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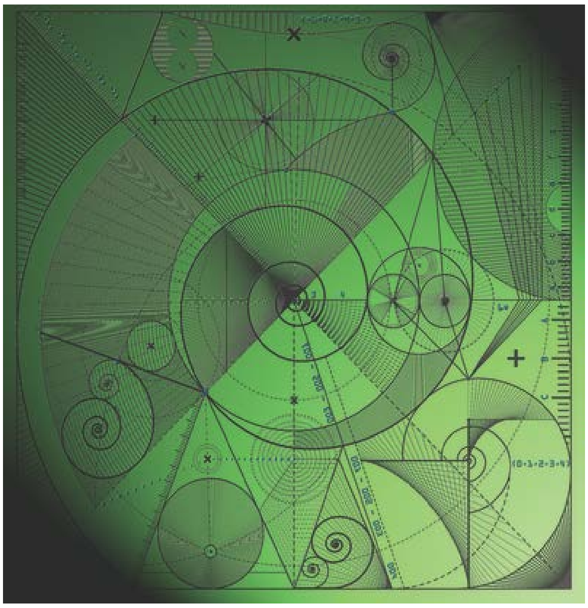

EXISTENCE OF DIM SOLUTIONS TO THE EQUATIONS OF VECTORIAL CALCULUS OF VARIATIONS IN L^∞

NIKOS KATZOURAKIS

Abstract. In the very recent paper [K] we introduced a new duality-free theory of generalised solutions which applies to fully nonlinear PDE systems of any order. As one of our first applications, we proved existence of vectorial solutions to the Dirichlet problem for the ∞-Laplace PDE system which is the analogue of the Euler-Lagrange equation for the functional $E_\infty(u,\Omega) = \|Du\|_{L^\infty(\Omega)}$. Herein we prove existence of a solution $u : \Omega \subseteq \mathbb{R} \longrightarrow \mathbb{R}^N$ to the Dirichlet problem for the system arising from the functional $E_\infty(u,\Omega) = \|H(\cdot,u,u')\|_{L^\infty(\Omega)}$. This is nontrivial even in the present $1D$ case, since the equations are non-divergence, highly nonlinear, degenerate, do not have classical solutions and standard approaches do not work. We further give an explicit example arising in variational Data Assimilation to which our result apply.

Contents

1. Introduction

Calculus of Variations in L^∞ has a long history and was pioneered by Aronsson in the 1960s [A1]-[A5]. In the simpler case of one space dimension, the basic object of study is the functional

$$E_\infty(u,\Omega) := \big\|H(\cdot,u,u')\big\|_{L^\infty(\Omega)}, \quad u : \Omega \subseteq \mathbb{R} \longrightarrow \mathbb{R}^N, \tag{1.1}$$

where $N \geq 1$ and $H \in C^2(\Omega \times \mathbb{R}^N \times \mathbb{R}^N)$ is a Hamiltonian function whose arguments will be denoted by (x,η,P). Aronssson was the first to note the locality problems associated to this functional and by introducing the appropriate minimality notion in L^∞, proved the equivalence between the so-called Absolute Minimisers

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and classical solutions of the "Euler-Lagrange" equation which is associated to the functional. The higher dimensional analogue $H(\cdot, u, Du)$ when $u : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ is a scalar function has also attracted considerable attention by the community, see e.g. [BEJ], [C] and for an elementary introduction [K8]. In particular, the Crandall-Ishii-Lions theory of Viscosity Solutions proved to be an indispensable tool in order to study the equations in L^∞ which are non-divergence, highly nonlinear and degenerate. Even in the simplest case where the Hamiltonian is the Euclidean norm, i.e. $H(P) = |P|^2$, in general the solutions are non-smooth and the corresponding PDE which is called ∞-Laplacian reads

$$\Delta_\infty \,:=\, Du \otimes Du : D^2 u \,=\, \sum_{i,j=1}^{N} D_i u\, D_j u\, D^2_{ij} u \,=\, 0. \tag{1.2}$$

However, until the early 2010s, the theory was essentially restricted to the scalar case $N = 1$. The main reason for this limitation was the absense of an effective theory of generalised solutions for non-divergence PDE system which would allow to study the counterparts of (1.2) emerging in the vectorial case. In a series of recent papers [K1]-[K7], the author has initiated the study of the vector-valued case, which except for its intrinsic mathematical interest it is also of paramount importance for applications. The results in the aforementioned papers include the study of the analytic properties of classical solutions to the fundamental equations and their connection to the supremal functional. In the case of

$$E_\infty(u, \Omega) \,=\, \big\| |Du|^2 \big\|_{L^\infty(\Omega)} \tag{1.3}$$

applied to Lipschitz mappings $u : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^N$ (where $|Du|$ denotes the Euclidean norm on $\mathbb{R}^{N\times n}$), the respective ∞-Laplace system is

$$\Delta_\infty u \,:=\, \Big(Du \otimes Du \,+\, |Du|^2 [Du]^\perp \otimes I \Big) : D^2 u \,=\, 0. \tag{1.4}$$

In (1.4), $[Du(x)]^\perp$ denotes the orthogonal projection on the nullspace of the linear map $Du(x)^\top : \mathbb{R}^N \longrightarrow \mathbb{R}^n$ and in index form (1.4) reads

$$\sum_{\beta=1}^{N} \sum_{i,j=1}^{n} \Big(D_i u_\alpha\, D_j u_\beta \,+\, |Du|^2 [Du]^\perp_{\alpha\beta}\, \delta_{ij} \Big) D^2_{ij} u_\beta \,=\, 0, \quad \alpha = 1, ..., N.$$

A further difficulty of (1.4) which is not present in the scalar case of (1.2) is that the coefficients may be discontinuous along interfaces even for C^∞ solutions because the term involving $[Du]^\perp$ measures the dimension of the tangent space of $u(\Omega) \subseteq \mathbb{R}^N$ (see [K1, K2]). This is a general vectorial phenomenon studied in some detail in [K3]. The appropriate minimality notion allowing to connect (1.4) to the functional (1.3) has been established in [K4]. In the $1D$ vectorial case of the supremal functional (1.1), the respective equations read

$$\begin{split} \Big(H_P \otimes H_P + H [H_P]^\perp H_{PP} \Big) u'' \;+\; H_P \big(H_\eta \cdot u' \,+\, H_x \big) \\ +\; H [H_P]^\perp \big(H_{P\eta} u' + H_{Px} - H_\eta \big) \;=\; 0 \end{split} \tag{1.5}$$

and apply to maps $u : \Omega \subseteq \mathbb{R} \longrightarrow \mathbb{R}^N$. In (1.5), the notation of subscripts denotes derivatives with respect to the respective variables and for the sake of brevity we have suppressed the arguments $\big(x, u(x), u'(x)\big)$ of $H_{PP}, H_{P\eta}, H_{Px}, H_P, H_\eta, H_x, H$. Moreover, $[H_P(x, \eta, P)]^\perp$ in this case reduces to the orthogonal projection on the hyperplane which is normal to the vector $H_P(x, \eta, P) \in \mathbb{R}^N$. The system (1.5) is a 2nd order ODE system which is quasilinear, non-divergence non-monotone and with discontinuous coefficients. Even in the scalar case of $N=1$, it is known since the work of Aronsson that (1.5) in general does not have solutions any more regular than just $C^1(\Omega,\mathbb{R}^N)$ and their "weak" interpretation is an issue.

Motivated in part by the necessity to study the nonlinear systems arising in L^∞, in the very recent paper [K] the author proposed a new theory of generalised solutions which applies to fully nonlinear PDE systems. In addition, this theory allows to interpret merely measurable mappings $u:\Omega\subseteq\mathbb{R}^n\longrightarrow\mathbb{R}^N$ as solutions of PDE systems which may even be defined by discontinuous nonlinearities and can be of any order:

$$F\big(\cdot,u,Du,D^2u...,D^pu\big)\,=\,0,\quad F\text{ Carathéodory map.}$$

In the above $Du,D^2u...,D^pu$ denote the derivatives of u of 1st, 2nd,... ,pth order respectively. Our approach is duality-free and bypasses the standard insuffiency of the theory of Distributions (and of weak solutions) to apply to even linear non-divergence equations with rough coefficients. The standing idea of the use of integration-by-parts in order to "pass derivatives to test functions" is replaced by a probabilistic description of the limiting behaviour of the difference quotients. This builds on the use of Young (Parameterised) measures over the compactification of the "state space", namely the space wherein the derivatives are valued. Background material on Young measures can be found e.g. in [P, FL, E, FG, M, V, CFV], but for the convenience of the reader we recall herein all the rudimentary properties we utilise in the paper.

The essential idea behind our new notion of solution is thoroughly explained later, but, at least for the case needed in this paper, it can be briefly motivated as follows. Assume that $u:\Omega\subseteq\mathbb{R}\longrightarrow\mathbb{R}^N$ is a strong a.e. solution of the system

$$F(\cdot,u,u',u'')\,=\,0,\qquad\text{on }\Omega. \tag{1.6}$$

We need a notion of solution which makes sense even if u is at most once differentiable. If u is twice differentiable, we have

$$F\Big(x,u(x),u'(x),\lim_{h\to0}D^{1,h}u'(x)\Big)\,=\,0,$$

for a.e. $x\in\Omega$, where $D^{1,h}$ stands for the difference quotient operator. By continuity, the limit commutes with the nonlinearity. Hence, we have

$$\lim_{h\to0}F\big(\cdot,u,u',D^{1,h}u'\big)\,=\,0,$$

a.e. on Ω. Note now that the above statement makes sense even if u is once differentiable, although as it stands does not look promising. In order to represent this limit in a convenient fashion, we embed the difference quotients $D^{1,h}u':\Omega\longrightarrow\mathbb{R}^N$ into the probability-valued maps (see the next section for the precise definitions) from Ω to the Alexandroff compactification $\overline{\mathbb{R}}^N$ and consider instead

$$\delta_{D^{1,h}u'}\;:\;\Omega\longrightarrow\mathscr{P}\big(\overline{\mathbb{R}}^N\big).$$

By the weak* compactness of this space, *regardless regularity of u there always exists a sequential limit $\mathcal{D}^2u$* of $\delta_{D^{1,h}u'}$ such that $\delta_{D^{1,h}u'}\overset{*}{\longrightarrow}\mathcal{D}^2u$ as $h\to0$. Then, it can be shown that for any "test function" $\Phi\in C^0_c\big(\mathbb{R}^N\big)$, we have

$$\int_{\overline{\mathbb{R}}^N}\Phi(X)\,F\big(x,u(x),u'(x),X\big)\,d\left[\mathcal{D}^2u(x)\right](X)\,=\,0,\quad\text{a.e. }x\in\Omega.$$

We emphasise that $\mathcal{D}^2 u$ always exists *independently of the twice differentiability of* u. If u'' happens to exist, then we have the extra information that $\mathcal{D}^2 u = \delta_{u''}$ a.e. on Ω and we reduce to strong solutions. This above property essentially consists the notion of **Dim Solutions** (in the special case of once differentiable solutions of 2nd order ODE systems) and will be taken as principal in this work.

As a first application of this new approach, in the paper [K] among other things we proved existence of Dim solutions to the Dirichlet problem for (1.4) when $n = N$. Herein we consider the relevant but different question of existence of Dim solutions to the Dirichlet problem for the ODE system (1.5). This is a non-trivial task even in the $1D$ case. In fact, it is not possible to be done in the generality of (1.1), (1.5) without structural conditions on H. The most important of these is that the Hamiltonian has to be radial in P. This means that there exist mappings

$$\begin{aligned} h\ &: \Omega \times \mathbb{R}^N \times [0,\infty) \longrightarrow \mathbb{R}, \\ V\ &: \Omega \times \mathbb{R}^N \longrightarrow \mathbb{R}^N, \end{aligned}$$

such that H can be written as

$$H(x,\eta,P)\ =\ h\left(x,\eta,\frac{1}{2}|P-V(x,\eta)|^2\right). \tag{1.7}$$

Unfortunately this condition is necessary, since as we proved in [K3] it is (roughly) both necessary and sufficient for the ODE system to be degenerate elliptic. Under the assumption (1.7), the system (1.5) becomes

$$\left\{ \begin{aligned} (h_p)^2\big|u'-V\big|^2\big(u''-V_\eta u'-V_x\big)\ =\ &-h_p\big(h_x+h_\eta\cdot u'\big)(u'-V) \\ &+\ h_p|u'-V|^2[u'-V]^\perp\Big(h_\eta\ -\ h_p(u'-V)^\top V_\eta\Big), \end{aligned} \right. \tag{1.8}$$

where for brevity we have omitted the arguments

$$\left(\cdot,u,\frac{1}{2}\big|u'-V(\cdot,u)\big|^2\right),\quad (\cdot,u)$$

of h_p, h_η, h_x, h and V, V_η, V_x respectively.

The table of contents gives an idea how this paper is organised. After developing some basic theory of Dim solutions in the special case of 2nd order ODE sytems required for the equations we are treating in this paper, we formally derive the equations (1.5) and (1.8) governing $1D$ vectorial problems in L^∞. A more detailed account of the Dim solutions' approach can be found in [K]. However, we chose to reproduce a large part of the relevant material of [K] herein as well because in the present case the machinery can be simplified largely.

Our central result Theorem 4.1 establishes existence of Dim solutions for (1.8) given any boundary conditions. Moreover, we obtain extra information regarding *partial regularity*: for, the Dim solution we obtain is actually C^2 on an open subset of the domain, whose complement is nowhere dense in $\mathbb{R}$ *but may not be a Lebesgue nullset*. This is a new type of partial regularity which seems to arise in L^∞. In addition, the solution is a limit of minimisers of the respective L^m functionals

$$E_m(u,\Omega)\ =\ \int_\Omega h\left(\cdot,u,\frac{1}{2}\big|u'-V(\cdot,u)\big|^2\right)^m$$

as $m \to \infty$. However, this does not prove that the Dim solution is any sort of minimiser to the supremal functional althoug it is a good candidate for this to be true. Moreover, this solution is a candidate for other useful properties, like uniqueness. This approach for existence is the "natural" one for L^∞ problems. This is in contrast to the method followed in [K]: therein we used an alternative approach for existence of solution to (1.4) based on the Dacorogna-Marcellini Baire category method [DM]. The Baire category method is an analytic counterpart to Gromov's geometric Convex Integration method. The idea in [K] was to construct Lipschitz continuous strong solutions to a 1st order differential inclusion of the form

$$Du(x) \in \mathcal{K} \subseteq \mathbb{R}^{n\times n}, \qquad \text{a.e. } x \in \Omega,$$

and subsequently use the machinery of Dim solutions to characterised it as a solution of (1.4).

We conclude this introduction by noting that the assumed form (1.7) of the Hamiltonian actually includes many interesting models. In particular, joint work with J. Bröcker [BK] suggests that one can replace standard models of variational Data Assimilation used in Meteorology (see [B, BS]) by the respective L^∞ counterparts (see Subsection 3.2). In principle this is very promising since minimisation in L^∞ excludes at the outset large spikes of the deviation and as a consequence it is expected to give better predictions.

2. Dim Solutions for fully nonlinear ODE systems

2.1. **Preliminaries and basics on Young measures.** Our notation is either standard (as e.g. in [E2, EG]) or self-explanatory. For example, the Lebesgue measure on $\mathbb{R}$ will be denoted by $|\cdot|$, the charasteristic function of the set A by χ_A, the standard Sobolev and L^p spaces of maps $u : \Omega \subseteq \mathbb{R} \longrightarrow \mathbb{R}^N$ by $L^p(\Omega, \mathbb{R}^N)$, $W^{m,p}(\Omega, \mathbb{R}^N)$, whilst if $N = 1$ we will write $L^p(\Omega, \mathbb{R}) = L^p(\Omega)$ etc. Moreover, we will follow the standard practice that while deriving estimates, constants may change from line to line but will be denoted by the same letter C.

In this paper, $N \in \mathbb{N}$ will always be the dimension of the range of our candidate solutions $u : \Omega \subseteq \mathbb{R} \longrightarrow \mathbb{R}^N$. Unless indicated otherwise, Greek indices $\alpha, \beta, \gamma, ...$ will run in $\{1, ..., N\}$ and the summation convention will be employed in products of repeated indices. The standard basis on $\mathbb{R}^N$ will be denoted by $\{e^\alpha\}$ and hence for the map u with components u_α we will write $u(x) = u_\alpha(x)e^\alpha$. The norm symbol $|\cdot|$ will always indicate the Euclidean one and the respective inner product will be denoted by "$\cdot$".

Given a vector $\xi \in \mathbb{R}^N \setminus \{0\}$, we consider the following orthogonal projections of $\mathbb{R}^N$ (on span$[\xi]$ and its normal hyperplane):

$$[\xi]^\top := \frac{\xi \otimes \xi}{|\xi|^2}, \quad [\xi]^\perp := I - \frac{\xi \otimes \xi}{|\xi|^2}. \tag{2.1}$$

If $\xi = 0$, we interpret the sign of zero as zero and hence then $[0]^\top = 0$, $[0]^\perp = I$.

Let now $E \subseteq \mathbb{R}$ be a (Lebesgue) measurable set and consider the Alexandroff 1-point compactification of the space $\mathbb{R}^N$:

$$\overline{\mathbb{R}}^N := \mathbb{R}^N \cup \{\infty\}.$$

Its topology will be the standard one which makes it homeomorphic to the sphere of the same dimension (via the stereographic projection which identifies $\{\infty\}$ with the north pole).

Definition 2.1 (Young Measures)**.** The space of Young (or Parameterised) Measures from E to $\overline{\mathbb{R}}^N$ is denoted by

$$\mathscr{Y}\big(E,\overline{\mathbb{R}}^N\big)$$

and is the set of probability-valued maps

$$\mathbb{R} \supseteq E \,\ni\, x \longmapsto \vartheta_x \,\in \mathscr{P}\big(\overline{\mathbb{R}}^N\big)$$

which are measurable with respect the x in the following sense: for any fixed open set $U \subseteq \overline{\mathbb{R}}^N$, then real function $E \ni x \mapsto \vartheta_x(U) \in [0,1]$ is (Lebesgue) measurable. This property is called *weak* measurability*.

The Young measures can be identified with a subset of the Banach space

$$L^\infty_{w^*}\big(E,\mathcal{M}\big(\overline{\mathbb{R}}^N\big)\big).$$

This space consists of weakly* measurable maps valued in the real (signed) Radon measures over $\overline{\mathbb{R}}^N$:

$$\mathbb{R} \supseteq E \,\ni\, x \longmapsto \vartheta_x \,\in \mathcal{M}\big(\overline{\mathbb{R}}^N\big).$$

The norm on the space $L^\infty_{w^*}\big(E,\mathcal{M}\big(\overline{\mathbb{R}}^N\big)\big)$ is

$$\|\vartheta\|_{L^\infty_{w^*}(E,\mathcal{M}(\overline{\mathbb{R}}^N))} \,:=\, \underset{x\in E}{\text{ess sup}}\,\|\vartheta_x\|\big(\overline{\mathbb{R}}^N\big)$$

where "$\|\cdot\|\big(\overline{\mathbb{R}}^N\big)$" is the total variation norm on $\mathcal{M}\big(\overline{\mathbb{R}}^N\big)$. For more details about this and relevant spaces we refer e.g. to [FL] (and references therein). Hence, the Young Measures are the subset of the unit sphere which consists of probability-valued weakly* measurable maps:

$$\mathscr{Y}(E,\overline{\mathbb{R}}^N) \,=\, \Big\{\vartheta \,\in\, L^\infty_{w^*}\big(E,\mathcal{M}\big(\overline{\mathbb{R}}^N\big)\big) \,:\, \vartheta_x \in \mathscr{P}\big(\overline{\mathbb{R}}^N\big),\text{ for a.e. } x\in E\Big\}.$$

It can be shown (see e.g. [FL]) that $L^\infty_{w^*}\big(E,\mathcal{M}\big(\overline{\mathbb{R}}^N\big)\big)$ is the dual space of the Banach space $L^1\big(E,C^0\big(\overline{\mathbb{R}}^N\big)\big)$:

$$\Big(L^1\big(E,C^0\big(\overline{\mathbb{R}}^N\big)\big)\Big)^* \,=\, L^\infty_{w^*}\big(E,\mathcal{M}\big(\overline{\mathbb{R}}^N\big)\big).$$

This L^1 space consists of strongly measurable maps valued in the (separable) space $C^0\big(\overline{\mathbb{R}}^N\big)$ of real continuous functions over $\overline{\mathbb{R}}^N$, in the standard Bochner sense. Its elements can be identified with a subset of the Carathéodory integrands

$$\Phi \;:\; E\times\overline{\mathbb{R}}^N \longrightarrow \mathbb{R}, \quad (x,X)\mapsto \Phi(x,X),$$

that is functions for which $x \mapsto \Phi(x,X)$ is measurable for evey $X \in \overline{\mathbb{R}}^N$ and $X \mapsto \Phi(x,X)$ is continuous for a.e. $x\in E$. The identification is given by considering Φ as a map $E \ni x \longmapsto \Phi(x,\cdot) \in C^0\big(\overline{\mathbb{R}}^N\big)$. The norm on this space is

$$\|\Phi\|_{L^1(E,C^0(\overline{\mathbb{R}}^N))} \,:=\, \int_E \big\|\Phi(x,\cdot)\big\|_{C^0(\overline{\mathbb{R}}^N)}\,dx.$$

The space $L^1\big(E,C^0\big(\overline{\mathbb{R}}^N\big)\big)$ is separable and the duality pairing

$$\langle\cdot,\cdot\rangle \;:\; L^\infty_{w^*}\big(E,\mathcal{M}\big(\overline{\mathbb{R}}^N\big)\big)\times L^1\big(E,C^0\big(\overline{\mathbb{R}}^N\big)\big) \longrightarrow \mathbb{R}$$

is given by

$$\langle \vartheta, \Phi \rangle := \int_E \int_{\overline{\mathbb{R}}^N} \Phi(x, X)\, d\vartheta_x(X)\, dx.$$

Since $L^1\big(E, C^0\big(\overline{\mathbb{R}}^N\big)\big)$ is separable, the unit ball of $L^\infty_{w^*}\big(E, \mathcal{M}\big(\overline{\mathbb{R}}^N\big)\big)$ is sequentially weakly* compact. Hence, for any bounded sequence $(\vartheta^m)_1^\infty \subseteq L^\infty_{w^*}\big(E, \mathcal{M}\big(\overline{\mathbb{R}}^N\big)\big)$, there is a limit map ϑ and a subsequence of m's along which $\vartheta^m \overset{*}{\longrightarrow} \vartheta$ as $m \to \infty$. Moreover, we have

$$\vartheta^m \overset{*}{\longrightarrow} \vartheta \quad \Longleftrightarrow \quad \langle \vartheta^m - \vartheta, \Phi \rangle \longrightarrow 0, \ \text{ for all } \Phi \in L^1\big(E, C^0\big(\overline{\mathbb{R}}^N\big)\big).$$

Further, by the density of the products of the form $\phi(x)\Phi(X)$, where $\phi \in L^1(E)$ and $\Phi \in C^0\big(\overline{\mathbb{R}}^N\big)$, for bounded sequences the weak* convergence $\vartheta^m \overset{*}{\longrightarrow} \vartheta$ is equivalent for any fixed $\Phi \in C^0\big(\overline{\mathbb{R}}^N\big)$ to

$$\int_{\overline{\mathbb{R}}^N} \Phi(X)\, d\big[\vartheta^m - \vartheta\big](X) \overset{*}{\longrightarrow} 0, \quad \text{ in } L^\infty(E).$$

Remark 2.2 (Properties of $\mathscr{Y}(E, \overline{\mathbb{R}}^N)$)**.** The set of Young measures is convex and by the compactness of $\overline{\mathbb{R}}^N$, it can be proved that it is sequentially weakly* compact in $L^\infty_{w^*}\big(E, \mathcal{M}\big(\overline{\mathbb{R}}^N\big)\big)$ (see e.g. [FG, CFV]). This property is essential in our setting. Moreover, the set of Lebesgue measurable functions $v : E \subseteq \mathbb{R}^n \longrightarrow \overline{\mathbb{R}}^N$ has weakly* dense image in $\mathscr{Y}\big(E, \overline{\mathbb{R}}^N\big)$ under the imbedding $v \mapsto \delta_v$ which is given by $\delta_v(x) := \delta_{v(x)}$.

The following lemma is a small variant of a standard result about Young measures but it plays an important role in our setting.

Lemma 2.3. *Suppose $v^m, v^\infty : \Omega \subseteq \mathbb{R} \longrightarrow \mathbb{R}^N$ are measurable maps, $m \in \mathbb{N}$, where Ω is open and $|\Omega| < \infty$. Then, there exists subsequences $(m_k)_1^\infty$, $(m_l)_1^\infty$ such that:*

$$\begin{aligned} v^m \longrightarrow v^\infty, \ \text{a.e. on } \Omega \quad &\Longrightarrow \quad \delta_{v^{m_k}} \overset{*}{\longrightarrow} \delta_{v^\infty}, \ \text{ in } \mathscr{Y}\big(\Omega, \overline{\mathbb{R}}^N\big), \\ \delta_{v^m} \overset{*}{\longrightarrow} \delta_{v^\infty} \ \text{ in } \mathscr{Y}\big(\Omega, \overline{\mathbb{R}}^N\big) \quad &\Longrightarrow \quad v^{m_l} \longrightarrow v^\infty, \ \text{a.e. on } \Omega. \end{aligned}$$

In the following subsection we motivate the notion of solution we will use in this work in the special case of locally Lipschitz continuous solutions of 2nd order fully nonlinear ODE systems.

2.2. **Motivation of the notion of solution.** Suppose $\Omega \subseteq \mathbb{R}$ is an open set and

$$F \ : \ \Omega \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$$

is a Carathéodory mapping. Assume that $u : \Omega \subseteq \mathbb{R} \longrightarrow \mathbb{R}^N$ is a $W^{2,\infty}_{\text{loc}}(\Omega, \mathbb{R}^N)$ strong a.e. solution of the system

$$F(\cdot, u, u', u'') \,=\, 0, \quad \text{ on } \Omega. \tag{2.2}$$

We are looking for a notion of generalised solution which makes sense even if $u \in W^{1,\infty}_{\text{loc}}(\Omega, \mathbb{R}^N)$ and hence when the solution is only once differentiable a.e. on Ω. Since u is assumed twice differentiable a.e. on Ω, we have

$$F\Big(x, u(x), u'(x), \lim_{h \to 0} D^{1,h} u'(x)\Big) \,=\, 0,$$

for a.e. $x \in \Omega$, where $D^{1,h}$ stands for the standard difference quotient operator. But since F is a Carathéodory map, the limit commutes with the coefficients:

$$\lim_{h\to 0} F\big(\cdot, u, u', D^{1,h}u'\big) \,=\, 0, \tag{2.3}$$

a.e. on Ω. Note now that this statement makes sense if u is once differentiable and the limit when taken outside may exist even when it may not exist when it is put back inside. However, the above as it stands does not look useful since we need a way to represent this limit. Going back to (2.2), we observe that u is a strong solution of (2.2) if and only it satisfies

$$\int_{\mathbb{R}^N} \Phi(X)\, F(\cdot, u, u', X)\, d[\delta_{u''}](X) \,=\, 0, \quad \text{a.e. on } \Omega,$$

for any compactly supported "test" function $\Phi \in C^0_c\big(\mathbb{R}^N\big)$. This gives the idea that we can embed the difference quotient map $D^{1,h}u' : \Omega \longrightarrow \mathbb{R}^N$ into the spaces of Young measures and consider instead the Dirac measure evaluated at the difference quotients of u':

$$\delta_{D^{1,h}u'} \;:\; \Omega \longrightarrow \mathscr{P}\big(\overline{\mathbb{R}}^N\big).$$

We use the Alexandroff compactification $\overline{\mathbb{R}}^N$ instead of $\mathbb{R}^N$ and attach the point at ∞ in order to avoid the loss of mass of the measures and retain weak* compactness for the space $\mathscr{Y}\big(\Omega, \overline{\mathbb{R}}^N\big)$. Thus, by weak* compactness, *regardless regularity of* u *there always exists* a sequential limit $\mathcal{D}^2u$ of $\delta_{D^{1,h}u'}$ in the Young measures such that, along a subsequence we have

$$\delta_{D^{1,h}u'} \overset{*}{\longrightarrow} \mathcal{D}^2u, \quad \text{in } \mathscr{Y}\big(\Omega, \overline{\mathbb{R}}^N\big), \tag{2.4}$$

as $h \to 0$. Then, by imbedding the ODE system into the space of Young measures, (2.3) is equivalent to

$$\int_{\overline{\mathbb{R}}^N} \Phi(X)\, F\big(\cdot, u, u', X\big)\, d\big[\delta_{D^{1,h}u'}\big](X) \,\longrightarrow\, 0, \quad \text{a.e. on } \Omega,$$

along a sequence as $h \to 0$, for any $\Phi \in C^0_c\big(\mathbb{R}^N\big)$. Hence, by passing to the limit, a simple argument involving Egoroff's theorem allows to deduce that for any $\Phi \in C^0_c\big(\mathbb{R}^N\big)$ we have

$$\int_{\overline{\mathbb{R}}^N} \Phi(X)\, F\big(\cdot, u, u', X\big)\, d[\mathcal{D}^2u](X) \,=\, 0, \quad \text{a.e. on } \Omega.$$

We stress now that this statement is *independent of the twice differentiability of the solution of* (2.2). In the event that u' is differentiable a.e. on Ω and u'' is measurable, we have $\mathcal{D}^2u = \delta_{u''}$ a.e. on Ω and by applying Lemma 4.4 we go back to strong solutions of (2.2).

2.3. **Main definitions and analytic properties.** Now we give the main definitions and some rudimentary properties of our notion of solution only in the special case which is needed in this paper for the equations in L^∞. For the general case we refer to [K].

Definition 2.4 (Dim 2nd Derivatives)**.** Let Ω be an open set in $\mathbb{R}$ and suppose $u : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^N$ is in $W^{1,\infty}_{\text{loc}}(\Omega, \mathbb{R}^N)$. We define the **Dim 2nd Derivatives** of

u as the subsequential limits $\mathcal{D}^2 u$ of the 1st difference quotients of the derivative u' in the space of Young measures $\mathscr{Y}\big(\Omega, \overline{\mathbb{R}}^N\big)$:

$$\delta_{D^{1,h_i}u'} \overset{*}{\longrightarrow} \mathcal{D}^2 u, \qquad , \text{ in } \mathscr{Y}\big(\Omega, \overline{\mathbb{R}}^N\big), \quad \text{as } i \to \infty.$$

Here $(h_i)_1^\infty \subseteq \mathbb{R} \setminus \{0\}$ is any infinitesimal sequence and

$$D^{1,h_i}u' \ : \ \Omega \subseteq \mathbb{R} \longrightarrow \mathbb{R}^N, \qquad i \in \mathbb{N},$$

is the 1st difference quotient of the derivative u', localised inside an h_i-inner neighbourhood of Ω:

$$D^{1,h_i}u'(x) \ := \ \chi_{\Omega^i}(x)\left(\frac{u'(x+h_i) - u'(x)}{h_i}\right),$$
$$\Omega^i \ := \ \big\{x \in \Omega \ : \ \mathrm{dist}(x, \partial\Omega) > h_i\big\}.$$

By analogy, one can define the Dim derivatives of any order in the obvious way but we will not need them in this work.

Remark 2.5 (Adaptive difference quotients)**.** In the present case the main concept of Dim derivative is largely simplified over the general case of [K]. Namely, for the current $1D$ problem we can obtain existence without being required to use special adaptive difference quotients, as it happens e.g. for the ∞-Laplace PDE system. The term *adaptivity* is borrowed from numerical analysis and in this $1D$ context it means that the step sizes of the difference quotients depend on the basepoint:

$$D^{1,h_i(x)}u'(x) \ = \ \frac{u'\big(x + h_i(x)\big) - u'(x)}{h_i(x)}, \quad (h_i)_1^\infty \subseteq L^\infty(\mathbb{R}),\ h_i > 0 \text{ a.e. on } \Omega.$$

This can be avoided for the fundamental ODE system of L^∞ we are considering herein, but *in general it may not be possible.* For the general case, the reader may see [K].

The weak* compactness of the set of Young measures implies the following result:

Lemma 2.6 (Existence of Dim Derivatives)**.** *Every locally Lipschitz mapping* $u : \Omega \subseteq \mathbb{R} \longrightarrow \mathbb{R}^N$ *possesses at least one Dim 2nd derivative. Moreover,* u *has at least one* $\mathcal{D}^2 u$ *for every choice of infinitesimal sequence* $(h_i)_1^\infty$*.*

We note that in general Dim derivatives may not be unique for non-differentiable maps. The next lemma claims the rather obvious fact that Dim derivatives are compatible with classical derivatives and is an immediate consequence of Lemma 4.4.

Lemma 2.7 (Compatibility of classical/strong with Dim derivatives)**.** *Let* $u : \Omega \subseteq \mathbb{R} \longrightarrow \mathbb{R}^N$ *be a map in* $W^{1,\infty}_{loc}(\Omega, \mathbb{R}^N)$*. If* u *is twice differentiable a.e. on* Ω *and* u'' *is measurable, then the Dim 2nd derivative* $\mathcal{D}^2 u$ *is unique and*

$$\delta_{u''} \ = \ \mathcal{D}^2 u, \quad \textit{a.e. on } \Omega.$$

The next notion of solution will be central in this work.

Definition 2.8 (Dim Solutions of 2nd order ODE systems)**.** Let $\Omega \subseteq \mathbb{R}$ be an open set and

$$F \ : \ \Omega \times \Big(\mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N\Big) \longrightarrow \mathbb{R}^N$$

a Carathéodory map. Consider the ODE system

$$F\big(\cdot, u, u', u''\big) \,=\, 0, \quad \text{on } \Omega. \tag{2.5}$$

We say that the $W^{1,\infty}_{\text{loc}}(\Omega,\mathbb{R}^N)$ map $u : \Omega \subseteq \mathbb{R} \longrightarrow \mathbb{R}^N$ is a Dim Solution of (2.5) when there exist a Dim 2nd derivative $\mathcal{D}^2 u$ with respect to an infinitesimal sequence $(h_i)_1^\infty$:

$$\delta_{D^{1,h_i}u'} \overset{*}{\longrightarrow} \mathcal{D}^2 u, \quad \text{in } \mathscr{Y}\big(\Omega, \overline{\mathbb{R}}^N\big), \quad \text{as } i \to \infty,$$

such that

$$\int_{\overline{\mathbb{R}}^{Nn^2}_s} \Phi(X)\, F\big(\cdot, u, u', X\big)\, d[\mathcal{D}^2 u](X) \,=\, 0, \quad \text{a.e. on } \Omega,$$

for any $\Phi \in C^0_c\big(\mathbb{R}^N\big)$.

Remark 2.9 (Absence of concentrations)**.** Fix a $\Phi \in C^0_c\big(\mathbb{R}^N\big)$ and let $u : \Omega \subseteq \mathbb{R} \longrightarrow \mathbb{R}^N$ be a locally Lipschitz map and F a Carathéodory map as above. Then, we have the estimate

$$\begin{aligned}\left|\int_{\overline{\mathbb{R}}^N} \Phi\big(X\big)\, F\big(\cdot, u, u', X\big)\, d[\mathcal{D}^2 u](X)\right| \,&\leq\, \int_{\overline{\mathbb{R}}^N} \Big|\Phi\big(X\big)\, F\big(\cdot, u, u', X\big)\Big|\, d[\mathcal{D}^2 u](X)\\ &\leq\, \|\Phi\|_{C^0(\mathbb{R}^N)} \max_{X \in \text{supp}(\Phi)} \big|F\big(\cdot, u, u', X\big)\big|.\end{aligned}$$

Hence, "... $= 0$ a.e. on Ω" in the definition above is equivalent to "... $= 0$ in $L^\infty(\Omega)$". Thus, the left hand side is always a measurable map and no lower-dimensional measures supported on nullsets can arise.

The following proposition is an easy consequence of Lemma 2.7 and asserts that Dim and strong solutions are compatible when the Dim solution is twice differentiable. For further details we refer to [K].

Proposition 2.10 (Compatibility of Dim Solutions with Strong/classical Solutions)**.** *Let $\Omega \subseteq \mathbb{R}$ be open and F a Carathéodory map. Consider the ODE system*

$$F\big(\cdot, u, u', u''\big) \,=\, 0, \quad \text{on } \Omega.$$

Suppose also that $u : \Omega \subseteq \mathbb{R} \longrightarrow \mathbb{R}^N$ is a $W^{1,\infty}_{loc}(\Omega,\mathbb{R}^N)$ map. Then, if u is Dim solution of the system which is twice differentiable a.e. on Ω and u'' is measurable, then u is a strong solution. Conversely, if u is a strong solution of the system on Ω in the sense that u'' exists a.e. and is measurable, then, u is a Dim solution.

In [K] the reader may find alternative equivalent formulations of the definitions of Dim solutions. In particular, it can be shown that the locally Lipschitz map $u : \Omega \subseteq \mathbb{R} \longrightarrow \mathbb{R}^N$ is a Dim solution of

$$F\big(\cdot, u, u', u''\big) \,=\, 0, \quad \text{on } \Omega,$$

if and only if for a.e. $x \in \Omega$, the restriction on $\mathbb{R}^N$ of the Dim 2nd derivative $\mathcal{D}^2_* u(x) := \mathcal{D}^2 u(x) \llcorner \mathbb{R}^N$ (i.e. off $\{\infty\}$) satisfies the differential inclusion

$$\text{supp}\big(\mathcal{D}^2_* u(x)\big) \,\subseteq\, \Big\{F\big(x, u(x), u'(x), \cdot\big) = 0\Big\},$$

where "'supp" denotes the support. We will not this equivalent formulation in this paper.

We conclude this elementary introduction to Dim solutions by noting that, clearly, our analytic approach is much more malleable and effective than the cumbersome algebraic theories of the type of "multiplication of distributions" (see e.g.

[Co]). The latter approaches present some serious inconsistencies when applied to differential equations.

3. Derivation of the fundamental equations in L^∞

In this section we formally derive the fundamental equations associated to variational problems for the supremal functional

$$E_\infty(u,\Omega) := \operatorname*{ess\,sup}_{x\in\Omega} H(x,u(x),u'(x)),$$

when $H \in C^2(\Omega\times\mathbb{R}^N\times\mathbb{R}^N)$ is a nonnegative Hamiltonian,$\Omega \subseteq \mathbb{R}$ is open and $N \geq 1$. The next derivation has been performed in [K3] in the general higher-dimensional case, but we include it here because it provides further insights since the method of proof makes the foregoing formal calculations rigorous. For the sake of completeness of the exposition, we give all the steps of the derivation.

We begin by noting that since the functional is not Gateaux differentiable, we can not obtain the equations by taking variations as in the integral case. Instad, we obtain the equations in the limit of the Euler-Lagrange equations related to the L^m integral functional

$$E_m(u,\Omega) := \int_\Omega \big(H(x,u(x),u'(x))\big)^m\,dx, \tag{3.1}$$

as $m\to\infty$. Here we suppose that $m\geq 2$ and u is a map $u:\Omega\subseteq\mathbb{R}\longrightarrow\mathbb{R}^N$. The Euler-Lagrange equation of (3.1) is the ODE system

$$\Big(H^{m-1}(\_,u,u')H_P(\_,u,u')\Big)' \;=\; H^{m-1}(\_,u,u')H_\eta(\_,u,u'). \tag{3.2}$$

Evidently, the subscripts denote derivatives with respect to the respective argument. By distrbuting derivatives and normalising, the equation (3.2) gives

$$\big(H(\_,u,u')\big)'H_P(\_,u,u') \;+\; \frac{H(\_,u,u')}{m-1}\Big(\big(H_P(\_,u,u')\big)' - H_\eta(\_,u,u')\Big) \;=\; 0, \tag{3.3}$$

on $\Omega\subseteq\mathbb{R}$. Then, by employing (2.1) applied to $\xi = H_P(\_,u,u')$ we expand the term in the bracket of (3.3) and obtain

$$\begin{aligned}\big(H(\_,u,u')\big)'H_P(\_,u,u') + \frac{H(\_,u,u')}{m-1}[H_P(\_,u,u')]^\top\Big(\big(H_P(\_,u,u')\big)' - H_\eta(\_,u,u')\Big)\\ = -\,\frac{H(\_,u,u')}{m-1}[H_P(\_,u,u')]^\perp\Big(\big(H_P(\_,u,u')\big)' - H_\eta(\_,u,u')\Big).\end{aligned} \tag{3.4}$$

By mutual orthogonality of the projections, the left and right hand side of (3.4) are normal to each other. Hence, they both have to vanish and (3.4) splits to two independent ODE systems. Therefore, by supressing for brevity the arguments $(\_,u,u')$, we get the pair of systems

$$\big(H\big)'H_P \;+\; \frac{H}{m-1}[H_P]^\top\big((H_P)' - H_\eta\big) \;=\; 0, \tag{3.5}$$

$$H[H_P]^\perp\big((H_P)' - H_\eta\big) \;=\; 0. \tag{3.6}$$

By expansion of derivatives, we further get

$$H_P\Big(H_P^\top u'' + H_\eta^\top u' + H_x\Big) + \frac{H[H_P]^\top}{m-1}\Big(H_{PP}u'' + H_{P\eta}u' + H_{Px} - H_\eta\Big) = 0, \tag{3.7}$$

$$H[H_P]^\perp\Big(H_{PP}u'' + H_{P\eta}u' + H_{Px} - H_\eta\Big) = 0. \tag{3.8}$$

As $m\to\infty$, we obtain the complete ODE system for a general Hamiltonian when we have dependence on all the arguments $(x, u(x), u'(x))$:

$$H_P\Big(H_P^\top u'' + H_\eta^\top u' + H_x\Big) = 0, \tag{3.9}$$

$$H[H_P]^\perp\Big(H_{PP}u'' + H_{P\eta}u' + H_{Px} - H_\eta\Big) = 0. \tag{3.10}$$

By mutual perpendicularity of the differential operators in (3.9), (3.10), we have that this pair of systems is equivalent to the single system

$$\Big(H_P\otimes H_P + H[H_P]^\perp H_{PP}\Big)u'' + H_P\big(H_\eta\cdot u' + H_x\big) + H[H_P]^\perp\big(H_{P\eta}u' + H_{Px} - H_\eta\big) = 0.$$

3.1. **The degenerate elliptic case of the system.** Unfortunately, it is not in general possible to obtain existence of solution of the equations (3.9), (3.10) without imposing structural conditions on the hamiltonian H. The problem is that the systems (3.9), (3.10) fail to be elliptic in the sense needed for existence. In particular, *the coefficient of* (3.10) *may be discontinuous at points where* $H_P(\cdot, u, u') = 0$. In this subection we derive the appropriate degenerate elliptic version. It was shown in [K3], actually in a more general setting, that we need to assume that H depends on u' in a radial fashion. This forces the matrices $[H_P]^\perp$ an H_{PP} to commute, making the coeffients of the system continuous and allowing them to "match" to a single ODE system. Hence, we will henceforth assume that H has the form

$$H(x,\eta,P) := h\Big(x,\eta,\frac{1}{2}\big|P - V(x,\eta)\big|^2\Big) \tag{3.11}$$

where

$$\begin{aligned} h &: \overline{\Omega}\times\mathbb{R}^N\times[0,\infty)\longrightarrow[1,\infty), \quad (x,\eta,p)\mapsto h(x,\eta,p),\\ V &: \overline{\Omega}\times\mathbb{R}^N\longrightarrow\mathbb{R}^N, \qquad\qquad (x,\eta)\mapsto V(x,\eta), \end{aligned} \tag{3.12}$$

are C^2 maps up to the boundary, while the p-partial derivative $h_p(x,\eta,p)$ of h is strictly positive. There is no loss of generality to assume as we have done that $h\geq 1$, since if h is bounded below, we can always add a positive constant to h and the equations remain the same because additive constants commute with the supremal functional (and this constant also regularises the L^m functional). By assuming (3.11), the respective functionals become

$$E_\infty(u,\Omega) = \Big\|h\Big(\cdot,u,\frac{1}{2}\big|u' - V(\cdot,u)\big|^2\Big)\Big\|_{L^\infty(\Omega)}, \tag{3.13}$$

$$E_m(u,\Omega) = \int_\Omega h\Big(\cdot,u,\frac{1}{2}\big|u' - V(\cdot,u)\big|^2\Big)^m. \tag{3.14}$$

A model case of this class of functionals is the one arising in variational Data Assimilation and is given in the next subsection. We now derive the equations correspoding to both the L^m functional and the L^∞ functional for the specific form of H as in (3.11). We first differentiate (3.11) and for the sake of brevity we suppress the arguments

$$\begin{aligned} H_{P_\alpha} &= h_p(P-V)_\alpha, \\ H_{\eta_\alpha} &= h_{\eta_\alpha} - h_p(P-V)_\gamma V_{\gamma\eta_\alpha}, \\ H_x &= h_x - h_p(P-V)_\gamma V_{\gamma x}, \\ H_{P_\alpha P_\beta} &= h_{pp}(P-V)_\alpha (P-V)_\beta + h_p \delta_{\alpha\beta} \end{aligned}$$

and

$$\begin{aligned} H_{P_\alpha x} &= -h_p V_{\alpha x} + \big[h_{px} - h_{pp}(P-V)_\gamma V_{\gamma x}\big](P-V)_\alpha, \\ H_{P_\alpha \eta_\beta} &= -h_p V_{\alpha\eta_\beta} + (P-V)_\alpha \big[h_{p\eta_\beta} - h_{pp}(P-V)_\gamma V_{\gamma\eta_\beta}\big]. \end{aligned}$$

Then, by using the identity

$$\big[h_p(P-V)\big]^\top = [P-V]^\top, \tag{3.15}$$

which is a consequence of the assumption we have made that $h_p > 0$ and by grouping terms, the equation (3.7) after a calculation gives

$$\begin{aligned} &\left\{ \frac{h[u'-V]^\top}{m-1}\Big(h_p I + h_{pp}(u'-V)\otimes(u'-V)\Big) \right. \\ &\quad \left. + (h_p)^2 (u'-V)\otimes(u'-V) \right\}\big(u'' - V_\eta u' - V_x\big) \\ &\qquad = -\frac{h[u'-V]^\top}{m-1}\left[h_p(u'-V)^\top V_\eta - h_\eta + \big(h_{p\eta}\cdot u' + h_{px}\big)(u'-V)\right] \\ &\qquad\quad - h_p\big(h_x + h_\eta \cdot u'\big)(u'-V). \end{aligned} \tag{3.16}$$

Similarly, by the identity (3.15) and since $h/(m-1) > 0$, (3.8) gives

$$\begin{aligned} [u'-V]^\perp &\left\{ \Big(h_p I + h_{pp}(u'-V)\otimes(u'-V)\Big)\big(u'' - V_\eta u' - V_x\big) \right. \\ &\quad \left. - h_\eta + h_p(u'-V)^\top V_\eta + \big(h_{p\eta}\cdot u' + h_{px}\big)(u'-V) \right\} = 0. \end{aligned} \tag{3.17}$$

Since the projection $[u'-V]^\perp$ annihilates $u'-V$, (3.17) simplifies to

$$h_p [u'-V]^\perp \big(u'' - V_\eta u' - V_x\big) = [u'-V]^\perp \Big(h_\eta - h_p(u'-V)^\top V_\eta\Big). \tag{3.18}$$

We now observe that in view of the identities (2.1), the two systems (3.16) and (3.18) can be matched and the discontinuous but mutually singular coefficients $[u'-V]^\top$ and $[u'-V]^\perp$ add to the identity. For, by multiplying (3.18) by $h_p|u'-V|^2$ and adding it to (3.16), we obtain

(3.19)
$$\left\{\begin{array}{l} \left[\dfrac{h\big(h_p + h_{pp}|u'-V|^2\big)[u'-V]^\top}{m-1} \,+\, (h_p)^2|u'-V|^2 I\right]\big(u''-V_\eta u' - V_x\big) \\ \qquad = -\dfrac{h[u'-V]^\top}{m-1}\left[-h_\eta \,+\, h_p(u'-V)^\top V_\eta \,+\, \big(h_{p\eta}\cdot u + h_{px}\big)(u'-V)\right] \\ \qquad\quad -\, h_p\big(h_x + h_\eta \cdot u'\big)(u'-V) \\ \qquad\quad +\, h_p|u'-V|^2[u'-V]^\perp\Big(h_\eta \,-\, h_p(u'-V)^\top V_\eta\Big). \end{array}\right.$$

The ODE system (3.19) is the Euler-Lagrange equation of the functional (3.14) in expanded form. The left hand side is the only one containing 2nd derivatives.

Remark 3.1 (Notation)**.** We recall that for the sake of brevity we have suppressed in the above ODE system the dependence on the arguments

$$\left(\cdot, u, \frac{1}{2}\big|u' - V(\cdot,u)\big|^2\right)$$

of the mappings h, h_p, h_η, h_x and $h_{pp}, h_{p\eta}, h_{px}$. The same is true for the arguments $(\cdot, u)$ of the mappings V, V_η, V_x. We also note that the coefficients which are of order $O\big(\frac{1}{m-1}\big)$ remain discontinuous, but this causes no problems since the terms invloving these will be annihilated as $m \to \infty$.

By letting $m \to \infty$ in (3.19) and by our assumption that $h_p > 0$, we (formally) get the fundamental ODE system in L^∞ which corresponds to the supremal functional (3.13):

$$\text{(3.20)} \qquad \left\{\begin{array}{l} (h_p)^2|u'-V|^2\big(u''-V_\eta u' - V_x\big) \,=\, -h_p\big(h_x + h_\eta\cdot u'\big)(u'-V) \\ \qquad\qquad\qquad +\, h_p|u'-V|^2[u'-V]^\perp\Big(h_\eta \,-\, h_p(u'-V)^\top V_\eta\Big). \end{array}\right.$$

We now rewrite the above systems (3.19), (3.20) in a more compact form. We define the mapping (recall Remark 3.1)

$$\text{(3.21)} \qquad \begin{aligned} F^\infty(\cdot,u,u') \,:=\, &-\, h_p\big(h_x + h_\eta\cdot u'\big)(u'-V) \\ &+\, (h_p)^2|u'-V|^2[u'-V]^\perp\Big(h_\eta \,-\, h_p(u'-V)^\top V_\eta\Big), \end{aligned}$$

and the mappings

$$\text{(3.22)} \qquad \begin{aligned} f^\infty(\cdot,u,u') \,&:=\, -h[u'-V]^\top\Big[-h_\eta \,+\, h_p(u'-V)^\top V_\eta \\ &\qquad\qquad +\, \big(h_{p\eta}\cdot u + h_{px}\big)(u'-V)\Big], \\ A^\infty(\cdot,u,u') \,&:=\, h\big(h_p + h_{pp}|u'-V|^2\big)[u'-V]^\top. \end{aligned}$$

Then, the Euler-Lagrange system (3.19) of the L^m functional (3.13) can be written as

(3.23)
$$\left\{\frac{A^\infty(\cdot,u,u')}{m-1} + h_p^2\Big(\cdot,u,\frac{1}{2}\big|u'-V(\cdot,u)\big|^2\Big)\big|u'-V(\cdot,u)\big|^2 I\right\}\Big[u'' - \big(V(\cdot,u)\big)'\Big] = \frac{f^\infty(\cdot,u,u')}{m-1} \,+\, F^\infty(\cdot,u,u')$$

and the L^∞ system (3.20) arising from the supremal functional (3.14) can be written as

$$h_p^2\Big(\cdot, u, \frac{1}{2}\big|u' - V(\cdot,u)\big|^2\Big)\big|u' - V(\cdot,u)\big|^2\Big[u'' - \big(V(\cdot,u)\big)'\Big] \,=\, F^\infty(\cdot,u,u') \tag{3.24}$$

where $F^\infty, f^\infty, A^\infty$ are given by (3.21), (3.22).

3.2. **A model of Data Assimilation in Meteorology.** (This subsection is a result of the discussions with J. Bröcker, who we would like to thank.) Suppose $\Omega \subseteq \mathbb{R}$ is a bounded interval and let us choose the Hamiltonian

$$H^*(x,\eta,P) \,:=\, 1 + \frac{1}{2}\big|k(x) - K(\eta)\big|^2 + \frac{1}{2}\big|P - V(x,\eta)\big|^2,$$

where $N, M \in \mathbb{N}$ and

$$\begin{aligned} k &: \Omega \subseteq \mathbb{R} \longrightarrow \mathbb{R}^M, \\ K &: \mathbb{R}^N \longrightarrow \mathbb{R}^M, \\ V &: \Omega \times \mathbb{R}^N \longrightarrow \mathbb{R}^N \end{aligned}$$

are all C^1 mappings. In the notation of the previous subsection, H^* corresponds to the choice of

$$h^*(x,\eta,p) \,:=\, 1 + \frac{1}{2}\big|k(x) - K(\eta)\big|^2 + p.$$

In standard variational Data Assimilation models (see [B, BS]), one seeks find minimisers $u : \Omega \subseteq \mathbb{R} \longrightarrow \mathbb{R}^N$ of the functional

$$E(u,\Omega) \,=\, \int_\Omega H^*(\cdot,u,u').$$

The motivation to study this particular problem comes from the applications to the Earth Sciences and especially Meterolory. In mathematical terms, the question is the following: let V be a time-dependent vector field describing the law of motion (e.g. Newtonian forces or finite-dimensional Galerkin approximation of the Euler equations). Let also k be a map of some partial "measurements" in continuous time along the trajectory. The map K is a submersion which corresponds to some component of the projection we are able to measure, for example some projection. Then, we wish to find the actual solution $u : \Omega \subseteq \mathbb{R} \longrightarrow \mathbb{R}^N$, which should satisfy the law of motion and should also be compatible with the measurements:

$$\left\{ \begin{aligned} u'(t) \,&=\, V\big(t,u(t)\big), \quad t \in \Omega, \\ K(u(t)) \,&=\, k(t), \qquad\quad\;\; t \in \Omega. \end{aligned} \right.$$

However, this problem is in general overdetermined (due to errors in the measurements etc) since we impose a pointwise manifold constraint to the solution of the ODE system. Hence, minimisation of E allows to find approximate solutions to this overdetermined problem, by minimising the deviation in L^2. But if instead of E we choose to use the respective supremal functional

$$E_\infty(u,\Omega) \,=\, \big\|H^*(\cdot,u,u')\big\|_{L^\infty(\Omega)},$$

then it is expected that minimisation in L^∞ will give better results: in this case, large "spikes" of the deviation from the actual solution (namely, the value of H^*) with small area are from the outset excluded (see [BK]). For the choice of $H =$

H^*, the fundamental equations in the space L^∞ arising from the model of Data Assimilation in Meteorology reads

$$\begin{aligned} &\big|u' - V(\cdot,u)\big|^2 \Big\{ u'' - \big(V(\cdot,u)\big)' \\ &\quad - \big[u' - V(\cdot,u)\big]^\perp \Big(\big(K(u)-k\big)^\top K_\eta(u) \, - \, \big(u'-V(\cdot,u)\big)^\top V_\eta(\cdot,u) \Big) \Big\} \\ &= \Big[K_\eta(u) : \big(K(u)-k\big) \otimes u' \, + \, \big(K(u)-k\big)\cdot k_x \Big] \big(u' - V(\cdot,u)\big). \end{aligned}$$

Our main existence result applies in particular to this ODE system. although the L^∞ equations are more complicated than the respective L^2 Euler-Lagrange equations, evidence [BK] suggest hat they provide better models.

4. Existence of Dim solutions to the equations in L^∞

In this section we prove existence of Dim solutions to the Dirichlet problem for the fundamental equation (3.20) arising from variational problems of the functional

$$E_\infty(u,\Omega) \,=\, \Big\| h\Big(\cdot, u, \frac{1}{2}\big|u' - V(\cdot,u)\big|^2\Big)\Big\|_{L^\infty(\Omega)},$$

where $\Omega \subseteq \mathbb{R}$ and $N \geq 1$. The following is the principal result of this work.

Theorem 4.1 (Existence of Dim solutions to the Dirichlet Problem)**.** *Let $\Omega \subseteq \mathbb{R}$ be a bounded interval and let*

$$\begin{aligned} h &: \overline{\Omega} \times \mathbb{R}^N \times [0,\infty) \longrightarrow [1,\infty), \\ V &: \overline{\Omega} \times \mathbb{R}^N \longrightarrow \mathbb{R}^N, \end{aligned}$$

be given maps, $N \geq 1$. We suppose that

$$\text{(4.1)} \qquad \left\{ \begin{array}{r} h \text{ is } C^2 \text{ up to the boundary,} \\ C(|\eta|) \,\geq\, h_p(x,\eta,p) \,\geq\, c_0, \\ 2h_{pp}(x,\eta,p)p \,+\, h_p(x,\eta,p) \,\geq\, c_0, \\ \big|h_x(x,\eta,p)\big|, \big|h_\eta(x,\eta,p)\big| \,\leq\, C(|\eta|)(1\,+\,p), \\ \big|h_{pp}(x,\eta,p)\big|, \big|h_{p\eta}(x,\eta,p)\big|, \big|h_{px}(x,\eta,p)\big| \,\leq\, C(|\eta|)(1\,+\,p^M), \end{array} \right.$$

$$\text{(4.2)} \qquad \left\{ \begin{array}{l} V \text{ is } C^1 \text{ up to the boundary,} \\ \big|V(x,\eta)\big| \,\leq\, (1/c_0)(1\,+\,|\eta|^\alpha), \end{array} \right.$$

for some constants $c_0, \alpha \in (0,1)$, some $M \in \mathbb{N}$, some positive continuous increasing function $C \in C^0([0,\infty))$ and all $(x,\eta,p) \in \Omega \times \mathbb{R}^N \times [0,\infty)$.

Then, for any affine map b the ODE system (3.20) has a Lipschitz continuous Dim solution $u^\infty : \overline{\Omega} \subseteq \mathbb{R} \longrightarrow \mathbb{R}^N$ with $u = b$ at the endpoints $\partial\Omega$. In addition, the following stronger properties holds:

(1) For ***any*** *infinitesimal sequence $0 < |h_i| \to 0$, there is a subsequence $(h_{i_j})_{j=1}^\infty$ and a Dim 2nd derivative arising from that subsequence*

$$\delta_{D^{1,h_{i_j}} u^{\infty\prime}} \overset{*}{\longrightarrow} \mathcal{D}^2 u^\infty, \quad \text{ in } \mathscr{Y}\big(\Omega, \overline{\mathbb{R}}^N\big), \quad \text{as } j \to \infty,$$

such that, for any $\Phi \in C^0_c(\mathbb{R}^N)$, the map u^∞ satisfies

$$\int_{\overline{\mathbb{R}}^N} \Phi(X)\Big\{h_p^2\Big(\cdot,u,\frac{1}{2}\big|u' - V(\cdot,u)\big|^2\Big)\big|u' - V(\cdot,u)\big|^2 \cdot \Big[X - \big(V(\cdot,u)\big)'\Big] \, - \, F^\infty(\cdot,u,u')\Big\}\, d[\mathcal{D}^2 u](X) \,=\, 0,$$

a.e. on Ω. Here F^∞ is given by (3.21).

(2) The Dim solution u^∞ is a sequential uniform limit on Ω of C^2 minimisers in $W^{1,2m}_b(\Omega,\mathbb{R}^N)$ of the integral functionals (3.14) and a fortiori of smooth solutions to (3.19).

(3) There is an open subset $\Omega_\infty \subseteq \Omega$ such that the Dim solution u^∞ is in $C^2(\Omega_\infty,\mathbb{R}^N)$. Moreover, Ω_∞ is given by

$$\Omega_\infty \,:=\, \big\{\, u^{\infty\prime} \neq V(\cdot,u^\infty)\,\big\} \cup \mathit{int}\big\{\, u^{\infty\prime} = V(\cdot,u^\infty)\,\big\}.$$

Remark 4.2. i) Note that the conclusion (1) above is stronger than the general Definition 2.8 of Dim solutions since actually the desired integral formula is satisfied for all infinitesimal sequences. Since this is appears to be an exception rather than the rule for general systems, we have not considered it useful to re-define Dim solutions in this stronger sense.

ii) The conclusion of (2) says that the Dim solution u^∞ is a uniform limit of L^m minimisers as $m \to \infty$, however this does *not* prove that the Dim solution is an L^∞ minimal map of the supremal functional in the sense of the variational characterisation of [K4]. Investigation of this question is left for future work. Notwithstanding, the solution we construct in this way is a "good" solution of the system, as opposed to the solutions we construct for the ∞-Laplace system in [K] via "analytic convex integration", namely the Dacorogna-Marcellini Baire category method (see the remarks in [K] about uniqueness to the Dirichlet problem).

iii) The conclusion of (3) is a partial regularity result which says that all possible Dim 2nd derivatives coincide on Ω_∞ and in addition

$$\mathcal{D}^2 u^\infty \,=\, \delta_{u^{\infty\prime\prime}} \quad \text{a.e. on } \Omega_\infty.$$

However, this differs from classical partial regularity results in that the complement of Ω_∞ is Ω is a closed nowhere dense set (a topological boundary) *but not necessarily of zero Lebesgue measure.* Moreover, it leaves open the possibility of lower dimensional concentration measures supported on $\Omega \setminus \Omega_\infty$.

The proof of Theorem 4.1 is split to several lemmas. The first one below shows that the minimisation problem of E_m in $W^{1,2m}_b(\Omega,\mathbb{R}^N)$ has a solution u^m, and also the minimisers u^m converge weakly to a candidate u^∞.

Lemma 4.3 (Existence of minimisers and convergence)**.** *Let h, V, Ω, b satisfy the assumptions of Theorem 4.1. Then, for any affine map b the functional (3.14) has a local minimisers in the space $W^{1,2m}_b(\Omega,\mathbb{R}^N)$. Moreover, we have the following estimate*

$$\|u\|_{W^{1,2m}(\Omega)} \,\leq\, C\Big(E_m(u,\Omega)^{\frac{1}{2m}} \,+\, \max_{\partial\Omega}|b| \,+\, 1\Big) \tag{4.3}$$

where $C > 0$ depends only on the assumptions and the length of Ω. In addition, there is a subsequence $(m_k)_1^\infty$ and $u^\infty \in W_b^{1,\infty}(\Omega, \mathbb{R}^N)$ such that

$$\begin{cases} u^m \longrightarrow u^\infty, & \text{in } C^0(\overline{\Omega}, \mathbb{R}^N), \\ u^{m\prime} \rightharpoonup u^{\infty\prime}, & \text{in } L^p(\Omega, \mathbb{R}^N), \text{ for any } p > 2, \end{cases}$$

along $m_k \to \infty$, and also

$$\|u^\infty\|_{W^{1,\infty}(\Omega)} \le C, \tag{4.4}$$

where the constant C depends only on the assumptions, b and Ω.

Proof of Lemma 4.3. Step 1. We begin by recording some elementary inequalities we will use in the sequel. For any $t \ge 0$, $0 < \alpha < 1$ and $\varepsilon > 0$, Young's inequality gives

$$t^\alpha \le \varepsilon t + \left(\frac{\alpha}{\varepsilon}\right)^{\frac{\alpha}{1-\alpha}} (1-\alpha). \tag{4.5}$$

Moreover, for any $P, V \in \mathbb{R}^N$ and $0 < \delta < 1$, we also have

$$(1-\delta)|P|^2 \le |P-V|^2 + \frac{1}{\delta}|V|^2. \tag{4.6}$$

Finally, for any $u \in W^{1,2m}(\Omega, \mathbb{R}^N)$, we have the following Poincaré inequality *which is uniform in $m \in \mathbb{N}$*:

$$\|u\|_{L^{2m}(\Omega)} \le 2(|\Omega|+1)\Big(\|u'\|_{L^{2m}(\Omega)} + \max_{\partial\Omega}|u|\Big). \tag{4.7}$$

Indeed, in order to see (4.7), suppose u is smooth and since $\big|u(x) - u(y)\big| \le \int_\Omega |u'|$, for $y \in \partial\Omega$ we have

$$\begin{aligned} |u(x)|^{2m} &\le \left(\int_\Omega |u'| + \max_{\partial\Omega}|u|\right)^{2m} \\ &\le 2^{2m-1}\left[\left(\int_\Omega |u'|\right)^{2m} + \max_{\partial\Omega}|u|^{2m}\right] \\ &\le (2(|\Omega|+1))^{2m-1}\left[\int_\Omega |u'|^{2m} + \max_{\partial\Omega}|u|^{2m}\right], \end{aligned}$$

which leads to (4.7).

Step 2. We now show that the functional E_m is weakly lower semicontinuous in $W^{1,2m}(\Omega, \mathbb{R}^N)$. Indeed, by setting

$$F(x, \eta, P) := h^m\Big(x, \eta, \frac{1}{2}\big|P - V(x,\eta)\big|^2\Big), \tag{4.8}$$

we have for the hessian with respect to P that (we suppress arguments again)

$$F_{PP} = mh^{m-2}\Big\{hh_p I + \big(hh_{pp} + (m-1)(h_p)^2\big)(P-V)\otimes(P-V)\Big\}.$$

By our assumptions on h and since the projection $[P-V]^\top$ satisfies $[P-V]^\top \le I$, we obtain the matrix inequality

$$\begin{aligned} F_{PP} &\ge mh^{m-2}\Big\{hh_p I + hh_{pp}(P-V)\otimes(P-V)\Big\} \\ &\ge mh^{m-1}\Big(h_p I + (c_0 - h_p)[P-V]^\top\Big) \\ &\ge mc_0 I. \end{aligned}$$

Since F is convex in P and nonnegative, the conclusion follows by standard lower semicontinuity results (see e.g. [D, GM]).

Step 3. Now we use Steps 1, 2 and derive the energy estimate, which will guarantee the coercivity of E_m. By our assumptions on h and the mean value theorem, there is a $\hat p \in [0,p]$ such that

$$h(x,\eta,p) \,=\, h_p(x,\eta,\hat p)p \,+\, h(x,\eta,0) \,\geq\, c_0 p \,+\, 1.$$

Hence, by using (4.6) the above gives

$$\begin{aligned} h\Big(x,\eta,\frac{1}{2}\big|P-V(x,\eta)\big|^2\Big) \,&\geq\, \frac{c_0}{2}\big|P-V(x,\eta)\big|^2 \\ &\geq\, \frac{c_0}{2}(1-\delta)|P|^2 \,-\, \frac{c_0}{2\delta}|V(x,\eta)|^2. \end{aligned} \tag{4.9}$$

Then, by our assumption on V and (4.5), (4.6), for $\sigma > 0$ small we have

$$\begin{aligned} h\Big(x,\eta,\frac{1}{2}\big|P-V(x,\eta)\big|^2\Big) \,&\geq\, \frac{c_0}{2}(1-\delta)|P|^2 \,-\, \frac{1}{2c_0\delta}(1\,+\,|\eta|^\alpha)^2 \\ &\geq\, \frac{c_0}{2}(1-\delta)|P|^2 \,-\, \frac{\sigma}{c_0\delta}|\eta|^2 \,-\, C(\delta,\sigma,\alpha), \end{aligned}$$

where $C(\delta,\sigma,\alpha)$ denotes a constant depending only on these numbers. We now select

$$\delta \,:=\, \frac{1}{2}, \quad \sigma \,:=\, 2c_0\varepsilon, \quad \varepsilon > 0,$$

to find

$$h\Big(x,\eta,\frac{1}{2}\big|P-V(x,\eta)\big|^2\Big) \,\geq\, \frac{c_0}{4}|P|^2 \,-\, \varepsilon|\eta|^2 \,-\, C(\varepsilon,\alpha).$$

Hence, for any $m \in \mathbb{N}$ by the Hölder inequality and the above estimate, we have the bound

$$\frac{1}{3^{m-1}}\Big(\frac{c_0}{4}\Big)^m|P|^{2m} \,\leq\, h^m\Big(x,\eta,\frac{1}{2}\big|P-V(x,\eta)\big|^2\Big) \,+\, \varepsilon^m|\eta|^{2m} \,+\, C(\varepsilon,\alpha)^{2m}.$$

Consequently, for any $u \in W^{1,2m}_b(\Omega,\mathbb{R}^N)$, by integrating over Ω and by utilising (4.7) and (3.14), we have

$$\begin{aligned} 3\Big(\frac{c_0}{12}\Big)^m\int_\Omega|u'|^{2m} \,&\leq\, E_m(u,\Omega) \,+\, \varepsilon^m\int_\Omega|u|^{2m} \,+\, C(\varepsilon,\alpha)^{2m}|\Omega| \\ &\leq\, E_m(u,\Omega) \,+\, C(\varepsilon,\alpha)^{2m}|\Omega| \\ &\quad\, +\, \varepsilon^m\big(2(|\Omega|+1)\big)^{2m}\left\{\max_{\partial\Omega}|b|^{2m} \,+\, \int_\Omega|u'|^{2m}\right\}. \end{aligned}$$

Hence, we obtain the estimate

$$\left\{\Big(\frac{c_0}{12}\Big)^m \,-\, \big(4(|\Omega|+1)^2\varepsilon\big)^m\right\}\int_\Omega|u'|^{2m} \,\leq\, E_m(u,\Omega) \,+\, C^{2m}\Big(\max_{\partial\Omega}|b|^{2m} \,+\, 1\Big)$$

where the constant C above depends on $\varepsilon,\alpha,\Omega$. By choosing

$$\varepsilon \,:=\, \frac{c_0}{3\,2^5(|\Omega|+1)^2},$$

we get

$$\left\{\frac{c_0}{12}\Big(1-\frac{1}{2^m}\Big)^{\frac{1}{m}}\right\}^m\int_\Omega|u'|^{2m} \,\leq\, E_m(u,\Omega) \,+\, C^{2m}\Big(\max_{\partial\Omega}|b|^{2m} \,+\, 1\Big)$$

and since $\lim_{m\to\infty}\big(1-2^{-m}\big)^{1/m} = 1$, the desired estimate (4.3) ensues.

Step 4. We finally show existence of minimisers and convergence. We have the a priori energy bounds

$$\begin{aligned} \inf\Big\{E_m(v,\Omega)^{\frac{1}{2m}}\,:\, v\in W^{1,2m}_b(\Omega,\mathbb{R}^N)\Big\} \,&\leq\, E_m(b,\Omega)^{\frac{1}{2m}} \\ &\leq\, \left(\int_\Omega h^m\Big(\cdot,b,\frac{1}{2}\big|b'-V(\cdot,b)\big|^2\Big)\right)^{\frac{1}{2m}} \\ &\leq\, |\Omega|^{\frac{1}{2m}}\left\|h\Big(\cdot,b,\frac{1}{2}\big|b'-V(\cdot,b)\big|^2\Big)\right\|^{\frac{1}{2}}_{L^\infty(\Omega)} \end{aligned}$$

and

$$E_m(v,\Omega)\,\geq\,0,\quad v\in W^{1,2m}_b(\Omega,\mathbb{R}^N).$$

Hence, by standard results (see e.g. [D, GM]), there exists a global minimiser u^m of the functional E_m in $W^{1,2m}_b(\Omega,\mathbb{R}^N)$. Moreover, by (4.3) and (4.11) we have the bound

$$\|u^m\|_{W^{1,2m}(\Omega)}\,\leq\, C\left(\left\|h\Big(\cdot,b,\frac{1}{2}\big|b'-V(\cdot,b)\big|^2\Big)\right\|^{\frac{1}{2}}_{L^\infty(\Omega)}+\max_{\partial\Omega}|b|\,+\,1\right). \tag{4.10}$$

Let $C(\Omega,b)$ denote the right hand side of (4.10). Then, for any $r\in(2,m)$, we have

$$\begin{aligned} \|u^m\|_{W^{1,2r}(\Omega)}\,&\leq\,|\Omega|^{\frac{1}{2r}-\frac{1}{2m}}\|u^m\|_{W^{1,2m}(\Omega)} \\ &\leq\,|\Omega|^{\frac{1}{2r}-\frac{1}{2m}}C(\Omega,b). \end{aligned}\tag{4.11}$$

Hence, for any $r>2$ fixed, the sequence $(u^m)_1^\infty$ is bounded in $W^{1,2r}_b(\Omega,\mathbb{R}^N)$. Hence, there exists a $u^\infty\in\cap_{r=1}^\infty W^{1,2r}_b(\Omega,\mathbb{R}^N)$ such that $u^m \longrightarrow u^\infty$ in $W^{1,2r}_b(\Omega,\mathbb{R}^N)$ along a subsequence $m_k\to\infty$. By letting $m\to\infty$ in (4.11) along the subsequence, the weak lower semicontinuity of the L^{2r} norm implies

$$\|u^\infty\|_{W^{1,2r}(\Omega)}\,\leq\,|\Omega|^{\frac{1}{2r}}C(\Omega,b).$$

By letting now $r\to\infty$, we derive the desired bound for u^∞. The lemma ensues. □

Next, we show that the minimisers just obtained actually are weak solutions of the respective Euler-Lagrange equations.

Lemma 4.4 (Weak solutions of the L^m equations)**.** *Let h,V,Ω,b satisfy the assumptions of Theorem 4.1 and let $(u^m)_1^\infty$ be the sequence of minimisers constructed in Lemma 4.3. Then, each u^m is a weak solution in $W^{1,2m}_b(\Omega,\mathbb{R}^N)$ of the Euler-Lagrange equation of (3.14):*

(4.12)

$$\left\{\begin{aligned} &\Big[h^{m-1}\Big(\cdot,u,\frac{1}{2}\big|u'-V\big|^2\Big)h_p\Big(\cdot,u,\frac{1}{2}\big|u'-V(\cdot,u)\big|^2\Big)\,(u'-V)\Big]' \\ &=\,h^{m-1}\Big(\cdot,u,\frac{1}{2}\big|u'-V\big|^2\Big)h_p\Big(\cdot,u,\frac{1}{2}\big|u'-V\big|^2\Big)\cdot \\ &\quad\cdot\Big[h_\eta\Big(\cdot,u,\frac{1}{2}\big|u'-V\big|^2\Big)\,-\,h_p\Big(\cdot,u,\frac{1}{2}\big|u'-V\big|^2\Big)\big(u'-V(\cdot,u)\big)^\top V_\eta(\cdot,u)\Big] \end{aligned}\right.$$

By omitting for brevity the arguments $\big(\cdot,u,\frac{1}{2}\big|u'-V\big|^2\big)$ of h,h_p,h_η and $(\cdot,u)$ of V,V_η, the ODE system (4.12) can be compactly written as

$$\Big(h^{m-1}h_p\,(u'-V)\Big)'\,=\,h^{m-1}h_p\,\Big(h_\eta\,-\,h_p(u'-V)^\top V_\eta\Big).$$

Proof of Lemma 4.4. Let F be given by (4.8). Then, by suppressing once again the arguments of h, h_p, h_η, we have that

$$\begin{aligned} F_P(x,\eta,P) \,&=\, mh^{m-1}h_p\,\big(P - V(x,\eta)\big),\\ F_\eta(x,\eta,P) \,&=\, mh^{m-1}h_p\,\Big(h_\eta \,-\, h_p\big(P - V(x,\eta)\big)^\top V_\eta(x,\eta)\Big) \end{aligned}$$

and the system (4.12) can be written compactly as

$$\big(F_P(\cdot,u,u')\big)' \,=\, F_\eta(\cdot,u,u'). \tag{4.13}$$

By our assumption on h, we have

$$h(x,\eta,p) \,\leq\, h(x,\eta,0) \,+\, \max_{0\leq \bar p\leq p} h_p(x,\eta,\bar p)\,p \,\leq\, C(|\eta|)(1\,+\,p).$$

Hence,

$$\begin{aligned} h^{m-1}\Big(x,\eta,\frac{1}{2}\big|P - V(x,\eta)\big|^2\Big) \,&\leq\, C(|\eta|)\,\Big(1\,+\,\big|P - V(x,\eta)\big|^2\Big)^{m-1}\\ &\leq\, C(|\eta|)\,\Big(1\,+\,\big|P - V(x,\eta)\big|^{2m-2}\Big). \end{aligned}$$

Further, by our assumptions on h and V, we have the bounds

$$\begin{aligned} \big|F_P(x,\eta,P)\big| \,&\leq\, C(|\eta|)\big(1\,+\,|P|^{2m-2}\big)\big|P - V(x,\eta)\big|\\ &\leq\, C(|\eta|)\big(1\,+\,|P|^{2m-2}\big)\big(|P| + C(|\eta|)\big)\\ &\leq\, C(|\eta|)\big(1\,+\,|P|^{2m-1}\big), \end{aligned} \tag{4.14}$$

and

$$\begin{aligned} \big|F_\eta(x,\eta,P)\big| \,&\leq\, C(|\eta|)\big(1\,+\,|P|^{2m-2}\big)\Big[C(|\eta|)\,\Big(1 + \big|P - V(x,\eta)\big|^2\Big)\\ &\qquad +\, C(|\eta|)\big|P - V(x,\eta)\big|\Big]\\ &\leq\, C(|\eta|)\big(1\,+\,|P|^{2m-2}\big)C(|\eta|)\big(|P|^2 + 1\big)\\ &\leq\, C(|\eta|)\big(1\,+\,|P|^{2m}\big). \end{aligned} \tag{4.15}$$

By standard results (see e.g. [D]), these bounds imply that the functional is Gateaux differentiable on $W^{1,2m}_b(\Omega,\mathbb{R}^N)$ and the lemma follows. ☐

Now we show that the weak solutions u^m of the respective Euler-Lagrange equations actually are smooth solutions. This will imply that the formal calculations of the previous section in the derivation of (3.19) make rigorous sense.

Lemma 4.5 (C^2 regularity)**.** *Let u^m be the sequence of minimisers of the Lemma 4.5, $m \geq 2$. Then, each u^m is a classical solution in $C^2(\Omega,\mathbb{R}^N)$ of the Euler-Lagrange equation (4.12), and hence of the expanded form (3.19) of the same equation.*

Proof of Lemma 4.5. Fix $m \geq 2$ and let us drop the superscripts and denote u^m by just u. The first step is to prove higher local integrability and then bound the difference quotients of u' in L^2. Let us fix $q \in \mathbb{N}$ and $\zeta \in C^\infty_c(\Omega)$ with $0 \leq \zeta \leq 1$. We set:

$$\phi(x) \,:=\, \zeta(x)\int_{\inf\Omega}^x \zeta\big|u' - V(\cdot,u)\big|^q\big(u' - V(\cdot,u)\big), \quad x\in\Omega. \tag{4.16}$$

Then, $\phi \in W^{1,1}_c(\Omega,\mathbb{R}^N)$ and

$$\begin{aligned}\phi'(x) \,=\, &\zeta^2(x)\Big|u'(x)-V\big(x,u(x)\big)\Big|^q \,\big(u'(x)-V\big(x,u(x)\big)\big)\\ &+\,\zeta'(x)\int_{\inf\Omega}^{x}\zeta\big|u'-V(\cdot,u)\big|^q\big(u'-V(\cdot,u)\big),\end{aligned}$$

for a.e. $x\in\Omega$. Suppose now that $q\leq 2m-1$. Then, since $u'\in L^{2m}(\Omega,\mathbb{R}^N)$, we have that $\phi\in W^{1,2m}_c(\Omega,\mathbb{R}^N)$. By inserting the test function ϕ in the weak formulation of the system (4.13) (i.e. (4.12)) and by suppressing again the arguments for the sake of brevity, we have

$$\begin{aligned}\int_\Omega &\left\{h^{m-1}h_p(u'-V)\cdot\left[\zeta^2|u'-V|^q(u'-V)\,+\,\zeta'\int_{\inf\Omega}\zeta|u'-V|^q(u'-V)\right]\right\}\\ &+\,\int_\Omega\left\{h^{m-1}h_p\left(h_\eta-h_p(u'-V)^\top V_\eta\right)\cdot\left[\zeta\int_{\inf\Omega}\zeta|u'-V|^q(u'-V)\right]\right\}\,=\,0.\end{aligned}$$

By our assumptions on h,V, we have that $h_p\geq c_0$ and $2h\geq c_0|u'-V|^2$. By using the bounds (4.14), (4.15) (when F is given by (4.8)) that $\zeta\leq 1$ and the elementary inequalities

$$\begin{aligned}\int_{\inf\Omega}^x|f|\,&\leq\,\int_\Omega|f|,\qquad x\in\Omega,\quad f\in L^1(\Omega),\\ t^{2m-1}\,&\leq\,t^{2m}+1,\quad t\geq 0,\end{aligned}$$

we have

$$\begin{aligned}\int_\Omega\zeta^2|u'-V|^{2m+q}\,\leq\,C\left(\int_\Omega\zeta|u'-V|^{q+1}\right)&\left\{\int_\Omega|\zeta'|\left(h^{m-1}h_p\,|u'-V|\right)\,+\right.\\ &\left.+\,\int_\Omega\zeta\left(h^{m-1}h_p\left|h_\eta-h_p(u'-V)^\top V_\eta\right|\right)\right\}\end{aligned}$$

which gives

$$\begin{aligned}\int_\Omega\zeta^2|u'-V|^{2m+q}\,\leq\,C\big(\|u\|_{L^\infty(\Omega)}\big)&\left(\int_\Omega\zeta|u'-V|^{q+1}\right)\cdot\\ &\cdot\int_\Omega\left\{|\zeta'|\Big(1+|u'-V|^{2m-1}\Big)\,+\,\zeta\Big(1+|u'-V|^{2m}\Big)\right\}.\end{aligned}$$

Hence, we have obtained

$$\int_\Omega\zeta^2|u'-V|^{2m+q}\,\leq\,C\big(\|u\|_{L^\infty(\Omega)}\big)\left(\int_\Omega\zeta|u'-V|^{q+1}\right)\int_\Omega 1+|u'-V|^{2m}. \tag{4.17}$$

In view of the estimate (4.17), by taking $q+1=2m$ we have that $u'-V\in L^{4m-1}_{\text{loc}}(\Omega,\mathbb{R}^N)$. Hence, we can go back and choose $q+1=4m-1$ and then the test function satisfies $\phi\in W^{1,4m-1}_c(\Omega,\mathbb{R}^N)$ which makes it admissible and we can repeat the process. Hence, by applying the estimate again we infer that $u'-V\in L^{6m-2}_{\text{loc}}(\Omega,\mathbb{R}^N)$. By continuing, the induction principle says that the estimate holds for all integers of the form

$$q\,=\,(2m-1)k,\quad k\in\mathbb{N}$$

and we obtain that $u'-V\in\cap_{r=1}^\infty L^r_{\text{loc}}(\Omega,\mathbb{R}^N)$. Thus, we conclude that u' is in $L^r_{\text{loc}}(\Omega,\mathbb{R}^N)$ for all $r\geq 1$.

The next step is to prove that $D^{1,t}u'$ is bounded in L^2_{loc}. The idea is classical, but we provide the arguments for the sake of completeness. To this end, we test in the weak formulation of (4.13) against difference quotients of the form

$$\phi := -D^{1,-t}\left(\zeta^2 D^{1,t}u\right), \quad \zeta \in C^\infty_c(\Omega), \quad D^{1,t}u(x) = \frac{u(x+t)-u(x)}{t}, \quad t \neq 0.$$

Let F be given by (4.8). Then, for $\varepsilon > 0$ and t small, we have

$$\begin{aligned} I &:= \left| \int_\Omega D^{1,t}\big(F_P(\cdot,u,u')\big)\cdot\Big(\zeta^2 D^{1,t}u' + 2\zeta\zeta' D^{1,t}u\Big)\right| \\ &\leq \int_\Omega \big|F_\eta(\cdot,u,u')\big|\Big(\zeta|D^{1,t}u| + \zeta^2|D^{1,t}D^{1,t}u|\Big) \\ &\leq K\left(\int_\Omega \zeta\big|F_\eta(\cdot,u,u')\big|^2 + \int_\Omega \zeta|u'|^2\right) + \varepsilon\int_\Omega \zeta^2|D^{1,t}u'|^2. \end{aligned} \tag{4.18}$$

for some constant $K > 0$ independent of t. By using the inequality $F_{PP} \geq c_0 I$ and the identity

$$\begin{aligned} &D^{1,t}\big(F_P(\cdot,u,u')\big)(x) \\ &= \int_0^1 \bigg\{ F_{PP}\Big(\cdot, \lambda u(x+t)+(1-\lambda)u(x), \lambda u'(x+t)+(1-\lambda)u'(x)\Big)D^{1,t}u'(x) \\ &\quad + F_{P\eta}\Big(\cdot, \lambda u(x+t)+(1-\lambda)u(x), \lambda u'(x+t)+(1-\lambda)u'(x)\Big)D^{1,t}u(x) \\ &\quad + F_{Px_{1,t}}\Big(\cdot, \lambda u(x+t)+(1-\lambda)u(x), \lambda u'(x+t)+(1-\lambda)u'(x)\Big)\bigg\}\, d\lambda \end{aligned}$$

(where $F_{Px_{1,t}}$ denotes difference quotient with respect to the x variable), we have the bound

$$I \geq \frac{1}{K}\int_\Omega \zeta^2|D^{1,t}u'|^2 - C\big(\|u\|_{L^\infty(\Omega)}\big)\int_\Omega \zeta\big|P(|u'|)\big| \tag{4.19}$$

where $K > 0$ is a constant independent of t, whilst P is a polynomial expression and it is a consequence of our growth assumptions on the Hamiltonian and its derivatives. Since $u \in C^0(\overline{\Omega},\mathbb{R}^N)$ and $u' \in L^p_{\text{loc}}(\Omega,\mathbb{R}^N)$ for all $p \geq 1$, by (4.18) and (4.19) we obtain that $u \in W^{2,2}_{\text{loc}}(\Omega,\mathbb{R}^N)$. Thus, the calculations in the derivation of the expanded form of the system make sense a.e. on Ω. Since F_{PP} is a strictly positive matrix, by a standard bootstrap argument in the system we obtain that $u \in C^2(\Omega,\mathbb{R}^N)$ and the lemma follows. $\square$

Now we may prove the main result.

Proof of Theorem 4.1. In view of Lemmas 4.3, 4.4, 4.5, let $(u^m)_2^\infty$ denote the subsequence of minimisers in $C^0(\overline{\Omega},\mathbb{R}^N)\cap C^2(\Omega,\mathbb{R}^N)$ of the functionals E_m over the spaces $W^{1,2m}_b(\Omega,\mathbb{R}^N)$ (given by (3.14)). Then, along this subsequence we have

$$\left\{ \begin{array}{ll} u^m \longrightarrow u^\infty, & \text{in } C^0(\overline{\Omega},\mathbb{R}^N), \\ u^{m\prime} \rightharpoonup u^{\infty\prime}, & \text{in } L^p(\Omega,\mathbb{R}^N), \text{ for all } p > 2, \end{array} \right. \tag{4.20}$$

as $m \to \infty$, and the limit satisfies $u^\infty \in W^{1,\infty}_b(\Omega,\mathbb{R}^N)$. Moveover, each u^m is a classical solution of the system (3.19), or equivalently of (3.23) with $f^\infty, F^\infty, A^\infty$ given by (3.21), (3.22). The goal is to show that the limit map u^∞ is a Dim solution of the system (3.20) (or equivalently (3.24)) and also $u^\infty = b$ on $\partial\Omega$. We begin by observing that the boundary condition is satisfied as a result of the uniform convergence on $\overline{\Omega}$. Moreover, by multiplying (3.23) with $u^{m\prime\prime} - \big(V(\cdot,u^m)\big)'$, we obtain

$$
\begin{aligned}
&\left\{\frac{A^\infty(\cdot,u^m,u^{m\prime})}{m-1} + h_p^2\Big(\cdot,u,\frac{1}{2}\big|u^{m\prime}-V(\cdot,u^m)\big|^2\Big)\big|u^{m\prime}-V(\cdot,u^m)\big|^2 I\right\} : \\
&: \Big[u^{m\prime\prime} - \big(V(\cdot,u^m)\big)'\Big] \otimes \Big[u^{m\prime\prime} - \big(V(\cdot,u^m)\big)'\Big] \\
&\qquad = \left(\frac{f^\infty(\cdot,u^m,u^{m\prime})}{m-1} + F^\infty(\cdot,u^m,u^{m\prime})\right)\cdot \Big[u^{m\prime\prime} - \big(V(\cdot,u^m)\big)'\Big] \\
&\qquad \leq \left|\frac{f^\infty(\cdot,u^m,u^{m\prime})}{m-1} + F^\infty(\cdot,u^m,u^{m\prime})\right| \Big|u^{m\prime\prime} - \big(V(\cdot,u^m)\big)'\Big|.
\end{aligned}
\tag{4.21}
$$

By assumption we have that $h_p \geq c_0$. In addition, by (3.22) the matrix map A^∞ is non-negative. Hence (4.21) gives the estimate

$$
c_0^2\big|u^{m\prime}-V(\cdot,u^m)\big|^2\Big|u^{m\prime\prime} - \big(V(\cdot,u^m)\big)'\Big| \leq \left|\frac{f^\infty(\cdot,u^m,u^{m\prime})}{m-1} + F^\infty(\cdot,u^m,u^{m\prime})\right|.
$$

Hence, we have the estimate

$$
\Big|\big|u^{m\prime}-V(\cdot,u^m)\big|^2\big(u^{m\prime}-V(\cdot,u^m)\big)'\Big| \leq \frac{1}{c_0^2}\left|\frac{f^\infty(\cdot,u^m,u^{m\prime})}{m-1} + F^\infty(\cdot,u^m,u^{m\prime})\right|.
\tag{4.22}
$$

By using the elementary inequality

$$
\Big|\big(|f|^3\big)'\Big| \leq 3\,\big||f|^2 f\big|, \quad f \in C^1(\Omega,\mathbb{R}^N),
$$

(4.22) gives the stimate

$$
\left|\Big(\big|u^{m\prime}-V(\cdot,u^m)\big|^3\Big)'\right| \leq \frac{3}{c_0^2}\left|\frac{f^\infty(\cdot,u^m,u^{m\prime})}{m-1} + F^\infty(\cdot,u^m,u^{m\prime})\right|.
\tag{4.23}
$$

By (4.23), (4.20) and the form of the right hand side given by (3.21), (3.22), we have that the sequence

$$
v^m := \big|u^{m\prime}-V(\cdot,u^m)\big|
\tag{4.24}
$$

is bounded in $W^{1,p}(\Omega)$, for any $p > 2$. Hence, by the compactness of the imbedding $W^{1,p}(\Omega) \Subset C^0(\overline{\Omega})$, there is a continuous non-negative function v^∞ such that

$$
v^m \longrightarrow v^\infty, \quad \text{in } C^0(\overline{\Omega}),
$$

along perhaps a further subsequence as $m \to \infty$. We claim that we have

$$
\big|u^{\infty\prime}-V(\cdot,u^\infty)\big| \leq v^\infty, \quad \text{a.e. on } \Omega.
\tag{4.25}
$$

Indeed, by the weak lower semi-continuity of the L^p norm, for every $x \in \Omega$ and $r > 0$ fixed we have that

$$
\begin{aligned}
\frac{1}{2r}\int_{x-r}^{x+r}\big|u^{\infty\prime}-V(\cdot,u^\infty)\big|^p &\leq \liminf_{m\to\infty}\frac{1}{2r}\int_{x-r}^{x+r}\big|u^{m\prime}-V(\cdot,u^m)\big|^p \\
&= \lim_{m\to\infty}\frac{1}{2r}\int_{x-r}^{x+r}(v^m)^p \\
&= \frac{1}{2r}\int_{x-r}^{x+r}(v^\infty)^p.
\end{aligned}
\tag{4.26}
$$

By passing to the limit as $r \to 0$ in (4.26), the Lebesgue differentiation theorem implies the inequality (4.25) is valid a.e. on Ω. We now set

$$\Omega^\infty := \big\{ x \in \Omega \ : \ v^\infty(x) > 0 \big\}.$$

By the continuity of v^∞, Ω^∞ is open in Ω, the set $\Omega \setminus \Omega^\infty$ is closed in Ω and

$$\Omega \setminus \Omega^\infty \, = \, \big\{ x \in \Omega \ : \ v^\infty(x) = 0 \big\}.$$

By (4.25), we have

$$\big| u^{\infty\prime} - V(\cdot, u^\infty) \big| \, = \, 0, \quad \text{a.e. on } \Omega \setminus \Omega^\infty. \tag{4.27}$$

On the other hand, since $v^m \longrightarrow v^\infty$ in $C^0(\overline{\Omega})$, for any $\Omega' \Subset \Omega^\infty$, there is a $\sigma_0 > 0$ and an $m(\Omega') \in \mathbb{N}$ such that for all $m \geq m(\Omega')$, we have

$$v^m \, \geq \, \sigma_0 \qquad \text{on } \Omega'. \tag{4.28}$$

By (4.28), (4.26) and (4.24), we have

$$\Big| u^{m\prime\prime} - \big(V(\cdot, u^m)\big)' \Big| \, \leq \, \frac{3}{(c_0 \sigma_0)^2} \left| \frac{f^\infty(\cdot, u^m, u^{m\prime})}{m-1} + F^\infty(\cdot, u^m, u^{m\prime}) \right|, \qquad \text{on } \Omega'. \tag{4.29}$$

By (4.29) and (4.20) we have that $u^{m\prime\prime}$ is bounded in $L^p_{\text{loc}}(\Omega^\infty, \mathbb{R}^N)$. Hence, we have that

$$\left\{ \begin{array}{ll} u^m \longrightarrow u^\infty, & \text{in } C^0(\Omega^\infty, \mathbb{R}^N), \\ u^{m\prime} \rightharpoonup u^{\infty\prime}, & \text{in } L^p_{\text{loc}}(\Omega^\infty, \mathbb{R}^N), \ \text{for all } p > 2, \\ u^{m\prime\prime} \rightharpoonup u^{\infty\prime\prime}, & \text{in } L^p_{\text{loc}}(\Omega^\infty, \mathbb{R}^N), \ \text{for all } p > 2. \end{array} \right.$$

Thus, by passing to the limit in the ODE system (3.19) as $m \to \infty$ along a subsequence, we have that the restriction of u^∞ over the open set Ω^∞ is a strong a.e. solution of (3.20) on Ω^∞. By bootstapping in the equation, we have that actually $u^\infty \in C^2(\Omega^\infty, \mathbb{R}^N)$. On the other hand, we have that

$$\big| u^{\infty\prime} - V(\cdot, u^\infty) \big| \, = \, 0, \qquad \text{a.e. on } \Omega \setminus \Omega^\infty.$$

Hence, if the set $\Omega \setminus \Omega^\infty$ has non-trivial topological interior, by differentiating the relation $u^{\infty\prime} = V(\cdot, u^\infty)$ we have that $u^{\infty\prime\prime}$ exists a.e. on the interior of the open (but possibly empty) set $\Omega \setminus \Omega^\infty$ and by bootstrapping again we see that u^∞ is C^2 on $\text{int}(\Omega \setminus \Omega^\infty)$. Putting the above together, we have that $u^{\infty\prime\prime}$ exists and is continuous on the open set Ω_∞ defined in the statement of the theorem which is the union of Ω^∞ and of the interior of $\Omega \setminus \Omega^\infty$:

$$u^\infty \in C^2(\Omega_\infty, \mathbb{R}^N), \quad \Omega_\infty \, = \, \Omega^\infty \cup \text{int}\,(\Omega \setminus \Omega^\infty).$$

We now conclude by showing that u^∞ is a Dim solution of (3.24) on Ω. Let $D^{1,h_i} u^{\infty\prime}$ be the first difference quotients of $u^{\infty\prime}$ along a sequence $h_i \to 0$ as $i \to \infty$ and let $\mathcal{D}^2 u^\infty$ be a Dim 2nd derivative of u^∞ arising from the subsequential weak* convergence of the difference quotients, that is

$$\delta_{D^{1,h_{i_j}} u^{\infty\prime}} \overset{*}{\rightharpoonup} \mathcal{D}^2 u^\infty, \qquad \text{in } \mathscr{Y}\big(\Omega, \overline{\mathbb{R}}^N\big),$$

as $j \to \infty$, in the space of Young measures from $\Omega \subseteq \mathbb{R}$ into the 1-point compactification of $\mathbb{R}^N$. By the regularity of u^∞ on Ω^∞ and Lemma 2.7, we have that the restriction of every Dim 2nd derivative on Ω^∞ is the Dirac mass at the second derivatives:

$$\mathcal{D}^2 u^\infty(x) \, = \, \delta_{u^{\infty\prime\prime}(x)}, \qquad \text{for a.e. } x \in \Omega^\infty. \tag{4.30}$$

Then, by Proposition 2.10 we have that u^∞ is Dim solution on Ω^∞, since it is a strong solution on this subdomain. Consequently, for any test function $\Phi \in C^0_c(\mathbb{R}^N)$ we have

$$\begin{split}\int_{\overline{\mathbb{R}}^N} \Phi(X)\Big\{ h_p^2\Big(\cdot, u^\infty, \frac{1}{2}\big|u^{\infty\prime} - V(\cdot,u^\infty)\big|^2\Big)\big|u^{\infty\prime} - V(\cdot,u^\infty)\big|^2 \cdot \\ \cdot \Big[X - \big(V(\cdot,u^\infty)\big)'\Big] \,-\, F^\infty\big(\cdot, u^\infty, u^{\infty\prime}\big)\Big\}\, d[\mathcal{D}^2 u^\infty](X) \,=\, 0,\end{split} \tag{4.31}$$

a.e. on $\Omega^\infty \subseteq \Omega$. On the other hand, since $\big|u^{\infty\prime} - V(\cdot,u^\infty)\big| = 0$, a.e. on $\Omega \setminus \Omega^\infty$, we have for any $\Phi \in C^0_c(\mathbb{R}^N)$ that

$$\begin{split}\int_{\overline{\mathbb{R}}^N} \Phi(X)\Big\{ h_p^2\Big(\cdot, u^\infty, \frac{1}{2}\big|u^{\infty\prime} - V(\cdot,u^\infty)\big|^2\Big)\big|u^{\infty\prime} - V(\cdot,u^\infty)\big|^2 \cdot \\ \cdot \Big[X - \big(V(\cdot,u^\infty)\big)'\Big]\Big\}\, d[\mathcal{D}^2 u^\infty](X) \,=\, 0,\end{split} \tag{4.32}$$

a.e. on $\Omega \setminus \Omega^\infty$. Also, by the equality (3.21) we see that the right hand side vanishes as well on this set:

$$F^\infty\big(\cdot, u^\infty, u^{\infty\prime}\big) \,=\, 0, \qquad \text{a.e. on } \Omega \setminus \Omega^\infty. \tag{4.33}$$

By putting (4.31), (4.32), (4.33) together, we conclude that u^∞ is indeed a Dim solution of the Dirichlet problem for the fundamental equations in L^∞, which is also a uniform sequential limit of minimisers of the respective L^m functionals as $m \to \infty$. The theorem ensues. □

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