_1 = \hat{r_0} + K\phi_1$ by finding

$$p_1 = \sum_{i=1}^{N} \beta_i \chi_i$$

where

$$\sum_{i=1}^{N} \beta_i \{(\chi_i, \chi_j) - (K\chi_i, \chi_j)\} = (\hat{r_0}, \chi_j), \quad j = 1, \ldots, N. \tag{2.7}$$

We are looking for an approximation in the same subspace as before, so the basis functions remain the same, which means we do not have to recalculate the inner-products on the left-hand-side of equation (2.7). The only new calculations are of the inner-products $(\hat{r_0}, \chi_j)$, $j = 1, \ldots, N$. In terms of the projection P_n, we have $p_1 = (I - K)^{-1} P_n \hat{r_0}$.

As before we can perform a single iteration of our approximation p_1 to ϕ_1, to obtain $\hat{p_1} = \hat{r_0} + Kp_1$, $(\hat{p_1} = (I - KP_n)^{-1}\hat{r_0})$. So our new approximation to ϕ is $\hat{p_0} + \hat{p_1}$.

Now

$$\begin{aligned}
\hat{p_0} + \hat{p_1} - \phi &= (I - KP_n)^{-1}\hat{r_0} + \hat{p_0} - \phi \\
&= -(I - KP_n)^{-1}(I - K)(\hat{p_0} - \phi) + (\hat{p_0} - \phi) \\
&= (I - KP_n)^{-1}(K - KP_n)(\hat{p_0} - \phi) \\
&= S(\hat{p_0} - \phi) \\
&= S^2(p_0 - \phi)
\end{aligned}$$

So $\hat{p_0} + \hat{p_1}$ is an improved approximation to ϕ, compared with $\hat{p}_0$, provided $\|S\| < 1$, which we have already assumed for the iterated Galerkin approximation.

This process has the advantage that it can be repeated. We can write

$$\phi = \hat{p_0} + \hat{p_1} + \phi_2$$

where ϕ_2 is our new correction term. Substituting this into $\phi = f + K\phi$ and rearranging gives

$$\phi_2 = \hat{r_1} + K\phi_2$$

where $\hat{r_1} = \hat{r_0} - (I - K)\hat{p_1}$, the residual error in $\hat{p_0} + \hat{p_1}$.

Again this can be solved approximately using the iterated Galerkin method to give an approximation $\hat{p_2}$ to ϕ_2. Our approximation to ϕ would then be $\hat{p_0}+\hat{p_1}+\hat{p_2}$, which has an error

$$\hat{p_0} + \hat{p_1} + \hat{p_2} - \phi = S(\hat{p_0} + \hat{p_1} - \phi) = S^3(p_0 - \phi).$$

In general, if we write $\phi \approx \hat{\sigma}^n_{gal} = \hat{p_0} + \ldots + \hat{p_n}$, then each correction term is determined by using Galerkin's method on $\phi_n = \hat{r}_{n-1} + K\phi_n$ to obtain $p_n = (I - P_nK)^{-1}P_n\hat{r}_{n-1}$ then iterating to get $\hat{p_n} = \hat{r}_{n-1} + Kp_n$, where $\hat{r}_{n-1} = f - (I - K)\hat{\sigma}^{n-1}_{gal}$. The error in the current approximation is given by $\hat{\sigma}^n_{gal} - \phi = S(\hat{\sigma}^{n-1}_{gal} - \phi) = S^{n+1}(p_0 - \phi)$.

An important point to note is that $\hat{p}_{n+1}$ is calculated using the error involved with the current approximation $\hat{\sigma}^n_{gal}$, so a small error at one stage is not propagated; rather a compensatory correction is introduced in subsequent terms. This suggests that, provided $\|S\| < 1$, the process produces a sequence of successively improved approximations to ϕ without having to enlarge the size of the subspace E_n.

The bound on the norm of S only provides a guide for the convergence of the re-iterated method. The value of $\|S\|$ may be estimated beforehand to give an indication of whether the method will converge for a particular choice of subspace (see Porter & Stirling [4] for details).

The iteration itself also provides a means for checking convergence. It is easy to see that what we require for convergence is for successive residuals to reduce in magnitude, ie. that

$$\frac{\|\hat{r_n}\|}{\|\hat{r}_{n-1}\|} < 1.$$

Taking n=1, and expressing the residuals in terms of the projection P_n, we see

$$\begin{aligned} \hat{r_0} &= f + K\hat{p_0} - \hat{p_0} & \text{and} \quad \hat{r_1} &= f + K(\hat{p_0} + \hat{p_1}) - (\hat{p_0} + \hat{p_1}) \\ &= f - (I-K)\hat{p_0} & &= \hat{r_0} - (I-K)\hat{p_1} \\ &= f - (I-K)(I-KP_n)^{-1}f & &= \hat{r_0} - (I-K)(I-KP_n)^{-1}\hat{r_0} \\ &= K(I-P_n)(I-KP_n)^{-1}f & &= K(I-P_n)(I-KP_n)^{-1}\hat{r_0} \end{aligned}$$

So

$$\frac{\|\hat{r_1}\|}{\|\hat{r_0}\|} \leq \|(K-KP_n)(I-KP_n)^{-1}\|. \tag{2.8}$$

We can in fact find a better condition for convergence. If the calculations can be performed exactly, it can be easily shown that $\hat{p}_n = S^n\hat{p}_0$. Now, $\sum \hat{p}_n = \sum S^n\hat{p}_0$ can be shown to converge (by the root test) if the spectral radius of S, denoted $\rho(S)$, is less than one. It follows, then, that the re-iterated Galerkin method will converge, in theory, if $\rho(S) < 1$. Furthermore, the ratio of successive residual norms, $\frac{\|\hat{r}_n\|}{\|\hat{r}_{n-1}\|}$, can be shown to converge to $\rho(S)$, giving a quick and easy check for convergence. Indeed, if we take a little extra trouble to calculate the residual error

in the initial (non-iterated) Galerkin approximations we find, (taking $n = 0$),

$$\begin{aligned} r_0 &= f + Kp_0 - p_0 \\ &= \hat{p_0} - p_0 \\ &= (I - KP)^{-1} f - (I - P_n K)^{-1} P_n K \\ &= (I - P_n)(I - KP_n)^{-1} f, \end{aligned}$$

which leads us to

$$\frac{\|\hat{r_n}\|}{\|r_n\|} \leq \|K\|, \tag{2.9}$$

ie. the ratio of the residual of the iterated Galerkin approximation, over the norm of the residual of the Galerkin approximation, provides a lower bound on the norm of the operator K. Whereas this quantity has no direct consequences for the convergence of the method, it is however valuable information to know about the kernel, and comes relatively free.

Chapter 3

Kantorovich Methods

3.1 The Kantorovich Method

The re-iterated Galerkin method provides a means of calculating an approximation to the solution of an integral equation, providing certain conditions are met, to any desired accuracy. The number of re-iterations required to reach a specific accuracy depends upon the rate of convergence of the method, which, in turn, depends upon the size of the subspace used. An increase in the size of E_n will result in faster convergence, but requires more calculations.

Is there a way in which the re-iteration process can be made more efficient, increasing the rate of convergence whilst minimising the number of calculations?

The Kantorovich method is an established adaptation of Galerkin's method. Suppose we wish to solve $(I-K)\phi = f$ (and that $(I-K)^{-1}$ exists), then the solution can be expressed as $\phi = f + K\phi = f + \psi$, where $\psi = K\phi$. By substituting $\phi = f + \psi$ into the original equation we see that ψ satisfies the equation

$$\psi = Kf + K\psi. \tag{3.1}$$

Equation(3.1) represents the application of the operator K to both sides of the original integral equation $\phi = f + K\phi$.

We get a clue that we may have improved matters by considering the Neumann series for our equation, given by

$$\phi = f + Kf + K^2 f + \ldots + K^n f + \ldots \tag{3.2}$$

The Kantorovich method gives $\phi = f + Kf + K\psi$, compared to $\phi = f + K\phi$ for Galerkin, and so contains an extra term of the Neumann series.

We approximate ψ by finding the Galerkin solution of the equation $\psi = Kf + K\psi$, ie.

$$\psi \approx q_n = \sum_{i=1}^{n} \beta_i \chi_i,$$

where $q_n \in E_n$, a subspace spanned by the orthonormal basis $\{\chi_1, \ldots, \chi_n\}$.

We want to choose q_n so that $A\psi - Kf$ is orthogonal to E_n, where $A = I - K$. Therefore

$$\begin{aligned}
(Aq_n - Kf, \chi_j) &= 0, \qquad j = 1, \ldots, n \\
\Rightarrow \qquad (Aq_n, \chi_j) &= (Kf, \chi_j), \quad j = 1, \ldots, n \\
\Rightarrow \quad \textstyle\sum_{i=1}^{n} \beta_i \{(\chi_i, \chi_j) - (K\chi_i, \chi_j)\} &= (Kf, \chi_j), \quad j = 1, \ldots, n
\end{aligned}$$

Notice that the β_i's given by the system of equations above differ from the α_i's given by Galerkin's method because we have a different free term in the integral equation, Kf instead of f. The solution to our original problem is therefore approximated by $\phi \approx \tilde{p_n} = f + q_n$, which is different from the $\phi \approx p_n$ of Galerkin's method.

The benefit of making the substitution in the Kantorovich method becomes clear when we compare the differences between the approximations and $\phi =$

$(I-K)^{-1}f$. If we denote by P_n the orthogonal projection onto E_n, we see that

$$p_n - \phi = (I - P_nK)^{-1}(P_n - I)(I-K)^{-1}f, \tag{3.3}$$

and

$$\tilde{p_n} - \phi = (I - P_nK)^{-1}(P_nK - K)(I-K)^{-1}f. \tag{3.4}$$

The difference between the two errors is that for the first, P_n has to be chosen such that $(P_n - I)(I-K)^{-1}f$ is small, so the choice depends on f, whereas, for the Kantorovich method, it is only necessary to choose P_n such that $\|P_nK - K\|$ is small, independently of f, which is more easily satisfied (see Sloan [6]).

Another advantage of the Kantorovich method is that it removes f from the calculations, replacing it by Kf. This is desirable if f is not a particularly smooth function. The application of K to f has the effect of smoothing the free term. Kf is then used in the calculations to approximate ψ, with f being added on afterwards to give $\tilde{p}$, the approximation to ϕ.

3.2 The Iterated Kantorovich Method

Sloan [6] demonstrated that the principle of iteration, which he first applied to Galerkin's method, could also be brought to bear on Kantorovich's method.

We wish to find the solution $\phi = f + \psi$, where ψ satisfies the integral equation $\psi = Kf + K\psi$. The Galerkin approximation to ψ gives

$$\psi \approx q_0 = (I - P_nK)^{-1}P_nKf \quad \Rightarrow \quad \phi \approx \tilde{p_0} = f + q_0.$$

If we apply the iteration process to q_0, by replacing ψ in the equation, we arrive at

$$\hat{q_0} = Kf + Kq_0$$

ie.

$$\hat{q}_0 = (I - KP_n)^{-1}Kf$$

$$\Rightarrow \qquad \phi \approx \hat{\tilde{p}}_0 = f + \hat{q}_0.$$

The error in the approximation of ψ is given by

$$\begin{aligned}
\hat{q}_0 - \psi &= (I - KP_n)^{-1}Kf - (I - K)^{-1}Kf \\
&= (I - KP_n)^{-1}(K - KP_n)(q_0 - \psi) \\
&= S(q_0 - \psi), \qquad \text{where } S = (I - KP_n)^{-1}(K - KP_n),
\end{aligned}$$

which in turn implies

$$\hat{\tilde{p}}_0 - \phi = S(p_0 - \phi).$$

The operator S is the same as that encountered in the iterated Galerkin method , which is as one might expect considering that the two methods differ only in the choice of free term. As with the Galerkin method, iteration will improve the Kantorovich approximation if the norm of the operator S is less than one, which can be guaranteed by a suitable choice of subspace. The advantage of the iterated Kantorovich method over the iterated Galerkin method lies in the possibility that $\psi = \phi - f$ will be smoother than ϕ, and thus easier to approximate.

3.3 The Modified Iterated Kantorovich Method

Although it has not previously been demonstrated, it seems fair to suppose that, since Kantorovich's method benefits from iteration, it might equally benefit from the process of re-iteration.

The iterated Kantorovich method is obtained by applying the iterated Galerkin method to the equation $\psi = Kf + K\psi$, where $\psi = \phi - f$. The iterated approximation $\hat{q}_0$ exhibits an error which can be compensated for by adding a correction term, ψ_1 say, so that

$$\psi = \hat{q}_0 + \psi_1.$$

By substituting ψ into the modified integral equation we see

$$\begin{aligned} \psi_1 &= Kf + K\hat{q}_0 - \hat{q}_0 + K\psi_1 \\ &= \hat{r}_0 + K\psi_1, \end{aligned}$$

where $\hat{r}_0$ is the residual error incurred in approximating ψ by $\hat{q}_0$.

We now have an integral equation to solve for ψ_1, with free term

$$\begin{aligned} \hat{r}_0 &= Kf - (I - K)\hat{q}_0 \\ &= K(Kf - (I - K)q_0) \\ &= Kr_0. \end{aligned}$$

As before, the ratio of the residual norms of the iterated and non-iterated approximations estimate $\|K\|$.

Once again we can use iterated Galerkin to find an approximation $\hat{q}_1$ to ϕ_1, and hence our new approximated solution to the modified equation becomes

$$\psi \approx \hat{q}_0 + \hat{q}_1,$$

where $\hat{q}_1 = (I - KP_n)^{-1}\hat{r}_0$, which has an error

$$\begin{aligned} \hat{q}_0 + \hat{q}_1 - \psi &= (I - KP_n)^{-1}\hat{r}_0 + (\hat{q}_0 - \psi) \\ &= -(I - KP_n)^{-1}(I - K)(\hat{q}_0 - \psi) + (\hat{q}_0 - \psi) \end{aligned}$$

$$
\begin{aligned}
&= (I - KP_n)^{-1}(K - KP_n)(\hat{q}_0 - \psi) \\
&= S(\hat{q}_0 - \psi) \\
&= S^2(q_0 - \psi),
\end{aligned}
$$

ie. $\hat{\tilde{p}}_1 - \phi = S^2(q_0 - \psi)$, where $\phi \approx \tilde{p}_1 = f + \hat{q}_0 + \hat{q}_1$. So $\hat{q}_0 + \hat{q}_1$ is an improved approximation to ψ if $\|S\| < 1$, which we have previously assumed.

As before, this process can be repeated by noting that

$$\psi = \hat{q}_0 + \hat{q}_1 + \psi_2,$$

where ψ_2 is a correction term which satisfies the equation

$$\psi_2 = \hat{r}_1 + K\psi_2,$$

where $\hat{r}_1 = \hat{r}_0 - (I - K)\hat{q}_1 = (K - KP_n)(I - KP_n)^{-1}\hat{r}_0$, the residual error in $\hat{q}_0 + \hat{q}_1$. As with the re-iterated Galerkin method, the ratio of successive residual norms, $\frac{\|\hat{r}_n\|}{\|\hat{r}_{n-1}\|}$, gives an underestimate for the norm of S.

In general, then, we have $\phi \approx \hat{\sigma}^n_{kant} = f + \hat{q}_0 + \ldots + \hat{q}_n$, where each correction term ψ_n is determined by using iterated Galerkin on $\psi_n = \hat{r}_{n-1} + K\psi_n$, giving $\hat{q}_n = (I - KP_n)^{-1}\hat{r}_{n-1}$, where $\hat{r}_{n-1} = f - (I - K)\hat{\sigma}^{n-1}_{kant}$. The error in the current approximation being given by $\hat{\sigma}^n_{kant} - \phi = S(\hat{\sigma}^{n-1}_{kant} - \phi) = S^{n+1}(q_0 - \psi)$.

The benefits of re-iteration for the Kantorovich method are the same as for the Galerkin method. Any errors involved in calculating a particular approximation are compensated for in the next iteration, and the process can be repeated to achieve, in theory, any desired accuracy, provided that $\rho(S) < 1$.

This modified iterated Kantorovich method has an advantage when $\psi = \phi - f$ is smoother than ϕ, since then the error in the initial approximation, $q_0 - \psi$, is

generally smaller than that for the re-iterated Galerkin method, $p_0 - \phi$. All subsequent approximations, however, improve by the same factor, $\|S\|$, for both methods. So the rates of convergence are virtually the same.

The improvement in the initial approximation, brought about by the Kantorovich "regularization" of the original equation, leads us to wonder whether subsequent approximations might also benefit from similar regularization.

3.4 The Re-iterated Kantorovich Method

Suppose, once again, that we wish to solve

$$\phi = f + K\phi \tag{3.5}$$

in a Hilbert space H, where K is a compact linear map on H, $f \in H$ is given, and ϕ is sought. We first perform a Kantorovich "regularization" on the above equation, whereby we write the solution as

$$\phi = f + \psi, \tag{3.6}$$

where $\psi = K\phi$ satisfies the equation

$$\psi = Kf + K\psi. \tag{3.7}$$

We seek a Galerkin approximation $\tilde{q}_0$ to ψ in a subspace E_n of H, so that $\tilde{q}_0 = (I - P_nK)^{-1}P_nKf$, where P_n is the orthogonal projection of H onto E_n. Our Kantorovich approximation, $\tilde{p}_0$, is therefore

$$\phi \approx \tilde{p}_0 = \tilde{q}_0 + f.$$

Next we form the iterated Galerkin solution of equation (3.7), ie.

$$\hat{\tilde{q}}_0 = Kf + K\tilde{q}_0 = (I - KP_n)^{-1}Kf,$$

giving us the iterated Kantorovich approximation

$$\phi \approx \hat{\tilde{p}}_0 = \hat{\tilde{q}}_0 + f,$$

where

$$\hat{\tilde{p}}_0 - \phi = S(\tilde{p}_0 - \phi).$$

As we have seen before, iteration improves the initial approximation by a factor $\|S\|$ for both the Kantorovich and Galerkin methods, the Kantorovich method having the possible advantage that $\tilde{p}_0$ has better convergence properties than p_0, since it relies upon $\|P_n K - K\| \to 0$ as $n \to \infty$, rather than $(P_n - I)(I - K)^{-1} \to 0$ as $n \to \infty$.

We can utilise this convergence property in a re-iterative method by performing a Kantorovich regularization at each iteration. Instead of finding a correction term ψ_1 to $\hat{\tilde{q}}_0$, to form $\psi = \hat{\tilde{q}}_0 + \psi_1$, as we did for the modified iterated Kantorovich method, we now look for a correction term for ϕ.

We know $\phi \approx \hat{\tilde{p}}_0 = f + \hat{\tilde{q}}_0$, so this implies that

$$\phi = \hat{\tilde{p}}_0 + \phi_1, \tag{3.8}$$

which in turn implies that

$$\phi_1 = \hat{r}_0 + K\phi_1, \tag{3.9}$$

where $\hat{r}_0 = f - (I - K)\hat{\tilde{p}}_0 = Kr_0$.

This method differs from the modified iterated Kantorovich method in that we now perform a second Kantorovich regularization, this time to equation (3.9).

Let $K\phi_1 = \psi_1$, then $\phi_1 = \hat{r}_0 + \psi_1$, where ψ_1 satisfies

$$\psi_1 = K\hat{r}_0 + K\psi_1.$$

The Galerkin solution to this is

$$\psi_1 \approx \tilde{q}_1 = (I - P_n K)^{-1} P_n K \hat{r}_0,$$

which we can iterate to obtain

$$\phi_1 \approx \hat{\tilde{q}}_1 = K\hat{r}_0 + K\tilde{q}_1 = (I - KP_n)^{-1} K\hat{r}_0.$$

The correction term ϕ_1 is therefore given by

$$\phi_1 \approx \hat{\tilde{p}}_1 = \hat{r}_0 + \hat{\tilde{q}}_1.$$

Substituting for ϕ_1 in equation (3.8) gives us a new approximation to ϕ, that is

$$\phi \approx \hat{\tilde{p}}_0 + \hat{\tilde{p}}_1. \tag{3.10}$$

To see if we have improved our approximation we need to look at the error.

$$\begin{aligned}
\hat{\tilde{p}}_0 + \hat{\tilde{p}}_1 &= \hat{r}_0 + \hat{\tilde{q}}_1 - (\phi - \hat{\tilde{p}}_0), \\
&= \hat{r}_0 + (I - KP_n)^{-1} K\hat{r}_0 - (I - K)^{-1}\hat{r}_0, \\
&= (I - KP_n)^{-1}(KP_nK - K^2)(I - K)^{-1}\hat{r}_0, \\
&= (I - KP_n)^{-1}(K - KP_n)K(\hat{\tilde{p}}_0 - \phi), \\
&= SK(\hat{\tilde{p}}_0 - \phi) = SKS(\tilde{p}_0 - \phi),
\end{aligned}$$

so that

$$\|\hat{\tilde{p}}_0 + \hat{\tilde{p}}_1 - \phi\| = \|SK(\hat{\tilde{p}}_0 - \phi)\| \leq \|SK\|\|\hat{\tilde{p}}_0 - \phi\|,$$

which is an improvement if $\|SK\| < 1$.

We may continue by setting

$$\phi = \hat{\tilde{p}}_0 + \hat{\tilde{p}}_1 + \phi_2,$$

where ϕ_2 is a correction term which satisfies the equation

$$\phi_2 = \hat{r}_1 + K\phi_2, \tag{3.11}$$

where $\hat{r}_1 = f - (I-K)(\hat{\tilde{p}}_0 + \hat{\tilde{p}}_1) = K\hat{r}_1$. Performing a Kantorovich regularization on the above equation gives us

$$\phi_2 = \hat{r}_1 + \psi_2,$$

where ψ_2 satisfies

$$\psi_2 = K\hat{r}_1 + K\psi_2.$$

Applying Galerkin's method gives us $\psi_2 \approx \tilde{q}_2 = (I - P_nK)^{-1}P_nK\hat{r}_1$, and upon iteration, $\psi_2 \approx \hat{\tilde{q}}_2 = K\hat{r}_1 + K\tilde{q}_2 = (I - KP_n)^{-1}K\hat{r}_1$.

So now

$$\phi_2 = \hat{\tilde{p}}_2 = \hat{r}_1 + \hat{\tilde{q}}_2,$$

which implies that

$$\phi \approx \hat{\tilde{p}}_0 + \hat{\tilde{p}}_1 + \hat{\tilde{p}}_2.$$

The error in this approximation can be shown to be

$$\begin{aligned} \hat{\tilde{p}}_0 + \hat{\tilde{p}}_1 + \hat{\tilde{p}}_2 - \phi &= (I - KP_n)^{-1}(K - KP_n)K(\hat{\tilde{p}}_0 + \hat{\tilde{p}}_1 - \phi) \\ &= SK(\hat{\tilde{p}}_0 + \hat{\tilde{p}}_1 - \phi) \end{aligned}$$

so that

$$\|\hat{\tilde{p}}_0 + \hat{\tilde{p}}_1 + \hat{\tilde{p}}_2 - \phi\| < \|\hat{\tilde{p}}_0 + \hat{\tilde{p}}_1 - \phi\|$$

provided, once again, that $\|SK\| < 1$.

In general, if $\phi \approx \hat{\sigma}^n_{rekant} = \hat{\tilde{p}}_0 + \ldots + \hat{\tilde{p}}_n$, and $\hat{r}_n = f - (I-K)\hat{\sigma}^n_{rekant}$ denotes the residual after the nth iteration, then the correction term $\phi_n = \hat{r}_{n-1} + \psi_n$, is

approximated by $\hat{\tilde{p}}_n = \hat{r}_{n-1} + \hat{\tilde{q}}_n$, where $\hat{\tilde{q}}_n = (I - KP_n)^{-1}K\hat{r}_{n-1}$ is the iterated solution of the regularized equation $\psi_n = K\hat{r}_{n-1} + K\psi_n$.

This gives

$$\begin{aligned} \|\hat{\sigma}^{n+1}_{rekant} - \phi\| &\leq \|SK\|\|\hat{\sigma}^{n}_{rekant} - \phi\| \\ &= \|SK\|^{n+1}\|\hat{\tilde{p}}_0 - \phi\| \\ &\leq \|SK\|^{n+1}\|S\|\|\tilde{p}_0 - \phi\| \end{aligned}$$

Note that the operator controlling the convergence of this method is $\|SK\|$, rather than $\|S\|$ produced by the re-iterated Galerkin and modified iterated Kantorovich methods. This gives us a new condition on the size of the subspace E_n. We now require that E_n is chosen such that $\|SK\| < 1$.

As commented on previously, these norms give a weaker bound on the condition for convergence than may be obtained by considering the spectral radius of the operators. Once again the method can be shown to converge (by the root test) if $\rho(SK) < 1$. Also, the ratio of successive residual norms,

$$\frac{\|\hat{r}_n\|}{\|\hat{r}_{n-1}\|} \leq \|(K - KP_n)(I - KP_n)^{-1}K\|,$$

can be shown to converge to the spectral radius of SK.

In order to better understand the differences between the three methods we can compare the solutions offered by them at each iteration. Let

$$\hat{\sigma}^i_{gal} = \hat{p}_0 + \ldots + \hat{p}_i, \tag{3.12}$$

$$\hat{\sigma}^i_{kant} = f + \hat{q}_0 + \ldots + \hat{q}_i, \tag{3.13}$$

$$\hat{\sigma}^i_{rekant} = \hat{\tilde{p}}_0 + \ldots + \hat{\tilde{p}}_i, \tag{3.14}$$

be the approximations to ϕ using the re-iterated Galerkin, modified iterated Kantorovich and re-iterated Kantorovich methods respectively.

Then, for i=0,

$$\begin{aligned}
\hat{\sigma}^0_{gal} &= f + Kp_0, \\
\hat{\sigma}^0_{kant} &= f + Kf + Kq_0, \\
\hat{\sigma}^0_{rekant} &= f + Kf + K\tilde{q}_0,
\end{aligned}$$

and for i=1,

$$\begin{aligned}
\hat{\sigma}^1_{gal} &= f + Kf + K^2p_0 + Kp_1, \\
\hat{\sigma}^1_{kant} &= f + Kf + K^2f + K^2q_0 + Kq_1, \\
\hat{\sigma}^1_{rekant} &= f + Kf + K^2f + K^3f + K^3\tilde{q}_0 + K\tilde{q}_1.
\end{aligned}$$

In general, for i=n,

$$\begin{aligned}
\hat{\sigma}^n_{gal} &= \sum_{j=0}^{n} K^j f + \sum_{j=0}^{n} K_{j+1} p_{n-j}, \\
\hat{\sigma}^n_{kant} &= \sum_{j=0}^{n+1} K^j f + \sum_{j=0}^{n} K_{j+1} q_{n-j}, \\
\hat{\sigma}^n_{rekant} &= \sum_{j=0}^{2n+1} K^j f + \sum_{j=0}^{n} K^{2j+1} \tilde{q}_{n-j}.
\end{aligned}$$

From this we can see that after n iterations, the re-iterated Galerkin approximation contains the first n+1 terms of the Neumann series for $\phi = f + K\phi$, the modified iterated Kantorovich approximation contains the first n+2 terms, and the re-iterated Kantorovich approximation contains the first 2n+2 terms of the Neumann series, twice as many as for the re-iterated Galerkin method. This property appears to suggest that, under suitable conditions, the re-iterated Kantorovich method will have a much faster rate of convergence than the other two methods. In reality, if $\|K\|$ is greater than one, the Neumann series diverges, and it is the other terms which have to compensate. The Neumann series itself cannot therefore be responsible for the rate of convergence.

In the following chapters we will be comparing the efficiency and accuracy of the re-iterated Galerkin method with the new re-iterated Kantotovich method. The modified iterated Kantorovich method has not been considered further due to the belief that any improvement over the re-iterated Galerkin method will be minimal.

Chapter 4

Numerical Considerations

4.1 Numerical Integration

The process of calculating successive corrections to previous approximations, a feature of all the re-iterative methods, ensures that the accuracy of the numerical solution can be increased until it is of the same order as the computational accuracy of the method. Once the residual $\hat{r}_n$ reaches this accuracy there is no point in continuing the iteration, as no further improvement can be made. The main factor governing the computational accuracy is the method of numerical integration used. In all the cases described here modified Gauss-Legendre quadrature was employed, using 10 points and 20 subintervals. In this way, the value of every function used in the method is known at each of the two hundred Gauss points in the interval.

Numerical integration routines tend to encounter difficulties when dealing with kernels containing slope discontinuities, such as

$$(Kf)(x) = \int_0^1 max(x,t)f(t)dt, \qquad 0 \leq x \leq 1,$$

which has such a discontinuity at $x = t$.

The problem may be remedied by noting that

$$\begin{aligned}(Kf)(x) &= \int_0^1 k(x,t)f(t)dt \\ &= \int_0^1 k(x,t)(f(t) - f(x))dt + f(x)\int_0^1 k(x,t)dt\end{aligned}$$

This reformulated equation has a first term with a continuous first derivative, and a second term that may be evaluated exactly. (See Chamberlain [1], who used this device in the case where $k(x,t) = \frac{1}{2\alpha}sin(\alpha|x-t|)\rho(t))$.

4.2 Choosing The Subspace

In order to keep calculations as simple as possible, attention is restricted to finding approximations in subspaces of minimal dimension. Provided that the subspace is of sufficient size to ensure the conditions $\rho(S) < 1$ and $\rho(SK) < 1$ are satisfied, the re-iteration process will cause the approximations to converge, even if the initial approximations are not very accurate. For the equations considered in the following chapter, one-dimensional subspaces suffice.

The next question is how to choose the subspace.

It would seem reasonable to suggest that we should seek a subspace which exhibits some relationship to the equation we are attempting to solve. We know that the Neumann series for the equation $\phi = f + K\phi$, which is given by

$$\phi = \sum_{n=1}^{\infty} K^{n-1}f,$$

converges if $\|K\| < 1$. This leads us to suppose that, if $\|K\|$ is small, then

$$p_n = f + Kf + K^2f + \ldots + K^{n-1}f,$$

would be a good approximation to ϕ. For this reason we conclude that we ought to be looking for approximations belonging to the n-dimensional subspace spanned by $f, Kf, K^2f, \ldots, K^{n-1}f$. Indeed, it has been shown (Porter & Stirling [3]) that this constitutes the optimal choice of subspace.

For our one-dimensional case, therefore, we are looking for Galerkin approximations of the form $p = \alpha_1\chi_1$, where χ_1 is chosen to be the free term of the integral equation. For the re-iterated Galerkin method this is obviously $\chi_1 = f$. What is perhaps not so obvious is what we mean by the free term for the Kantorovich method.

Whilst it is true we are trying to find ϕ satisfying $\phi = f + K\phi$, which would suggest taking $\chi_1 = f$, the Kantorovich regularization effectively removes f from the approximation, so that we are instead seeking to approximate ψ, where ψ satisfies $\psi = Kf + K\psi$. Here the free term is not f, but Kf, so we should perhaps take our trial function to be $\chi = Kf$ instead.

Since the aim of this project is to give a comparison of the different methods, it seems sensible that they should use the same subspaces. On the other hand, each method should be illustrated utilising its optimal choice of subspace. For the one-dimensional examples given in the following chapter, therefore, the Kantorovich method is shown using both $\chi = f$ and $\chi = Kf$.

Chapter 5

Examples and Results

5.1 The Kernel 2max(x,t)

The first equation under consideration is

$$\phi(x) = 1 + \int_0^1 2max(x,t)\phi(t)dt, \qquad 0 \leq x \leq 1, \tag{5.1}$$

which has been chosen because many of its properties are already known.

An exact solution can be found by converting the equation to an equivalent boundary value problem, ie.

$$\phi''(x) = 2\phi(x),$$

$$\phi'(x) = 0, \qquad \phi(1) - \phi'(1) = 1,$$

which has the solution

$$\phi(x) = \frac{\cosh x\sqrt{2}}{(\cosh\sqrt{2} - \sqrt{2}\sinh\sqrt{2})}.$$

The operator K, given by $(K\phi)(x) = \int_0^1 2max(x,t)\phi(t)dt$, is compact and self-adjoint, and can be shown to have one positive eigenvalue, lying between

	x_0	x_1	x_2	x_3	x_4	$\frac{\|\hat{r}_n\|}{\|\hat{r}_{n-1}\|}$	$\frac{\|\hat{r}_n\|}{\|r_n\|}$
$\hat{\sigma}^0_{gal}$	-2.000001	-2.188480	-2.751958	-3.690437	-4.996087		0.356
$\hat{\sigma}^3_{gal}$	-1.771604	-1.877493	-2.210471	-2.820908	-3.791033	0.462	0.357
$\hat{\sigma}^6_{gal}$	-1.792672	-1.907131	-2.263463	-2.905859	-3.907931	0.462	0.357
$\hat{\sigma}^9_{gal}$	-1.790601	-1.904216	-2.258252	-2.897505	-3.896435	0.462	0.357
$\hat{\sigma}^{12}_{gal}$	-1.790804	-1.904503	-2.258764	-2.898326	-3.897565	0.462	0.357
$\hat{\sigma}^{15}_{gal}$	-1.790784	-1.904475	-2.258714	-2.898245	-3.897454	0.462	0.357
$\hat{\sigma}^{18}_{gal}$	-1.790786	-1.904478	-2.258719	-2.898253	-3.897465	0.462	0.357
$\hat{\sigma}^{21}_{gal}$	-1.790786	-1.904477	-2.258718	-2.898253	-3.897464	0.462	0.357
$\phi(x)$	-1.790786	-1.904477	-2.258718	-2.898253	-3.897464		

Table 5.1: Re-iterated Galerkin Approximations

1·38 and 1·42, and infinitely many negative eigenvalues, all of modulus less than 0·3.

Table (5.1) shows the results of applying the re-iterated Galerkin method (hereby denoted the RIG method) to equation (5.1), using the one-dimensional subspace spanned by $\chi = f = 1$. The successive approximations, $\hat{\sigma}^n_{gal} = \hat{p}_0 + \cdots + \hat{p}_n$, are given at the points x_j $(j = 0, \ldots, 4)$, where the x_j's are the Gauss points closest to $x = (0.25)j$. The exact solution, $\phi(x)$, is also given, as well as the values $\frac{\|\hat{r}_n\|}{\|\hat{r}_{n-1}\|}$ and $\frac{\|\hat{r}_n\|}{\|r_n\|}$, which are the underestimates for $\rho(S)$ and $\|K\|$ respectively.

The approximations are seen to converge to the exact solution after twenty-two iterations (to an accuracy of six decimal places). The underestimate of the spectral radius of $S = (I - KP_n)^{-1}(K - KP_n)$, settles immediately to 0·462, and hence satisfies our convergence criterion, whilst the underestimate of $\|K\|$ settles to 0·357.

The results for the re-iterated Kantorovich method (hereby denoted the RIK

	x_0	x_1	x_2	x_3	x_4	$\frac{\|\hat{r}_n\|}{\|\hat{r}_{n-1}\|}$	$\frac{\|\hat{r}_n\|}{\|r_n\|}$
$\hat{\sigma}^0_{rekant}$	-2.500001	-2.627627	-3.032719	-3.785712	-4.996088		0.356
$\hat{\sigma}^2_{rekant}$	-1.813364	-1.927510	-2.283396	-2.926562	-3.932512	0.179	0.354
$\hat{\sigma}^4_{rekant}$	-1.791510	-1.905215	-2.259509	-2.899160	-3.898587	0.179	0.354
$\hat{\sigma}^6_{rekant}$	-1.790809	-1.904501	-2.258744	-2.898282	-3.897500	0.179	0.354
$\hat{\sigma}^8_{rekant}$	-1.790787	-1.904478	-2.258719	-2.898254	-3.897465	0.179	0.354
$\hat{\sigma}^9_{rekant}$	-1.790786	-1.904477	-2.258718	-2.898253	-3.897464	0.179	0.354
$\phi(x)$	-1.790786	-1.904477	-2.258718	-2.898253	-3.897464		

Table 5.2: Re-iterated Kantorovich approximations ($\chi = f$)

method) for the same equation, again using the one-dimensional subspace spanned by $\chi = f = 1$, are given in Table (5.2). The ratio $\frac{\|\hat{r}_n\|}{\|\hat{r}_{n-1}\|}$ now represents the spectral radius of SK, the convergence factor for the RIK method. Its value, 0· 179, is much smaller than the underestimate of $\rho(S)$, which is inherent in the RIG method. This is why the RIK($\chi = f$) method converges to the exact solution after only ten iterations. The ratio $\frac{\|\hat{r}_n\|}{\|r_n\|}$, which underestimates $\|K\|$ for all of the methods given here, has a similar value to that of the RIG method.

Finally, Table (5.3) shows the results for the RIK method using the one-dimensional subspace spanned by $\chi = Kf$. Now $\rho(SK)$, which depends upon

	x_0	x_1	x_2	x_3	x_4	$\frac{\|\hat{r}_n\|}{\|\hat{r}_{n-1}\|}$	$\frac{\|\hat{r}_n\|}{\|r_n\|}$
$\hat{\sigma}^0_{rekant}$	-1.801170	-1.919602	-2.286479	-2.942667	-3.958169		0.260
$\hat{\sigma}^2_{rekant}$	-1.790821	-1.904529	-2.258813	-2.898404	-3.897670	0.058	0.260
$\hat{\sigma}^4_{rekant}$	-1.790786	-1.904478	-2.258719	-2.898253	-3.897464	0.058	0.260
$\hat{\sigma}^6_{rekant}$	-1.790786	-1.904477	-2.258718	-2.898253	-3.897464	0.058	0.260
$\phi(x)$	-1.790786	-1.904477	-2.258718	-2.898253	-3.897464		

Table 5.3: Re-iterated Kantorovich approximations ($\chi = Kf$)

the projection P_n, and thus the choice of subspace, is approximated as 0·058. This is smaller still than is obtained by taking $\chi = f$. As we might expect, therefore, the approximations to ϕ, given by $\hat{\sigma}^n_{rekant} = \hat{\hat{p}}_0 + \cdots + \hat{\hat{p}}_n$, converge to an accuracy of six decimal places after only seven iterations. (Naturally the rate of convergence for all three methods would be improved by taking a higher dimensional subspace.)

The value given here for $\|K\|$ is noticeably different from that given in the two previous tables. It isn't immediately obvious why this should be; K hasn't changed, so the reason must be linked to the choice of subspace. In order to see what is happening we need to look at the Rayleigh quotient, denoted $R(\phi)$.

If T is a bounded, self-adjoint linear map from a Hilbert space, H, to itself, then the Rayleigh quotient

$$R_T(\phi) = \frac{(T\phi, \phi)}{(\phi, \phi)},$$

gives a lower bound on the largest positive eigenvalue of T, denoted $\mu_1^+(T)$. That is

$$\begin{aligned} \mu_1^+(T) &= \sup\{R_K(\phi) : \phi \neq 0\} \\ &= \max\{R_K(\phi) : \phi \neq 0\} : \mu_1^+(T) \neq 0. \end{aligned}$$

Also

$$\|T\| = \max\{|R_K(\phi)| : \phi \neq 0\}.$$

(See Porter & Stirling [3] Lemma 5.1)

If we take $\phi = r_n$, and $T = K^2$, we see that

$$R_K(r_n) = \frac{(Kr_n, Kr_n)}{(r_n, r_n)} \leq \|K\|^2 = \mu_1^+(K^2),$$

where μ_1^+ denotes the largest positive eigenvalue of K, with ϕ_1^+ being the corresponding eigenvector. This implies that

$$\frac{\|Kr_n\|^2}{\|r_n\|^2} = \frac{\|\hat{r}_n\|^2}{\|r_n\|^2} \leq \|K\|^2.$$

So the closer r_n is to a constant multiple of the eigenvector ϕ_1^+, the closer the ratio $\frac{\|\hat{r}_n\|}{\|r_n\|}$ is to $\|K\|$. Now, r_n depends ultimately on f, K and either p_0 or q_0. Since f and K are fixed by the problem, it is the initial Galerkin approximation, determined by the choice of subspace, that defines r_n.

It would appear from the example that since

$$0.260 < 0.354 < 0.357 \leq \|K\|,$$

taking $\chi = f$ causes r_n to be a better approximation to a multiple of ϕ_1^+, which is why it gives a closer approximation to $\|K\|$ (for this example).

5.2 The $\frac{1}{2\kappa_0}\sin(\kappa_0|x-t|)$ Kernel

The second problem under consideration is the equation

$$\phi(x) = \cos(\kappa_0 x) + \int_0^1 \frac{1}{2\kappa_0}\sin(\kappa_0|x-t|)\phi(t)dt, \qquad 0 \leq x \leq 1, \qquad (5.2)$$

where κ_0 is a real parameter.

The motivation for using this equation comes from its use in examining wave scattering using the mild-slope equation, as investigated by Chamberlain [2].

Chamberlain noted that the solution to the boundary value problem he wished to solve could be expressed as a linear combination of the solutions of two real-valued integral equations, given by

$$(I - LP)\phi_i = f_i, \qquad i = 1, 2$$

where

$$f_i(x) = \begin{cases} \cos(\kappa_0 x) & ,\text{if } i = 1 \\ \sin(\kappa_0 x) & ,\text{if } i = 2 \end{cases},$$

$$\begin{aligned} (L\phi)(x) &= \int_0^1 \frac{1}{2\kappa_0} \sin(\kappa_0 |x-t|)\phi(t)dt, \\ (P\phi)(x) &= \rho(x)\phi(x), \end{aligned}$$

and where $\rho(x)$ is a real-valued function representing the effects of perturbations in the still water depth.

In our example, we are taking $i = 1$, $\rho(x) = 1$, and κ_0, the scaled wave number, to be equal to 1·5. For this special case we can once again find an exact solution by noting that equation (5.2) is equivalent to the following boundary value problem:

$$\phi''(x) + (\kappa_0^2 - 1)\phi(x) = 0,$$

$$\begin{aligned} \kappa_0\phi(1) &= 2\kappa_0 \cos\kappa_0 - \sin\kappa_0\phi'(0) - \kappa_0\cos\kappa_0\phi(0), \\ \phi'^{(1)} &= -2\kappa_0 \sin\kappa_0 - \cos\kappa_0\phi'(0) + \kappa_0\sin\kappa_0\phi(0), \end{aligned}$$

which yields the solution

$$\phi(x) = C\cos(\omega x) + D\sin(\omega x),$$

where

$$\begin{aligned} C &= \frac{2\kappa_0}{\Delta}\{\omega + \omega\cos\kappa_0\cos\omega + \kappa_0\sin\kappa_0\sin\omega\}, \\ D &= \frac{2\kappa_0}{\Delta}\{\omega\cos\kappa_0\sin\omega - \kappa_0\sin\kappa_0\cos\omega\}, \\ \omega &= \sqrt{\kappa_0^2 - 1}\,, \qquad \kappa_0 > 1, \end{aligned}$$

and

$$\Delta = (\kappa_0^2 + \omega^2)\sin\kappa_0\sin\omega + 2\kappa_0\omega(1 + \cos\kappa_0\cos\omega), \qquad \Delta \neq 0.$$

	x_0	x_1	x_2	x_3	x_4	$\frac{\|\hat{r}_n\|}{\|\hat{r}_{n-1}\|}$	$\frac{\|\hat{r}_n\|}{\|r_n\|}$
$\hat{\sigma}^0_{rekant}$	1.124034	1.008170	0.815989	0.560542	0.258385		0.095
$\hat{\sigma}^1_{rekant}$	1.136099	1.018510	0.821792	0.561365	0.259203	0.026	0.082
$\hat{\sigma}^2_{rekant}$	1.135752	1.018261	0.821738	0.561428	0.259193	0.047	0.075
$\hat{\sigma}^3_{rekant}$	1.135769	1.018272	0.821739	0.561424	0.259193	0.051	0.073
$\hat{\sigma}^4_{rekant}$	1.135768	1.018271	0.821739	0.561424	0.259193	0.052	0.073
$\phi(x)$	1.135768	1.018271	0.821739	0.561424	0.259193		

Table 5.4: Re-iterated Galerkin Approximations

Tables (5.4)-(5.6) show the results of approximating the solution of equation (5.2) using the RIG, RIK($\chi = f$) and RIK($\chi = Kf$) methods respectively.

Once again the Kantorovich methods display faster convergence than the re-iterated Galerkin method, with the choice of subspace, $\chi = Kf$, giving the best results of all, with a convergence factor of $\rho(SK)$=0·007. For the first time it is possible to see, in Table (5.4), the underestimate of the spectral radius of the S operator converging. This illustrates the importance of not using the ratio of the first two residual norms to give an estimate of the convergence factor. Its value may increase, as it does in this case, and could exceed the convergence limit $\rho(S) = 1$.

The very small values given by the ratio $\frac{\|\hat{r}_n\|}{\|\hat{r}_{n-1}\|}$ for all three methods in this

	x_0	x_1	x_2	x_3	x_4	$\frac{\|\hat{r}_n\|}{\|\hat{r}_{n-1}\|}$	$\frac{\|\hat{r}_n\|}{\|r_n\|}$
$\hat{\sigma}^0_{rekant}$	1.135032	1.017839	0.821061	0.560244	0.257597		0.095
$\hat{\sigma}^1_{rekant}$	1.135762	1.018268	0.821732	0.561413	0.259177	0.010	0.096
$\hat{\sigma}^2_{rekant}$	1.135768	1.018271	0.821739	0.561424	0.259193	0.010	0.096
$\phi(x)$	1.135768	1.018271	0.821739	0.561424	0.259193		

Table 5.5: Re-iterated Kantorovich Approximations ($\chi = f$)

	x_0	x_1	x_2	x_3	x_4	$\frac{\|\hat{r}_n\|}{\|\hat{r}_{n-1}\|}$	$\frac{\|\hat{r}_n\|}{\|r_n\|}$
$\hat{\sigma}^0_{rekant}$	1.135681	1.018201	0.821707	0.561426	0.259188		0.084
$\hat{\sigma}^1_{rekant}$	1.135768	1.018271	0.821738	0.561424	0.259193	0.007	0.085
$\hat{\sigma}^2_{rekant}$	1.135768	1.018271	0.821739	0.561424	0.259193	0.007	0.085
$\phi(x)$	1.135768	1.018271	0.821739	0.561424	0.259193		

Table 5.6: Re-iterated Kantorovich Approximations ($\chi = Kf$)

example, results in the need for far fewer iterations than was required for the previous equation. The RIK($\chi = Kf$) method, in particular, achieves an accuracy of around four decimal places after the first iteration, even though it eventually requires the same number of iterations as the RIK($\chi = f$) method to reach an accuracy of six decimal places.

Chapter 6

Convergence Regions and Eigenvalue Bounds

The results in the previous chapter lead us to wonder in what cases the Kantorovich based re-iterative methods are better, in the sense of convergence rates, than the re-iterated Galerkin method, and whether taking $\chi = Kf$ is always preferable than $\chi = f$ for such methods.

A simple way to test this proposition experimentally would be to apply the three methods to a series of different kernels, and measure $\frac{\|\hat{r}_n\|}{\|\hat{r}_{n-1}\|}$ for each. By introducing a parameter, λ, to the kernel, we can achieve the desired changes in a controlled way. The integral equations we investigate, therefore, are of the more general form

$$\phi = f + \lambda K\phi.$$

If we now make incremental changes in the value of λ, and apply the methods, noting the value of the ratio of successive residual norms, we can build up a picture of how the convergence factors, $\rho(S)$ and $\rho(SK)$, vary with λ.

6.1 The Kernel 2λmax(x,t)

The first equation we consider is

$$\phi(x) = 1 + \lambda \int_0^1 2\max(x,t)\phi(t)dt, \qquad 0 \leq x \leq 1. \tag{6.1}$$

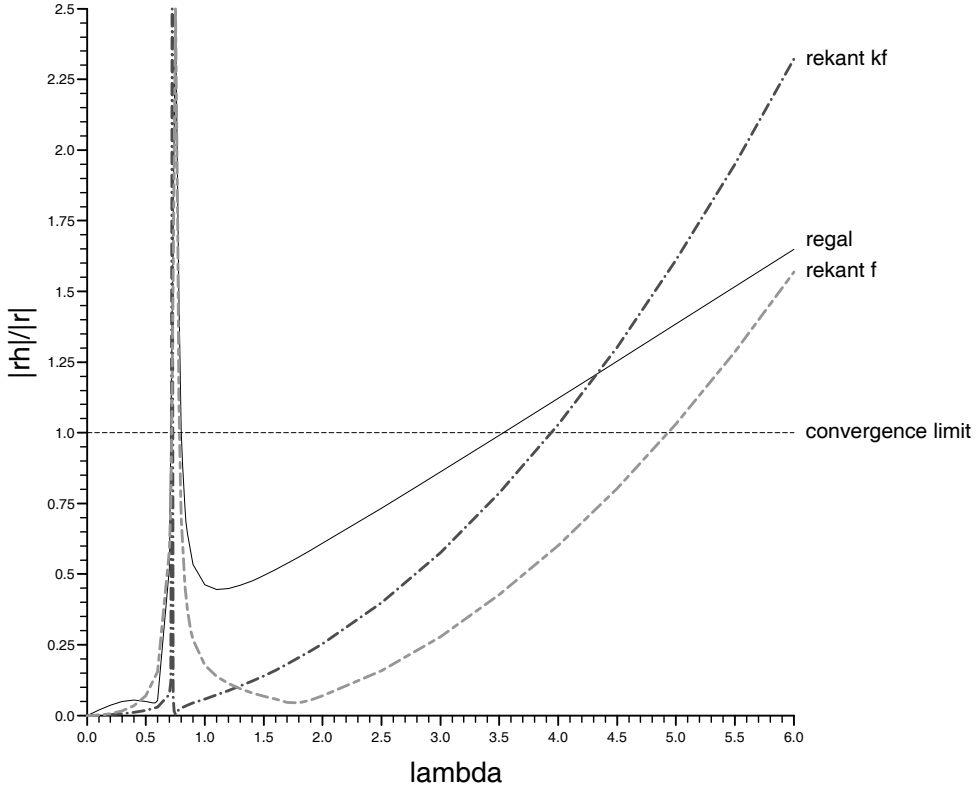

Figure 6.1: Convergence of methods for $k(x,t) = 2\lambda\max(x,t)$

Figure (6.1) illustrates the variation in $\frac{||\hat{r}_n||}{||\hat{r}_{n-1}||}$ (written as $|rh|/|r|$ in the Figure) for each of the methods, with λ taken in the range, $0 \leq \lambda \leq 6$. The most striking feature of the graph is the spike which occurs at around $\lambda = 0 \cdot 7$. It seems reasonable to suppose that this spike in some way represents the presence of an eigenvalue, since the ratio of the residual norms approximates the spectral radii of the operators S and SK.

There is a process by which it is occasionally possible to calculate, analytically, the value of the spectral radius of an operator. This process has been described in the following section to calculate $\rho(S)$ for the RIG method, in a one-dimensional subspace, using the kernel $k(x,t) = 2\lambda \max(x,t)$.

6.1.1 Calculating $\rho(S)$

Let $\phi = f + K\phi$ in $L_2(0,1)$, where $(K\phi)(x) = \int_0^1 2\lambda \max(x,t)\phi(t)dt$ and $\lambda > 0$ (recall that K has only one positive eigenvalue and infinitely many negative eigenvalues). We seek the Galerkin approximation in the space of constant functions, ie. let $\phi = p_0$ be the Galerkin approximation, where $p_0 = \alpha g$ and $g(x) = 1$, so that

$$(p_0 - f - Kp_0, g) = 0, \qquad p_0 = \alpha g,$$

$$\Rightarrow \qquad \alpha\{(g,g) - (Kg,g)\} = (f,g).$$

Now $(g,g) = 1$, and

$$(Kg)(x) = \int_0^x 2\lambda x dt + \int_x^1 2\lambda t dt = \lambda(1+x^2)$$

$$\Rightarrow \qquad (Kg,g) = \int_0^1 \lambda(1+x^2)dx = \frac{4\lambda}{3}.$$

Therefore, $\alpha = -c(f,g)$, where $-c = \frac{3}{3-4\lambda}$, provided $3 \neq 4\lambda$,

$$\Rightarrow \qquad p_0 = -c(f,g)g.$$

[So $((I - P_nK)^{-1}P_nf)(x) = -c\int_0^1 f(t)g(t)g(x)dt = -c\int_0^1 f(t)dt$.]

Now, $\hat{p}_0 = f + Kp_0 = f - c(f,g)Kg$,

ie.

$$\hat{p}_0(x) = f(x) - \int_0^1 c\lambda(1+x^2)f(t)dt = ((I - KP_n)^{-1}f)(x).$$

Finally,

$$S = (I - KP_n)^{-1}(K - KP_n) = I - (I - KP_n)^{-1}(I - K),$$

so

$$\begin{aligned}(Sf)(x) &= f(x) - (f(x) - (Kf)(x)) + \int_0^1 c\lambda(1+x^2)\{f(t) - (Kf)(t)\}dt \\ &= (Kf)(x) + c\lambda(1+x^2)\{\int_0^1 f(t)dt - \int_0^1 f(t)dt \int_0^1 2\lambda \max(s,t)ds\} \\ &= \int_0^1 \{2\lambda \max(x,t) + c\lambda(1+x^2) - c\lambda^2(1+x^2)(1+t^2)\}f(t)dt,\end{aligned}$$

$$\text{ie. } (Sf)(x) = \int_0^1 \{2\lambda \max(x,t) + c\lambda(1+x^2)(1-\lambda-\lambda t^2)\}f(t)dt.$$

To find the eigenvalues of S, let ϕ now satisfy $\phi = \Lambda S\phi$,

$$\begin{aligned}ie. \quad \phi(x) &= 2\Lambda\lambda \int_0^x x\phi(t)dt + 2\Lambda\lambda \int_x^1 t\phi(t)dt - c\Lambda\lambda(1+x^2)\int_0^1 (\lambda t^2 + \lambda - 1)\phi(t)dt \\ \Rightarrow \quad \phi'(x) &= 2\Lambda\lambda \int_0^x \phi(t)dt - 2xc\Lambda\lambda \int_0^1 (\lambda t^2 + \lambda - 1)\phi(t)dt \\ \Rightarrow \quad \phi''(x) &= 2\Lambda\lambda\phi(x) - 2c\Lambda\lambda \int_0^1 (\lambda t^2 + \lambda - 1)\phi(t)dt\end{aligned}$$

where $\phi'(0) = 0, \qquad \phi'(1) = \phi(1).$

Now,

$$\begin{aligned}\int_0^1 \phi(x)dx &= \Lambda \int_0^1 \phi(t)dt \int_0^1 \{2\lambda \max(x,t) - c\lambda(\lambda t^2 + \lambda - 1)(1+x^2)\}dx \\ &= \frac{1}{3}c\Lambda\lambda \int_0^1 (1+3t^2)\phi(t)dt.\end{aligned}$$

Hence

$$(1 - \frac{1}{3}c\Lambda\lambda)\int_0^1 \phi(t)dt = -c\Lambda\lambda \int_0^1 t^2\phi(t)dt,$$

and so,

$$\phi''(x) - 2\Lambda\lambda\phi(x) = 2\lambda(1 - \frac{1}{3}c\Lambda\lambda)\int_0^1 \phi(t)dt - 2c\Lambda\lambda(\lambda - 1)\int_0^1 \phi(t)dt = 2\xi,$$

where $\xi = \lambda(1-\Lambda)\int_0^1 \phi(t)dt$.

Suppose $\Lambda < 0$, and write $2\Lambda\lambda = -\omega^2$ ($\omega \in \mathbb{R}$).

Then

$$\begin{aligned}\phi(x) &= a\cos(\omega x) + b\sin(\omega x) - \frac{\xi}{\Lambda\lambda} \\ &= a\cos(\omega x) - \frac{\xi}{\Lambda\lambda},\end{aligned}$$

(since $\phi'(0) = 0$). Therefore

$$\int_0^1 \phi(x)dx = \frac{a}{\omega}\sin\omega - \frac{\xi}{\Lambda\lambda},$$

$$\xi = \lambda(1-\Lambda)\tfrac{a}{\omega}\sin\omega - \lambda(1-\Lambda)\tfrac{\xi}{\Lambda\lambda}$$

$$\Rightarrow \qquad \xi = \Lambda\lambda(1-\Lambda)\tfrac{a}{\omega}\sin\omega.$$

Now, $\phi'(1) = \phi(1)$

$$\begin{aligned}\Rightarrow \quad -a\omega\sin\omega &= a\cos\omega - \frac{\xi}{\Lambda\lambda} \\ &= a\cos\omega - (1-\Lambda)\frac{a}{\omega}\sin\omega,\end{aligned}$$

So $(\frac{1-\Lambda}{\omega} - \omega)\sin\omega = \cos\omega$, but $\Lambda = -\frac{1}{2\lambda}\omega^2$, so

$$(1 + \frac{1}{2\lambda}\omega^2 - \omega^2)\sin\omega = \omega\cos\omega$$

$$\Rightarrow \qquad \tan\omega = \frac{2\lambda\omega}{2\lambda(1-\omega^2)+\omega^2}. \tag{6.2}$$

Finally, let the roots of the above equation be $\pm\hat{\omega}_n$ ($\hat{\omega}_n > 0$) for $n \in \mathbb{N}$. Then the eigenvalues of S are given by $\mu_n^{-1} = -\frac{1}{2\lambda}\hat{\omega}_n^2$, ie. $\mu_n = \frac{-2\lambda}{\hat{\omega}_n^2}$ ($n \in \mathbb{N}$). Let $\hat{\omega}_1$ denote the smallest such root.

Now, if we take the case when $\Lambda > 0$ so that $2\Lambda\lambda = \omega^2$, then ω satisfies

$$\tanh\omega = \frac{2\lambda\omega}{2\lambda(1+\omega^2)-\omega^2}. \tag{6.3}$$

Let the roots of this equation be $\pm\tilde{\omega}_n$ ($\tilde{\omega}_n > 0$) for $n \in \mathbb{N}$, and denote the smallest root by $\tilde{\omega}_1$.

The spectral radius of the operator S is therefore given by

$$\rho(S) = \frac{2\lambda}{\omega_1^2}, \tag{6.4}$$

where $\omega_1 = \min(\hat{\omega}_1, \tilde{\omega}_1)$.

In order to determine the minimum of $\hat{\omega}_1$ and $\tilde{\omega}_1$ it is necessary to explore the behavior of the roots of equations (6.2) and (6.3). Such analysis is quite involved and so only the results are given here.

For $0 \leq \lambda \leq 0{\cdot}5$ equation (6.3) has no non-trivial roots, unlike equation (6.2), and so we take $\omega_1 = \hat{\omega}_1$, giving $\rho(S) = \frac{2\lambda}{\hat{\omega}_1}$ which increases linearly as λ tends to 0·5.

For $0 \cdot 5 < \lambda < 0 \cdot 75$ equation (6.3) has one root which starts off being very large, but which tends to zero like $\frac{1}{\epsilon}$ (where $\epsilon \to \infty$) as $\lambda \to 0 \cdot 75$. This results in $\rho(S) \to \infty$ as $\tilde{\omega}_1 \to 0$, giving $\rho(S) = \frac{2\lambda}{\tilde{\omega}_1}$.

Finally, for $\lambda > 0 \cdot 75$, equation (6.3) once again has no non-trivial roots, whereas equation (6.2) has a smallest root which increases rapidly from zero to π as λ increases. For this region, $\rho(S)$ is therefore given by $\frac{2\lambda}{\hat{\omega}_1}$.

The values obtained for $\rho(S)$ correspond almost exactly with the values of $\frac{||\hat{r}_n||}{||\hat{r}_{n-1}||}$ obtained via the RIG method, and so confirms our view that the ratio of successive residual norms converges to the spectral radius of the convergence operator. Similar analysis to calculate $\rho(SK)$ for the RIK method proves too difficult, but the process of re-iteration provides a means of obtaining an accurate approximation.

It would appear reasonable to suggest that since no solution exists at an

eigenvalue of the operator K (ie. for $\lambda = \frac{1}{\mu_n}$, where μ_n is an eigenvalue of K), the numerical method would not be able to find an approximation and hence would not converge. This would result in the ratio of the residual norms, and therefore $\rho(S)$ and $\rho(SK)$, exceeding the convergence limit, and in fact, becoming infinite. If so, the spikes in Figure (6.1) would appear to mark the presence of an eigenvalue of the kernel operator K, given by $\mu = \frac{1}{\lambda}$. Since λ is positive in the examples, and K has only one positive eigenvalue, the graphs seem to provide a means of obtaining the maximum positive eigenvalue of K.

Unfortunately the graphs do not tend to infinity at the same point. The RIG and RIK($\chi = f$) methods both become infinite at $\lambda = 0 \cdot 75$, whilst the RIK($\chi = Kf$) method becomes infinite at a value closer to $\lambda = 0 \cdot 72$. The problem is that we are not actually solving $\phi = f + K\phi$, but rather an approximate equation lying in a reduced space. The eigenvalues we are locating, therefore, are not of the original equation, but belong instead to the approximate equation.

The approximate eigenvalues arise from the initial Galerkin method and are unaltered by iteration or re-iteration. Thus, in the one-dimensional case with $\chi = f$, the Galerkin equation is $\alpha\{\|f\|^2 - \lambda(Kf, f)\} = \|f\|^2$, which produces the approximate eigenvalue $\tilde{\mu}_1 = \frac{(Kf,f)}{\|f\|^2}$. Since K is self-adjoint we can deduce that $\mu_1^- \leq \tilde{\mu}_1 \leq \mu_1^+$ by considering the Rayleigh quotient. The eigenvalue apparent in the graphs is therefore an underestimate of the maximum positive eigenvalue of K. When we change the subspace, as for the RIK($\chi = Kf$) method, we also change the value of $\tilde{\mu}_1$, in this case to $\tilde{\mu}_1 = \frac{\|Kf\|^2}{(Kf,f)}$. Since $\frac{(Kf,f)}{\|f\|^2} \leq \frac{\|Kf\|^2}{(Kf,f)}$, this second $\tilde{\mu}_1$ will be no less than (and in practice greater than) the first $\tilde{\mu}_1$. Hence, taking $\chi = Kf$ produces a closer underestimate to the maximum positive

eigenvalue of K than can be achieved by taking $\chi = f$.

This result is echoed in the graphs, where the RIK($\chi = f$) and RIG (also with $\chi = f$) methods offer $\tilde{\mu}_1 = 1 \cdot 33$, corresponding to $\lambda = 0 \cdot 75$, whilst the RIK($\chi = Kf$) method gives $\tilde{\mu}_1 = 1 \cdot 39$, since $1 \cdot 38 \leq \mu_1^+ \leq 1 \cdot 42$ can be obtained by standard eigenvalue estimation methods, as mentioned earlier.

Away from the eigenvalue, both of the RIK methods exhibit better convergence properties than the RIG method, as illustrated by their lower $\frac{||\hat{r}_n||}{||\hat{r}_{n-1}||}$ values, which in turn lead to increased intervals of convergence. Although the RIK graphs intersect the RIG graph, they do not do so until all the methods have exceeded the limit for convergence. For this particular problem we can conclude that the RIK methods exhibit consistently better convergence properties than the RIG method, and that, away from the eigenvalue, taking $\chi = f$ constitutes a better choice of subspace than $\chi = Kf$.

6.2 The Kernel $\frac{\lambda}{2\kappa_0} sin(\kappa_0|x-t|)$

The second equation we consider is the general form of equation (5.2), ie.

$$\phi(x) = \cos(\kappa_0 x) + \lambda \int_0^1 \frac{1}{2\kappa_0} \sin(\kappa_0|x-t|)\phi(t)dt, \qquad 0 \leq x \leq 1 \tag{6.5}$$

where κ_0 is a parameter (taken to be $\kappa_0 = 1 \cdot 5$ in the following example).

Unlike equation (6.1), finding both $\rho(S)$ and$\rho(SK)$ analytically proves too difficult, so we have to rely entirely on the numerical results in order to gain information about the convergence properties of the methods for this kernel. Figure (6.2) shows the graphs of $\frac{||\hat{r}_n||}{||\hat{r}_{n-1}||}$ against λ for the three methods when

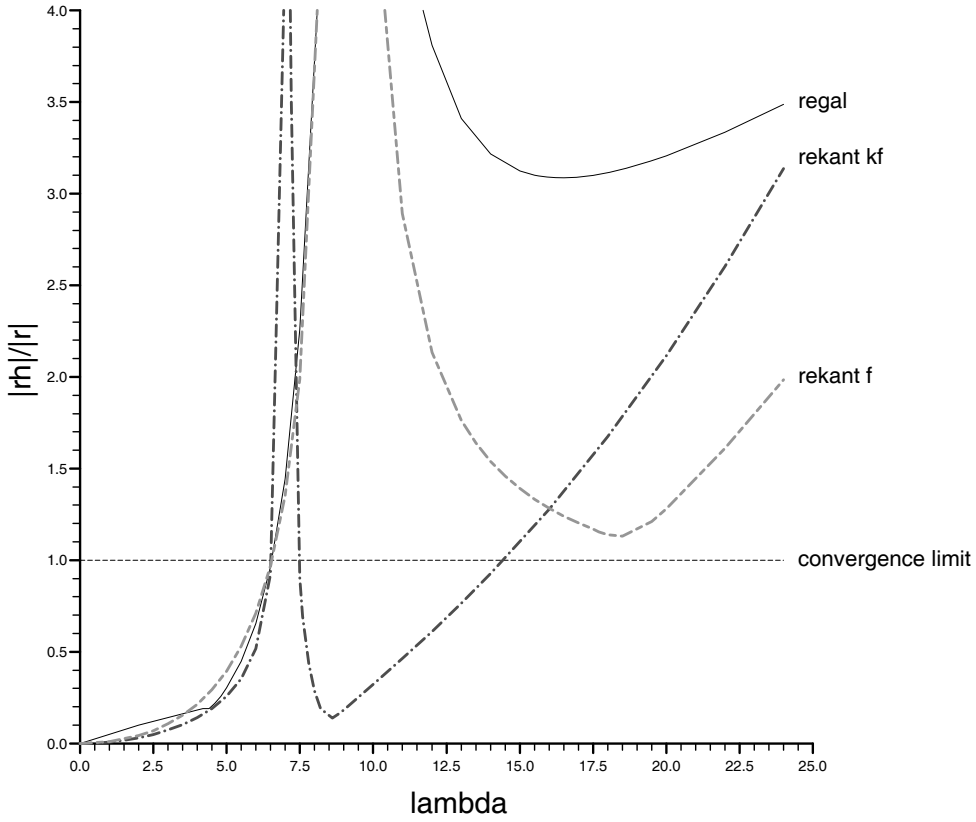

Figure 6.2: Convergence of methods for $k(x,t) = \frac{\lambda}{2\kappa_0}\sin(\kappa_0|x-t|)$

approximating equation (6.5). As before we notice discontinuities in the results, pointing to the presence of approximate eigenvalues. This time, however, the singularities occur at markedly different values for the methods with different subspaces. The $\chi = f$ methods become infinite at around $\lambda = 9 \cdot 1$, whilst the $\chi = Kf$ method becomes infinite at approximately $\lambda = 7 \cdot 1$. As in the previous example, we can show that the latter choice of subspace provides the closest underestimate to the maximum positive eigenvalue of K, giving us $\tilde{\mu}_1 = \frac{1}{\lambda} \approx 0 \cdot 14 \leq \mu_1^+$.

We can compare this result with the following bounds on the largest positive and negative eigenvalues of K, as proposed by Chamberlain [1] (Lemma 4.2), who

gives

$$\begin{aligned}\mu_1^+ &\leq \min(0\cdot 173704 + \frac{\kappa_0^2}{24\sqrt{7}}\ ,\ \frac{1}{2\sqrt{2\kappa_0^2}}\sqrt{\kappa_0^2 - \sin^2\kappa_0}\ ),\\ \mu_1^- &\geq \max(-\pi^{-2}\ ,\ -\frac{1}{2\sqrt{2\kappa_0^2}}\sqrt{\kappa_0^2 - \sin^2\kappa_0}\ ),\end{aligned}$$

ie. taking $\kappa_0 = 1\cdot 5$ we have

$$\begin{aligned}\mu_1^+ &\leq 0\cdot 17604,\\ \mu_1^- &\geq -0\cdot 10132.\end{aligned}$$

Our approximation, $\tilde{\mu}_1$, certainly lies below Chamberlain's upper bound, and since it is an underestimate, provides us with a lower bound on μ_1^+.

All three methods, in this example, demonstrate convergence up to $\lambda \approx 6\cdot 5$, where the effect of the approximate eigenvalue becomes too great. Neither the RIG method, nor the RIK($\chi = f$) method, manage to recover sufficiently from the eigenvalue to provide additional regions of convergence. The RIK($\chi = Kf$) method, on the other hand, recovers relatively quickly, recrossing the convergence limit at $\lambda \approx 7\cdot 5$, and providing convergence up to $\lambda \approx 16\cdot 0$.

The RIK($\chi = Kf$) method is clearly the best method for this problem using a one-dimensional subspace.

Chapter 7

Extending to 2-D

The theoretical convergence limit, $\frac{||\hat{r}_n||}{||\hat{r}_{n-1}||} = 1$, does not prove to be very useful when approximating solutions to integral equations. The number of iterations required to reach a specified accuracy tends to infinity as the limit is approached. A much more sensible practical limit would be $\frac{||\hat{r}_n||}{||\hat{r}_{n-1}||} = 0 \cdot 5$, corresponding to roughly forty iterations. If the methods require more iterations than this it is time to consider enlarging the size of the subspace.

Up to now we have been looking for approximations in a one-dimensional subspace spanned by either $\chi = f$ or $\chi = Kf$. By extending the subspace to two dimensions we are introducing additional degrees of freedom to the approximations, leading to increased accuracy. We should find that, as a result, the number of iterations required for convergence will decrease, leading to an overall reduction in the values of $\frac{||\hat{r}_n||}{||\hat{r}_{n-1}||}$, and thus to larger intervals of convergence.

As outlined in section (4.2), the optimal choice of subspace is based upon the terms of the Neumann series. For a two-dimensional subspace, therefore, we want E_n to be spanned by f and Kf. Rather than take f and Kf as they are, we

form an orthonormal basis $\{\chi_1, \chi_2\}$, where

$$\chi_1 = \frac{f}{\sqrt{(f,f)}},$$

$$\chi_2 = \frac{\zeta}{\sqrt{(\zeta,\zeta)}},$$

and

$$\zeta = Kf - (Kf, \chi_1)\chi_1.$$

Since we know that

$$(\chi_m, \chi_n) = \begin{cases} 1 & (m = n) \\ 0 & (m \neq n) \end{cases}$$

this also saves us some calculations.

Figure (7.1) shows the effect of enlarging the subspace on the graph of $\frac{||\hat{r}_n||}{||\hat{r}_{n-1}||}$ against λ for equation (6.1) . Both the RIK and RIG methods illustrated use the trial space $\{\chi_1, \chi_2\}$ described above. The approximate eigenvalue peak has narrowed considerably for both methods. As can be seen more clearly in the

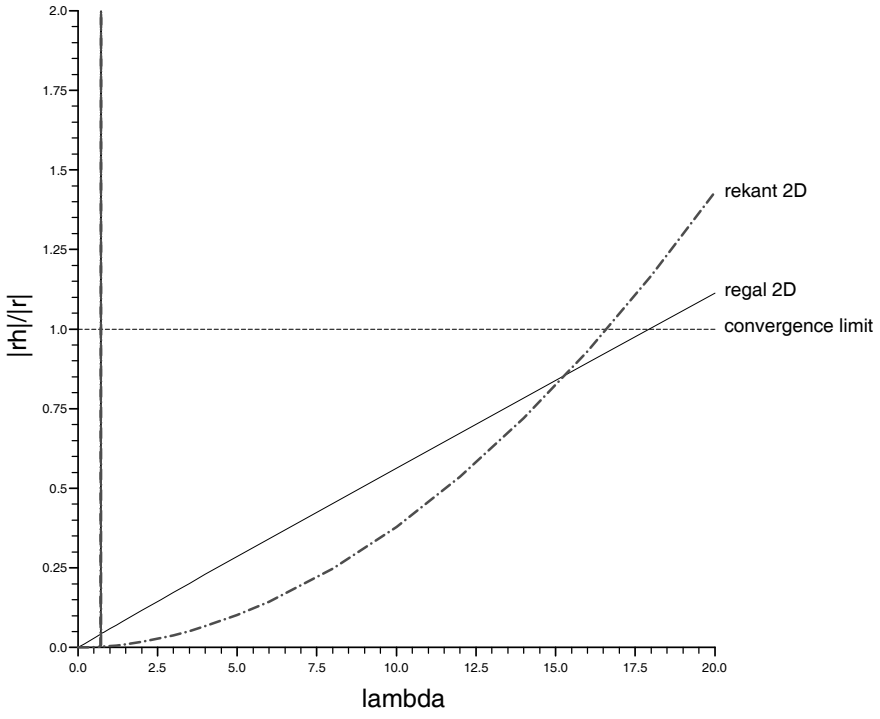

Figure 7.1: Approximating $k(x,t) = 2\lambda \max(x,t)$ in 2-D

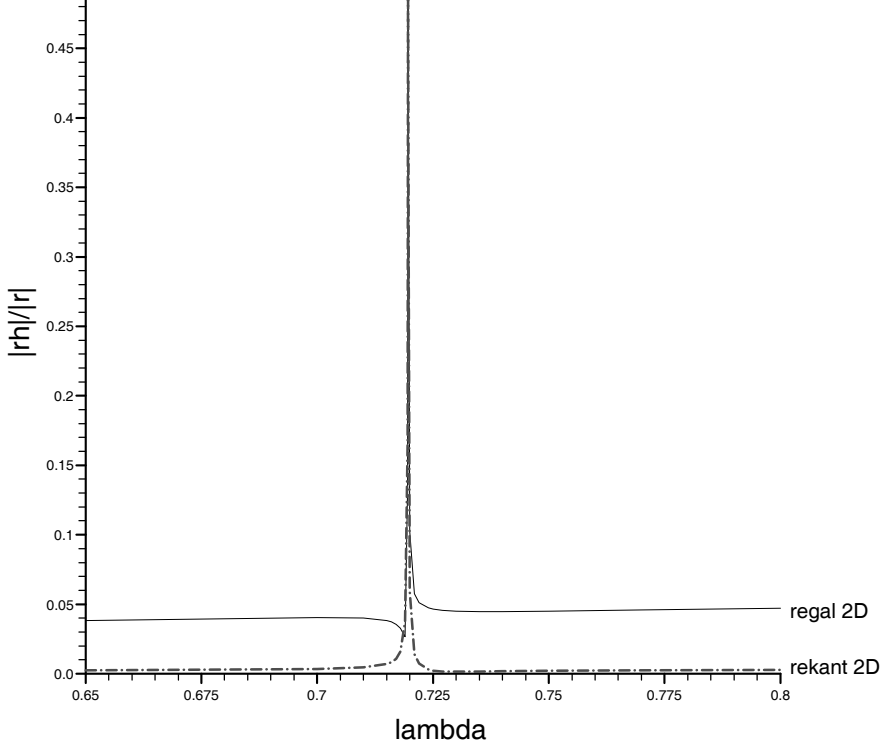

Figure 7.2: The eigenvalue of $k(x,t) = 2\lambda \max(x,t)$ for a 2-D subspace

magnification in Figure (7.2), the value of λ at which the approximate eigenvalue occurs can be pinpointed at $\lambda = 0 \cdot 71967$, corresponding to $\tilde{\mu}_1 \approx 1 \cdot 38949$. The disruption caused to the convergence of the methods by the presence of the eigenvector is greatly reduced, allowing us to see more clearly the linear nature of the RIG method, and the quadratic nature of the RIK method. It also demonstrates that the RIG method, which was the most disrupted in the one-dimensional case, may converge faster than the RIK method for certain values of λ, and hence for certain kernels. The point of cross-over in this example does not, however, occur until $\frac{\|\hat{r}_n\|}{\|\hat{r}_{n-1}\|}$ is significantly greater than $0 \cdot 75$, a value at which, in a practical situation, we would be looking to utilise an even higher dimensional subspace.

Finally, Figure (7.3) illustrates the effect of moving to a two-dimensional subspace for equation (6.5). In the one-dimensional case the eigenvalue proved rather

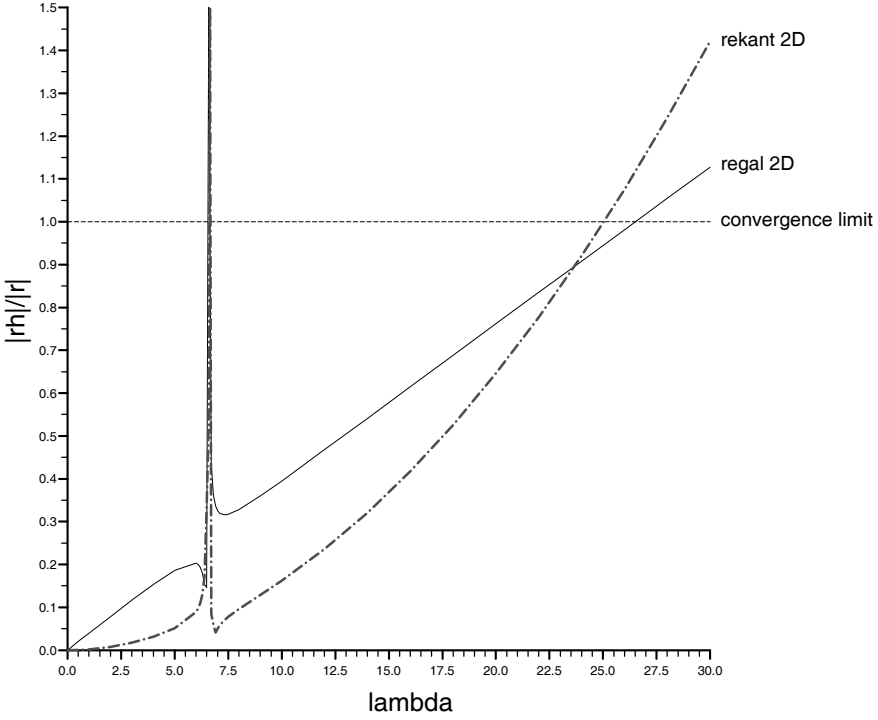

Figure 7.3: Approximating $k(x,t) = \frac{\lambda}{2\kappa_0}\sin(\kappa_0|x-t|)$ in 2-D

disastrous for the convergence of the RIG method, preventing it from providing any convergence after $\lambda = 6 \cdot 5$. Using a two-dimensional subspace significantly reduces the effect of the approximate eigenvalue, which we can now estimate, with some accuracy, to be in the region of $\tilde{\mu}_1 \approx 0 \cdot 15224$ ($\lambda \approx 6 \cdot 5686$). As before, the RIK method demonstrates better convergence properties in general, the point of cross-over with the RIG method being at a value of $\frac{||\hat{r}_n||}{||\hat{r}_{n-1}||}$ too high to be of practical use.

Chapter 8

Conclusions

The results in the previous two Chapters suggest that the new re-iterated Kantorovich method will converge, and do so faster than the re-iterated Galerkin method, if

$$\rho(SK) < \rho(S) \leq 1.$$

When $\|K\|$ is large, ie. when λ is large in the previous graphs, the RIG method can display the better convergence properties. However, if the view is taken that values of $\frac{\|\hat{r}_n\|}{\|\hat{r}_{n-1}\|}$ greater than $0 \cdot 5$, corresponding to roughly forty iterations, signal the need for a higher dimensional subspace in which to seek a solution, then the RIK method represents the best practical method of approximation. Naturally, increasing the size of the subspace, and therefore introducing more degrees of freedom into the approximation, has the effect of increasing the rate of convergence, leading to increased regions of convergence.

The presence of approximate eigenvalues can cause a fair amount of disruption to the methods. This is especially true when using a subspace of low dimension, where the number of terms in the expansion of the approximation available to

compensate for the rapid growth of the residuals, is low. It has been shown that choosing a subspace spanned by $\chi = Kf$, rather than by $\chi = f$, for the one-dimensional case, not only gives a closer underestimate to the exact eigenvalue μ_1^+ of the problem, but also reduces the effect of the approximate eigenvalue on convergence. The position of the approximate eigenvalue may be calculated beforehand and so be avoided. This is done by considering the initial Galerkin equation, given by (in 1-D case)

$$\alpha_1\{\|\chi_1\|^2 - \lambda(K\chi_1, \chi_1)\} = (f, \chi_1),$$

hence the eigenvalue, $\tilde{\mu}_1$, is given by

$$\tilde{\mu}_1 = \frac{(K\chi_1, \chi_1)}{\|\chi_1\|^2},$$

where χ_1 is the choice of trial function. Also to be avoided, naturally, are the exact eigenvalues of the equation. Since the numerical methods approximate the solution by solving an approximate equation, they may be seen to converge for values of λ, corresponding to the exact eigenvalues, for which we know the exact equation has no solution. In these cases the numerical results will be entirely spurious. Avoidance of the approximate and exact eigenvalues should not, in practice, cause too much disruption to the regions of convergence of the methods since the approximate and exact eigenvalues should be reasonably close together.

The ratio of successive residual norms for the RIG method has been seen to converge to the spectral radius of S. A similar result has not been proved for the RIK method since, for the equations given in the examples, an analytical calculation of $\rho(SK)$ proved to be too difficult. The convergence properties of the method, however, lead us to the conclusion that a similar result will hold.

Whereas the estimates of $\rho(S)$ proved to be very accurate, a similar degree of accuracy could not be attributed to the underestimates of $\|K\|$, given by $\frac{\|\hat{r}_n\|}{\|r_n\|}$ for both methods. These were shown to be very poor in comparison, depending upon the residual, r_n, being a close approximation to a constant multiple of the eigenvector ϕ_1^+.

There is obviously a need for more exhaustive testing of the re-iterated Kantorovich method, as well as more in-depth analysis, before its strengths and weaknesses are fully understood. In the limited time available, it has only been possible to explore some basic ideas in relation to a few particular examples and many issues have arisen which require closer analysis and examination. Some obvious features that have not been explored in this report include: a closer study of the analytical properties of the spectral radii of S and SK ; the difference in the way the two methods converge to the solution of the equation, ie. the Galerkin method oscillates about the solution, whereas the Kantorovich method approaches the solution monotonically from above; and the way in which the ratio of successive residual norms tends to oscillate when it reaches a local minimum after encountering an eigenvalue.

In the example using the kernel $k(x,t) = \frac{1}{2\kappa_0}\sin(\kappa_0|x-t|)$, κ_0 was fixed at $1\cdot 5$. Due to its use as a kernel in the practical application of modelling wave scattering, it would be interesting to investigate the effect of changing κ_0 on the convergence properties of the two re-iterative methods. This could lead to the methods being used to find a solution to the practical problem which Chamberlain [2] solved approximately by variational techniques.

Finally, Porter & Stirling [4] demonstrated how the properties of the re-

iterated Galerkin method could be exploited in variational principles. It would seem plausible that the re-iterated Kantorovich method could be similarly utilised, and may, under certain conditions, once again yield improved results.

Bibliography

[1] Chamberlain, P.G. 1991. *Wave propagation on water of uneven depth.* Ph.D. thesis, University of Reading.

[2] Chamberlain, P.G. 1993. *Wave scattering over uneven depth using the mild-slope equation.* Wave Motion 17, pp 267-285.

[3] Porter, D. & Stirling, D.S.G. 1990. *Integral Equations.* Cambridge Univ. Press.

[4] Porter, D. & Stirling, D.S.G. 1993. *The Re-iterated Galerkin Method.* IMAJ Num. Anal. 13, pp 125-139.

[5] Schock, E. 1985. *Arbitrarily slow convergence, uniform convergence and superconvergence of Galerkin-like methods.* IMAJ Num. Anal. 5, pp 153-160.

[6] Sloan, I.H. 1984. *Four varients of the Galerkin method for integral equations of the second kind.* IMAJ Num. Anal. 4.

[7] Sloan, I.H. 1976. *Improvement by iteration for compact operator equations.* Math. Comp. 30, pp 758-764.

[8] Sloan, I.H., Burn, B.J. & Datyner, N. 1975. *A new approach to the numerical solution of integral equations.* J. Comp. Physics 18, pp 92-105.