School of Mathematical
and Physical Sciences

University of Reading

Department of Mathematics and Statistics

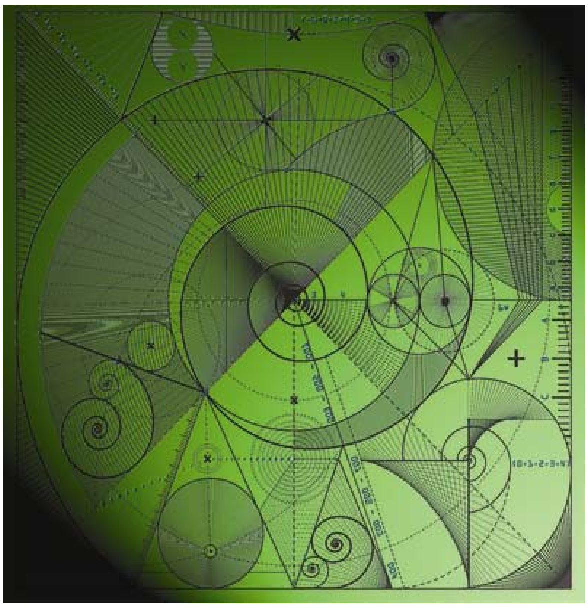
Preprint MPS_2010-29

31 August 2010

Wave trapping in a two-dimensional sound-soft acoustic waveguide of slowly-varying width

by

Nicholas R.T. Biggs



Wave trapping in a two-dimensional sound-soft acoustic waveguide of slowly-varying width

Nicholas R. T. Biggs

Department of Mathematics and Statistics, University of Reading,
P. O. Box 220, Whiteknights, Reading, RG6 7AX, U.K.

Received ?

Abstract

In this paper we derive novel approximations to trapped waves in a two-dimensional Dirichlet acoustic waveguide whose walls vary slowly along the guide. The guide contains a single smoothly bulging region, but is otherwise straight, and the modes are trapped within this localised increase in width.

A WKBJ-type expansion yields an approximate expression for the modes which can be present. Crucially, these modes can display either propagating or evanescent behaviour. Using matched asymptotic expansions, connection formulae are determined which bridge the gap across the cut-off between propagating and evanescent solutions in a tapering region of the waveguide. A uniform expansion is then derived, and it is shown that appropriate zeros of this expansion correspond to trapped mode wavenumbers. Numerical results are then compared to results of the full linear problem calculated using a spectral method, and the two are shown to be in excellent agreement.

Keywords: slowly-varying width; waveguide; quasi-modes; perturbation methods; turning point; WKBJ.

1. Introduction

It is well-established that vibrational energy can become trapped within waveguides by local changes in the guide's width (e.g., [1-2]) or curvature (e.g. [2-5]), resulting in what are termed trapped modes: localised solutions of the homogeneous time-harmonic boundary-value problem. In this paper, we focus on the trapping that can occur within a straight two-dimensional acoustic waveguide with a localised increase in width (i.e., a *bulge*). Physically, this geometry is capable of trapping waves since a particular mode may be evanescent in the narrower uniform region to either side of the bulge, but propagating in a uniform guide of width equal to the bulge width. Thus there may be a solution which is oscillatory within the bulge, and evanescent outside it, i.e., a trapped wave, the trapped mode frequency then lying between the cut-off frequencies associated with the width of the bulge and the width of the surrounding straight region.

Analytical determination of trapped wave frequencies and the associated modal structure is generally difficult, but if the width of the waveguide is slowly-varying, in the sense that the length-scale ϵ^{-1} over which the width changes satisfies $0 < \epsilon \ll 1$, then this small parameter can be used to develop an asymptotic scheme. In a recent series of papers ([1-2]), an asymptotic procedure is developed which allows calculation of the trapped wave frequencies (or, more precisely, the $O(\epsilon)$ correction away from the cut-off frequencies) as solutions of a simple ODE eigenvalue problem, given the additional geometrical constraint that the amplitude of the bulge is $O(\epsilon^2)$. This ODE eigenvalue problem is then accurately and efficiently solved using a spectral method.

A complementary problem to determining trapped modes within a slowly-varying waveguide is to instead determine an approximation to the types of *propagating* modes which can exist therein. This is commonly-achieved using variations on the general WKBJ ansatz $\phi = Ae^{iP/\epsilon}$ (see [6-8] for examples of using the approximation in problems in which the curvature rather than the width varies slowly). The approximation was used in [9] to model surface gravity water waves above a slowly-varying bed, and modified in [10] to include an expansion of the phase P in powers of ϵ. In [6-8], A and P are expanded in powers of ϵ and are functions of both longitudinal and lateral coordinates, which allows A and P to be identified as the real amplitude and phase, respectively. The expressions derived are referred to as *quasi-modes* in [6-8], are uncoupled, and to first order coincide with the adiabatic approximation in which gradients in waveguide width are ignored, and the modes are given locally by the separable solutions which exist in a uniform guide of the same local width.

In this paper, we use an expansion similar to that which yields the quasi-mode expressions to instead examine the trapping problem. We first derive a version of quasi-modes which allows both propagating and evanescent behaviour, depending on if the wavenumber is greater or less than the local cut-off. These expressions stem from a WKBJ-type ansatz $\phi = Ae^P$, with A and P both functions of longitudinal and lateral coordinates, and $A = A_0 + \epsilon A_1 + \ldots$ and $P = \epsilon^{-1}P_{-1} + \epsilon P_1 + \ldots$. The "phase" P is allowed to be real or imaginary to produce either propagating or evanescent behaviour. This asymptotic scheme is inconsistent in that two unknowns, A and P, replace the original single unknown ϕ, and there are insufficient equations to determine all unknowns. Thus only the first few terms in the expansions for A and P can be determined before this inconsistency hinders calculation of the higher order terms, and in particular means that the expressions are not as accurate as the (purely propagating) quasi-modes of [6-8] (the final approximations to the trapped wave frequencies still prove extremely accurate, however).

Now, a mode trapped within a bulge is oscillatory in nature within the centre of the bulge, and then changes in character to an evanescent wave as the narrower portion of the guide is reached, the point at which this character change occurs being an example of a turning point (see, for example [11]), at which in particular the expressions derived for the quasi-modes are not valid. However, formulae which connect the propagating and evanescent waves across the turning point can be obtained via the method of matched asymptotic expansions (in a fashion similar to that used in [12]). Motivated by the solution appropriate in the vicinity of the turning point, a uniformly valid expansion is then derived which includes the two quasi-mode forms as limiting behaviour. The form of this uniform expansion is very similar to that determined via the Langer transformation for ODEs (see [11]). Determining appropriate zeros of this uniform expansion via a standard iterative procedure then furnishes highly accurate approximations to the trapped wave frequencies and modal structures.

The paper proceeds as follows. In section 2 we derive the new quasi-mode expressions which allow both propagating and evanescent behaviour. Then we consider the reflection of one such propagating quasi-mode at a taper in a waveguide, first using a matched asymptotics procedure to connect the propagating and evanescent expansions, and then via a uniform expansion.

In section 3, we show how the uniform approximation to the taper problem can be used to derive approximations to the corresponding eigenvalue problem, and then compare a selection of these results to numerical approximations to solution of the full linear problem, calculated using a spectral method. Finally, in section 4 we conclude and offer some suggestions for further work.

2. Waves in a two-dimensional acoustic waveguide of slowly-varying width

2.1. Preliminaries

We wish to determine the values of $\bar{k}$ giving a non-trivial solution to the homogeneous Dirichlet boundary-value problem

$$\bar{\phi}_{\bar{x}\bar{x}} + \bar{\phi}_{\bar{y}\bar{y}} + \bar{k}^2\bar{\phi} = 0 \qquad (-\infty < \bar{x} < \infty,\ -\bar{h}_-(\bar{x}) < \bar{y} < \bar{h}_+(\bar{x})) \tag{2.1}$$

$$\bar{\phi} = 0 \qquad (-\infty < \bar{x} < \infty,\ y = \pm\bar{h}_\pm(\bar{x})) \tag{2.2}$$

$$\bar{\phi} \to 0 \qquad \text{as } \bar{x} \to \pm\infty. \tag{2.3}$$

Here $\bar{k} = \omega/c$ is the wavenumber, with ω the prescribed wave frequency, and c the sound speed. The time-dependence $e^{-i\omega t}$ is implied but omitted throughout.

The duct walls $\bar{h}_\pm(\bar{x})$ are slowly-varying in the sense that

$$\bar{h}_\pm(\bar{x}) = h_0 h_\pm(x) \tag{2.4}$$

for some positive h_0, where $x = \epsilon\bar{x}/h_0$ (with $0 < \epsilon \ll 1$) is a slow variable, and the functions $\bar{h}_\pm(\bar{x})$ are continuous and differentiable. The walls straighten at infinity, so that $h_\pm(x) \to 1$ as $x \to \pm\infty$. Near $x = 0$ they bulge outwards, so that $h_\pm(0) > 1$, and they are symmetric about $x = 0$, so that $h_\pm(x) = h_\pm(-x)$ for all $x \geq 0$, and in particular $h'_\pm(0) = 0$.

We non-dimensionalise by writing $\bar{\phi}(\bar{x}, \bar{y}) = \phi(x, y)$, where $y = \bar{y}/h_0$, so that we have

$$\epsilon^2\phi_{xx} + \phi_{yy} + k^2\phi = 0 \qquad (-\infty < x < \infty,\ -h_- < y < h_+) \tag{2.5}$$

$$\phi = 0 \qquad (-\infty < x < \infty,\ y = \pm h_\pm(x)) \tag{2.6}$$

$$\phi \to 0 \qquad \text{as } x \to \pm\infty \tag{2.7}$$

where $k = \bar{k}h_0$ is the dimensionless wavenumber.

We first investigate what type of modes exist as solutions to the boundary-value problem consisting of just (2.5) and (2.6), the *quasi-modes*, and consider the reflection of one of these quasimodes at a taper in a duct.

2.2. Quasi-modes

We use the WKBJ-type ansatz

$$\phi = Ae^{P} \tag{2.8}$$

where

$$A = A_0(x,y) + \epsilon A_1(x,y) + \epsilon^2 A_2(x,y) + \cdots \tag{2.9}$$

and

$$P = \epsilon^{-1} P_{-1}(x,y) + \epsilon P_1(x,y) + \epsilon^2 P_2(x,y) + \cdots \tag{2.10}$$

An $O(\epsilon^0)$ term is not included in the expansion for P since it can be subsumed into A_0. Note that P is allowed to be complex-valued to ensure that (2.8) can include both propagating (P imaginary) and evanescent ($\mathrm{Re}(P) \neq 0$) modes. The replacement of a single unknown, ϕ, by two unknowns, A and P, clearly leads to an under-determined scheme since we now have twice as many unknowns as equations. However, it turns out that the first few terms in the series (2.9) and (2.10) can be calculated before this inconsistency halts further progress.

The expressions (2.8), (2.9) and (2.10) are substituted into (2.5), and terms at each order are equated. At $O(\epsilon^{-2})$, we have

$$A_0 {P_{-1}}_y^2 = 0$$

whose solution we write as

$$P_{-1} = f(x), \tag{2.11}$$

where f is to be determined. The $O(\epsilon^{-1})$ equation is then trivially satisfied, and at $O(\epsilon^0)$ we have

$$A_{0yy} + (k^2 + f'^2)A_0 = 0. \tag{2.12}$$

The appropriate solution of this equation is

$$A_0 = a_0(x)S(x,y), \qquad S = (2/w)^{1/2}\sin[\alpha(y + h_-)], \tag{2.13}$$

where

$$w(x) = h_+(x) + h_-(x) \tag{2.14}$$

is the duct width,

$$\alpha^2(x) = k^2 + f'^2(x), \tag{2.15}$$

and we must choose

$$\alpha(x) \equiv \alpha_n(x) = \frac{n\pi}{w(x)} \tag{2.16}$$

for $n \in \mathbb{N}$ to ensure that the boundary conditions (2.6) are satisfied.

To solve (2.15) for f, we must be careful to distinguish cases for which $\alpha_n^2(x) - k^2 \geq 0$ and for which $\alpha_n^2(x) - k^2 \leq 0$. Thus

$$f(x) \equiv f_n(x) = \begin{cases} \pm i \displaystyle\int^x \left(k^2 - \alpha_n^2(x_0)\right)^{1/2} \,\mathrm{d}x_0, & k \geq \alpha_n(x), \\ \pm \displaystyle\int^x \left(\alpha_n^2(x_0) - k^2\right)^{1/2} \,\mathrm{d}x_0, & k \leq \alpha_n(x). \end{cases} \tag{2.17}$$

If k is larger than the local cut-off $\alpha_n(x)$ the mode is thus propagating; if k is smaller, the mode is evanescent.

Returning to the equation hierarchy, the $O(\epsilon)$ equation is

$$A_{1yy} + \alpha_n^2 A_1 = -(f_n'' + P_{1yy})A_0 - 2P_{1y}A_{0y} - 2f_n' A_{0x}. \tag{2.18}$$

Multiplying this expression by A_0 and then integrating from $y = -h_-$ to $y = h_+$ leads to the solvability condition

$$\int_{-h_-}^{h_+} A_0 \left\{(f_n'' + P_{1yy})A_0 + 2P_{1y}A_{0y} + 2f_n' A_{0x}\right\} \mathrm{d}y = 0, \tag{2.19}$$

which is used to determine a_0. Because $A_0 = 0$ on $y = \pm h_\pm$, this condition reduces to

$$0 = \int_{-h_-}^{h_+} \frac{\mathrm{d}}{\mathrm{d}x}\left(f_n' A_0^2\right)\, \mathrm{d}y = \frac{\mathrm{d}}{\mathrm{d}x}\left(\int_{-h_-}^{h_+} f_n' A_0^2\, \mathrm{d}y\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(f_n' a_0^2\right), \tag{2.20}$$

from which

$$a_0 = c|f'|^{-1/2} \tag{2.21}$$

for constant c. This gives the quasi-modes

$$\phi \equiv \phi_n = c_n |f_n'|^{-1/2} e^{\epsilon^{-1} f_n} S_n(x,y) + O(\epsilon) \tag{2.22}$$

for each $n \in \mathbb{N}$, where the c_n are constants,

$$S_n(x,y) = (2/w)^{1/2} \sin[\alpha_n(y + h_-)] \tag{2.23}$$

in which α_n is found from (2.16), f_n is given by (2.17), and

$$|f_n'| = |k^2 - \alpha_n^2|^{1/2}. \tag{2.24}$$

2.3. Wave reflection in a tapering duct

We now consider what happens when one of the propagating quasi-modes, given by (2.22) with f_n by the first of (2.17), reaches a monotonically tapering section of waveguide, and in particular reaches a region in which it can no longer propagate, and is reflected. We fix $n \in \mathbb{N}$, and for clarity suppress its appearance in the notation, writing α for α_n, S for S_n, and so on.

2.3.1. Matched asymptotics

Suppose there is an x_* such that $\alpha(x_*) = k$, and that $w'(x_*) < 0$ in a neighbourhood of $x = x_*$. The n-th mode $\phi(x,y)$ is then propagating in $x < x_*$ but evanescent in $x > x_*$, so we write

$$\phi = \phi^{(\pm)}(x) S(x,y) \qquad \text{in } x \gtrless x_*$$

where

$$\phi^{(-)} = \frac{I \exp\left\{-i\epsilon^{-1}\int_x^{x_*} \left(k^2 - \alpha^2(x_0)\right)^{1/2} \mathrm{d}x_0\right\} + R\exp\left\{i\epsilon^{-1}\int_x^{x_*} \left(k^2 - \alpha^2(x_0)\right)^{1/2} \mathrm{d}x_0\right\}}{\left(k^2 - \alpha^2(x)\right)^{1/4}} \tag{2.25}$$

and

$$\phi^{(+)} = \frac{T\exp\left\{-\epsilon^{-1}\int_{x_*}^{x} \left(\alpha^2(x_0) - k^2\right)^{1/2}\, \mathrm{d}x_0\right\}}{\left(\alpha^2(x) - k^2\right)^{1/4}}. \tag{2.26}$$

Here I is the prescribed amplitude of an incoming wave ϕ from $x < x_*$, and R and T are the unknown reflection and transmission coefficients, respectively. Energy conservation considerations dictate that $|R| = |I|$.

This representation is not valid in a small neighbourhood of $x = x_*$. To see this, note that

$$k^2 - \alpha^2(x) = -\Delta(x - x_*) + O\left((x - x_*)^2\right),$$

where $\Delta = -2k^3 w'(x_*)/(n\pi) > 0$. Thus in the limit $x \to x_*^-$,

$$\begin{aligned} \exp\left\{i\epsilon^{-1}\int_x^{x_*} \left(k^2 - \alpha^2(x_0)\right)^{1/2} \mathrm{d}x_0\right\} \quad &\sim \quad \exp\left\{i\epsilon^{-1}\Delta^{1/2}\int_x^{x_*} (x_* - x_0)^{1/2}\, \mathrm{d}x_0\right\} \\ &= \quad \exp\left\{i\frac{2}{3}\epsilon^{-1}\Delta^{1/2}(x_* - x)^{3/2}\right\} \end{aligned} \tag{2.27}$$

so that there is a nonuniformity when $(x_* - x)^{3/2} = O(\epsilon)$ i.e., when $(x_* - x) = O(\epsilon^{2/3})$. A similar analysis can be carried out for the evanescent exponential term in (2.26), and we see that in the limit $x \to x_*^+$,

$$\exp\left\{-\epsilon^{-1}\int_{x_*}^{x}\left(\alpha^2(x_0) - k^2\right)^{1/2}\mathrm{d}x_0\right\} \sim \exp\left\{-\frac{2}{3}\epsilon^{-1}\Delta^{1/2}(x - x_*)^{3/2}\right\}. \tag{2.28}$$

Having determined the size of the nonuniformity, we now return to the governing equation (2.5), and write $\phi(x, y) = \chi(\xi)S(x, y)$ where the stretched variable $\xi = \epsilon^{-2/3}(x - x_*)$ is $O(1)$ in the region of interest. This substitution yields the equation

$$\epsilon^{2/3}\left(\chi'' - \Delta\xi\chi\right)S = -\epsilon^{4/3}(\xi\chi'' + 2\chi' - \Delta\xi^2\chi)\frac{\partial S}{\partial x}(x_*, y) + O(\epsilon^2),$$

so to leading order $\chi \approx \chi_0$, where

$$\chi_0'' - \Delta\xi\chi_0 = 0. \tag{2.29}$$

This has bounded solution

$$\chi_0 = F\mathrm{Ai}(\Delta^{1/3}\xi), \tag{2.30}$$

for constant F, where Ai denotes Airy's function of the first kind.

Finally, we determine R, T and F by matching the appropriate expansions of the representations (2.25), (2.26) and (2.30) across the boundary layer centred at $x = x_*$. Using the standard large argument form of the Airy function Ai, we have from (2.30) that

$$\phi \sim F\pi^{-1/2}\Delta^{-1/12}(-\xi)^{-1/4}\sin\left(\frac{\pi}{4} + \frac{2}{3}\Delta^{1/2}(-\xi)^{3/2}\right) \qquad \text{as } \xi \to -\infty,$$

which agrees with the expansion

$$\phi^{(-)} \sim \frac{I\exp\left\{-i\frac{2}{3}\Delta^{1/2}(-\xi)^{3/2}\right\} + R\exp\left\{i\frac{2}{3}\Delta^{1/2}(-\xi)^{3/2}\right\}}{\Delta^{1/4}\epsilon^{1/6}(-\xi)^{1/4}} \qquad \text{as } x \to x_*^-,$$

of (2.25) provided

$$R = e^{-i\pi/2}I \tag{2.31}$$

and

$$F = \frac{2e^{-i\pi/4}\pi^{1/2}}{\Delta^{1/6}\epsilon^{1/6}}I. \tag{2.32}$$

Similarly,

$$\phi \sim \frac{1}{2}F\pi^{-1/2}\Delta^{-1/12}\xi^{-1/4}\exp\left(-\frac{2}{3}\Delta^{1/2}\xi^{3/2}\right) \qquad \text{as } \xi \to \infty,$$

which agrees with the expansion

$$\phi^{(+)} \sim \frac{T\exp\left(-\frac{2}{3}\Delta^{1/2}\xi^{3/2}\right)}{\Delta^{1/4}\epsilon^{1/6}\xi^{1/4}} \qquad \text{as } x \to x_*^+$$

of (2.26), provided

$$T = \frac{1}{2}F\pi^{-1/2}\Delta^{1/6}\epsilon^{1/6} = e^{-i\pi/4}I. \tag{2.33}$$

To summarise, the solution is given by (2.25) for $x \lesssim x_*$ with R found from (2.31), by (2.26) for $x \gtrsim x_*$ with T found from (2.33), and in the boundary layer region by

$$\phi \sim F\mathrm{Ai}(\epsilon^{-2/3}\Delta^{1/3}(x - x_*))S(x, y), \tag{2.34}$$

from (2.30), with F given by (2.32).

Composite expansions valid for $x \lesssim x_*$ and $x \gtrsim x_*$ are readily constructed, but to derive a uniformly valid expansion we now adopt a different approach.

2.3.2. A uniform expansion

Bearing in mind the form of the boundary layer solution (2.34), we propose an expansion of the form

$$\phi(x,y) = B(x,y)\mathrm{Ai}(\epsilon^{-2/3}g(x)) + C(x,y)\mathrm{Ai}'(\epsilon^{-2/3}g(x)), \tag{2.35}$$

where

$$\begin{aligned} B &= B_0 + \epsilon^{2/3}B_1 + \epsilon^{4/3}B_2 + \ldots \\ C &= \epsilon^{2/3}C_1 + \epsilon^{4/3}C_2 + \ldots \\ g &= g_0 + \epsilon^{2/3}g_1 + \epsilon^{4/3}g_2 + \ldots, \end{aligned} \tag{2.36}$$

and g is real-valued. An $O(\epsilon^0)$ term is not included in the expansion of C since we know from equation (2.34) that only the Airy function Ai, and not its derivative Ai$'$, appears in the boundary layer expansion at leading order.

Substituting the ansatz (2.35) into equation (2.5) we find that

$$\begin{aligned} &\left[\epsilon^2 B_{xx} + (k^2 + g{g'}^2)B + B_{yy} + \epsilon^{2/3}({g'}^2 C + gg''C + 2gg'C_x)\right]\mathrm{Ai}(\epsilon^{-2/3}g) \\ &+ \left[\epsilon^2 C_{xx} + (k^2 + g{g'}^2)C + C_{yy} + \epsilon^{4/3}(g''B + 2g'B_x)\right]\mathrm{Ai}'(\epsilon^{-2/3}g) = 0, \end{aligned} \tag{2.37}$$

and now equating the coefficients of $\mathrm{Ai}(\epsilon^{-2/3}g)$ and $\mathrm{Ai}'(\epsilon^{-2/3}g)$ to zero yields the pair of equations

$$\epsilon^2 B_{xx} + (k^2 + g{g'}^2)B + B_{yy} + \epsilon^{2/3}({g'}^2 C + gg''C + 2gg'C_x) = 0 \tag{2.38}$$

and

$$\epsilon^2 C_{xx} + (k^2 + g{g'}^2)C + C_{yy} + \epsilon^{4/3}(g''B + 2g'B_x) = 0. \tag{2.39}$$

The expansions (2.36) are substituted into this pair of equations, yielding a hiearchy of differential equations for the functions B_j, C_j and g_j. (Clearly, the scheme (2.35) is inconsistent since there are two sets of equations (2.38) and (2.39) from which to determine the three sets of functions, B_j, C_j and g_j. However, in common with the inconsistent scheme described in section 2.2, it turns out that the first few terms can be calculated, and they are all we require.)

The $O(\epsilon^0)$ coefficient of Ai is

$$B_{0yy} + (k^2 + g_0{g'_0}^2)B_0 = 0,$$

which has solution

$$B_0 = b_0(x)S(x,y), \tag{2.40}$$

with $S(x,y) = (2/w)^{1/2}\sin[\alpha(y+h_-)]$, and where as before $\alpha(x)$ and $S(x)$ are shorthand for $\alpha_n(x)$ and $S_n(x)$. Here we've chosen

$$k^2 + g_0{g'_0}^2 = \alpha^2 \tag{2.41}$$

in order to satisfy the boundary conditions (2.6).

Now, the solution of (2.41) will clearly be related to the solution of (2.15). Indeed, if we assume again that there is an x_* such that $\alpha(x_*) = k$, and that $w'(x) < 0$ in a neighbourhood of $x = x_*$, then if $x \geq x_*$ we have $\alpha(x) \geq k$, and we can write $g_0 = (3f/2)^{2/3}$ where f satisfies (2.15). Similarly, if $x \leq x_*$ then $\alpha(x) \leq k$, and now if we write $g_0 = -(3if/2)^{2/3}$, then f again satisfies (2.15). Recalling that we seek a real-valued function g, the appropriate solution is thus

$$g_0(x) = \begin{cases} -\left(\dfrac{3}{2}\displaystyle\int_x^{x_*} \left(k^2 - \alpha^2(x_0)\right)^{1/2}\,\mathrm{d}x_0\right)^{2/3} & x \leq x_*, \\ \left(\dfrac{3}{2}\displaystyle\int_{x_*}^{x} \left(\alpha^2(x_0) - k^2\right)^{1/2}\,\mathrm{d}x_0\right)^{2/3} & x \geq x_*. \end{cases} \tag{2.42}$$

Returning to the hierarchy of equations, the $O(\epsilon^{2/3})$ coefficient of Ai$'$ is

$$C_{1yy} + \alpha^2 C_1 = 0,$$

which is solved by

$$C_1 = c_1(x)S(x,y). \tag{2.43}$$

The $O(\epsilon^{2/3})$ coefficient of Ai is then the equation

$$B_{1yy} + \alpha^2 B_1 = -g_0'(g_0'g_1 + 2g_0g_1')B_0. \tag{2.44}$$

Multiplying (2.44) by B_0 and integrating from $y = -h_-$ to $y = h_+$ shows that the solvability condition $g_0'g_1 + 2g_0g_1' = 0$ must be satisfied, from which $g_1 = G_1 g_0^{-1/2}$ for constant G_1; the solution of (2.44) is then $B_1 = b_1(x)S(x,y)$.

Finally, the $O(\epsilon^{4/3})$ coefficient of Ai' is

$$C_{2yy} + \alpha^2 C_2 = -2g_0'B_{0x} - g_0''B_0. \tag{2.45}$$

Again, multiplying (2.45) by B_0 and integrating from $y = -h_-$ to $y = h_+$ shows that we require

$$\int_{-h_-}^{h_+} (2g_0'B_{0x} + g_0''B_0)\, B_0 \,\mathrm{d}y = 0,$$

which reduces to

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(b_0^2 g_0'\right) = 0.$$

This has solution

$$b_0 = G|g_0'|^{-1/2} \tag{2.46}$$

for constant G, and where from (2.41),

$$|g_0'| = |k^2 - \alpha^2|^{1/2}|g_0|^{-1/2}.$$

To leading order, the uniform approximation to the n-th quasi-mode reflection/transmission problem is thus

$$\phi(x,y) \sim \frac{G|g_0(x)|^{1/4}\mathrm{Ai}(\epsilon^{-2/3}g_0(x))S(x,y)}{|k^2 - \alpha^2(x)|^{1/4}} \tag{2.47}$$

where g_0 is given from (2.42). The form of this expression bears an obvious similarity to the corresponding ODE expansion derived via the Langer transformation (see [11]).

It only remains to determine the constant G, which we do by ensuring that as $x \to -\infty$, this representation of ϕ agrees with (2.25), where R is now again regarded as unknown. Now, as $x \to -\infty$, $k > \alpha_*(x)$ and $g_0(x) < 0$, so that

$$\mathrm{Ai}(\epsilon^{-2/3}g_0(x)) \sim \pi^{-1/2}\epsilon^{1/6}(-g_0(x))^{-1/4} \sin\left(\frac{\pi}{4} + \epsilon^{-1}\int_x^{x_*} \left(k^2 - \alpha^2(x_0)\right)\,\mathrm{d}x_0\right),$$

and thus

$$\begin{aligned}
\phi/S \;&\sim\; G|k^2 - \alpha^2(x)|^{-1/4}\pi^{-1/2}\epsilon^{1/6} \sin\left(\frac{\pi}{4} + \epsilon^{-1}\int_x^{x_*} \left(k^2 - \alpha^2(x_0)\right)\,\mathrm{d}x_0\right) \\
&=\; -\frac{i}{2}G|k^2 - \alpha^2(x)|^{-1/4}\pi^{-1/2}\epsilon^{1/6} \left\{\exp\left(\frac{\pi i}{4} + i\epsilon^{-1}\int_x^{x_*} \left(k^2 - \alpha^2(x_0)\right)\,\mathrm{d}x_0\right)\right. \\
&\qquad\qquad \left. - \exp\left(-\frac{\pi i}{4} - i\epsilon^{-1}\int_x^{x_*} \left(k^2 - \alpha^2(x_0)\right)\,\mathrm{d}x_0\right)\right\}.
\end{aligned}$$

This agrees with (2.25) provided

$$G = 2\pi^{1/2}\epsilon^{-1/6}e^{-i\pi/4}I \tag{2.48}$$

and also

$$R = -\frac{1}{2}iG\pi^{-1/2}\epsilon^{1/6}e^{i\pi/4}$$

so that

$$R = e^{-i\pi/2}I,$$

as already determined in (2.31) via the matched asymptotic expansions approach.

3. Trapping of waves in a symmetric duct

In this section, we revive the subscript notation denoting dependence on $n \in \mathbb{N}$, writing $\alpha_n(x)$ for $\alpha(x)$, $S_n(x,y)$ for $S(x,y)$, and so on.

Now, the uniformly-valid approximation to the reflection/transmission process in a tapering duct, equation (2.47), behaves as a decaying wave in $x > x_*^{(n)} > 0$, and as a propagating wave in $0 < x < x_*^{(n)}$, where $x_*^{(n)}$ is the root of $\alpha_n(x_*^{(n)}) = k$. In particular, if the duct narrows monotonically from its maximum width at $x = 0$, and k lies in $(\alpha_n(0), n\pi/2)$ (where $n\pi/2 = \lim_{x\to\infty} \alpha_n(x)$), then the n-th quasi-mode (2.47) will behave as a propagating wave for $x \in [0, x_*^{(n)})$, and as a decaying wave in $(x_*^{(n)}, \infty)$. Furthermore, if we now choose k so that in addition $\phi_n(0,y) = 0$ for $y \in (-h(0), h(0))$, then, by extending the solution antisymmetrically via the identification $\phi_n(x,y) = -\phi_n(-x,y)$ for $x < 0$, this solution represents an antisymmetric trapped mode. Similarly, if we choose k so that $\phi_{nx}(0,y) = 0$ for $y \in (-h(0), h(0)$, then by extending the solution symmetrically via $\phi_n(x,y) = \phi_n(-x,y)$ for $x < 0$, this solution represents a symmetric trapped mode.

Since $h'_\pm(0) = 0$ implies that $w'(0) = 0$ and $\alpha'_n(0) = 0$, we thus seek roots of

$$\mathrm{Ai}(\epsilon^{-2/3} g_{0,n}(0;k)) = 0 \quad \text{(antisymmetric modes)}, \tag{3.1}$$

and

$$\mathrm{Ai}'(\epsilon^{-2/3} g_{0,n}(0;k)) = 0 \qquad \text{(symmetric modes)}, \tag{3.2}$$

where, in a notation slightly modified from (2.42),

$$g_{0,n}(0;k) = \begin{cases} -\left(\dfrac{3}{2}\displaystyle\int_0^{x_*^{(n)}(k)} \left(k^2 - \alpha_n^2(x_0)\right)^{1/2} \,\mathrm{d}x_0\right)^{2/3} & x_*^{(n)}(k) \geq 0, \\ \left(\dfrac{3}{2}\displaystyle\int_{x_*^{(n)}(k)}^0 \left(\alpha_n^2(x_0) - k^2\right)^{1/2} \,\mathrm{d}x_0\right)^{2/3} & x_*^{(n)}(k) \leq 0, \end{cases} \tag{3.3}$$

and $x_*^{(n)}(k)$ is the unique positive root of $\alpha_n(x_*^{(n)}(k)) = k$. Note that in practice only the first line of (3.3) is used, since it is clear that $x_*^{(n)}(k) > 0$.

Rather than searching for the multiple roots of (3.2) and (3.1), it is more efficient to fix roots of $\mathrm{Ai}'(z) = 0$ or $\mathrm{Ai}(z) = 0$ and then solve for $\epsilon^{-2/3} g_{0,n}(0;k)$ equal to these values. Denoting the m-th roots of $\mathrm{Ai}'(z) = 0$ and $\mathrm{Ai}(z) = 0$ by z'_m and z_m ($m \in \mathbb{N}$) respectively (ordered so that $0 > z'_1 > z'_2 > \ldots$ and $0 > z_1 > z_2 > \ldots$), the symmetric wavenumbers can then be found from

$$\epsilon^{-2/3} g_{0,n}(0;k) = z'_m \qquad (m, n \in \mathbb{N}) \tag{3.4}$$

and the antisymmetric wavenumbers from

$$\epsilon^{-2/3} g_{0,n}(0;k) = z_m \qquad (m, n \in \mathbb{N}). \tag{3.5}$$

We denote solutions of (3.4) and (3.5), by $k^s_{m,n}$ and $k^a_{m,n}$ ($m, n \in \mathbb{N}$) respectively.

Once the wavenumbers are determined, the corresponding trapped mode is found from (2.47). The structure of the Airy function $\mathrm{Ai}(z)$ for $z < 0$ implies that the wavenumber $k^s_{m,n}$ corresponds to a trapped mode with $m - 1/2$ oscillations in the x-direction and $n/2$ oscillations in the y-direction. Similarly, the wavenumber $k^a_{m,n}$ corresponds to a trapped mode with m oscillations in the x-direction and again $n/2$ oscillations in the y-direction.

3.1. Numerical results

To verify the efficacy of our asymptotic method, we also solve the full linear problem (2.5)–(2.7) using a Chebyshev-Laguerre spectral method, as advocated in [1-2] and described in detail in [13]. The application of the boundary conditions is made easier by transforming the equations onto the uniform width semi-infinite

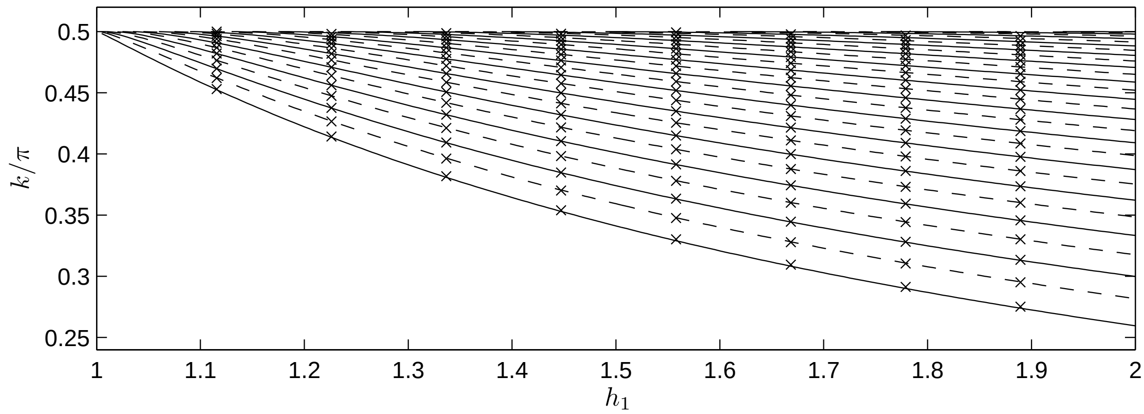


Figure 1: Wavenumbers for the symmetric (solid line) and antisymmetric (dashed line) trapped modes for the duct profi le (3.6), plotted as a function of the bulge half-width h_1. Here, $\epsilon = 0.1$. Symbols denote solutions of the full problem determined via the spectral method.

rectangular domain $R = \{(x,\eta) : x > 0, \ -1 < \eta < 1\}$, where $\eta = -1 + 2(y + h_-)/w$. Then $\psi(x,\eta) = \phi(x,y)$ satisfies

$$\epsilon^2(\psi_{xx} + \eta_x^2\psi_{\eta\eta} + 2\eta_x\psi_{x\eta} + \eta_{xx}\psi_\eta) + \eta_y^2\psi_{\eta\eta} + k^2\psi = 0$$

in R, together with the boundary conditions $\psi(x,\pm 1) = 0$ for $x > 0$ and $\psi \to 0$ as $x \to \infty$ for $-1 < y < 1$. Here $\eta_x = w^{-1}(2h'_- - (\eta+1)w')$, $\eta_{xx} = w^{-1}(2h''_- - (\eta+1)w'' - \eta_x w') - w^{-2}(2h'_- - (\eta+1)w')$, and $\eta_y = 2w^{-1}$.

A Chebyshev expansion is used in the η-direction, with Dirichlet conditions imposed at $\eta = \pm 1$, and a Laguerre expansion is used in the x-direction, with either a Neumann or Dirichlet condition imposed at $x = 0$ to determine either the symmetric or antisymmetric modes. The required decay as $x \to \infty$ is automatically accommodated by the Laguerre expansion. Using 15 points in the η-direction and 25 in the x-direction is sufficient to ensure 5 decimal places of accuracy for all numerical results presented here.

The symmetric duct wall profiles

$$h_\pm(x) = 1 + (h_1 - 1)\mathrm{sech}x \tag{3.6}$$

are chosen for illustrative purposes. This duct widens from a uniform width of $w = 2$ in the limit $x \to \infty$ to a symmetric bulge centred on $x = 0$ of total width $2h_1$.

Figure 1 displays the trapped symmetric and antisymmetric wavenumbers (solid and dashed lines respectively) for a fixed value of $\epsilon = 0.1$, with $n = 1$, and as the bulge half-width h_1 is increased. These results are obtained by solving (3.4) and (3.5) using a Newton-Raphson iterative scheme. The lowest solid line corresponds to $m = 1$ in (3.4), the next to $m = 2$, and so on; similarly for the dashed lines and values of m in (3.5). Numerical results using the spectral method are also shown (symbols), with which the approximate results are clearly in excellent agreement.

As the bulge half-width h_1 increases from unity, all trapped mode wavenumbers originate at the (non-dimensional) cut-off wavenumber $\pi/2$ and then decrease in magnitude. The number of trapped modes which can exist is a rapidly increasing function of the bulge half-width h_1, because the slowly-varying nature of the duct ensures that if the width of the bulge increases, then the length of the bulge must also increase, and this means that more along-duct oscillations can be “fitted in”. A similar picture emerges if a higher value of n is taken, but instead the trapped mode wavenumbers originate at the cut-off wavenumber $n\pi/2$ and then decrease in magnitude.

Figure 2 shows how the accuracy of the approximation varies as the slowness parameter ϵ is increased to unity. The approximations are surprisingly good: the least accurate approximation is that to the first (fundamental) symmetric mode $k^s_{1,1}$, but even for this mode the relative error averages 0.7% across the range of ϵ, with a maximum of 0.9% for $\epsilon = 1$. Thus the approximation appears to be of some use outside its asymptotic regime $0 < \epsilon \ll 1$.

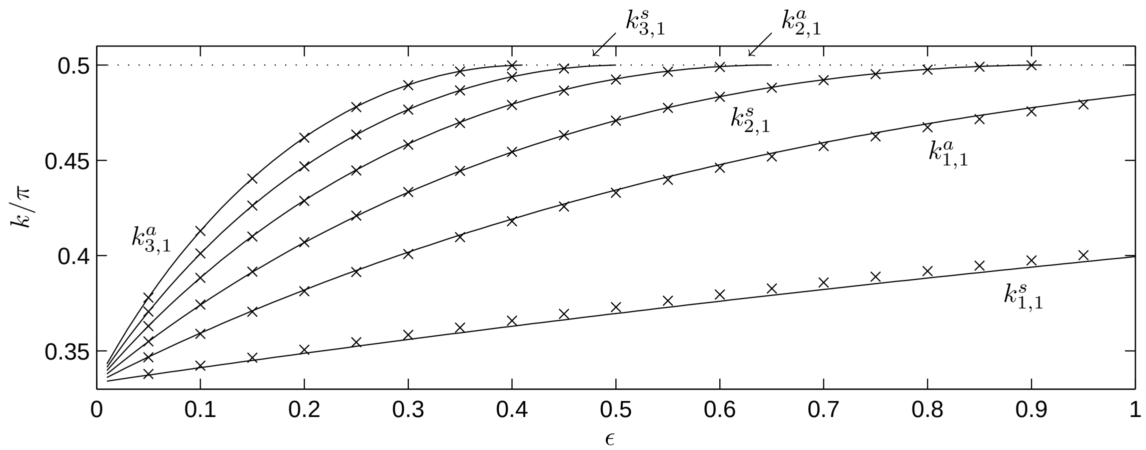


Figure 2: Trapped mode wavenumbers for the duct profi le (3.6) with $h_1 = 1.5$, plotted as a function of ϵ. Solid lines denote solutions of (3.4) and (3.5), as labelled; symbols denote solutions of the full problem determined via the spectral method.

Finally, we make use of the fact that once the trapped wavenumbers are determined, the modal structure is immediately available from the uniform expansion (2.47). We display in figure 3 the eigensolutions corresponding to the first eight eigenvalues for a particular example of the wall profiles (3.6). The oscillations in the x- and y-directions are clearly seen.

4. Conclusions and further work

In this paper, we have derived a new method for determining the wave frequencies and eigenmode structures for waves trapped within a bulging Dirichlet waveguide whose walls vary slowly along the guide.

The method presented here has three key strengths. First, there is no restriction placed on the *amplitude* of the wall bulge. Second, the trapped wave frequencies are ultimately found as solutions of a simple nonlinear algebraic equation, which can readily be solved approximately using standard iterative methods. Third, once the trapped wave frequencies are determined, an explicit expression for the corresponding approximate trapped mode structure is available.

The method can clearly be applied to many trapped wave problems in which the trapping mechanism is a slowly-varying property change. This change in property is the width of the waveguide in the present paper, but another example is that of a bent waveguide with slowly-varying curvature. Trapping within bulging three-dimensional waveguides can also be examined using the current technique, and this work is in progress.

Acknowledgements The author thanks Tahnia Appasawmy for pointing out some typographical errors in the text.

References

[1] J. Postnova, R. V. Craster, Trapped modes in topographically varying elastic waveguides, Wave Motion, 44, (2008) 205–221.

[2] J. Postnova, R. V. Craster, Trapped modes in elastic plates, ocean and quantum waveguides, Wave Motion, 45, (2008) 565–579.

[3] P. Duclos, P. Exner, Curvature-induced bound states in quantum waveguides in two and three dimensions, Rev. Maths. Phys. 7 (1995) 73–102.

[4] D. Gridin, R. V. Craster, A. T. I. Adamou, Trapped modes in curved elastic plates, Proc. R. Soc. Lond. A 461 (2005) 1181–1197.

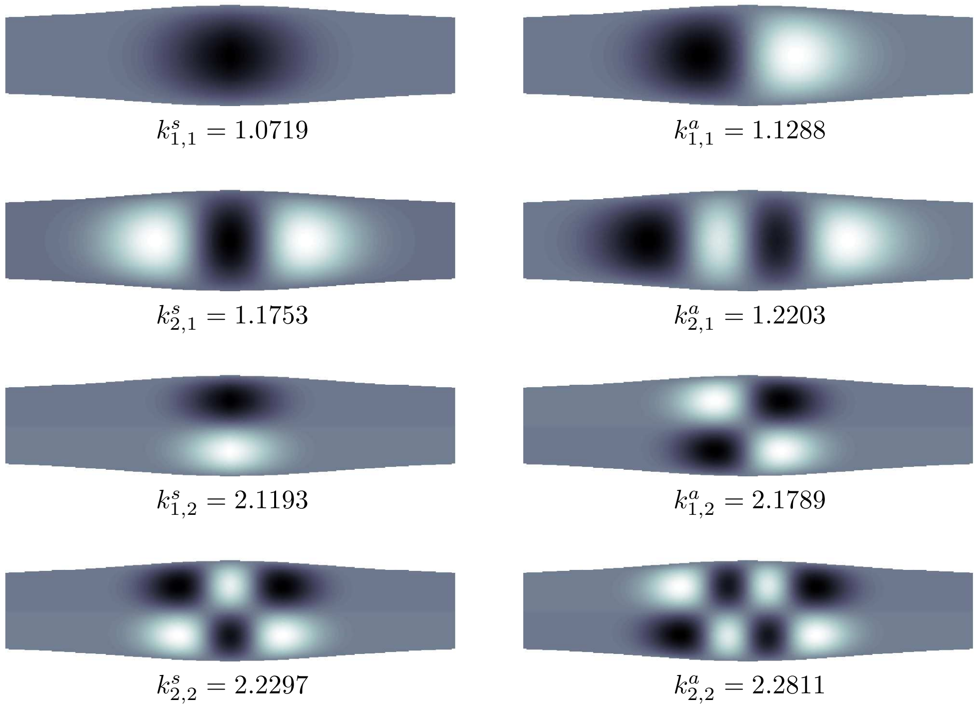


Figure 3: Structure of the first eight trapped modes, given from the uniform expansion (2.47), for the duct profile (3.6) with $\epsilon = 0.1$ and $h_1 = 1.5$.

[5] D. Gridin, A. T. I. Adamou, R. V. Craster, Trapped modes in bent elastic rods, Wave Motion, 42, (2005) 352–366.

[6] D. Gridin, R. V. Craster, Quasi-modes of a weakly curved waveguide, Proc. R. Soc. A, 459, (2003) 2909–2931.

[7] D. Gridin, R. V. Craster, Lamb quasi-modes in curved plates, Proc. R. Soc. A, 460, (2004) 1831–1847.

[8] A. T. I. Adamou, D. Gridin, R. V. Craster, Acoustic quasi-modes in slowly varying cylindrical tubes, Quart. J. Mech. Appl. Math., 58(3) (2005) 419–438.

[9] J. B. Keller, Surface waves on water of non-uniform depth, J. Fluid Mech., 4, (1958) 607–614.

[10] R. Smith, Propagation in slowly-varying wave-guides, Proc. R. Soc. A 302 (1968) 555–576.

[11] A. H. Nayfeh, Perturbation methods. John Wiley & Sons, 1973.

[12] S. W. Rienstra, Sound propagation in slowly varying lined flow ducts of arbitrary cross-section, J. Fluid Mech., 495, (2003) 157–173.

[13] J. A. C. Weidman, S. C. Reddy A MATLAB Differentiation Matrix Suite, ACM Trans. Math. Soft., 26(4) (2000) 465–519.