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by

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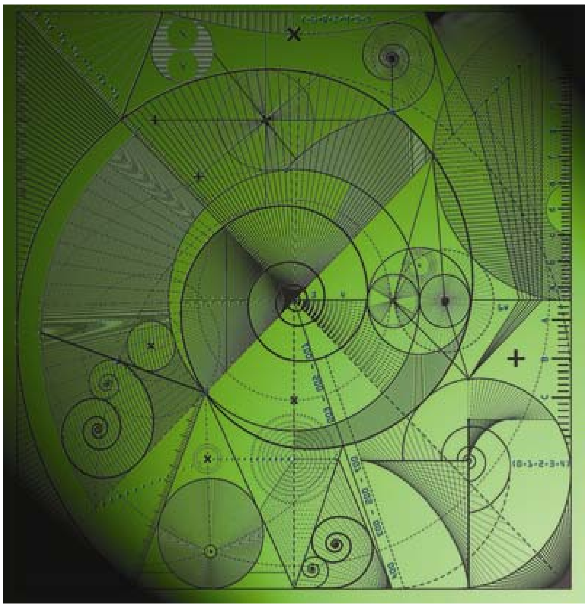

Two Maps and Worldwide Ipod Interest

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Abstract. There is often a desire to determine if the dynamics of interest are chaotic or not. Since positive Lyapunov exponents are a signature for chaos, they are often used to determine this. Reliable estimates of Lyapunov exponents should demonstrate evidence of convergence; but literature abounds in which this evidence lacks. This paper presents two maps through which it highlights the importance of providing evidence of convergence of Lyapunov exponent estimates. Worldwide ipod interest is then used as a practical example and the results bear semblance to both maps.

INTRODUCTION

The average rate of separation of initially nearby trajectories of a dynamical system is measured by Lyapunov exponents. In forecasting, Lyapunov exponents are used to quantify how predictable the underlying system is (e. g. [1]). They can also be used to merely characterise the system as has been done in diagnostic studies (e. g. [2, 3]). For these reasons, literature abounds in which numerical computations of Lyapunov exponents are sought with diverse applications. Nonetheless, finite data only affords finite time Lyapunov exponents. By Oseledec's theorem [4, 5], finite time Lyapunov exponents can converge to global Lyapunov exponents. It is therefore necessary to establish convergence before reporting any estimates; but there are many examples where convergence is, at the best, not adequately established (e.g. [1, 2, 3, 6, 7, 8, 9]).

With the aid of two nonlinear maps, the aim of this paper is to highlight the importance of establishing convergence of Lyapunov exponent estimates. We emphasise a distributional approach to this end and suggest a way to deal with finite real data. Worldwide interest on ipod is taken as a real world example for illustrative purposes. The two maps have contrasting dynamics, but they both bear semblance to the ipod interest.

LYAPUNOV EXPONENTS

Lyapunov exponents measure the average rate of separation of two trajectories that are initially infinitely close to each other. Consider a map

$$x_{t+1} = F(x_t),$$

where $x_t \in R^m$. For an initial state x_0, the dynamics of its small perturbation, δx_0, are governed by the linear propagator, $\mathscr{M}(x_0,N)$, which is a product of Jacobians so that $\mathscr{M}(x_0,N) = DF(x_{N-1})\cdots DF(x_1)DF(x_0)$. The finite time average separation/growth rate of two initially nearby trajectories is then given by

$$\lambda_N = \frac{1}{N}\log||\mathscr{M}\delta x_0||.$$

Oseledec [4] provides a theorem that guarantees that the limit $\lim_{N\to\infty}\lambda_N$ is unique. Denote this limit by Λ. If δx_0 is a member of the right singular vectors of $\mathscr{M}$, then λ_N is a finite time Lyapunov exponent and Λ is a global Lyapunov exponent.

Since, for a given N, finite time Lyapunov exponents are functions of the initial states, it is useful to report values corresponding to a distribution of initial states to determine convergence. For an assessment of convergence of finite time Lyapunov exponents, one can sample the initial states from the invariant distribution, $\rho(x)$. In numerical computations, the invariant distribution is often not accessible in closed form. Nonetheless one can iterate forward an

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initial distribution $\rho_0(x)$ so that after N iterations $\rho_N(x)$ is an estimate of the invariant distribution. One would then sample initial states from $\rho_N(x)$ to estimate λ_N's. The aim here is to sample the initial states according to the invariant measure, if it exists. As a consequence of Oseledec's theorem [4], the distribution of the λ_N's will converge to a delta distribution centred at Λ as $N \to \infty$. The aforementioned approach is used in the numerical computations of the two maps in the next section.

It is ill-advised to use a single initial state to estimate Λ, even if the graph of λ_N versus N appears to approach a horizontal line. When one is faced with real data, the approach of the previous paragraph cannot be used unless there is a reliable mathematical model of the system. Nonetheless, one may use bootstrap approaches suggested in [10] for various values of N to assess convergence in the distribution of the λ_N's. In the ipod example, we consider only the maximal Lyapunov exponent and compute each λ_N as discuessed in [11].

THE TWO MAPS

Two maps and numerical computations of their Lyapunov exponents are discussed in this section. There is clear convergence of distributions of finite time Lyapunov exponents in the first map, but no convergence in the second.

Infinitely Piecewise Linear Map

Let us first consider the infinitely piece-wise linear map

$$\phi_k(x) = \begin{cases} \frac{2}{2^k}(x - \frac{1}{2}), & x \in [\frac{1}{2}, 1], \\ \\ 2^i(x - \frac{1}{2^i}), & x \in [2^{-i}, 2^{1-i}), \quad i = 2, 3, \ldots \end{cases} \tag{1}$$

defined for fixed $k \in \{0, 1, \ldots\}$. The invariant distribution of this map is the standard uniform distribution when $k = 0$ and can indeed be expressed in closed form for all k. Moreover, for any $k \in \{0, 1, \ldots\}$, it can be shown that the global Lyapunov exponent of the map is $\Lambda^{(k)} = 2$.

Numerical computations were performed for the case $k = 4$. We considered trajectories of length $N = 2^n$ starting from 2^{10} initial conditions, with $n = 0, \ldots, 20$. The resulting distributions of finite time Lyapunov exponents are shown in Fig 1 on the left. Evidently, with every doubling of the length of trajectories, the distribution of finite time Lyapunov exponents converges to a delta distribution centred at 2.

Paradoxical Map

The other map considered here is

$$\varphi(x) = \begin{cases} \frac{x}{1-x}, & x \in (0, 1/2), \\ \\ 2(1-x), & x \in (1/2, 1). \end{cases}$$

It has an unstable fixed point at the origin. The stability of the origin is determined using $\varphi'(x) = (1-x)^{-2}$. Clearly, $\varphi'(x) > 1$ for any $x > 0$. However, when x is close to zero then $\varphi'(x)$ is close to 1. Hence the origin is a weak repeller as pointed out in [12]. Indeed it is this property that makes the dynamics of this map paradoxical as was found in [12, 13] on a similar map that differs only by the linear part. If $\varphi^j(x) \in (0, 1/2)$ for all $j = 0, 1, \ldots, n-1$, then $\varphi^n(x) = x/(1-nx)$, where $\varphi^0(x) = x$. On the hand, if $\varphi^j(x) \in (1/2, 1)$ for all $j = 0, 1, \ldots, n-1$, then

$$\varphi^n(x) = 2\left[\frac{1-(-2)^n}{3} - (-2)^{n-1}x\right].$$

It is, therefore, evident that with probability one trajectories with initial conditions sampled according to the Lebesgue measure on $[0, 1]$ do not converge to the origin. However, it can be shown that iterates of distributions that are initially uniform converge to a delta distribution centred at the origin. These two contrasting behaviours of trajectories and densities are paradoxical and are due to the weak repeller at the origin.

Distributions of finite time Lyapunov exponents for varying N are shown in Fig 1 on the right. The initial conditions of trajectories used to estimate each λ_N were sampled from $U[0, 1]$. Notice that even when $N = 2^{25}$, the distribution still has not converged to a delta distribution. Those only familiar with behaviour reminiscent to the previous example could mistakenly attribute the bell shape here to uncertainty.

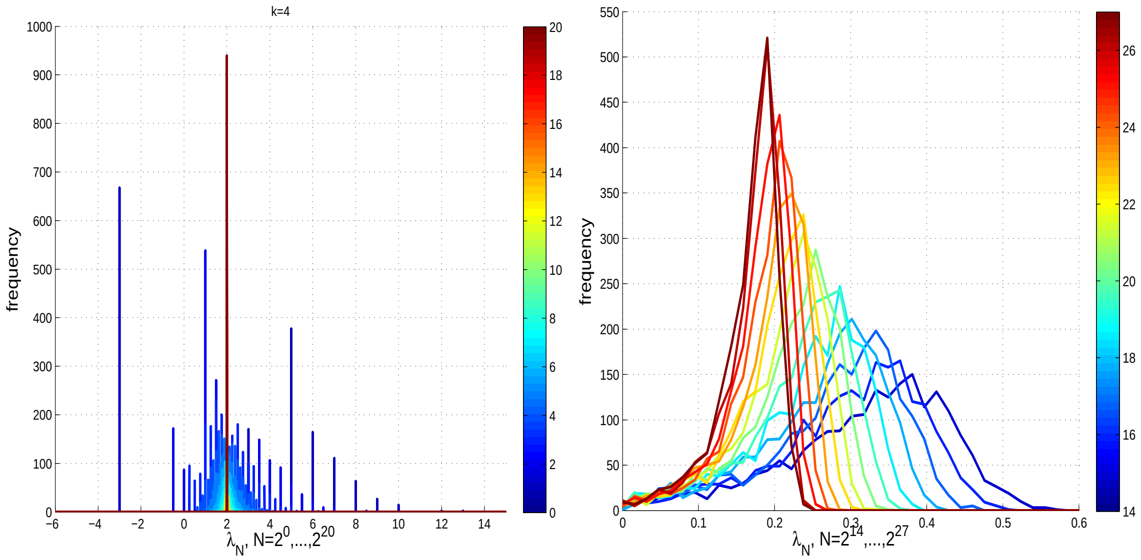

FIGURE 1. Distributions of finite time Lyapunov exponents for the infinitely piecewise map (left) and paradoxical map (right)

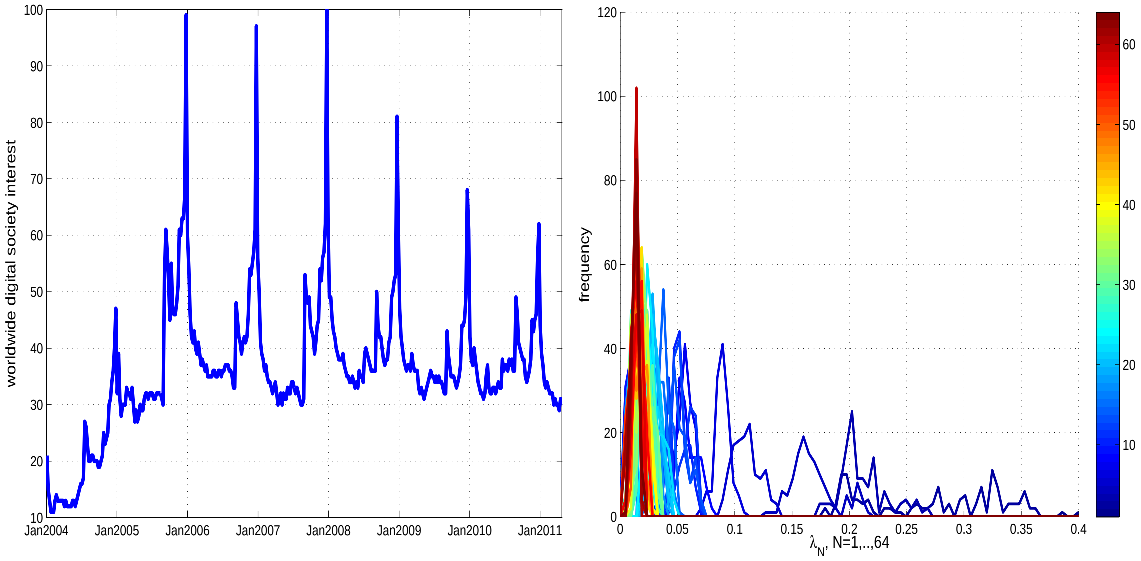

FIGURE 2. (Left) Time series of interest on ipod. (Right) Distributions of maximal finite time Lyapunov exponents.

WORLDWIDE IPOD INTEREST

As a practical example, we considered worldwide interest on ipod as captured by Google Insights. Google Insights is a search tool that provides insights on the searches that people enter on the Google search engine. Ipod is a media player by Apple that was launched in 2001. We entered the word *ipod* into Google Insights to obtain a time series of counts per week of the searches that people made since 2004. The counts provide a proxy of people's interest on the ipod. We will refer to the people who make searches on Google, the *digital society*. Therefore, Google Insights may be considered to track the interest of the digital society on a particular theme. Its url is http://www.google.com/insights/search/#. On Google Insights, one can search by country, or worldwide. For this study, we performed a worldwide search.

The time series of ipod interest is shown in Fig 2 on the left panel. Notice that from 2004, interest on ipod increased almost steadily until it assumed oscillatory behaviour. It was because of this oscillatory nature that ipod was chosen for this study. For further analyses, we discarded the time series prior to 2005 as transient behaviour. In order to compute distributions of finite time Lyapunov exponents, there are two parameters that have to be determined first. One of the parameters to determine is an appropriate dimension for embedding the scalar time series into higher dimensions. In order to determine the dimension, we first need the time delay. We selected the time delay as discussed in [14] and the dimension by the method of false nearest neighbours discussed in [15]. Suitable time delay and embedding dimension were found to be 9 weeks and 6 respectively.

To determine if the dynamics are chaotic or not, it suffices to estimate the maximal Lyapunov exponent. To this end, we appealed to the algorithm of Sato et al. [11], which can also be found in [16]. To obtain a distribution of finite time Lyapunov exponents, we performed block resampling as suggested in [10], each block being a quarter of the time series in length. The corresponding distributions of finite time Lyapunov exponents are shown in Fig 2 on the right panel. Like in the infinitely piecewise linear map, the distributions seem to converge to a delta distribution with the mode at $\lambda = 0.0141$ as N increases, after which they to diverge slightly. Such a value of the maximal Lyapunov exponent would mean that initial condition perturbations would, on the average, double after 48 weeks. At the same time, the distributions seem to be shifting to the left as was the case with the paradoxical map. If the dynamics do not change, more data could shed more light on the convergence properties of these distributions.

DISCUSSION

We have highlighted the importance of establishing convergence of finite time Lyapunov exponents to global Lyapunov exponents. In particular, we argued for a distributional approach to establishing convergence. Numerical computations were performed on two maps to illustrate the points. While there is clear convergence on one map, there is no convergence on the other: the reasons may be traced to the presence and absence of invariant distributions respectively.

Lessons learnt from the two maps may be applied to the dynamics of worldwide interest of the digital society on the ipod. For these dynamics, distributions of estimates of the maximal Lyapunov exponents appear to converge to a delta distribution. The estimated value suggests that the dynamics are very predictable. In terms of sales and marketing, this is very useful information. It affords forward planning with some assurance of stability in market behaviour. However, the time series is short and a lot more could be learnt when more data is accrued. In fact, semblance of the distributions to those of the map whose estimates showed no convergence suggests a cautious approach.

In previous studies, Gencay [17] performed a distributional study that was concerned with assessing statistical significance of the estimates but not convergence. Hence the distributional approach presented here is novel.

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