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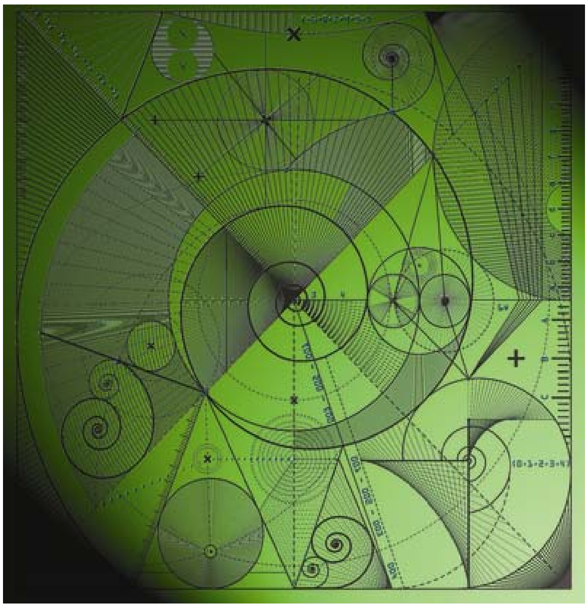

The elliptic sine-Gordon equation in a half plane

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Abstract

We consider boundary value problems for the elliptic sine-Gordon equation posed in the half plane $y > 0$. This problem was considered in [10] using the classical inverse scattering transform approach. Given the limitations of this approach, the results obtained rely on a nonlinear constraint on the spectral data derived heuristically by analogy with the linearized case.
We revisit the analysis of such problems using a recent generalization of the inverse scattering transform, known as the Fokas method, and show that the nonlinear constraint of [10] is a consequence of the so-called global relation. We also show that this relation implies a stronger constraint on the spectral data, and in particular that no choice of boundary conditions can be associated with a decaying (possibly mod 2π) solution analogous to the pure soliton solutions of the usual, time-dependent sine-Gordon equation.
We also briefly indicate how, in contrast with the evolutionary case, the elliptic sine-Gordon equation posed in the half plane does not admit linearisable boundary conditions.

1 Introduction

The elliptic sine-Gordon equation

$$q_{xx} + q_{yy} = \sin q, \qquad q = q(x, y), \tag{1.1}$$

is an integrable PDE in two variables. From the point of view of the modelling of physical phenomena, the motivation for the study of this equation comes from its applications in several areas of mathematical physics including the theory of Josephson effects, superconductors and spin waves in ferromagnets, see e.g. [10]. However this equation is also of considerable interest from a purely mathematical point of view. Indeed, while an example of a nonlinear integrable PDE, equation (1.1) is a time-independent PDE of elliptic type, and therefore it differs from most other one-dimensional nonlinear integrable models, that describe a temporal evolution process.

The inverse scattering method has been used to analyse this equation in $\mathbb{R}^2$; namely, the problem with prescribed periodic behaviour at infinity was considered in [5], while special solutions for the problem posed in the whole of $\mathbb{R}^2$ were found in the 80's, see the references in [5]. However, the classical inverse scattering transform cannot be used in general to derive a solution representation without adapting it to allow for the treatment of boundary conditions.

Such an extension of the classical inverse scattering transform has recently been proposed and applied to solve a variety of boundary value problems for integrable evolution PDE, see the monograph [6]. In this paper, we use this extension to analyse boundary value problems for (1.1) posed in the half plane $\{(x, y) : x \in \mathbb{R},\ y > 0\}$. Such problems were also considered in [10] under the assumption that the boundary data satisfy a certain nonlinear equation, deduced heuristically by analogy with the linearized case. We show here that this nonlinear equation is obtained as a consequence of the so-called *global relation*. The global relation is derived rigorously in our approach, and it is shown to imply a stronger constraint on the boundary data than the one imposed in [10]. By requiring its validity, we characterize all pairs of functions that can occur as boundary values of decaying (mod 2π) solutions of equation (1.1) in the half plane.

We also briefly consider the problem of solving a generic well-posed boundary value problem, when one boundary condition is prescribed and a second one must be determined. An example of such problem arises when the Dirichlet datum $q(x, 0)$ is given while the Neumann datum $q_y(x, 0)$ is unknown and must be obtained as part of the solution. The characterization of the unknown boundary value relies on the analysis of the global relation. The experience of linear elliptic problems, and research currently in progress, indicate that this is a difficult problem in general. However, all the usual integrable evolution PDEs (when considered e.g. for $\{(x, t) : x > 0, t > 0\}$) admit special types of boundary conditions, called *linearisable*, for which the dependence of the solution on the unknown boundary values can be eliminated [6]. Such boundary conditions have in most cases appeared previously and independently in the literature, as they correspond to boundary value problems that can be solved as effectively as their linear reductions. In particular, the usual sine-Gordon equation posed for $x > 0$, $t > 0$ admits two such linearisable conditions [8]. However, we show here that there do not exist any analogues of the latter linearisable conditions for (1.1) posed on the half plane $y > 0$.

The paper is organized as follows. In section 2, we set down the notation and state the main results. In section 3, we solve the Dirichlet boundary value problem for the corresponding linear model, the modified Helmholtz equation. The solution is obtained by the Fokas' approach, following a series of steps that can be generalized for the nonlinear case. This is done to illustrate the main ideas in a simpler setting. In particular, we derive the global relation and show how its invariance properties yield an expression of the solution in terms of the known boundary condition only. In section 4, we perform the spectral analysis of equation (1.1) posed on the half plane, and derive the global relation. We then derive a formal representation of the solution under the assumption that the given boundary values are admissible, i.e, that they satisfy the global relation. Finally, in section 5, we show how the condition for boundary conditions to be linearisable implies that the solution is zero (mod 2π). This implies that for any given well-posed boundary value problem, the characterization of the unknown boundary data is an essentially nonlinear process. This characterization, known as the generalized Dirichlet to Neumann map, is work in progress, and will be presented elsewhere.

2 Setting of the problem and statement of results

We start with defining the notation and conventions we use throughout the paper.

- The matrices σ_i are the usual Pauli matrices, given by:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

- The expressions $\omega_1(\lambda)$ and $\omega_2(\lambda)$ denote the following functions of the complex parameter λ :

$$\omega_1(\lambda) = \frac{1}{2}\left(\lambda + \frac{1}{\lambda}\right), \quad \omega_2(\lambda) = \frac{1}{2i}\left(\lambda - \frac{1}{\lambda}\right). \tag{2.1}$$

  Since $Re\,(\omega_1(\lambda)) = \frac{1}{2}Re\,(\lambda)\,(1 + \frac{1}{|\lambda|^2})$ and $Re\,(\omega_2(\lambda)) = \frac{1}{2}Im\,(\lambda)\,(1 + \frac{1}{|\lambda|^2})$ the exponentials $\mathrm{e}^{\omega_1(\lambda)}$ and $\mathrm{e}^{\omega_2(\lambda)}$ are bounded as $|\lambda| \to \infty$ as follows:

$$\mathrm{e}^{\omega_1(\lambda)} \text{ bounded for } Re\,(\lambda) < 0; \qquad \mathrm{e}^{\omega_2(\lambda)} \text{ bounded for } Im\,(\lambda) < 0.$$

- The notation $\widehat{\sigma_3}$ denotes the matrix commutator with σ_3, hence

$$\widehat{\sigma_3}M = [\sigma_3, M], \qquad \mathrm{e}^{\widehat{\sigma_3}}M = \mathrm{e}^{\sigma_3}M\mathrm{e}^{-\sigma_3} = \begin{pmatrix} m_{11} & m_{12}\mathrm{e}^2 \\ m_{21}\mathrm{e}^{-2} & m_{22} \end{pmatrix}.$$

Admissible sets

Let g_0, $g_1 : \mathbb{R} \to \mathbb{C}$ be given functions with the following regularity and decay properties :

- $g_0,\ g_1 \in \mathbf{C}^2(\mathbb{R})$ (p1)
- $g_0(x) + 2\pi m$ (for some $m \in \mathbb{Z}$), $g_0'(x), g_1(x) \to 0$ as $|x| \to \infty$. (p2)

We are not presently interested in issues of regularity, and the above assumptions are sufficient to guarantee that our solution procedure is rigorously justified at all steps.

Given $g_0(x)$, $g_1(x)$ as above, for $\lambda \in \mathbb{C}$ we define a matrix-valued function $Q_0(x, \lambda)$ by

$$Q_0(x,\lambda) = \frac{ig_0'(x) + g_1(x)}{4}\sigma_1 - \frac{i}{4\lambda}(\sin g_0(x))\sigma_2 + \frac{i}{4\lambda}(1 - \cos g_0(x))\sigma_3. \tag{2.2}$$

The matrix-valued function Q_0 satisfies the symmetry properties

$$Q_0(x,\lambda)_{22} = Q_0(x,-\lambda)_{11}; \qquad Q_0(x,\lambda)_{12} = Q_0(x,-\lambda)_{21} \tag{2.3}$$

These properties guarantee that for $\lambda \in \mathbb{R}$ the matrix $m(x,\lambda)$ is well defined as the unique solution of the linear integral equation

$$m(x,\lambda) = I - \int_x^\infty \mathrm{e}^{-\frac{\omega_2(\lambda)}{2}(x-\xi)\widehat{\sigma_3}} \left[Q_0(\xi,\lambda)m(\xi,\lambda)\right] d\xi. \tag{2.4}$$

Indeed, the existence and uniqueness of the solution of this linear integral equation, under the given assumptions and given the symmetry properties of Q_0, can be established by adapting the proofs of the more general results in [3, 4]. See also the section on the sine-Gordon equation in [7].

Definition 2.1 *Let g_0, $g_1 : \mathbb{R} \to \mathbb{R}$ satisfy the two properties* (p1) *and* (p2)*, and let $m(x,\lambda)$ be the matrix defined as the unique solution of the linear integral equation (2.4), with Q_0 given by (2.2). Define the matrix-valued function $R(\lambda) : \mathbb{R} \to \mathbb{C}$ by*

$$R(\lambda) = \lim_{x\to-\infty} \left[\mathrm{e}^{\frac{\omega_2(\lambda)}{2}x\widehat{\sigma_3}} m(x,\lambda)\right]. \tag{2.5}$$

The set $\{g_0(x), g_1(x)\}$ is called an admissible set for equation (1.1), posed in the half plane $\{y \geq 0\}$, if $R(\lambda)$ has the form

$$R(\lambda) = \begin{pmatrix} 1 & b(-\lambda) \\ b(\lambda) & 1 \end{pmatrix}, \quad \lambda \in \mathbb{R}, \tag{2.6}$$

for some function $b(\lambda) : \mathbb{R} \to \mathbb{C}$ whose support is contained in the negative real half line:

$$b(\lambda) = 0 \quad for \;\; \lambda \in \mathbb{R}^+.$$

The function $b(\lambda)$ is called the spectral function associated with the set $\{g_0, g_1\}$.

Remark 2.1 The matrix-valued function $R(\lambda)$ given by (2.6) is explicitly of the form

$$R(\lambda) = \begin{cases} \begin{pmatrix} 1 & 0 \\ b(\lambda) & 1 \end{pmatrix}, & \lambda \in \mathbb{R}^-, \\[2ex] \begin{pmatrix} 1 & b(-\lambda) \\ 0 & 1 \end{pmatrix}, & \lambda \in \mathbb{R}^+. \end{cases} \tag{2.7}$$

The admissibility of a set of two functions g_0, g_1 as above is a necessary and sufficient condition for them to realize the two boundary values of a solution of (1.1) at $y = 0$. Namely, if $q(x,y)$ is a solution of equation (1.1) posed in the half plane, with suitable smoothness and decay, then its boundary values $q(x,0)$ and $q_y(x,0)$ form an admissible set. Conversely, any two functions g_0 and g_1 that form an admissible set can be realized as the boundary values ($g_0(x) = q(x,0)$, $g_1(x) = q_y(x,0)$) of a solution of equation (1.1). This is the content of the two propositions we now state.

Proposition 2.1 *Let q be a solution of equation (1.1) in the half plane $\{y \geq 0\}$, such that $q \in \mathbf{C^2}(\mathbb{R} \times \mathbb{R}^+)$ and satisfies $q + 2\pi m$, $q_y \to 0$ when $|y| + |x| \to \infty$ ($m \in \mathbb{Z}$). Let*

$$g_0(x) = q(x, 0), \qquad g_1(x) = q_y(x, 0).$$

Then the functions $\{g_0(x), g_1(x)\}$ form an admissible set.

Proposition 2.2 *Let $\{g_0(x), g_1(x)\}$ be an admissible set, and let $b(\lambda)$ be its associated spectral function.*
Let $M(x, y, \lambda)$ be the unique solution of the following Riemann-Hilbert problem :

$$M_-(x, y, \lambda) = M_+(x, y, \lambda) J(x, y, \lambda), \quad \lambda \in \mathbb{R}, \qquad \det(M_\pm) = 1, \tag{2.8}$$

where $M_\pm$ are analytic functions of λ in $\mathbb{C}^\pm$ respectively, and

$$J(x, y, \lambda) = \begin{pmatrix} 1 & b(-\lambda)\mathrm{e}^{-\theta(x,y,\lambda)} \\ -b(\lambda)\mathrm{e}^{\theta(x,y,\lambda)} & 1 \end{pmatrix}, \qquad \theta(x, y, \lambda) = \omega_1(\lambda) y + \omega_2(\lambda) x. \tag{2.9}$$

This Riemann-Hilbert problem is uniquely solvable, and the function $q(x, y)$ defined by

$$q_x - i q_y = -(\tilde{M})_{12}, \qquad \cos q(x, y) = 1 - 4i(\frac{\partial}{\partial x}\tilde{M}_{22}) + 2(\tilde{M}_{12})^2, \tag{2.10}$$

where

$$\tilde{M} = \lim_{\lambda \to \infty} (\lambda(M - I)), \qquad I = diag(1, 1),$$

solves the following boundary value problem for the elliptic sine-Gordon equation:

$$q_{xx} + q_{yy} = \sin q, \qquad x \in \mathbb{R},\ y \geq 0, \tag{2.11}$$

$$q(x, 0) = g_0(x), \quad q_y(x, 0) = g_1(x), \qquad x \in \mathbb{R}. \tag{2.12}$$

The proof of these two propositions is given in section 4. In the next section we illustrate the main steps of our Riemann-Hilbert approach in the simpler linear case.

3 The modified Helmholtz equation in the half plane

We consider the linear version of the elliptic sine-Gordon equation (1.1), an important equation in its own right known as the *modified Helmholtz* equation. This equation, in the context of the method we use in this paper, is studied in [6], where full details can be found. For concreteness, we briefly analyse a concrete, Dirichlet boundary value problem for this equation in the half plane:

$$q_{xx} + q_{yy} = q, \qquad x \in \mathbb{R}, \quad y > 0, \tag{3.1}$$

$$q(x, 0) = g_0(x), \qquad x \in \mathbb{R}, \tag{3.2}$$

where the function $g_0(x)$ is assumed to have appropriate smoothness and decay at infinity (for example, to satisfy the properties (p1) and (p2)).

Proposition 3.1 *The solution of the boundary value problem (3.1) admits the integral representation*

$$q(x,y) = \frac{1}{2\pi i}\int_0^\infty \mathrm{e}^{-\omega_2(\lambda)x-\omega_1(\lambda)y}\omega_1(\lambda)\rho(\lambda)\frac{d\lambda}{\lambda}, \tag{3.3}$$

where ω_1, ω_2 are given by (2.1), and the spectral function $\rho(\lambda)$ is defined by

$$\rho(\lambda) = \int_0^\infty \mathrm{e}^{\omega_2(\lambda)x} g_0(x)dx, \quad \lambda \geq 0. \tag{3.4}$$

We note a solution of the form (3.3) was obtained in [11]. Here, we present an alternative approach for the solution of this problem, based on a Riemann-Hilbert formulation.

Sketch of proof: Equation (3.1) can be written as the compatibility condition of the following ODEs

$$\mu_x + \omega_2(\lambda)\mu = -q_y + \omega_1(\lambda)q, \qquad \mu_y + \omega_1(\lambda)\mu = q_x - \omega_2(\lambda)q, \tag{3.5}$$

known as the Lax pair of the equation. The compatibility condition ($\mu_{xy} = \mu_{yx}$) of the Lax pair (3.5) is equivalent to the condition that the differential form W, given by

$$W(x,y,\lambda) = e^{\omega_1(\lambda)y+\omega_2(\lambda)x}[(-q_y + \omega_1(\lambda)q)dx + (q_x - \omega_2(\lambda)q)dy],$$

is exact. Note that this form is well defined only when $\lambda < 0$, since $Re(\omega_1(\lambda)y + \omega_2(\lambda)x) \leq 0$ for all $x \in \mathbb{R}$, $y > 0$ only if $\lambda \in \mathbb{R}^-$.

Our approach is based on the simultaneous spectral analysis of both equations (3.5) (or of the equivalent differential form) . Namely, we consider solutions of the system (3.5) and request that they are bounded with respect to the spectral parameter λ, and that the respective region of boundedness cover the whole complex λ plane. Assuming that $q + 2\pi m$, $q_y \to 0$ when $|y| + |x| \to \infty$, one such set of solutions is given by

$$\mu_1(x,y,\lambda) = \int_{-\infty}^x e^{-\omega_2(\lambda)(x-\xi)}(-q_y + \omega_1(\lambda)q)(\xi,y)d\xi, \quad \lambda \in \mathbb{C}^+, \tag{3.6}$$

$$\mu_2(x,y,\lambda) = \int_x^\infty e^{-\omega_2(\lambda)(x-\xi)}(-q_y + \omega_1(\lambda)q)(\xi,y)d\xi, \quad \lambda \in \mathbb{C}^-. \tag{3.7}$$

These functions are bounded where indicated, and satisfy

$$\mu_j \backsim -q + O\left(\frac{1}{\lambda}\right), \quad |\lambda| \to \infty, \quad j = 1,2 \tag{3.8}$$

$$(\mu_2 - \mu_1)(x,y,\lambda) = \mathrm{e}^{-\omega_1(\lambda)y-\omega_2(\lambda)x}\int_{-\infty}^\infty e^{\omega_2(\lambda)\xi}(-q_y + \omega_1(\lambda)q)(\xi,0)d\xi, \quad \lambda \in \mathbb{R}. \tag{3.9}$$

These data determine a Riemann-Hilbert problem on $\mathbb{R}$. This problem can be solved explicitly to yield a function $\mu(x,y,\lambda)$ defined globally, for all $\lambda \in \mathbb{C}$. In turn, we obtain for $q(x,y) = \mu_x + \omega_2(\lambda)\mu$ an expression analogous to (3.3), but with $2\omega_1(\lambda)\rho(\lambda)$ replaced by

$$\tilde{g}(\lambda) = -\tilde{g}_1(\lambda) + \omega_1(\lambda)\tilde{g}_0(\lambda), \quad \lambda \geq 0, \quad \tilde{g}_i(\lambda) = \int_{-\infty}^\infty e^{\omega_2(\lambda)x}\partial_y^{(i)} q(x,0)dx. \tag{3.10}$$

In this expression, the function $\tilde{g}_0(\lambda)$ can be computed from the given Dirichlet boundary condition, but the function $\tilde{g}_1(\lambda)$ is unknown. The remaining problem is the determination of the transform $\tilde{g}_1(\lambda)$ of the unknown boundary value $q_y(x,0)$.
To this end, we consider the differential form $W(x,y,\lambda)$, which is bounded in λ for all (x,y) in the half plane when $\lambda < 0$. This form is exact, hence it is also closed in the half plane $y > 0$, and therefore

$$\int_{-\infty}^{\infty} \mathrm{e}^{\omega_2(\lambda)x}(-q_y + \omega_1(\lambda)q)(x,0)dx = 0, \quad \lambda < 0.$$

This identity is the *global relation*. It has two consequences. Firstly, it implies that the jump (3.9) is nonzero only for $\lambda \geq 0$. Secondly, it yields the following relation between the two boundary data:

$$\tilde{g}_1(\lambda) = \omega_1(\lambda)\tilde{g}_0(\lambda), \qquad \lambda < 0. \tag{3.11}$$

Equation (3.11) is the consequence of the global relation associated with this boundary value problem in terms of the spectral functions $\tilde{g}_i$ (see [6] for a general discussion). Letting $\lambda \to -1/\lambda$ and using the invariance of the spectral functions under this transformation, we obtain

$$\tilde{g}_1(\lambda) = -\omega_1(\lambda)\tilde{g}_0(\lambda), \;\; \lambda \geq 0 \Rightarrow \tilde{g}(\lambda) = 2\omega_1(\lambda)\tilde{g}_0(\lambda), \;\; \lambda \geq 0.$$

We will not consider how to generalize this last step, namely the explicit determination of $\tilde{g}_1$ in terms of $\tilde{g}_0$ through the resolution of the global relation, in the nonlinear case. Instead, we will prescribe two boundary conditions, but require that they satisfy the global relation, hence that they are consistent with the solvability of the boundary value problem.

4 The elliptic sine-Gordon equation

The Lax pair formulation

The elliptic sine-Gordon equation (1.1) can be written as the compatibility condition of the following Lax pair

$$M_x(x,y,\lambda) + \frac{w_2(\lambda)}{2}[\sigma_3, M](x,y,\lambda) \;\; = \;\; Q(x,y,\lambda)M(x,y,\lambda), \tag{4.1}$$

$$M_y(x,y,\lambda) + \frac{w_1(\lambda)}{2}[\sigma_3, M](x,y,\lambda) \;\; = \;\; iQ(x,y,-\lambda)M(x,y,\lambda), \tag{4.2}$$

where the matrix $Q_(x,y,\lambda)$ is given by in terms of the solution $q(x,y)$ of (1.1) and its derivatives by

$$Q = \frac{i(q_x - iq_y)}{4}\sigma_1 - \frac{i}{4\lambda}(\sin q)\sigma_2 + \frac{i}{4\lambda}(1-\cos q)\sigma_3. \tag{4.3}$$

This Lax pair is equivalent, but not equal, to the one considered in [10, 11]. We derived this Lax pair from the requirement that the linear limit coincides with (3.5). Indeed, at each step of our construction, the solution of the linear problem presented in the previous section can be used as a guideline, as well as providing a check of the correctness of the results.

As in the linear case, the Lax pair is equivalent to the condition that the (matrix-valued) differential form Ω, given by

$$\Omega(x,y,\lambda) = \mathrm{e}^{(\frac{\omega_1(\lambda)y}{2}+\frac{\omega_2(\lambda)x}{2})\widehat{\sigma_3}}\left[Q(x,y,\lambda)M(x,y,\lambda)dx + iQ(x,y,-\lambda)M(x,y,\lambda)dy\right]. \quad (4.4)$$

is exact, hence that there exists a matrix $M(x,y,\lambda)$ such that

$$d\left(\mathrm{e}^{-\left(\frac{\omega_1(\lambda)}{2}y+\frac{\omega_2(\lambda)}{2}x\right)\widehat{\sigma_3}}M(x,y,\lambda)\right) = \Omega(x,y,\lambda). \quad (4.5)$$

The matrix-valued form Ω is a bounded function of λ, for all (x,y) in the half plane $\{y>0\}$, when $\lambda\in\mathbb{R}^+$ for the elements in the first column, and when $\lambda\in\mathbb{R}^-$ for the elements in the second column.

The spectral analysis

Every matrix solution of (4.5) has unit determinant and, in the domain where it is bounded, satisfies the asymptotic estimate

$$M(x,y,\lambda) = I + \frac{\tilde{M}(x,y)}{\lambda} + O\left(\frac{1}{\lambda^2}\right), \quad |\lambda|\to\infty. \quad (4.6)$$

A particular choice of solutions, in analogy with the solutions (3.6)-(3.7) of the linear case, is

$$M_1(x,y,\lambda) = I + \int_{-\infty}^{x} e^{-\frac{\omega_2(\lambda)}{2}(x-\xi)\widehat{\sigma_3}}(QM_1)(\xi,y,\lambda)d\xi, \quad (4.7)$$

$$M_2(x,y,\lambda) = I - \int_{x}^{\infty} e^{-\frac{\omega_2(\lambda)}{2}(x-\xi)\widehat{\sigma_3}}(QM_2)(\xi,y,\lambda)d\xi, \quad (4.8)$$

We write

$$M_1 = (M_1^-, M_1^+), \qquad M_2 = (M_2^+, M_2^-),$$

where $M_i^{\pm}$, $i=1,2$, indicates a column vector, with $\pm$ denoting that the elements of the vector are functions of λ bounded and analytic in $C^{\pm}$ respectively.
Any two solutions M, M^* of the Lax pair (4.1)-(4.2) are related in their common domain by an expression of the form

$$M^{-1}M^*(x,y,\lambda) = \mathrm{e}^{-(\frac{\omega_1(\lambda)y}{2}+\frac{\omega_2(\lambda)x}{2})\widehat{\sigma_3}}R(\lambda), \quad (4.9)$$

for some $R(\lambda)$ which is a function of λ only. In particular, for $M=M_1$, $M^*=M_2$, we have for $\lambda\in\mathbb{R}$,

$$M_2(x,y,\lambda) = M_1(x,y,\lambda)\mathrm{e}^{-(\frac{\omega_1(\lambda)y}{2}+\frac{\omega_2(\lambda)x}{2})\widehat{\sigma_3}}R(\lambda), \quad \lambda\in\mathbb{R} \quad (4.10)$$

and using that $\lim_{x\to-\infty} M_1(x,0,\lambda) = I$, we find that $R(\lambda)$ is given by

$$R(\lambda) = \lim_{x\to-\infty}\left[\mathrm{e}^{\frac{\omega_2(\lambda)x}{2}\widehat{\sigma_3}}M_2(x,0,\lambda)\right] = I - \int_{-\infty}^{\infty} e^{\frac{\omega_2(\lambda)}{2}\xi\widehat{\sigma_3}}(QM_2)(\xi,0,\lambda)d\xi, \quad \lambda\in\mathbb{R}. \quad (4.11)$$

Since $M_2(x,0,\lambda)$ is defined in terms of the boundary values of q at $y=0$, namely $q(x,0)$ and $q_y(x,0)$, our definition of $R(\lambda)$ depends on these two functions.
Two properties of $R(\lambda)$ follow immediately from this definition:
(1) $\det(R)=1$.
(2) $R(\lambda)$ grows exponentially as $|\lambda|\to\infty$ along any direction off the real axis;
In addition, using the symmetry properties of the matrix $Q(x,0,\lambda)$, which are the same as (2.3), we find
(3) the matrix R is of the form

$$R(\lambda)=\begin{pmatrix} a(\lambda) & b(-\lambda) \\ b(\lambda) & a(-\lambda)\end{pmatrix}, \qquad a(\lambda)a(-\lambda)-b(\lambda)b(-\lambda)=1. \tag{4.12}$$

The global relation

The differential form $\Omega(x,y,\lambda)$ given by (4.5) is exact, and the elements of its first column vector are bounded in λ for $\lambda\in\mathbb{R}^+$, while those of its second column vector are bounded in λ for $\lambda\in\mathbb{R}^-$. Hence for these values of λ, the integral of $\Omega(x,y,\lambda)$ along the boundary of the half plane vanishes:

$$\int_{-\infty}^{\infty} \mathrm{e}^{\frac{\omega_2(\lambda)x}{2}\widehat{\sigma_3}}(QM)(x,0,\lambda)dx=0. \tag{4.13}$$

In particular, choosing $M=M_2$, where M_2 is the solution defined by (4.8), expression (4.13) depends on $q(x,0)$ and $q_y(x,0)$, and becomes the *global relation*:

$$\int_{-\infty}^{\infty} \mathrm{e}^{\frac{\omega_2(\lambda)x}{2}\widehat{\sigma_3}}(QM_2)(x,0,\lambda)dx=0. \tag{4.14}$$

This relation is well defined for $\lambda\in\mathbb{R}^+$ for the first column vector, and for $\lambda\in\mathbb{R}^-$ for the second column vector.

Proof of Proposition 2.1

We now assume that $q(x,y)$ is a solution of (1.1) with the stated properties, and consider the function $R(\lambda)$ defined by (2.5), with $g_0(x)=q(x,0)$, $g_1(x)=q_y(x,0)$. Then equations (2.4) and (4.8) imply that the two functions $m(x,\lambda)$ and $M_2(x,0,\lambda)$ are identical, and the global relation (4.14) then implies that $R(\lambda)$ satisfies

$$R(\lambda)=\begin{cases}\begin{pmatrix} 1 & b(-\lambda) \\ 0 & a(-\lambda)\end{pmatrix} & \lambda\in\mathbb{R}^+ \\ \\ \begin{pmatrix} a(\lambda) & 0 \\ b(\lambda) & 1\end{pmatrix} & \lambda\in\mathbb{R}^-\end{cases} \tag{4.15}$$

Since $\det(R)=1$ we find also that $a(\lambda)=1\ \forall\ \lambda\in\mathbb{R}$. Hence R takes the form (2.6). We conclude that, by definition, the set $\{q(x,0),q_y(x,0)\}$ is admissible.
QED

The Riemann-Hilbert problem

We continue our analysis, under the assumption that q is a solution of (1.1) with appropriate smoothness and decay. Our aim is to give a representation of q in terms of its boundary values. To this end, in this section we use the outcome of the spectral analysis to determine explicitly and to solve a Riemann-Hilbert problem. We then show that q can be expressed in terms of the solution of this problem.
The condition (4.10) and the asymptotic condition (4.6) determine uniquely a matrix Riemann-Hilbert problem on $\mathbb{R}$. Indeed, defining

$$\begin{cases} M_+(x,y,\lambda) = (M_2^+, M_1^+) & \lambda \in \mathbb{C}^+, \\ M_-(x,y,\lambda) = (M_1^-, M_2^-) & \lambda \in \mathbb{C}^-. \end{cases}$$

we find by rearranging (4.10) that

$$M_-(x,y,\lambda) = M_+(x,y,\lambda) J(x,y,\lambda), \qquad \det(M_\pm) = 1,$$

where the jump matrix $J(x,y,\lambda)$ is given by (2.9). Since the global relation implies that $b(\lambda) = 0$ for $\lambda \in \mathbb{R}^+$, the matrix J is either upper or lower triangular for each $\lambda \in \mathbb{R}$.
Rewriting the jump condition, we obtain

$$M_+ - M_- = M_+ - M_+ J - = M_+(I - J) \Rightarrow M_+ - M_- = M_+ \tilde{J} \tag{4.16}$$

where $\tilde{J}$=$I - J$.
The solution of this Riemann-Hilbert problem is now given by a standard Cauchy-type formula, see e.g. [1]. For example, the second column of the solution $M(x,y,\lambda)$ of this RH problem is given by

$$\begin{pmatrix} M_{12}(x,y,\lambda) \\ M_{22}(x,y,\lambda) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{1}{2\pi i} \int_0^\infty \begin{pmatrix} (M_+\tilde{J})_{12}(x,y,\lambda') \\ (M_+\tilde{J})_{22}(x,y,\lambda') \end{pmatrix} \frac{d\lambda'}{\lambda' - \lambda} = \tag{4.17}$$

$$= \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{1}{2\pi i} \int_0^\infty \begin{pmatrix} \big((M_+)_{11}(x,y,\lambda') b(-\lambda') \mathrm{e}^{\theta(x,y,\lambda')} + (M_+)_{12}(x,y,\lambda')\big) \\ \big((M_+)_{21}(x,y,\lambda') b(-\lambda') \mathrm{e}^{\theta(x,y,\lambda')} + (M_+)_{22}(x,y,\lambda')\big) \end{pmatrix} \frac{d\lambda'}{\lambda' - \lambda}.$$

and similarly for the first column (with integral along the negative real axis).

The characterization of $q(x,y)$

To characterize q in terms of the solution of the Riemann-Hilbert problem, we consider the asymptotic estimate (4.6) and let

$$\tilde{M} = \lim_{\lambda \to \infty} (\lambda(M - I)).$$

Substituting (4.6) into the ODE (4.1), we find that the coefficient of λ^0 yields

$$-\frac{i}{4}[\sigma_3, \tilde{M}] = i \frac{q_x - i q_y}{4} \sigma_1 \quad \Longrightarrow \quad q_x - i q_y = 2(\tilde{M})_{21}$$

To obtain an expression in terms of q rather than its derivatives, we consider the coefficient of the $\frac{1}{\lambda}$ term. The (1,1) element of this coefficient yields

$$\cos q(x,y) = 1 + 4i(\tilde{M}_x)_{11} + 2(\tilde{M}_{21})^2. \tag{4.18}$$

These two expressions characterize $q(x,y)$ uniquely.

Proof of Proposition 2.2

Up to now we have assumed that $q(x, y)$ was a suitable solution of equation (1.1) posed in the half plane, and have shown that this function can be represented through the solution of the associated Riemann-Hilbert problem (4.16). The data of this Riemann-Hilbert problem are constructed in terms of the boundary values of q at $y = 0$.
We now start from a pair of such data, assuming that they form an admissible set. Using the spectral funcion $b(\lambda)$ associated with this set, we can determine a Riemann-Hilbert problem as in the statement of the proposition, with $q(x, 0)$ and $q_y(x, 0)$ replaced by the given admissible boundary conditions $g_0(x)$, $g_1(x)$. This Riemann-Hilbert problem is exactly of the form of the one associated starting with a solution of (1.1).
To prove this proposition, it remains to show that the function $q(x, y)$ given by (2.10) can be uniquely defined, and that it satisfies the elliptic sine-Gordon equation and the given boundary conditions. This proof is standard, and follows the lines of Theorem (16.1) of [6]. We do not repeat it here, and only mention that the unique solvability of the given Riemann-Hilbert problem follows from the symmetry properties of Q_0 which ensures the validity of the so-called vanishing lemma. Hence $q(x, y)$ can be defined, and the proof that it satisfies the equation and the boundary conditions is essentially based on the well-known dressing method.
QED

5 Linearisable boundary conditions

Equation (1.1) is of second order in y, and a boundary value problem is well posed in the half plane $y > 0$ when *one* condition is prescribed at $y = 0$. Our main result is proved under the assumption that *both* boundary values $q(x, 0)$ and $q_y(x, 0)$ are prescribed. However, the two data cannot be prescribed independently, and indeed we have to impose the further condition of admissibility.
The next natural step is to investigate whethere it is possible to determine any *linearisable* boundary conditions. These are a special class of boundary conditions, characterized by the fact that the global relation can be solved explicitly for one of the two boundary data when the other one is given, in analogy with what done in section 3 for the linear problem. More specifically, for such boundary conditions the spectral function $b(\lambda)$ that determines the integral representation of the solution $q(x, y)$ can be computed explicitly in terms of the one given boundary condition only. Hence when a linearisable boundary condition is prescribed, the boundary value problem can be solved as effectively as the corresponding linear problem.

It was proved in [8] that linearisable boundary conditions exist for the usual sine-Gordon equation. It is then natural to seek whether they exist in the present case. We follow the approach of [8], and show that the answer in the elliptic case is negative.

Consider the function $\tilde{m}(x, \lambda) = m(x, \lambda)\mathrm{e}^{-\frac{\omega_2(\lambda)}{2}x\sigma_3}$ where m is given by (2.4). This function satisfies the ODE

$$\tilde{m}_x = \left(Q(x, 0, \lambda) - \frac{\omega_2(\lambda)}{2}\sigma_3\right)\tilde{m}, \qquad \lim_{x\to\infty}\left(\mathrm{e}^{\frac{\omega_2(\lambda)}{2}x\sigma_3}\tilde{m}(x, \lambda)\right) = I. \tag{5.1}$$

Let

$$U(x,\lambda) = Q(x,0,\lambda) - \frac{w_2(\lambda)}{2}\sigma_3 = \frac{iq_x + q_y}{4}\sigma_1 - \frac{i}{4\lambda}(\sin q)\sigma_2 + \frac{i}{4}\left(\lambda - \frac{1}{\lambda}\cos q\right)\sigma_3,$$

where q, q_x q_y are evaluated at $y = 0$. It is immediate to verify that $\det(U)$ is a function of λ only through $w_2(\lambda)$. Since $w_2(-\frac{1}{\lambda}) = w_2(\lambda)$, we have $\det(U(-\frac{1}{\lambda})) = \det(U(\lambda))$. Hence it is natural to seek a matrix $T = T(\lambda)$, a function of λ *only*, satisfying

$$U(x,-\frac{1}{\lambda})T(\lambda) = T(\lambda)U(x,\lambda), \quad x \in \mathbb{R}. \tag{5.2}$$

If such a matrix $T(\lambda)$ exists, then it must also satisfy

$$\tilde{m}(x,-\frac{1}{\lambda})T = T\tilde{m}(x,\lambda), \quad \forall x \in \mathbb{R}. \tag{5.3}$$

Since

$$b(\lambda) = \left[\lim_{x\to-\infty}\left(e^{\frac{\omega_2(\lambda)}{2}x\sigma_3}\tilde{m}(x,\lambda)\right)\right]_{21}$$

the relation (5.3), in the limit as $x \to -\infty$, implies a relation between $b(\lambda)$ and $b(-\frac{1}{\lambda})$. This relation, and the global relation, could then be used to express $b(\lambda)$ is terms of only one boundary condition.
Thus we have reduced the problem to finding a matrix T that satisfies the condition (5.2). Imposing this condition, we find (up to multiples)

$$T = T(x,\lambda) = \begin{pmatrix} 1 & \frac{1+\lambda^2}{1-\lambda^2}\frac{1-\cos q(x,0)}{i\sin q(x,0)} \\ \frac{1+\lambda^2}{1-\lambda^2}\frac{1-\cos q(x,0)}{i\sin q(x,0)} & 1 \end{pmatrix} \tag{5.4}$$

The matrix-valued function T is independent of x only if $q(x,0) = constant$. However, the decay requirement on $q(x,0)$ (mod 2π) as x tends to infinity then implies that $q(x,0) = 2\pi m$, for some $m \in \mathbb{Z}$, and the unique solution of the problem is the zero solution (mod 2π). Hence there are no nontrivial linearisable conditions associated with the Lax pair (4.1)-(4.2).

Remark 5.1 Linearisable conditions are associated with a specific choice of Lax pair, and their construction could be based on some other Lax pair. The Lax pair used in [10] is equivalent to the one we use here, and yields the same conclusion. However it can be shown that equation (1.1) admits also a different, alternative Lax pair, analogous to the one used in [8] for the usual sine-Gordon equation. However, the analysis analogous to the one performed above yields that no non-trivial linearisable boundary conditions exist in connection with this Lax pair either, essentially because the decay constraint similarly implies that such conditions must vanish (mod 2π). The alternative Lax pair and the details of the computation, for the more difficult case of boundary value problems posed on a semistrip, are given in [9].

6 Conclusions

In this paper, we have used the Fokas method to study the elliptic sine-Gordon equation (1.1) posed on the half plane $y \geq 0$. In particular, we have corrected and extended the

results of [10], finding a representation of the solution of (1.1) under the assumption that all boundary values are prescribed in such a way that the global relation is satisfied.
Two properties of this problem are worthy of mention, as they appear to be specific to this model and differ from the analogous properties of the usual sine-Gordon equation, which describes a time evolution process.
Firstly, our results imply that the "nonreflecting"case $b(\lambda) = 0$ considered in [10] is not compatible with any choice of admissible boundary conditions. It is natural to consider the "nonreflecting" case in the context of evolution equations, when it is then possible to compute explicitly pure soliton solutions. The formal computation can be performed for the abstract Riemann-Hilbert problem defined in this paper, as done in [10], but it does not correspond to a well-posed boundary value problem of the kind examined here (indeed, the explicit soliton solution obtained in [10] does not decay in all directions). Assuming that the solution decays (mod 2π), we have shown that, for admissible boundary values, $a(\lambda) = 1$, $\lambda \in \mathbb{R}$, and if in addition $b(\lambda) = 0$ then the Riemann-Hilbert problem is trivial ($J = I$) and $M = I$, implying that $q(x, y) = 0$ (mod 2π). This is a significant property, to our knowledge previously undetected, specific to the present problem.
Secondly, it appears that it is not possible to prescribe boundary conditions for which the solution representation is explicit. Such conditions exist for other integrable evolution PDEs (the NLS and KdV equations, as well as sine-Gordon) and are called *linearisable*. In the present case, the conjugating matrices that characterize linearisable boundary conditions are trivial. We showed this here for the construction associated with the Lax pair (4.1)-(4.2). However, it can be shown that the same holds true when basing the construction of linearisable boundary conditions on the alternative Lax pair used in [8]. Hence it appears that no linearisable boundary conditions can be prescribed for the elliptic sine-Gordon equation posed on a half plane.

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