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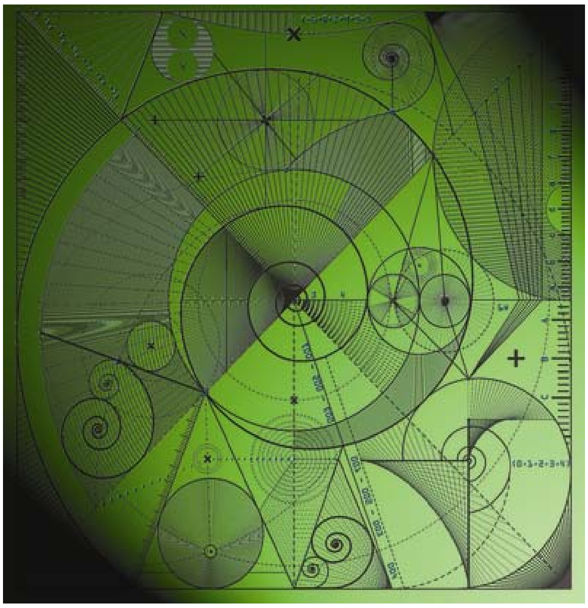

Flow-Dependent Balance Conditions for Incremental Data Assimilation: Elliptic Operators

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Abstract

We present a family of flow-dependent balance conditions for variational data assimilation (VAR). In the context of shallow water theory, the conditions are based upon using either relative vorticity or potential vorticity as the balanced variable, and incorporate both rotational and divergent components of the horizontal wind field. From this wind field we obtain a balanced height field, which can be used as a control variable in VAR. The balance conditions are defined using nonlinear Monge Ampère equations, which are required to be elliptic. A linear, incremental scheme is derived, and the ellipticity criterion becomes a condition on the linearization state. The schemes are tested using Rossby-Haurwitz waves for a range of Burger numbers, thereby enabling flow-dependency to be analyzed.

Keywords: Ellipticity; Monge-Ampère Equations; Linearisation; Rossby-Haurwitz Waves

1 Introduction

The importance of projecting information onto balanced states in the assimilation scheme of any numerical weather prediction (NWP) model is fundamental, and generating spurious interia-gravity waves is to be avoided. As pointed out by Lorenc (2003, §3(b)), physical arguments are frequently used to select a set of variables (so-called 'control variables' in NWP) for use in the assimilation schemes, which decompose states into balanced and unbalanced components. This strategy enables the background error covariances to be more readily applied than would be the case if, for example, the model variables such as horizontal momentum and pressure were used. The essence of the algorithms followed by many centres (see Lorenc *op. cit* and references therein) is to choose one new variable to represent the slowly-evolving, or balanced (Rossby wave) component of the flow, and two other variables to represent the unbalanced component of the flow (i.e. the gravity waves). Three such variables are sufficient to specify the initial conditions for a model based on the dry hydrostatic primitive equations.

Naturally, there are many possible choices of variables that we can employ; for example (see Section 2.1), vorticity (or stream function), divergence (or velocity potential) and an unbalanced pressure (obtained as a residual from a linear balance equation). In this decomposition the vorticity is considered as representing the balanced component of the flow, while the divergence and unbalanced pressure are considered to span the portion of phase space populated by high-frequency motions. A shortcoming of this choice of variables is that they correspond to a purely kinematic decomposition of the state, with all divergent motion being treated as unbalanced.

Representing the balanced component of the flow by the vorticity is certainly a reasonable first approximation for most global and mesoscale models, but it ignores the fact that a portion of the balanced flow may reside in the divergence. This may be especially important when attempting to relate increments in the horizontal momentum to increments in the vertical motion and vice-versa, which is an important coupling when moisture *is* considered.

In this paper, we introduce a method for deriving a set of variables for use in data assimilation in which the balanced divergence is incorporated into the single variable representing the slowly-evolving components of the flow. Our method involves elliptic partial differential equations (pdes) of Monge-Ampère type, and we investigate the conditions under which they are amenable to solution. An advantage of our scheme is that these pdes have variable coefficients, and consequently our method incorporates some elements of flow-dependency into the data assimilation.

In Section 2 we give a brief review of the current method for decomposing the flow into a balanced variable and an unbalanced variable as set out by Lorenc *et al* (2000). Following this review we provide a clear explanation of the Hamiltonian framework and the balanced relationships that arise from this approach (McIntyre and Roulstone (1996) and (2002), MR96 and MR02 hereafter). We conclude this section with an outline of the non-linear balance equations that arise from the Hamiltonian approach and derive the sets of 'control variables' that are naturally associated with these balance conditions.

Section 3 sets out the non-linear balance equations in terms of spherical geometry. We introduce the condition for the non-linear pdes to be elliptic (Courant and Hilbert 1962), and show that this condition gives rise to flow-dependency. One of the motivations for this work is to introduce higher order balance conditions into an incremental variational data assimilation scheme, VAR. The incremental version of VAR uses linear equations for small increments to the state variables of the model under consideration. In most of the operational numerical weather prediction centres a version of incremental variational data assimilation is used (Rawlins *et al.*, 2007), and so with this in mind in Section 4 we introduce an increment to our balanced variable and re-derive the balanced equations in terms of a small increment. We then have to consider the ellipticity condition for linear pdes, which is also presented in Section 4, and show an interesting result linking the linear pdes' ellipticity condition to that of the non-linear's pdes' conditions.

In Section 5 we perform a scale analysis for both the ellipticity condition and the terms in the linear pdes to explore the impact of the flow dependency with different Rossby-Haurwitz waves (Williamson *et al.*, 1992) which are parameterised by the Burger number. The values come from the Met Office's 2D shallow water equations model in a non-inertial framework on the sphere (Malcolm 1996).

2 Balance and Decompositions into Balanced and Unbalanced Variables

The aim of this section is to give an overview firstly of the current method for decomposing the horizontal wind fields into two new uncorrelated variables, Section 2.1, and secondly to provide an outline of the theory which enables us to define an alternative form for the balanced variable and the associated decomposition, Section 2.2, which comes from MR96 and MR02. The final subsection, Section 2.3, gives a brief outline of the alternative balanced wind field and the non-linear decomposition in coordinate free vector form.

2.1 A Decomposition Based on the Kinematics of Vorticity and Divergence

The current method in use at the Met Office is outlined in Lorenc *et al.* (2000). There they describe how the 'control variables' (that is, the variables used in the assimilation scheme as opposed to the variables used in the NWP model) are defined in terms of a Helmholtz decomposition of the horizontal momentum together with a linear balance condition. Their basic strategy is to define a (projective) transformation $(u, v, p, \rho) \mapsto (\psi, \chi, {}^{A}p)$, where the variables on the left are the two horizontal components of the wind field, the pressure, and the density, and the three variables on the right are a stream function, a velocity potential and an 'unbalanced' pressure. Then the horizontal wind field, $\mathbf{u}$, is decomposed into a balanced and an 'unbalanced' variable through the pair of Poisson equations

$$\xi \equiv \mathbf{k} \cdot \nabla \times \mathbf{u} = \nabla^2 \psi, \tag{1a}$$

$$\delta \equiv \nabla \cdot \mathbf{u} = \nabla^2 \chi, \tag{1b}$$

where $\mathbf{k}$ is the local unit vertical vector, ξ is the relative vorticity, δ is the divergence and ∇ is the gradient operator in the horizontal.

The unbalanced pressure, ${}^{A}p$, is defined by subtracting the pressure, p, calculated by solving the linear balance equation

$$\nabla \cdot (f\mathbf{k} \times \mathbf{u}) + \nabla \cdot (\gamma \nabla p) = 0 \tag{2}$$

for p given $\mathbf{u}$ and γ, from the total pressure. In (2), $\gamma = \rho^{-1}$ is the specific volume, which is usually approximated by $\gamma = \gamma(z)$ so that (2) is a constant-coefficient equation on each level, and f is the Coriolis parameter, defined as $f = 2\Omega \sin\theta$, where Ω is the Earth's rotation rate and θ is the angle of latitude such that $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. The inverse transformation involves the Helmholtz decomposition on each level

$$\mathbf{u} \equiv \mathbf{k} \times \nabla\psi + \nabla\chi \equiv \mathbf{u}_r + \mathbf{u}_d, \tag{3}$$

where $\mathbf{u}_r$ is the 'balanced' rotational part of the wind field and $\mathbf{u}_d$ is the 'unbalanced' divergent part, and then a 'balanced' pressure is again calculated from (2), and added to the unbalanced control variable ${}^{A}p$.

2.2 Balance Relations

In McIntyre and Roulstone (1996, 2002) an alternative Hamiltonian framework is derived for studying balanced models in atmosphere and ocean dynamics. One of their motivations is to study conservation laws and, in particular, models that conserve potential vorticity. They find that a simple balance relation, which can be thought of as a generalisation of geostrophic balance, can be defined and, using this definition, it is then easy to see how a hierarchy of conservation laws for potential vorticity for three different balanced models are related to one another. In the context of shallow water theory (see, for example, Rossby 1940), the potential vorticity, q, is defined in terms of the depth, $h(\mathbf{x}, t)$ and fluid velocity, $\mathbf{u}(\mathbf{x}, t)$, as

$$q = \frac{1}{h}\left(f + \nabla \times \mathbf{u}\right). \tag{4}$$

McIntyre and Roulstone noted that if we work on an f-plane and define a vector wind, which we shall write in the form

$$\mathbf{u}^b = \mathbf{u}_g - \frac{\alpha}{f}\left(\mathbf{u}_g \cdot \nabla\right)\left(\mathbf{k} \times \mathbf{u}_g\right), \tag{5}$$

where $\mathbf{u}_g = (g/f)\mathbf{k} \times \nabla h$ is the geostrophic wind, then substituting $\mathbf{u}^b$ for $\mathbf{u}$ in (4) yields the following expression for the potential vorticity

$$q^b = \frac{1}{h}\left(f + \frac{g}{f}\nabla^2 h + \frac{ag^2}{f^3}\mathrm{Hess}(h)\right). \tag{6}$$

The term $\mathrm{Hess}(h)$ is the determinant of the Hessian which is the matrix of the second order derivatives of h.

In (5), α is a real number, and is related to a in (6) by $a = -2\alpha$. If $\alpha = 0$, then $a = 0$ and the potential vorticity, q^b, involves the leading order contributions from the planetary rotation and from geostrophy. If $\alpha = -\frac{1}{2}$, then $a = 1$, and q^b is the expression for the potential vorticity for the semi-geostrophic equations for shallow water flow. If $\alpha = 1$ then $a = -2$ and the expression for the potential vorticity corresponds to Charney-Bolin, or nonlinear, balance.

Therefore, the generalisation of geostrophy defined by (5), can be used to define a balanced wind field that has a definite interpretation in terms of a potential vorticity. McIntyre and Roulstone interpreted $\mathbf{u}^b$ as a so-called constraint in their Hamiltonian models, and the reader is referred to their papers for more details on this topic. It is sufficient for our purposes to interpret this vector field in the way we have illustrated above.

2.3 Alternative Decompositions into Balanced and Unbalanced Variables

In this section we present non-linear balance relations derived from the balanced wind field, $\mathbf{u}^b$, for the height field given either a relative vorticity, RV hereafter, or a potential vorticity, PV hereafter. The reason for choosing these two variables is that it is known that the RV is a balanced variable in the atmosphere. The RV is also used in the current decomposition to define the balanced stream function. See Wlasak (2002), Wlasak *et al.* (2006), for discussion of PV as a control variable.

We firstly consider the RV approach, (1a), but instead of inserting the full wind field, $\mathbf{u}$, we substitute the balanced wind field, $\mathbf{u}^b$. The result is

$$\xi^b \equiv \mathbf{k} \cdot \nabla \times \mathbf{u}^b = \mathbf{k} \cdot \nabla \times \left(\mathbf{u}_g - \frac{\alpha}{f} \left(\mathbf{u}_g \cdot \nabla \right) \left(\mathbf{k} \times \mathbf{u}_g \right) \right), \tag{7}$$

where ξ^b is the balanced RV associated with this balanced wind field.

If we consider the PV approach, we use the definition given in Section 2.2 for q^b,

$$q^b \equiv \frac{f + \xi^b}{h} = \frac{f + \mathbf{k} \cdot \nabla \times \left(\mathbf{u}_g - \frac{\alpha}{f} \left(\mathbf{u}_g \cdot \nabla \right) \left(\mathbf{k} \times \mathbf{u}_g \right) \right)}{h}. \tag{8}$$

Solutions of equations (7) or (8) can be used to represent the balanced 'control variable', but we still have to determine the two unbalanced components. One way would be to evaluate (5) with the balanced height field that arises from inverting either (7) or (8). Given the balanced wind field, $\mathbf{u}^b$, we can then calculate the residual unbalanced component of the wind field, $\mathbf{u}^s$, as indicated in MR96, MR02, by

$$\mathbf{u}^s = \mathbf{u} - \mathbf{u}^b. \tag{9}$$

From this residual we apply a Helmholtz decomposition, similar to that currently done, to find the balanced and unbalanced variables to calculate a form of unbalanced stream function, ψ^s, and an unbalanced velocity potential, χ^s. This pair of equations would be

$$\xi^s \equiv \mathbf{k} \cdot \nabla \times \mathbf{u}^s = \nabla^2 \psi^s, \tag{10}$$

$$\delta^s \equiv \nabla \cdot \mathbf{u}^s = \nabla^2 \chi^s, \tag{11}$$

where ξ^s is the residual vorticity and δ^s is the residual divergence.

3 Balance equations in spherical geometry

The main aim of this section is to address how (7) and (8) appear in spherical coordinates if we are to consider these new balance equations for an operational data assimilation system. In the next subsection we briefly summarise the component form of the vector differential operators in spherical coordinates. From these definitions we expand $\mathbf{u}_g = (g/f)\mathbf{k} \times \nabla h$ into spherical coordinates along with $\mathbf{u}^b$ and hence (7) and (8). The resulting non-linear balance equations in spherical coordinates are elliptic non-linear pdes and therefore there is a condition for a solution to exisit. In Section 3.3 we outline the theory for the solutions to exist and derive this associated condition.

3.1 Vector Operators

The operators we require are as follows:

$$\nabla h = \left(\frac{1}{a\cos\theta}\frac{\partial h}{\partial \lambda}, \quad \frac{1}{a}\frac{\partial h}{\partial \theta} \right)^T,$$

$$\nabla \cdot \mathbf{u} \equiv \left(\frac{1}{a\cos\theta}\frac{\partial u}{\partial \lambda} - \frac{1}{a}\frac{\partial \cos\theta v}{\partial \theta} \right),$$

and

$$\nabla^2 h = \frac{1}{a^2\cos^2\theta}\frac{\partial^2 h}{\partial \lambda^2} + \frac{1}{a^2}\frac{\partial^2 h}{\partial \theta^2} - \frac{1}{a^2}\tan\theta\frac{\partial h}{\partial \theta}.$$

The $\mathbf{k}$th component of the curl is defined by

$$\mathbf{k} \cdot \nabla \times \mathbf{u} \equiv \frac{1}{a\cos\theta}\left(\frac{\partial v}{\partial \lambda} - \frac{\partial}{\partial \theta}(\cos\theta u) \right).$$

In all of the definitions a is the radius of the Earth, $6371220m$ and λ is the angle of longitude, $\lambda \in [0, 2\pi)$.

An important difference between the Cartesian coordinates and the spherical counterparts is the differentiation of the directional unit vectors with respect to the spherical coordinates (Bachelor 1967). The two we require are

$$\frac{\partial \mathbf{i}}{\partial \lambda} = \sin\theta\mathbf{j}, \quad \frac{\partial \mathbf{j}}{\partial \lambda} = -\sin\theta\mathbf{i}. \tag{12}$$

3.2 Higher order balance equations

The two components of the geostrophic wind are thus

$$u_g = -\frac{g}{af}\frac{\partial h}{\partial \theta}, \quad \text{and} \quad v_g = \frac{g}{af\cos\theta}\frac{\partial h}{\partial \lambda}. \tag{13}$$

Recalling the definition for $\mathbf{u}^b$,

$$\mathbf{u}^b \equiv \mathbf{u}_g - \frac{\alpha}{f}\left(\mathbf{u}_g \cdot \nabla\right)\left(\mathbf{k} \times \mathbf{u}_g\right),$$

and expanding using the operators defined earlier gives

$$u^b \equiv u_g - \frac{\alpha}{f}\left(\frac{u_g}{a\cos\theta}\frac{\partial v_g}{\partial \lambda} + \frac{v_g}{a}\frac{\partial v_g}{\partial \theta} + \frac{\tan\theta}{a}u_g^2\right), \tag{14}$$

$$v^b \equiv v_g + \frac{\alpha}{f}\left(\frac{u_g}{a\cos\theta}\frac{\partial u_g}{\partial \lambda} + \frac{v_g}{a}\frac{\partial u_g}{\partial \theta} - \frac{\tan\theta}{a}u_g v_g\right), \tag{15}$$

where the terms involving $\tan\theta$ occur as a result of the metric terms, (12).

We now expand the geostrophic wind components in (14) and (15) using (13), to give

$$\begin{aligned} u^b &= -\frac{g}{af}\frac{\partial h}{\partial \theta} - \frac{g^2\alpha}{a^3 f^3 \cos^2\theta}\left(\frac{\partial h}{\partial \lambda}\frac{\partial^2 h}{\partial \theta \partial \lambda} - \frac{\partial h}{\partial \theta}\frac{\partial^2 h}{\partial \lambda^2}\right. \\ &+ \tan\theta\left(\left(\frac{\partial h}{\partial \lambda}\right)^2 + \cos^2\theta\left(\frac{\partial h}{\partial \theta}\right)^2\right) \\ &- \left.\frac{\beta}{f}\left(\frac{\partial h}{\partial \lambda}\right)^2\right), \end{aligned} \tag{16}$$

$$\begin{aligned} v^b &= \frac{g}{af\cos\theta}\frac{\partial h}{\partial \lambda} + \frac{g^2\alpha}{a^3 f^3 \cos\theta}\left(\frac{\partial h}{\partial \theta}\frac{\partial^2 h}{\partial \theta \partial \lambda}\right. \\ &- \left.\frac{\partial h}{\partial \lambda}\frac{\partial^2 h}{\partial \theta^2} + \tan\theta\frac{\partial h}{\partial \theta}\frac{\partial h}{\partial \lambda} + \frac{\beta}{f}\frac{\partial h}{\partial \lambda}\frac{\partial h}{\partial \theta}\right), \end{aligned} \tag{17}$$

where

$$\beta \equiv \frac{df}{d\theta}. \tag{18}$$

The final step to arrive at the non-linear pdes depends on which variable we choose for the balanced component. If we use the RV then we obtain the following Monge-Ampère equation for h

$$\begin{aligned}
\xi^b &= \frac{g}{f}\nabla^2 h + \frac{g\beta}{f^2a^2}\frac{\partial h}{\partial \theta} + \frac{2g^2\alpha}{f^3a^4\cos^2\theta}\left(\left(\frac{\partial^2 h}{\partial\theta\partial\lambda}\right)^2\right. \\
&- \frac{\partial^2 h}{\partial\lambda^2}\frac{\partial^2 h}{\partial\theta^2} + 2\tan\theta\frac{\partial h}{\partial\lambda}\frac{\partial^2 h}{\partial\theta\partial\lambda} + \tan^2\theta\left(\frac{\partial h}{\partial\lambda}\right)^2 \\
&+ \sin\theta\cos\theta\frac{\partial h}{\partial\theta}\frac{\partial^2 h}{\partial\theta^2} \\
&+ \frac{1}{2}\left(\left(\frac{\partial h}{\partial\lambda}\right)^2 + \cos^2\theta\left(\frac{\partial h}{\partial\theta}\right)^2\right) \\
&+ \frac{\beta}{f}\left(2\frac{\partial h}{\partial\theta}\frac{\partial^2 h}{\partial\lambda^2} - 2\frac{\partial h}{\partial\lambda}\frac{\partial^2 h}{\partial\lambda\partial\theta} - 2\tan\theta\left(\frac{\partial h}{\partial\lambda}\right)^2\right. \\
&+ \left.2\frac{\beta}{f}\left(\frac{\partial h}{\partial\lambda}\right)^2 - \frac{3}{2}\sin\theta\cos\theta\left(\frac{\partial h}{\partial\theta}\right)^2\right) \\
&- \left.\frac{1}{2f}\frac{\partial\beta}{\partial\theta}\left(\frac{\partial h}{\partial\lambda}\right)^2\right).
\end{aligned} \tag{19}$$

An important feature of (19) is that all higher order derivatives of h – which enter when differentiating the balanced wind fields (14) and (15) – cancel above second order. Equation (19) is the full version of the pde at the heart of our scheme; in Section 5 we shall perform a scale analysis to see when it is possible to remove certain terms.

If we consider that the balanced component of the flow is determined by the PV instead of the relative vorticity, then we obtain another Monge-Ampère equation of the form

$$q^b = \frac{1}{h}\left(f + \xi^b\right) \tag{20}$$

Both equations (19) and (20) are non-linear elliptic pdes and as such there are certain conditions for their solutions to exist. In the following subsection we introduce the conditions for the solutions to (19) and (20) to exist, and show that these conditions are flow dependent.

3.3 Ellipticity Theory: Part I

The condition that we use here comes from Houghton (1968), which is for spherical coordinates. The Cartesian version can be found in Courant and Hilbert (1962). To derive the spherical version of this solvability condition we use the change of variable as set out in Houghton (1968).

The general definition of a Monge-Ampère equation is

$$A + B\rho + 2C\nu + D\mu + E\left(\rho\mu - \nu^2\right) = 0, \tag{21}$$

where

$$\rho = \frac{\partial^2 h}{\partial \lambda^2}, \qquad \mu = \frac{\partial^2 h}{\partial \theta^2}, \qquad \nu = \frac{\partial^2 h}{\partial \lambda \partial \theta}. \tag{22}$$

The coefficients A, B, C, D and E in (21) are given functions of $(\lambda, \theta, h, p, q)$, where $p = \partial h/\partial \lambda,\ q = \partial h/\partial \theta$.

The classification of (19) and (20) is determined by transforming the pde into its canonical form. This then determines the characteristics of the pde. For elliptic problems we require the characteristics to be complex and therefore

$$BD - C^2 - AE > 0. \tag{23}$$

The above condition is the ***ellipticity condition***, (Courant and Hilbert 1962).

As in Houghton (1968), we let $x = \lambda$ and $y = \theta$ and then for equation (19) we have

$$\begin{aligned}
A &= -\xi^b - \frac{\tan\theta g}{a^2 f}\frac{\partial h}{\partial \theta} + \frac{2g^2\alpha}{f^3 a^4 \cos^2\theta}\tan^2\theta\left(\frac{\partial h}{\partial \lambda}\right)^2 \\
&+ \frac{g^2\alpha}{f^3 a^4 \cos^2\theta}\left(\left(\frac{\partial h}{\partial \lambda}\right)^2 + \cos^2\theta\left(\frac{\partial h}{\partial \theta}\right)^2\right) \\
&+ \frac{4\beta^2\alpha g^2}{f^5 a^4 \cos^2\theta}\left(\frac{\partial h}{\partial \lambda}\right)^2 - \frac{4\beta\alpha g^2 \tan\theta}{f^4 a^4 \cos^2\theta}\left(\frac{\partial h}{\partial \lambda}\right)^2 \\
&- \frac{3\beta\alpha g^2 \tan\theta}{f^4 a^4 \cos^2\theta}\left(\frac{\partial h}{\partial \theta}\right)^2 - \frac{\partial \beta}{\partial \theta}\frac{\alpha g^2}{f^4 a^4 \cos^2\theta}\left(\frac{\partial h}{\partial \lambda}\right)^2, \\
&+ \frac{g\beta}{f^2 a^2}\frac{\partial h}{\partial \theta} \\
B &= \frac{g}{f a^2 \cos^2\theta} + \frac{4\beta\alpha g^2}{f^4 a^4 \cos^2\theta}\frac{\partial h}{\partial \theta}, \\
C &= \frac{2g^2\alpha \tan\theta}{f^3 a^4 \cos^2\theta}\frac{\partial h}{\partial \lambda} - \frac{2\beta g^2 \alpha}{f^4 a^4 \cos^2\theta}\frac{\partial h}{\partial \lambda}, \\
D &= \frac{g}{f a^2} + \frac{2g^2\alpha}{f^3 a^4}\tan\theta\frac{\partial h}{\partial \theta}, \\
E &= -\frac{2g^2\alpha}{f^3 a^4 \cos^2\theta}.
\end{aligned}$$

Recalling the definition of the geostrophic winds, (13), the ellipticity condition is

$$\begin{aligned}
&\frac{f}{2} + \frac{\alpha^2}{f a^2}\left(v_g^2 + u_g^2\right) - \frac{3\beta\alpha}{f a}u_g + \frac{\beta\alpha^2 \tan\theta}{f^2 a^2}u_g^2 \\
&\qquad + \left(\frac{2\beta^2\alpha}{f^3 a^2} - \frac{\partial \beta}{\partial \theta}\frac{2\alpha^2}{f^2 a^2}\right)v_g^2 > \alpha\xi^b.
\end{aligned} \tag{24}$$

The condition in (24) is true for all choices of α as at no point have we divided by α. It is clear to see that for the geostrophic case, $\alpha = 0$, the ellipticity condition is that for a Poisson equation, which is always satisfied since $1 > 0$, and therefore (20) with $\alpha = 0$ is robustly elliptic.

For the PV balance equation, (20), the coefficients are similar to those for (19), with a small modification to A. As a consequence of the additional terms in A, the ellipticity condition for (20) becomes

$$\frac{(1+2\alpha)\,f}{2} + \frac{\alpha^2}{fa^2}\left(v_g^2 + u_g^2\right) - \frac{3\beta\alpha}{fa}u_g + \frac{\beta\alpha^2\tan\theta}{f^2a^2}u_g^2 + \left(\frac{2\beta^2\alpha}{f^3a^2} - \frac{\partial\beta}{\partial\theta}\frac{2\alpha^2}{f^2a^2}\right)v_g^2 > \alpha h q^b = \alpha\left(f + \xi^b\right).$$

Due to the αf terms being present on both sides of the equation we therefore have the same condition for the PV as we had for the RV approach.

We also require boundary conditions to ensure that the elliptic equations have a solution. For these problems on the sphere we have periodicity of h in the λ direction, whilst in the θ direction there is a choice of conditions. The first is that because the λ direction is not defined at the poles, we cannot have any change in h in the direction of λ, that is, $\partial h/\partial\lambda = 0$. An alternative is to assume periodicity of h with respect to θ across the poles. Given these boundary conditions it follows that as long as the ellipticity condition (24) is satisfied, then there exist solutions to equations (19) and (20).

Most operational weather prediction centres use an incremental variational data assimilation scheme, such as the one developed at the Met Office (Rawlins *et al.*, 2007). For the higher order balance conditions, (19) and (20), to be of use in an incremental VAR scheme we require linear equations for perturbations. In the next section we address how to linearise all of the non-linear relations that we have used to derive (19) and (20), and we derive the associated higher order linear balance equations.

4 Linearisation

This section is comprised of two parts. The first part deals with the consequences of introducing a linearisation to the height field, and hence to $\mathbf{u}^b$ and the balance equations (19) and (20). The second part deals with the ellipticity conditions for the linear pdes. There is an interesting feature of the ellipticity conditions for the linear versions of (19) and (20), which relates the base state we choose to linearise about to the full non-linear pdes, and we discuss this at the end of this section.

4.1 Higher Order Linear Balance Equations

We seek a balanced height increment, h'. This is accomplished by introducing a base state height, $\bar{h}$, such that $h = \bar{h} + h'$. As a consequence of this the geostrophic winds can be defined as $\mathbf{u}_g = \bar{\mathbf{u}}_g + \mathbf{u}'_g$, or

$$\bar{u}_g = -\frac{g}{fa}\frac{\partial \bar{h}}{\partial \theta}, \qquad u'_g = -\frac{g}{fa}\frac{\partial h'}{\partial \theta}, \tag{25}$$

$$\bar{v}_g = \frac{g}{fa\cos\theta}\frac{\partial \bar{h}}{\partial \lambda}, \qquad v'_g = \frac{g}{fa\cos\theta}\frac{\partial h'}{\partial \lambda}. \tag{26}$$

This then enables (5) to be linearised as

$$\mathbf{u}^{b\prime} \equiv \mathbf{u}'_g - \frac{\alpha}{f}\left(\left(\bar{\mathbf{u}}_g \cdot \nabla\right)\left(\mathbf{k} \times \mathbf{u}'_g\right) + \left(\mathbf{u}'_g \cdot \nabla\right)\left(\mathbf{k} \times \bar{\mathbf{u}}_g\right)\right). \tag{27}$$

In spherical coordinates, the component form for (27) is

$$\begin{aligned} u^{b\prime} = u'_g &- \frac{\alpha}{f}\left(\frac{\bar{u}_g}{a\cos\theta}\frac{\partial v'_g}{\partial \lambda} + \frac{u'_g}{a\cos\theta}\frac{\partial \bar{v}_g}{\partial \lambda} + \frac{\bar{v}_g}{a}\frac{\partial v'_g}{\partial \theta} \right.\\ &\left. + \frac{v'_g}{a}\frac{\partial \bar{v}_g}{\partial \theta} + \frac{2\tan\theta}{a}u'_g\bar{u}_g\right) - 2\frac{\alpha\beta}{f^2}\bar{v}_g v'_g, \end{aligned} \tag{28}$$

$$\begin{aligned} v^{b\prime} = v'_g &+ \frac{\alpha}{f}\left(\frac{\bar{u}_g}{a\cos\theta}\frac{\partial u'_g}{\partial \lambda} + \frac{u'_g}{a\cos\theta}\frac{\partial \bar{u}_g}{\partial \lambda} + \frac{\bar{v}_g}{a}\frac{\partial u'_g}{\partial \theta} \right.\\ &\left. + \frac{v'_g}{a}\frac{\partial \bar{u}_g}{\partial \theta} - \frac{\tan\theta}{a}\left(u'_g\bar{v}_g + \bar{u}_g v'_g\right)\right) \\ &- \frac{\alpha\beta}{f^2}\left(\bar{u}_g v'_g + u'_g\bar{v}_g\right). \end{aligned} \tag{29}$$

The full expressions are given here to highlight the fact that this alternative approach is flow-dependent. An important feature of the expression (29) is that if the geostrophic winds are changing over short distances, then their derivatives are large and could make the extra terms, apart from the Laplacian, significant and these should therefore not be ignored.

If we compute the divergence of the balanced wind field, we find

$$\begin{aligned} \nabla \cdot \mathbf{u}^{b\prime} &\equiv \nabla \cdot \left(\mathbf{u}'_g - \frac{\alpha}{f}\left(\left(\bar{\mathbf{u}}_g \cdot \nabla\right)\left(\mathbf{k} \times \mathbf{u}'_g\right)\right.\right. \\ &= -\frac{\alpha}{f}\left(\left(\bar{\mathbf{u}}_g \cdot \nabla\right)\xi'_g + \left(\mathbf{u}'_g \cdot \nabla\right)\bar{\xi}_g\right), \end{aligned}$$

and we note that $\mathbf{u}^{b\prime}$ contains a balanced divergent component. (This can also be easily verified for the full field version as well.)

To derive the linearised versions of (19) we substitute $\mathbf{u}^{b\prime}$ for $\mathbf{u}$ in (1a). The resulting linear pde for the balanced height increment h' is

$$
\begin{aligned}
\xi^{b\prime} \equiv \frac{g}{f}\nabla^2 h' + \frac{g\beta}{f^2 a^2}\frac{\partial h'}{\partial \theta} + \frac{2g^2\alpha}{a^4 f^3 \cos^2\theta}\Bigg(& 2\frac{\partial^2 \bar{h}}{\partial\theta\partial\lambda}\frac{\partial^2 h'}{\partial\theta\partial\lambda} \\
& - \frac{\partial^2 \bar{h}}{\partial\lambda^2}\frac{\partial^2 h'}{\partial\theta^2} - \frac{\partial^2 \bar{h}}{\partial\theta^2}\frac{\partial^2 h'}{\partial\lambda^2} + 2\tan\theta\frac{\partial^2 \bar{h}}{\partial\theta\partial\lambda}\frac{\partial h'}{\partial\lambda} \\
& + 2\tan\theta\frac{\partial \bar{h}}{\partial\lambda}\frac{\partial^2 h'}{\partial\theta\partial\lambda} + 2\tan^2\theta\frac{\partial \bar{h}}{\partial\lambda}\frac{\partial h'}{\partial\lambda} + \cos^2\theta\frac{\partial \bar{h}}{\partial\theta}\frac{\partial h'}{\partial\theta} \\
& + \frac{\partial \bar{h}}{\partial\lambda}\frac{\partial h'}{\partial\lambda} + \sin\theta\cos\theta\left(\frac{\partial^2 \bar{h}}{\partial\theta^2}\frac{\partial h'}{\partial\theta} + \frac{\partial \bar{h}}{\partial\theta}\frac{\partial^2 h'}{\partial\theta^2}\right) \\
& + \frac{\beta}{f}\Bigg(2\frac{\partial \bar{h}}{\partial\theta}\frac{\partial^2 h'}{\partial\lambda^2} + 2\frac{\partial h'}{\partial\theta}\frac{\partial \bar{h}}{\partial\lambda^2} - 2\frac{\partial \bar{h}}{\partial\lambda}\frac{\partial^2 h'}{\partial\lambda\partial\theta} \\
& - 2\frac{\partial h'}{\partial\lambda}\frac{\partial^2 \bar{h}}{\partial\lambda\partial\theta} + \ 4\frac{\beta}{f}\frac{\partial \bar{h}}{\partial\lambda}\frac{\partial h'}{\partial\lambda} - 3\sin\theta\cos\theta\frac{\partial \bar{h}}{\partial\theta}\frac{\partial h'}{\partial\theta} \\
& - \frac{1}{f}\frac{\partial\beta}{\partial\theta}\frac{\partial \bar{h}}{\partial\lambda}\frac{\partial h'}{\partial\lambda}\Bigg)\Bigg). \qquad (30)
\end{aligned}
$$

For the PV approach we have to modify (8) slightly as it is non-linear in terms of h'. The linearised version for q^b is given by

$$
q^{b\prime} \equiv \frac{\xi^{b\prime}}{\bar{h}} - \frac{\left(f + \bar{\xi}^b\right)}{\bar{h}^2}h', \qquad (31)
$$

where $\bar{\xi}^b$ is (1a) evaluated with $\bar{u}^b$ which is given by

$$
\bar{\mathbf{u}}^b \equiv \bar{\mathbf{u}}_g - \frac{2\alpha}{f}\left(\bar{\mathbf{u}}_g \cdot \nabla\right)\left(\mathbf{k} \times \bar{\mathbf{u}}_g\right). \qquad (32)
$$

In its component form this is

$$
\begin{aligned}
\bar{u}^b \equiv \bar{u}_g - \frac{2\alpha}{f}&\left(\frac{\bar{u}_g}{a\cos\theta}\frac{\partial\bar{v}_g}{\partial\lambda} + \frac{\bar{v}_g}{a}\frac{\partial\bar{v}_g}{\partial\theta} + \frac{\tan\theta}{a}\bar{u}_g^2\right) \\
&- \frac{\alpha\beta}{f^2}\bar{v}_g^2, \\
\bar{v}^b \equiv \bar{v}_g + \frac{2\alpha}{f}&\left(\frac{\bar{u}_g}{a\cos\theta}\frac{\partial\bar{u}_g}{\partial\lambda} + \frac{\bar{v}_g}{a}\frac{\partial\bar{u}_g}{\partial\theta} - \frac{\tan\theta}{a}\bar{u}_g\bar{v}_g\right) \\
&- \frac{\alpha\beta}{f^2}\bar{u}_g\bar{v}_g.
\end{aligned}
$$

This then gives us two linear pdes, (30) for the RV method and (31) evaluated with (30) for the PV method.

As with the non-linear pdes (19) and (20), there is a similar condition for (30) and (31) to have solutions. In the next subsection we introduce the theory for linear pdes and the condition for solvability.

4.2 Ellipticity Theory: Part II

The general form for linear pdes differs slightly from that of the non-linear Monge-Ampère equation, as shown below

$$A\frac{\partial^2 h'}{\partial\lambda^2} + B\frac{\partial^2 h'}{\partial\theta\partial\lambda} + C\frac{\partial^2 h'}{\partial\theta^2} + D\frac{\partial h'}{\partial\lambda} + E\frac{\partial h'}{\partial\theta} + Fh' + G = 0, \tag{33}$$

where $A,\ldots,G$ are functions of $\left(\theta,\lambda,\bar{h}\right)$ and the derivatives of $\bar{h}$. From this, the linear pde is said to be elliptic if the condition

$$B^2 - 4AC < 0, \tag{34}$$

holds (Garabedian 1967).

As for the nonlinear case, we also require boundary conditions to ensure that the equations have a solution. For this problem we have periodicity in the λ direction, whilst in the θ direction we have two choices: the first is $\partial h'/\partial\lambda = 0$; the second is periodicity of h' with respect to θ across the poles.

Given these boundary conditions it can be shown, after some manipulations (see Fletcher (2004)), that the ellipticity condition for (30) is

$$\alpha\xi^b\left(\bar{h}\right) < \frac{f}{2} + \frac{\alpha^2}{a^2 f}\left(\bar{v}_g^2 + \bar{u}_g^2\right) - \frac{3\beta\alpha}{fa}\bar{u}_g + \frac{\beta\alpha^2\tan\theta}{f^2a^2}\bar{u}_g^2 + \left(\frac{2\beta^2\alpha}{f^3a^2} - \frac{\partial\beta}{\partial\theta}\frac{2\alpha^2}{f^2a^2}\right)\bar{v}_g^2, \tag{35}$$

where $\xi^b\left(\bar{h}\right)$ is (19) evaluated with $\bar{h}$.

For equation (31) it can be shown, as an extension to the ellipticity condition for (30), that the ellipticity condition for the PV equation is

$$\alpha q^b\left(\bar{h}\right)\bar{h} < \frac{(1+2\alpha)f}{2} + \frac{1}{a^2 f}\left(\bar{v}_g^2 + \bar{u}_g^2\right) - \frac{3\beta\alpha}{fa}\bar{u}_g + \frac{\beta\alpha^2\tan\theta}{f^2a^2}\bar{u}_g^2 + \left(\frac{2\beta^2\alpha}{f^3a^2} - \frac{\partial\beta}{\partial\theta}\frac{2\alpha^2}{f^2a^2}\right)\bar{v}_g^2, \tag{36}$$

where $q^b\left(\bar{h}\right)$ is (20) evaluated with $\bar{h}$.

We observe that the ellipticity conditions for the variable coefficient linear equations are identical in functional form to the ellipticity conditions for the nonlinear Monge-Ampère equations in Section 3, evaluated with $\bar{h}$. Therefore the ellipticity conditions, (35) and (36), require the base state height field to satisfy the ellipticity conditions for the non-linear forms, (24) and (25), in order for a solution for the increment to exist. In other words, as one might expect, because of the flow dependencies of the transformation, the base state height field, $\bar{h}$, plays a crucial role in the use of the transformation.

5 Scale Analysis

Given the result indicated in the last section we now perform a scale analysis on the new balance equations to identify for which types of flows these balance equations give us extra information. We also consider the terms in (30) and (31) to see if all the terms are significant.

The values we use to assess the effects the flow dependency has on the ellipticity conditions and on the coefficients come from three different runs of the Met Office's 2-D shallow water equations model, initialised by different Rossby-Haurwitz waves (Williamson *et al.*, 1992). The Met Office's shallow water model runs on an Arakawa C-grid and more details of the model can be found in Malcolm (1996).

The initial conditions for a Rossby-Haurwitz wave are defined by

$$\begin{aligned}
h &= \frac{1}{g}\big(gh_0 + a^2 A(\theta) + a^2 B(\theta)\cos R\lambda \\
&\quad + a^2 C(\theta)\cos 2R\lambda\big), \\
u &= a\omega\cos\theta + aK\cos^{R-1}\theta\left(R\sin^2\theta - \cos^2\theta\right) \\
&\quad \times \cos R\lambda, \\
v &= -aKR\cos^{R-1}\theta\sin\theta\sin R\lambda, \\
\zeta &= 2\omega\sin\theta - K\sin\theta\cos^R\theta\left(R^2+3R+2\right) \\
&\quad \times \cos R\lambda,
\end{aligned}$$

where h_0 is the height at the poles and $A(\theta), B(\theta)$ and $C(\theta)$ are given by

$$\begin{aligned}
A(\theta) &= \frac{\omega}{2}(2\Omega+\omega)\cos^2\theta + \frac{1}{4}K^2\cos^{2R}\theta\,[(R+1) \\
&\times \cos^2\theta + \left(2R^2 - R - 2\right) - 2R^2\cos^{-2}\theta], \\
B(\theta) &= \frac{2(\Omega+\omega)K}{(R+1)(R+2)}\cos^R\theta\left[\left(R^2+2R+2\right)\right. \\
&- \left.(R+1)^2\cos^2\theta\right], \\
C(\theta) &= \frac{K^2}{4}\cos^{2R}\theta\left[(R+1)\cos^2\theta - (R+2)\right],
\end{aligned}$$

and ω, K and R are three parameters that determine the characteristics of the Rossby-Haurwitz wave along with the initial height at the poles, h_0. The parameter R is the wavenumber and, for the results that are shown here, is taken to be 4 as this is the highest stable wave number for this type of wave in the shallow water model (Hoskins, 1973). The parameter h_0, determines the shortest height for the wave, ω determines the underlying zonal flow from West to East and K controls the amplitude of the wave.

We consider three sets of values for the parameters that generate flows with different Burger numbers, B_u. This number is given by

$$B_u \equiv \frac{\sqrt{gh}}{fL} = \frac{L_R}{L}, \tag{37}$$

where L is the horizontal length scale and L_R is the Rossby radius of deformation.

The three test cases generate different values for B_u at different latitudes. A more detailed study of the change in the Burger number for each of the following three test cases can be found in Wlasak (2002).

The first test case, TC1 hereafter, has the parameters $h_0 = 50m$ and $\omega = K = 7.848 \times 10^{-7}s^{-1}$. The type of flow that is generated has a low Burger number and a difference in the height from the pole to the equator of $200m$.

The second test case, TC2, has the parameters $h_0 = 8000m$ and $\omega = K = 7.848 \times 10^{-6}s^{-1}$. This flow has a high Burger number and is quite fast. The difference in the height from the pole to the equator of $2500m$.

The final test case, TC3, uses the parameters $h_0 = 8000m$ and $\omega = K = 7.848 \times 10^{-7}s^{-1}$. The flow this wave generates has a high Burger number with the same difference in the height profile from the pole to the equator as TC1 but the wave is slower than that of TC1.

The values that are used to perform the scale analysis are taken from the computed h field from each of the three test cases 72hrs after the start of the model run. Given the height field, a central differencing is used to calculate the geostrophic winds at the h points on the grid and the relevant derivatives. From these values we obtain a zonal average at 45°N. We also use the value for f at this latitude, $f = 1.0313 \times 10^{-4}s^{-1}$. The resulting values are displayed in Table 1 for all three test cases.

Before we perform the major parts of the scale analysis we need to consider the β term in the equations, to see if it should be retained. We assume that the value for β in the mid-latitudes is 10^{-11} (Browning and Kreiss, 2002). The main point to note here is that in the derivations of the equations we have taken the derivative of f^{-1} and not f itself. This has given us an extra f^{-1} in the equations. We start by considering a general term in the Hessian component (which includes all the terms after the first β term following the Laplacian) in (30). This is

$$\frac{g^2}{f^3a^4} \approx \frac{10^2}{10^{-12}10^{24}} \approx \frac{10^2}{10^{12}} \approx 10^{-10}.$$

We now consider the terms in the Hessian that involve β, these are

$$\frac{g^2\beta}{f^4a^3} \approx \frac{10^2 10^{-11}}{10^{-16}10^{18}} \approx \frac{10^{-9}}{10^2} \approx 10^{-11}.$$

Therefore the β terms are an order of magnitude smaller than those that are in the rest of the Hessian part of (30). As we show in the scale anaysis that follows, the equations are dominated by the Laplacian terms and therefore the

Table 1: Scale Analysis 1: Table of average values for the scale analysis, where $\bar{h}$ has units m, $\bar{u}_g$ and $\bar{v}_g$ have units ms^{-1} and their derivatives are in s^{-1}.

Test Case	$\bar{h}$	$\bar{u}_g$	$\bar{v}_g$	$\frac{\partial \bar{u}_g}{\partial \lambda}$	$\frac{\partial \bar{v}_g}{\partial \lambda}$	$\frac{\partial \bar{u}_g}{\partial \theta}$
1	165.79	3.48	2.73	6.72×10^{-7}	1.69×10^{-6}	1.14×10^{-6}
2	9.10×10^{3}	27.94	26.72	5.83×10^{-6}	1.66×10^{-6}	5.30×10^{-6}
3	8.10×10^{3}	3.65	2.22	6.45×10^{-7}	1.38×10^{-6}	2.78×10^{-7}

β terms being an order smaller than the Hessian components means that for the time being we shall drop these terms from the equations.

The aim of the scale analysis of the remaining terms is to answer the following three questions

- For what types of flows may the ellipticity conditions fail?
- When, if at all, are the extra terms significant with respect to the Laplacian?
- Is there any difference between the RV and the PV approach?

In the next two subsections we summarise the scale analysis and calculate the values for the coefficients in both the ellipticity condition and in the equations themselves, and then we shall return to the questions posed above in Section 6.

5.1 Scale Analysis of the Ellipticity Conditions

In this subsection we are concerned with the coefficients of the ellipticity conditions. We present four tables that contain approximations for all three test cases for both the RV and PV approach. Tables 2 and 4 contain approximations to the B^2 term in the ellipticity condition for the RV and PV methods respectively. Tables 3 and 5 contain the approximations for the coefficients in the $4AC$ term for the RV and PV methods respectively. All of these tables do not include the β terms which we have just shown to be an order of magnitude smaller than those with the Hessian component.

5.1.1 RV Approach

A clear feature in Table 2 is the difference, in magnitude, of the coefficients for TC2 compared to the other two test cases. The importance of this feature is that it highlights the possibility that flows similar to TC2 could have problems satisfying the ellipticity condition.

Before we look at the terms that make up $4AC$ we must note that the first row in both Table 3 and Table 5 are the values of the Laplacian term. Given

Table 2: Scale Analysis 2: Scale Analysis of the Coefficients in B^2 for the RV Method.

Coefficients $\left(m^2 s^{-6}\right)$	TC1	TC2	TC3
$16g^2\left(\dfrac{\partial \bar{u}_g}{\partial \lambda}\right)^2$	6.95×10^{-10}	5.23×10^{-8}	6.41×10^{-10}
$-\dfrac{32g^2 \sin\theta}{a}\dfrac{\partial \bar{u}_g}{\partial \lambda}\bar{v}_g$	-6.27×10^{-10}	-5.32×10^{-8}	-4.89×10^{10}
$\dfrac{16g^2 \sin^2\theta}{a^2}\bar{v}_g^2$	1.41×10^{-10}	1.35×10^{-8}	2.46×10^{-10}
B^2	1.82×10^{-10}	1.26×10^{-8}	2.46×10^{-10}

Table 3: Scale Analysis 3: Coefficients in $4AC$ for the RV Method.

Coefficients $\left(m^2 s^{-6}\right)$	TC1	TC2	TC3
$4g^2 f^2 \cos^2\theta$	2.05×10^{-6}	2.05×10^{-6}	2.05×10^{-6}
$\dfrac{8g^2 f \cos\theta}{a}\dfrac{\partial \bar{u}_g}{\partial \theta}$	4.53×10^{-8}	2.10×10^{-7}	1.10×10^{10}
$-8g^2 f \cos\theta \dfrac{\partial \bar{v}_g}{\partial \lambda}$	-9.48×10^{-8}	-9.32×10^{-7}	-7.76×10^{-8}
$-16g^2 \cos\theta \dfrac{\partial \bar{u}_g}{\partial \theta}\dfrac{\partial \bar{v}_g}{\partial \lambda}$	-2.10×10^{-9}	-9.56×10^{-8}	-4.16×10^{-10}
$-\dfrac{16g^2 \sin\theta\cos\theta}{a}\dfrac{\partial \bar{u}_g}{\partial \theta}\bar{u}_g$	-4.80×10^{-10}	-1.79×10^{-8}	-1.23×10^{-8}
$-\dfrac{8g^2 f \sin\theta\cos\theta}{a}\bar{u}_g$	-2.17×10^{-8}	-1.74×10^{-7}	-8.83×10^{-8}
$4AC$	1.97×10^{-6}	1.04×10^{-6}	1.89×10^{-6}

Table 4: Scale Analysis 4: Scale Analysis of the Coefficients in B^2 for the PV Method.

Coefficients $\left(s^{-6}\right)$	TC1	TC2	TC3
$\dfrac{16g^2\left(\dfrac{\partial \bar{u}_g}{\partial \lambda}\right)^2}{\bar{h}^2}$	2.53×10^{-14}	6.31×10^{-16}	9.73×10^{-18}
$-\dfrac{32g^2 \sin\theta \dfrac{\partial \bar{u}_g}{\partial \lambda} \bar{v}_g}{a\bar{h}^2}$	-2.28×10^{-14}	-6.42×10^{-16}	-7.42×10^{18}
$\dfrac{16g^2 \sin^2\theta \bar{v}_g^2}{a^2\bar{h}^2}$	5.13×10^{-15}	1.63×10^{-16}	1.42×10^{-18}
B^2	7.63×10^{-15}	1.52×10^{-16}	3.73×10^{-18}

this, we see that for TC1 and TC3 this term is a factor of 10^2 larger than all of the other coefficients in the $4AC$ term. However, for TC2 this term is only a factor of 10 larger than the others. We can therefore tentatively say that for TC2 the Laplacian term is affected by the extra terms.

If we now consider the approximate values for $4AC$, last row in Table 3, we see that for TC1 and TC3 this does not differ much from the value for the Laplacian. However, for TC2 we see a significant change between the two lines. Therefore we are seeing signs that for flows similar to TC2 then the extra terms involved in the balance equation (30) are having an affect on the ellipticity condition.

Finally if we compare the approximate value for the B^2 term, bottom row of Table 2, we see that for TC1 and TC3 the $4AC$ term is a factor of 10^4 larger than the B^2 term suggesting that the extra terms may not be that significant with respect to the ellipticity condition. However, this does not appear to be the case for TC2 where we only have a difference of 10^2 suggesting that flows similar to TC2 do require the extra terms but also that flows much faster or rapidly changing could violate the condition.

5.1.2 PV Approach

If we now consider the B^2 term for the PV approach, Table 4, we see an affect due to the inclusion of the height in the calculation of the balanced variable. We see this impact on all three terms in the B^2 component of the ellipticity condition with TC1 now having a larger value than TC2. This may indicate that for the PV approach more of the flow dependicy is introduced into the ellipticity condition. Therefore there is an extra component now, the base state height, which could cause the condition to be violated.

The impact of the introduction of the height field in the denominator is also

Table 5: Scale Analysis 5: Scale Analysis of the Coefficients in $4AC$ for the PV Method

Coefficients $\left(s^{-6}\right)$	TC1	TC2	TC3
$\dfrac{4g^2 f^2 \cos^2\theta}{\bar{h}^2}$	7.45×10^{-11}	2.47×10^{-14}	3.11×10^{-14}
$\dfrac{8g^2 f\cos\theta\dfrac{\partial\bar{u}_g}{\partial\theta}}{\bar{h}^2}$	1.65×10^{-14}	2.53×10^{-15}	1.67×10^{16}
$-\dfrac{8g^2 f\cos\theta\dfrac{\partial\bar{v}_g}{\partial\lambda}}{\bar{h}^2}$	-3.45×10^{-14}	-1.13×10^{-14}	-1.18×10^{-15}
$-\dfrac{16g^2\cos\theta\dfrac{\partial\bar{u}_g}{\partial\theta}\dfrac{\partial\bar{v}_g}{\partial\lambda}}{\bar{h}^2}$	-7.65×10^{-16}	-1.15×10^{-15}	-6.37×10^{-18}
$-\dfrac{16g^2\sin\theta\cos\theta\dfrac{\partial\bar{u}_g}{\partial\theta}\bar{u}_g}{a\bar{h}^2}$	-1.75×10^{-16}	-2.16×10^{-16}	-1.87×10^{-18}
$-\dfrac{8g^2 f\sin\theta\cos\theta\bar{u}_g}{a\bar{h}^2}$	-7.89×10^{-13}	-2.10×10^{-15}	-3.47×10^{-16}
$4AC$	7.36×10^{-11}	1.25×10^{-14}	2.97×10^{-14}

clear in the $4AC$ terms displayed in Table 5. Here we see that the Laplacian is still dominating the condition for TC1 but for TC2 this is not so. For the flows similar to TC3 we see that the effect is minimal and that even for the PV approach the ellipticity condition is dominated by the term involving the scaled Laplacian.

If we now compare the approximate values for B^2 and $4AC$ for the PV method we see that for TC1 and TC3, $4AC$ is a factor of 10^4 larger than B^2 but for TC2 this difference is approximately 10^2, suggesting that for TC2 we are seeing a possible affect on the ellipticity condition from the extra terms.

The important feature to note for this analysis is that we only took one value for the height field. Over the whole globe the values of h change quite significantly indicating that the ellipticity condition could be violated by heights not much larger than those for TC2.

5.2 Scale Analysis of the Linear Partial Differential Equations

In this subsection there are two tables, Table 6 contains the values for the coefficients in (30), and Table 7 has the values for (31), both with the β terms removed. In Tables 6 and 7 we have highlighted which coefficients are multiplying second order derivatives; this is to remind us that these are the coefficients in the ellipticity condition. The first three rows of both tables are the coeffi-

Table 6: Scale Analysis 6: Scale Analysis of the Coefficients in the Differential Equation for the RV Method.

Coefficient $\left(ms^{-3}\right)$	Term	TC1	TC2	TC3
gf	$\boldsymbol{h_{\lambda\lambda}}$	1.01×10^{-3}	1.01×10^{-3}	1.01×10^{-3}
$\sin\theta\cos\theta fg$	h_θ	5.06×10^{-4}	5.06×10^{-4}	5.06×10^{-4}
$\cos^2\theta fg$	$\boldsymbol{h_{\theta\theta}}$	5.06×10^{-4}	5.06×10^{-4}	5.06×10^{-4}
$2g\frac{\partial\bar{u}_g}{\partial\lambda}$	$\boldsymbol{h_{\theta\lambda}}$	2.62×10^{-5}	2.29×10^{-4}	2.53×10^{-5}
$2g\cos\theta\frac{\partial\bar{v}_g}{\partial\lambda}$	$\boldsymbol{h_{\theta\theta}}$	2.34×10^{-5}	2.30×10^{-5}	1.94×10^{-5}
$2g\frac{\partial\bar{u}_g}{\partial\theta}$	$\boldsymbol{h_{\lambda\lambda}}$	2.24×10^{-5}	1.04×10^{-4}	5.45×10^{-6}
$\frac{4g\sin\theta}{a}\bar{v}_g$	$\boldsymbol{h_{\theta\lambda}}$	1.19×10^{-5}	1.16×10^{-4}	9.67×10^{-6}
$\frac{2g\sin\theta\cos\theta}{a}\bar{u}_g$	$\boldsymbol{h_{\theta\theta}}$	5.36×10^{-6}	4.30×10^{-5}	5.62×10^{-6}
$\frac{(4g\tan\theta\sin\theta + 2g\cos\theta)}{a}\bar{v}_g$	h_λ	1.78×10^{-5}	1.75×10^{-4}	1.45×10^{-5}
$\frac{2g\cos^2\theta}{a}\bar{u}_g$	h_θ	5.36×10^{-6}	4.30×10^{-6}	5.62×10^{-6}
$4g\tan\theta\frac{\partial\bar{u}_g}{\partial\lambda}$	h_λ	2.64×10^{-5}	2.29×10^{-4}	2.53×10^{-5}
$2g\sin\theta\cos\theta\frac{\partial\bar{u}_g}{\partial\theta}$	h_θ	1.12×10^{-5}	5.20×10^{-5}	2.73×10^{-6}

cients of the Laplacian and it is with these values that we compare the other coefficients.

Starting with TC1 we see that the Laplacian terms are a factor of 10 larger than any of the terms arising from the higher order balance. However, of the extra nine terms only two are a factor of 10^2 smaller than the Laplacian. However, for TC3 we see that there are five terms that are factor of a 10^2 smaller than the Laplacian. This suggests that the extra terms are large enough to be considered of some significance for TC1. For TC3, however, only four terms are a factor of 10 smaller, suggesting that it may not be necessary to include all of the extra terms.

For TC2, however, there are four of the nine terms which are the same size as that of Laplacian, clearly showing that for these types of flows the extra terms are quite important and should be used. Of the remaining five terms there is only one that is a factor of 10^2 smaller than the Laplacian and is multiplying a first order derivative term.

If we now consider the coefficients of the pde for the PV approach we have an extra term in this table, Table 7 last row, which is due to the linearisation of the PV.

The first noticeable feature of this table is that the term from the linearisation is the largest coefficient for TC1. This suggests that there could be a difference in the results using the PV instead of the RV for this test case. For the other two test cases it is camparable in magnitude to that of the Laplacian; this suggests that this is a significant part of the approximation.

If we consider the remaining terms for the three test cases then we see that for TC1 there are five terms that are a factor of 10 smaller than the Laplacian and four that are a factor of 10^2 smaller. The same is also true for TC3. For TC2 we have five terms that are the same magnitude as the smallest terms in the Laplacian, with the remaining four terms only a factor of 10 smaller. This would suggest that we again would have to consider all the terms when using the PV approach for TC2, but for TC1 and TC3 then there are some terms that could be dropped. An important thing to remember here is that if we do drop any terms from the equation we have to be consistent and remove the same terms from $\xi^b\left(\bar{h}\right)$ and this may affect the summaries that we have presented here.

In the next section we use these summaries to answer the three questions which were posed at the beginning.

6 Conclusions and Further Work

In this paper we have derived two alternative methods for capturing the balanced part of atmospheric flows. These two methods are based upon results from MR96 and MR02 where a balanced wind field, $\mathbf{u}^b$, is defined which has an associated conserved potential vorticity. These two alternative methods are based upon considering either the relative vorticity or the potential vorticity associated with these balanced wind fields. These conditions result in either a

Table 7: Scale Analysis 7: Scale Analysis of the Coefficients in the Differential Equation for the PV Method.

Coefficient, s^{-3}	Term	TC1	TC2	TC3
$\frac{gf}{\bar{h}}$	$\boldsymbol{h_{\lambda\lambda}}$	6.09×10^{-6}	1.11×10^{-7}	1.25×10^{-7}
$\frac{\sin\theta\cos\theta fg}{\bar{h}}$	h_θ	3.05×10^{-6}	5.56×10^{-8}	6.24×10^{-8}
$\frac{\cos^2\theta fg}{\bar{h}}$	$\boldsymbol{h_{\theta\theta}}$	3.05×10^{-6}	5.55×10^{-8}	6.23×10^{-8}
$\frac{2g\frac{\partial\bar{u}_g}{\partial\lambda}}{\bar{h}}$	$\boldsymbol{h_{\lambda\theta}}$	1.59×10^{-7}	2.51×10^{-8}	3.12×10^{-9}
$\frac{2g\cos\theta\frac{\partial\bar{v}_g}{\partial\lambda}}{\bar{h}}$	$\boldsymbol{h_{\theta\theta}}$	1.44×10^{-7}	2.53×10^{-9}	2.36×10^{-9}
$\frac{2g\frac{\partial\bar{u}_g}{\partial\theta}}{\bar{h}}$	$\boldsymbol{h_{\lambda\lambda}}$	1.35×10^{-7}	1.14×10^{-8}	6.73×10^{-10}
$\frac{4g\sin\theta\bar{v}_g}{a\bar{h}}$	$\boldsymbol{h_{\lambda\theta}}$	7.17×10^{-8}	1.28×10^{-8}	1.19×10^{-9}
$\frac{2g\sin\theta\cos\theta\bar{u}_g}{a\bar{h}}$	$\boldsymbol{h_{\theta\theta}}$	3.23×10^{-8}	4.73×10^{-9}	6.94×10^{-10}
$\frac{(4g\tan\theta\sin\theta + 2g\cos\theta)\,\bar{v}_g}{a\bar{h}}$	h_λ	1.08×10^{-7}	1.92×10^{-8}	1.79×10^{-9}
$\frac{2g\cos^2\theta\bar{u}_g}{a\bar{h}}$	h_θ	3.23×10^{-8}	4.73×10^{-9}	6.94×10^{-10}
$\frac{4g\tan\theta\frac{\partial\bar{u}_g}{\partial\lambda}}{\bar{h}}$	h_λ	1.59×10^{-7}	2.50×10^{-8}	3.12×10^{-9}
$\frac{2g\sin\theta\cos\theta\frac{\partial\bar{u}_g}{\partial\theta}}{\bar{h}}$	h_θ	6.76×10^{-8}	5.71×10^{-9}	3.37×10^{-10}
$a^2 f^2 \cos^2\theta \left(\frac{f + \mathbf{k}\cdot\nabla\times\bar{\mathbf{u}}_g^c}{\bar{h}^2}\right)$	h	8.68×10^{-4}	2.58×10^{-7}	3.10×10^{-7}

Monge-Ampère equation that determines a balanced height field if we consider the non-linear version of $\mathbf{u}^b$, or a linear elliptic pde for a balanced height increment if we consider the linearised version of the balanced wind field, $\mathbf{u}^{b\prime}$. We showed that the ellipticity conditions for both types of pdes had a physical significance . However, for the linearised equation we still required the base state height, $\bar{h}$, to satisfy the nonlinear ellipticity condition for a solution to exist.

In the last section we performed a scale analysis of the ellipticity condition using three different Rossby-Haurwitz waves with the Met Office's 2D shallow water equations model on the sphere. At the beginning of Section 5 we posed three questions about the ellipticity condition and the linear pdes. We now present conclusions to these questions.

The first question was: for what types of flows may the ellipticity conditions fail? The main answer to this question is the same for both the RV and PV method. Flows much faster than TC2, but also flows which are rapidly changing over small distances, could cause the B^2 term to grow, which we require to be smaller than the $4AC$ term. If not, then the ellipticity condition would be violated. We must also recall that these derivatives are in the $4AC$ term, besides the Laplacian coefficients, and could counteract the effect in the B^2 term.

Question two was: when, if at all, are the extra terms significant with respect to the Laplacian? We answer this separately for the RV and PV method. For the RV method the results for TC2, Table 6, show that we should use all the extra terms for flows of this type, flows that are not dominated by the geostrophic flow, as these are comparable with the Laplacian. This is not the case for TC3 where we have geostrophic flow dominating and the extra terms are quite small compared to the Laplacian. For TC1 we see that some of the terms are comparable to the Laplacian and so suggests including all the terms.

For the PV method we have the same conclusions that we had for the RV method for TC2. Unlike for the RV method for TC3, half of the extra nine terms are a factor of 10 smaller than the Laplacian compared to only two for the RV method therefore we would have to consider using the extra terms when using the PV. TC1's results are quite similar to those for the RV method and so we say that all of the terms are significant to the Laplacian.

The final question was: is there any difference between the RV and PV approach? The main answer is clearly visable in the bottom row of Table 7 where we have the extra terms arising from the linearisation to the PV. For TC1 this dominates the scale analysis suggesting that for this type of flow we would have different results between the RV and the PV method. For the other two test cases we have this term the same size as the Laplacian, suggesting that there may not be much difference between the two.

Therefore our overall conclusions are that for flow similar to TC2, high Burger number with fast travelling, tall waves, we would have to use all of the terms from the higher order balance and for flows much faster or rapidly changing in height over short distances then the ellipticity condition could be violated.

For flows similar to TC1, low Burger number with short heights, then we suggest using the extra terms but we also strongly suggest using the PV approach as the extra factor from the linearisation of the potential vorticity is significantly larger than the Laplacian term.

The third test case, high Burger number, high height but slow wave, shows signs that the extra terms may not be that significant when considering the RV method but are slightly more significant if considering the PV approach. The Laplacian could, however, be sufficient for this type of near geostrophic flow.

In summary in this paper we have presented a hierachy of balance equations in both non-linear and linearised form which have come from a Hamiltonian framework derived in McIntyre and Roulstone (1996) and (2002). The advantage of these balance relationships is that they are flow dependent and are compatible for certain flow patterns with a data assimilation framework.

The non-linear balance relationships are in the form of Monge-Ampère equations which have an associated ellipticity condition which guarantees a solution if this conidtion is not violated. The ellipticity condition has enabled us to investigate when the flow dependency of the associated balanced decomposition may fail. The non-linear balance equations are second order approximations to geostrophy on the sphere and we have been able to test these conditions with idealised data to see the scale of the second order terms and when they may have an influence on the balance equations.

It should be noted that the ellipticity condition for the non-linear balance equations is similar in form to those shown in Knox (1997), but here we have not ignored the metric terms referred to in Knox (1997) as we have shown these to be a vital part of the ellipticity condition for certain types of flows.

One final remark about the methods presented in this paper is again associated with the ellipticity property, but now with the physical significance if the conditions fail. The ellipticity conditions presented in Sections 3 and 4 are in terms of a general α that could take the values 0, -0.5 or 1 depending on which type of balance is sought. It was remarked that for the case $\alpha = 0$ (geostrophic balance), the solution is robust, since the ellipticity condition is always satisfied; however, for the other two situations, $\alpha = -0.5$ (semi-geostrophic balance) and $\alpha = 1$ (Charney-Bolin balance), the ellipticity conditions are flow dependent and the solutions are not always guaranteed to exist. Our understanding is that when the ellipticity condition is not satisfied for either of the flow dependent cases, then this is simply informing us that the flow is not in Charney-Bolin balance or semi-geostrophic balance; however, there may be some part of the flow that is in geostrophic balance, albeit small.

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