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by

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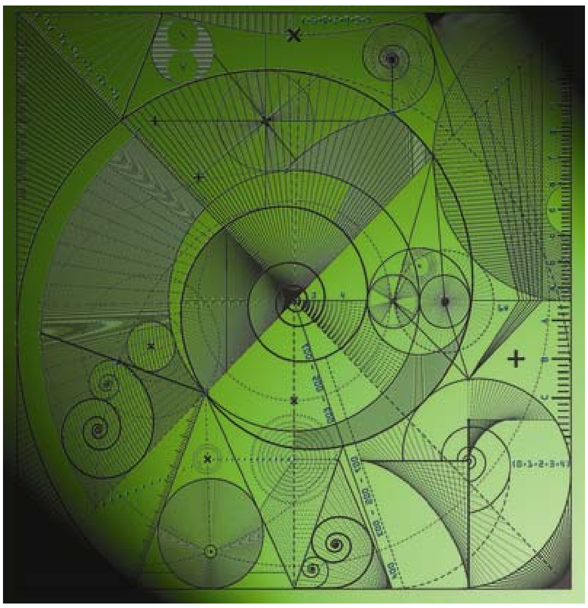

A moving mesh approach to ice sheet modelling

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Abstract

A moving mesh approach to the numerical modelling of problems governed by nonlinear time-dependent partial differential equations (PDEs) is applied to the numerical modelling of glaciers driven by ice diffusion and accumulation/ablation. The primary focus of the paper is to demonstrate the numerics of the moving mesh approach applied to a simple standard PDE model in reproducing the main features of glacier flow, including tracking the moving boundary (snout). A secondary aim is to investigate waiting time conditions under which the snout moves.

Keywords: Moving mesh, Ice sheets, Finite differences

1. Introduction

Computational studies of glaciers are particularly challenging to the modeller. Although ice sheet models are well-established, prediction of profiles and grounding movement are infeasible analytically and difficult to achieve numerically Payne and Vieli [5]. Ice moves in a similar manner to a viscous fluid, though with a very high viscosity approximately 10^{15} times that of water [2]. However, viscous theory cannot solely be used to describe flow, since glaciers are unique in experiencing *basal sliding.* This can be caused in two ways, via friction where the ice makes contact with the ground as it is flowing, or geothermal heat below the surface.

In order for glaciers to form they first need enough snow over the winter period to be able to survive through the summer, i.e. more accumulation of snow than is lost through melting and evaporation. This needs to be repeated over a number of succesive years, and as more snow builds up, the weight increases and pressure compresses the firn (old snow) into ice. Once this ice is thick enough, gravity, amongst other forces, causes the ice to flow. This is a long, complex process which takes less time in regions where temperature changes quicker, such as the Alps and North America. [2]

On a global scale, ice quantities vary considerably. At present glaciers make up around 2% of the Earths water, but during an ice age this vastly increases. Either way they have a large impact on the climate system, and are becoming increasingly affected by climate change. If all this ice melted into the oceans, there would be a sea level rise of around 70m. We are interested in glaciers for more than just the climate change reasons, as they can have a large effect on the local terrain, causing events such as landslides and flash floods.

We consider a simple PDE model of glacier movement in a moving frame of reference, discretise it and use a local mass balance principle to define a velocity in order to move the mesh. We note that, as in other nonlinear diffusion problems, glaciers experience a waiting time before they begin to move. We suggest a mechanism whereby waiting ends and the snout moves. Finally, consideration is given to ways the model may be extended, and the impacts that these extensions may have on the results we have obtained, leading the way to potential further work to be undertaken on the problem.

One of the main concepts to take into consideration when modelling glaciers is the idea of mass balance, and where on the glacier mass is gained or lost. Generally, near the source of the glacier, the accumulation of snow is greater than the ablation (melting/evaporation), so the mass increases. Further away the ablation becomes greater than the accumulation, and the mass decreases. However ice can build up in the lower zone due to ice flow coming from the glacier's upper zone. The front-most end of the glacier is known as the snout, which rarely moves straight away; it waits until the velocity behind it is great enough to push it down the mountain. It is this feature which is of special interest.

2. Model Description

Consider a glacier on a flat bed occupying the region $x \in [0, b(t)]$ as shown in Fig.1. Let $H(x,t)$ represent the thickness of the ice. At the ends of this domain we have two boundary conditions, $H = 0$ at the moving bounary $x = b(t)$, and $\frac{\partial H}{\partial x} = 0$ at the fixed point $x = 0$.

We consider a simple PDE model for glaciers proposed by Oerlemans [3] in 1984.

2.1. Model Derivation

In one dimension the continuity equation for ice can be written as

$$\frac{\partial H}{\partial t} = -\frac{\partial (Hu)}{\partial x} + s(x), \tag{1}$$

where H is the ice thickness, $s(x) = s_a(x) - s_b(x)$, with s_a the accumulation rate of snow and s_b the basal melting rate. Also u is defined as the mean

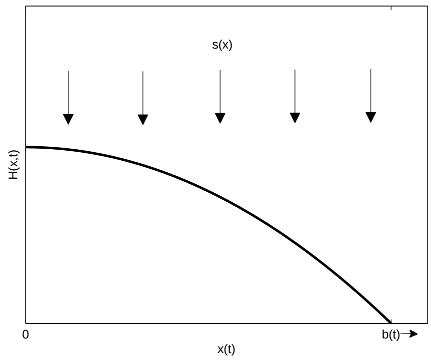


Figure 1: One-Dimensional Domain

depth integrated horizontal velocity given by [7]

$$u = \frac{2AH}{n+2}\tau_{dx}^{n}, \tag{2}$$

with τ_{dx} the stress term, and parameters A and n taken from Glen's flow law, an established general law for steady state ice deformation [6]. From Van Der Veen[8] the driving stress is given by

$$\tau_{dx} = -\rho g H \frac{\partial h}{\partial x}, \tag{3}$$

with ρ the ice density, g representing gravity, and h equal to ice thickness plus the surface elevation. On a flat bed there is no surface elevation so we may put $h = H$. From (2) and (3) we get an equation for the depth integrated horizontal velocity

$$u = -\frac{2AH}{n+2}\rho^n g^n H^n \left(\frac{\partial H}{\partial x}\right)^n. \tag{4}$$

The parameters A, n, ρ and g are set as constant to simplify the model, giving

$$u = -cH^{n+1}\left(\frac{\partial H}{\partial x}\right)^n, \tag{5}$$

where c is a single positive constant parameter.

Expressing the velocity in this form may present problems when dealing with the boundary condition $H = 0$ at $x = b(t)$, apparently giving a zero velocity at the right hand boundary and resulting in a glacier that will never move, which we know physically is not the case. However it is perfectly possible for u to be non-zero as long as $H^4 H_x^3$ is finite, which requires H_x to be infinite. From Roberts [7] we set $c = 0.000022765$ in standard SI units, and $n = 3$. For the most part though we are not concerned with physical values for the variables, but more with the methodology and the theory behind why we might see a certain behaviour, hence all the variables are non-dimensionalised. Substituting the velocity into equation (1) we get the model equation

$$\frac{\partial H}{\partial t} = c \frac{\partial}{\partial x} \left[H^{n+2} H_x^n \right] + s(x), \tag{6}$$

which incorporates non-linear diffusion and a source term. In this paper we set $s_b(x) \equiv 0$, making $s(x) = s_a(x)$, although the non-zero basal sliding case is considered in [4].

2.2. Mass Balance

An important physical property concerns the integral of the ice thickness over the whole domain (the volume), i.e.

$$\int_0^{b(t)} H(x,t)dx = \theta(t), \text{ say.} \tag{7}$$

From (1), using Leibniz's integral rule, and applying the boundary conditions

$$\begin{aligned}
\frac{d}{dt}\int_0^{b(t)} H(x,t)dx &= \int_0^{b(t)} \frac{\partial H}{\partial t}dx + H(b(t),t)\frac{db(t)}{dt} \\
&= -\int_0^{b(t)} \frac{\partial}{\partial x}[Hu]dx + \int_0^{b(t)} s(x)dx \\
&= -[Hu]_0^{b(t)} + \int_0^{b(t)} s(x)dx \\
&= \int_0^{b(t)} s(x)dx, \qquad (8)
\end{aligned}$$

the physical equivalent of which states that any change in the integral of ice thickness over the whole glacier, or equivalently any change in the ice volume, is due only to the snow term, which represents the net accumulation/ablation of snow over the whole glacier.

3. Snout Behaviour

From (5), with $n = 3$ we derive the useful form

$$u = -c(H^{4/3}H_x)^3 = -\frac{27}{343}c\left[(H^{7/3})_x\right]^3, \qquad (9)$$

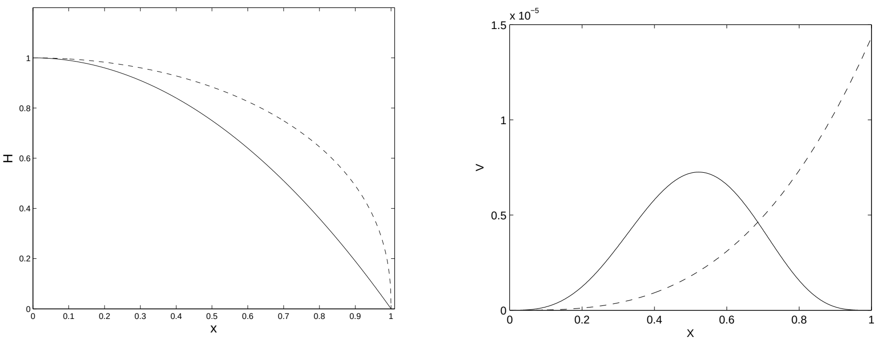
with $n = 3$.

When expressing the velocity in this manner it is interesting to substitute an expression for H that has the right general shape and satisfies the boundary conditions, i.e.

$$H = (1 - x^2)^\alpha \qquad (10)$$

where $\alpha > 0$, for which

$$\begin{aligned}
H^{\frac{7}{3}} &= (1 - x^2)^{\frac{7\alpha}{3}} \\
(H^{7/3})_x &= -2x.\frac{7\alpha}{3}(1 - x^2)^{\frac{7\alpha}{3}-1}. \qquad (11)
\end{aligned}$$



(a) (b)

Figure 2: Initial values for $\alpha = 1$ (solid) and $\alpha = 3/7$ (dashed). (a) shows the ice thickness with evidence of the infinite gradient in the $3/7ths$ case while (b) shows initial velocity, with an obvious difference at the right hand boundary

The velocity (9) then has some interesting properties as $x \to 1$, depending on the value of α.

$$\text{Case 1: } \frac{7\alpha}{3} > 1, \quad \Rightarrow \quad (H^{7/3})_x \text{ is zero} \tag{12}$$

$$\text{Case 2: } \frac{7\alpha}{3} < 1, \quad \Rightarrow \quad (H^{7/3})_x \text{ is infinite} \tag{13}$$

$$\text{Case 3: } \frac{7\alpha}{3} = 1, \quad \Rightarrow \quad (H^{7/3})_x \text{ is finite} \tag{14}$$

In case 1, from equation (9), the initial velocity of the snout of the glacier is zero, and it remains stationary. In case 2 the velocity is infinite, which is not physical. In case 3 we get a finite velocity value at the snout when $\alpha = \frac{3}{7}$, and the right hand boundary moves. From this analysis we can expect H to be asymptotically of the form $H = (1 - x^2)^{3/7}$ for $(1 - x)$ small at the moment of initial movement, i.e. if H approaches case 3 asymptotically at the snout then the boundary will move.

4. Subdomain mass balance (SDMB)

Since the problem involves a moving boundary, a natural description is to use a moving framework, for which we define a velocity $v(x,t)$ at any arbitrary point $\widehat{x}$ [1]. To define this velocity we assume that equation (8) holds in any moving subdomain $[0, \widehat{x}(t)]$ of $[0, b(t)]$. In physical terms this velocity v is such that the ice volume changes only due to the local accumulation/ablation of snow. In equation form we assume that

$$\frac{d}{dt}\int_0^{\widehat{x}(t)} H(x,t)dx = \int_0^{\widehat{x}(t)} s(x)dx \tag{15}$$

for each subdomain $(0, \widehat{x}(t))$.

By Leibniz's integral rule, making use of (6) and the boundary conditions given in Section 2,

$$\begin{aligned}\frac{d}{dt}\int_0^{\widehat{x}(t)} H(x,t)dx &= \int_0^{\widehat{x}(t)} \frac{\partial H}{\partial t}dx + H(\widehat{x}(t),t)\frac{d\widehat{x}(t)}{dt}\Bigg|_0^{\widehat{x}} \\ &= cH^5H_x^3|_{\widehat{x}} + \int_0^{\widehat{x}(t)} s(x)dx + H(\widehat{x}(t),t)\frac{d\widehat{x}(t)}{dt}. \end{aligned} \tag{16}$$

Therefore the assumption (15) is equivalent to

$$\int_0^{\widehat{x}(t)} \left\{ cH^5H_x^3 + H(\widehat{x}(t),t)\frac{d\widehat{x}(t)}{dt} \right\} dx = 0$$

which, since $\widehat{x}(t)$ is arbitrary, gives

$$cH^5H_x^3 + H(\widehat{x}(t),t)\frac{d\widehat{x}(t)}{dt} = 0.$$

Hence the velocity $v = d\widehat{x}/dt$ is driven only by the diffusion term and we obtain

$$v = \frac{d\widehat{x}(t)}{dt} = -\frac{cH^5H_x^3}{H} = -cH^4H_x^3 = -c_2\left[(H^{7/3})_x\right]^3 \tag{17}$$

where $c_2 = (3/7)^3 c$. Note that this velocity is the same as taking v to be the model velocity (5) at each of the nodes. Reversing the argument implies that the assumption (15) and the velocity (5) are equivalent.

5. Numerical Method

Equation (6) is generally impossible to solve analytically, so we seek a numerical approximation via a mesh. To do this we discretise (15) and (17). The mesh positions are updated at every time step.

Computation is performed with initial conditions (10) with α set to 1. The snow term is approximated for all time by the linear function (as in Van Der Veen [8])

$$s(x) = e(1 - dx), \tag{18}$$

where d and e are the snow parameters, set to be 0.5 and 0.05 respectively. The model is run for a sufficient length of time for the boundary to wait, then move. The initial mesh is chosen to be evenly spaced.

5.1. Explicit time-stepping

To advance the node positions $\widehat{x}_i$ from the velocity (17) we use an explicit Euler scheme. Letting k denote the time discretisation level,

$$\frac{\widehat{x}_i^{k+1} - \widehat{x}_i^k}{\Delta t} = -c_2 \left[\left[(H_i^{7/3})_x \right]^3 \right]^k , \tag{19}$$

Therefore, dropping the hat notation for convenience, at each time step we update each mesh point by:

$$x_i^{k+1} = x_i^k - c_2 \Delta t \left[\left[(H_i^{7/3})_x \right]^3 \right]^k \tag{20}$$

Since we do not seek high levels of accuracy Euler time-stepping is sufficient, provided the time step is suitably small to ensure stability.

To determine the updated ice thickness we use the same time-stepping scheme on equation (15). Note that the limits have been chosen to give an incremental form.

$$\frac{\left[\int_{x_{j-1}}^{x_{j+1}} H dx\right]^{k+1} - \left[\int_{x_{j-1}}^{x_{j+1}} H dx\right]^{k}}{\Delta t} = \int_{x_{j-1}}^{x_{j+1}} s dx.$$

Using the midpoint rule we obtain the approximation

$$(x_{j+1}^{k+1} - x_{j-1}^{k+1}) H_j^{k+1} - (x_{j+1}^k - x_{j-1}^k) H_j^k = \Delta t (x_{j+1}^k - x_{j-1}^k) s_j^k,$$

giving

$$H_j^{k+1} = \frac{(x_{j+1}^k - x_{j-1}^k)}{(x_{j+1}^{k+1} - x_{j-1}^{k+1})} (H_j^k + \Delta t s_j^k). \tag{21}$$

5.2. Results

The model is run with 51 mesh points ($\Delta x = 0.02$), with a time step $\Delta t = 0.005$.

Varying the parameter α in the initial conditions shows the snout profile behaviour of Section 3. In Fig. 2(a) we see that when $\alpha = 3/7$ the gradient is effectively infinite at the boundary, while a comparison case of $\alpha > 3/7$ shows a finite gradient. Similarly, looking at the initial velocities for each of the two cases we see in Fig. 2(b) that the boundary does not move when $\alpha > 3/7$, while in the comparison case it does. Also of note is that the peak velocity is at the snout of the glacier under the required condition, compared with near the centre for the stationary boundary case. This suggests that over time the peak velocity moves towards the boundary before the the glacier begins to move.

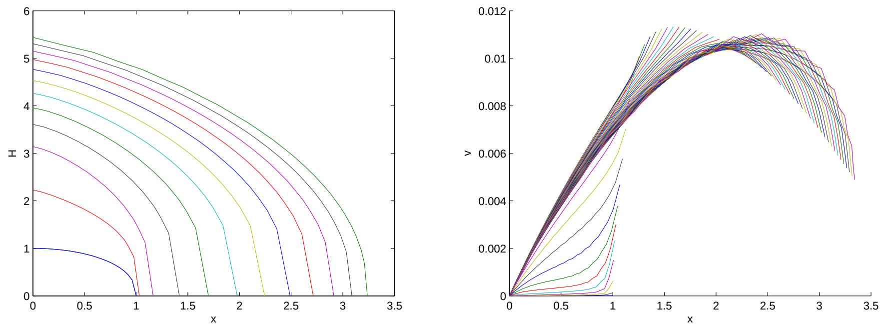
We may also look at the results over a period of time. For the experiments shown, the value of α has been set to 3/7 and 60000 steps were taken, or 300 time units. The results are outputted every 5000 steps. From Fig.3(a), the change in ice thickness is initially dominated by the build-up of snow. Even though the boundary is already moving, it is not until a larger amount of snow has fallen that the speed notably increases and the boundary moves further away. It is here that the diffusion term begins to dominate the flow. Similarly, the velocity (Fig. 3(b)) builds up more on the boundary initially, then as the outward movement speeds up, the peak velocity starts to return to the centre of the glacier. This is due to the glacier entering a region with less accumulation of snow, slowing down the end points. Eventually the model will reach an equilibium where the accumulation/ablation balances the diffusion.

6. Waiting Time

We now discuss in more detail the mechanism whereby the snout waits and then moves. We define b_0 as the initial location of the boundary, i.e. $b(0)$.

Since initally $H > 0$ for $x < b_0$ and $H = 0$ at $x = b_0$ $\forall$ t, the function $H(x)$ in the vicinity of $x = b$ is of the form

$$H(x) = (b_0 - x)^{\alpha} g(x) \tag{22}$$



(a) (b)

Figure 3: Numerical results using explicit time-stepping. (a) shows the evolution of the ice thickness and (b) shows the corresponding evolution of velocity.

to leading order in $(b_0 - x)$, where α is positive and $g(x) > 0$. Note that for the initial conditions in the computations $g(x) = (1 + x)^\alpha$. Hence to leading order in $(b_0 - x)$,

$$H(x)^{7/3} = (b_0 - x)^{7\alpha/3} G(x) \tag{23}$$

where $G(x) = (g(x))^{7/3} > 0$.

The velocity (17) then takes the form

$$v(x) = c\alpha^3 (b_0 - x)^{7\alpha - 3} \{G(x)\}^3 \tag{24}$$

to leading order in $(b_0 - x)$. When $\alpha = 3/7$ the velocity reduces to

$$v(x) = c(3/7)^3 \{G(x)\}^3 . \tag{25}$$

We see from (24) that $v(b_0) = \lim_{x \to 1} v(x) = 0$ when $\alpha > 3/7$, while from (25) $v(b_0)$ is non-zero when $\alpha = 3/7$. The discrepancy indicates a discontinuity in $v(x)$ at $\alpha = 3/7$ as $x \to b_0$. Fig. 4 shows a schematic plot of $v(x)$, normalised by $c\{G(x)\}^3$, against x for small $(b_0 - x)$ and various α, showing how the discontinuity forms. Since the velocity must remain continuous for physical reasons, the limit $\alpha = 3/7$ cannot be attained and the snout must move.

We now consider the transition in time of $v(x,t)$ to $\alpha = 3/7$ from above for small $(b_0 - x)$ under the subdomain mass balance assumption (15). To leading order in $(b_0 - x)$ the time-varying form of (23) is

$$H(x,t) = (b_0 - \widehat{x})^\alpha g(x,t) \tag{26}$$

where $\widehat{x} = x(t)$, while that of (24) is

$$v(x,t) = c\alpha^3 (b_0 - \widehat{x})^{7\alpha-3} \left\{G(\widehat{x},t)\right\}^3 . \tag{27}$$

We consider the evolution of the velocity under the subdomain mass balance assumption (15) applied to the interval (x, b_0), where $(b_0 - \widehat{x})$ is small. Let $\theta(t)$ be the positive mass in the triangle consisting of points $(\widehat{x}, H(x,t))$, $(\widehat{x}, 0)$ and the fixed point $(b_0, 0)$. To leading order in $(b_0 - \widehat{x})$, therefore,

$$\theta(t) = \frac{1}{2}(b_0 - \widehat{x}) H(x,t). \tag{28}$$

Hence, from (26), to leading order in $(b_0 - \widehat{x})$,

$$\frac{1}{2}(b_0 - \widehat{x})^{\alpha+1} g(\widehat{x},t) = \theta(t)$$

so that on the trajectory given by (28) from (27) the velocity at time t can be written

$$v(t) = c\alpha^3 (b_0 - \widehat{x})^{7\alpha-3} \left(\frac{2\theta(t)}{(b_0 - x)^{\alpha+1}}\right)^7 = c\alpha^3 (2\theta(t))^7 (b_0 - \widehat{x})^{-10} \tag{29}$$

and thus on the trajectory given by (28)

$$\alpha^3 = \frac{1}{c} \frac{1}{(2\theta(t))^7} (b_0 - \widehat{x})^{10} v(t)$$

Since $d\widehat{x}/dt = v(t)$, differentiating α^3 with respect to time along this trajectory,

$$3\alpha^2 \frac{d\alpha}{dt} = -\frac{1}{c}\frac{10}{(2\theta(t))^7}(b_0 - \widehat{x})^9 v(t)^2$$

to leading order in $(b_0 - \widehat{x})$, assuming that $d\theta/dt$ is of the same order as $\theta(t)$ and dv/dt is of the same order as $v(t)$. Thus $d\alpha/dt$ is strictly negative. Hence, for sufficiently small $(b_0 - \widehat{x})$, α is strictly decreasing and when α reaches 3/7 the boundary moves.

It is possible to interpret $v(x,t)$ in terms of (curved) characteristics. Although v is given explicitly by (17) it is useful to consider a PDE satisfied by v. Differentiating v we get

$$v_x = -3cH^4H_x^2H_{xx} - 4cH^3H_x^4 \tag{30}$$

$$v_t = -3cH^4H_x^2H_{xt} - 4cH^3H_x^3H_t \tag{31}$$

To keep equation (31) in terms of space derivatives, we can substitute equation (17) into the original equation (1), with u replaced by v, giving

$$\begin{aligned} H_t &= s - (Hv)_x \\ &= s - H_xv - Hv_x, \end{aligned} \tag{32}$$

which we can differentiate to get

$$H_{tx} = s_x - H_{xx}v - 2H_xv_x - Hv_{xx}. \tag{33}$$

The expressions for H_t and H_{tx} can be substituted into (31) to give

$$v_t = 3cH^5H_x^2v_{xx} - 10vv_x - v_xv - 3cH^4H_x^2s_x - 4cH^3H_x^3s,$$

and finally rearranged in the form of a Burgers-like equation with additional source terms,

$$v_t + 11vv_x = 3cH^5H_x^2v_{xx} - 3cH^4H_x^2s_x - 4cH^3H_x^3s. \qquad (34)$$

Hence

$$\frac{dv}{dt} = 3cH^5H_x^2v_{xx} - 3cH^4H_x^2s_x - 4cH^3H_x^3s \qquad (35)$$

along the (curved) characteristics given by

$$\frac{dx}{dt} = 11v(x,t) \qquad (36)$$

where H is obtained from $dH/dt = H_t + (dx/dt)H_x$ using (36). The presence of the dissipative terms in dv/dt prevents a shock forming in the interior but from (36) the characteristics become infinitely steep as $v(x,t) \to 0$ at the boundary $x = b_0$, consistent with the formation of the discontinuity in $v(x,t)$ at the snout.

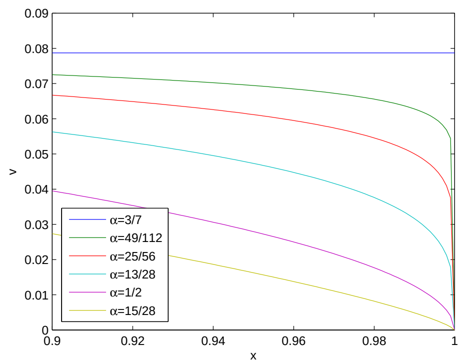


Figure 4: Plots of v against x for $\alpha = 15/28, 1/2, 13/28, 25/56, 49/112, 3/7$ showing formation of the discontinuity

7. Discussion

We have seen that a moving mesh method based on local mass balance works well for a simple one-dimensional glacier. We also analysed the conditions required for the snout to move, which required an infinite slope condition to be met asymptotically at the boundary in order for the velocity at this point to be non-zero. For an initial condition in the form of the quadratic function (10) with $\alpha > 3/7$, we were able to simulate the qualitative waiting time behaviour of the glacier as the power of α evolved to 3/7.

The model (6) works well for glaciers that are advancing, but since the velocity is proportional to the negative slope at the snout the glacier cannot retreat.

However, it is observed that glaciers also break-up, where large sections of ice completely break off from the main part and drift away or melt which, although presenting problems for a depth-averaged model such as the one considered here with the ice thickness vanishing at points other than at the boundary, might see a positive H_x gradient in certain areas, which would generate a negative velocity. A mechanism to allow these events to occur needs investigating.

A number of other avenues await exploration. The model we have been analysing throughout is only a 1D depth-averaged model, so a logical next step is to take the model into more dimensions. This could mean one of two things. The first is to consider a fully 3D model of the glacier instead of depth averaging. When the velocity varies with height we should see quite different movement which would require a far more complex model with mesh points moving in the vertical in addition to the x direction. We would expect the top and bottom of the glacier to be moving quicker than the middle section, and the impact of basal sliding could then be analysed more effectively. Alternatively we could keep the depth averaged vertically and consider the domain in the (x, y) plane, as opposed to a cross-section in the x-plane as is modelled currently. This will require additional boundary conditions at the sides of the glacier, the type depending on whether the boundary is a solid wall (no flux condition) or whether the glacier just curves to the ground $(H = 0)$. The model itself then takes the form

$$H_t(x, y, t) = \underline{\nabla}. \left[H(x, y, t)^5 \underline{\nabla} H(x, y, t)^3\right] + s(x, y). \tag{37}$$

A viable 2D numerical model using the same moving mesh approach is possible using finite element approximation.

Steering away from $H = 0$ at the snout, Payne et. al. [5] consider the different boundary conditions which occur when a glacier reaches the edge of a cliff or enters the ocean, as well as different representations of u. For cases where the glacier mostly sits on the water (ice shelf) there is the added problem of buoyancy. Payne et. al. propose a maritime boundary condition of the form

$$\left.\frac{\partial v}{\partial x}\right|_{shelffront} = A\left[\frac{1}{4}\rho_i g\left(1 - \frac{\rho_i}{\rho_w}\right)\right]^n h^n, \tag{38}$$

where ρ_w is an additional variable introduced for the density of the water.

A further objective in this ongoing work is to introduce the concepts of Data Assimilation. Here the aim is to set up an inverse problem to predict internal variables and model forcing using observations (mostly taken remotely) to further improve the model accuracy.

References

[1] C. Budd, W. Huang, and R. Russell. Adaptivity with moving grids. *Acta Numerica*, 18:111–241, 2009.

[2] M. Hambrey and J. Alean. *Glaciers*. Cambridge University Press, 1992.

[3] J. Oerlemans. http://www.geo.cornell.edu/geology/eos/iceflow/model_description.html, last checked 23rd November 2009.

[4] D. Partridge. Analysis and computation of a simple glacier model using moving grids. Master's thesis, Department of Mathematics, University of Reading, 2009.

[5] A. J. Payne and A. Vieli. Assessing the ability of numerical ice sheet models to simulate grounding line migration. *J. Geophysical Research*, 110, 2005.

[6] B. Rae, D. Irving, B. Hubbard, and J. McKinley. Preliminary investigations of centrifuge modelling of polycrystalline ice deformation. *Annals of Glaciology*, 31, 2000.

[7] R. Roberts. Modelling glacier flow. Master's thesis, Department of Mathematics, University of Reading, 2007.

[8] C. J. Van Der Veen. *Fundamentals of Glacier Dynamics*. Taylor and Francis, 1999.