School of Mathematical, Physical
and Computational Sciences

University of Reading

Department of Mathematics and Statistics

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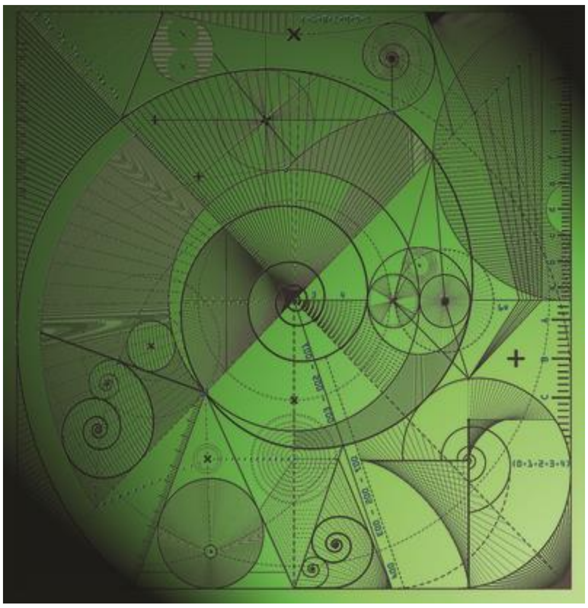
by

Nikos Katzourakis



AN L^∞ REGULARISATION STRATEGY TO THE INVERSE SOURCE IDENTIFICATION PROBLEM FOR ELLIPTIC EQUATIONS

NIKOS KATZOURAKIS

Abstract. In this paper we utilise new methods of Calculus of Variations in L^∞ to provide a regularisation strategy to the ill-posed inverse problem of identifying the source of a non-homogeneous linear elliptic equation, satisfying Dirichlet data on a domain. One of the advantages over the classical Tykhonov regularisation in L^2 is that the approximated solution of the PDE is uniformly close to the noisy measurements taken on a compact subset of the domain.

1. Introduction

Let $n \in \mathbb{N}$ and $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with $C^{1,1}$ regular boundary $\partial\Omega$. Let also L be the linear non-divergence differential operator

$$\mathrm{L}[u] := A : \mathrm{D}^2 u + b \cdot \mathrm{D}u + cu \tag{1.1}$$

which is assumed to be uniformly elliptic with bounded continuous coefficients:

$$\begin{cases} A \in (C^0 \cap L^\infty)(\Omega; \mathbb{R}^{n\times n}_s),\ b \in (C^0 \cap L^\infty)(\Omega; \mathbb{R}^n),\ c \in (C^0 \cap L^\infty)(\Omega), \\ \qquad\qquad \text{and exists } a_0 > 0 : \quad A : \xi \otimes \xi \geq a_0|\xi|^2, \text{ for all } \xi \in \mathbb{R}^n. \end{cases} \tag{1.2}$$

In the above, the notations ":" and "$\cdot$" symbolise the Euclidean inner products in the space of symmetric matrices $\mathbb{R}^{n\times n}_s$ and in $\mathbb{R}^n$ respectively, whilst $\mathrm{D}u = (\mathrm{D}_i u)_{i=1\dots n}$, $\mathrm{D}^2 u = (\mathrm{D}^2_{ij} u)_{i,j=1\dots n}$ and $\mathrm{D}_i \equiv \partial/\partial x_i$. The direct (or forward) Dirichlet problem for the above operator has the form

$$\begin{cases} \mathrm{L}[u] = f, & \text{in } \Omega, \\ \quad u = g, & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

and asks to determine u, given a source f and boundary data g. This is a classical problem which is essentially textbook material, see e.g. [19, Ch. 9]. In particular, it is well-posed (in the sense of Hadamard) and, given $f \in L^\infty(\Omega)$ and $g \in W^{2,\infty}(\Omega)$, there exists a unique solution u in the locally convex (Fréchet) space

$$\mathcal{W}^{2,\infty}_g(\Omega) := \bigcap_{1<p<\infty} \Big\{ u \in \big(W^{2,p} \cap W^{1,p}_g\big)(\Omega) : \quad \mathrm{L}[u] \in L^\infty(\Omega) \Big\}. \tag{1.4}$$

Note that due to the failure of the L^p elliptic estimates when $p = \infty$ (see e.g. [18]), in general $u \notin W^{2,\infty}(\Omega)$. Let us also note with the assumptions (1.2) on L, the case of divergence operators with C^1 matrix coefficient A is included as a special case:

$$\mathrm{L}'[u] = \mathrm{div}(A\mathrm{D}u) + b \cdot \mathrm{D}u + cu.$$

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The *inverse problem* associated to (1.3) consists of the question of finding f, given the boundary data g and some *partial information on the solution u, typically obtained through noisy (i.e. approximate) experimental measurements known only up to some error.* This problem is *severely ill-posed*, as the noisy data measured on a subset of the domain might either not be compatible with *any* exact solution, or even if they do, they may not suffice to determine a unique source f from it.

The above inverse problem is particularly important for several applications, especially in the model case of the Laplace operator $\mathrm{L} = \Delta$ and the Poisson equation, see e.g. [1, 8, 14, 20, 28, 29, 31, 33, 34, 35, 36, 37]. Herein we will assume that the noisy measurements on the solution take the form

$$\mathrm{Q}[u] \,=\, q^{\delta} \quad \text{on } \Gamma, \tag{1.5}$$

where Q is the (nonlinear differential) *observation operator*

$$\mathrm{Q}[u] \,:=\, K(\cdot, u, \mathrm{D}u) \tag{1.6}$$

with K satisfying

$$K \in C^0(\Gamma \times \mathbb{R} \times \mathbb{R}^n) \text{ and } K(x, \cdot, \cdot) \in C^1(\mathbb{R} \times \mathbb{R}^n) \text{ for any } x \in \Gamma. \tag{1.7}$$

Here Γ is the set on which we take measurements. It will be assumed it satisfies

$$\Gamma \subseteq \overline{\Omega} \text{ is compact and } \mathcal{H}^{\gamma}(\Gamma) < \infty, \text{ for some } \gamma \in [0, n]. \tag{1.8}$$

In the above, $\mathcal{H}^{\gamma}$ denotes the Hausdorff measure of dimension γ. Our general measure theory and function space notation will be either self-explanatory or otherwise standard, as e.g. in [13, 15, 26]. Finally, $q^{\delta} \in L^{\infty}(\Gamma, \mathcal{H}^{\gamma})$ is the function of noisy (deterministic) measurements taken on Γ, at noise level at most $\delta > 0$, that is

$$\|q^{\delta} - q^0\|_{L^{\infty}(\Gamma, \mathcal{H}^{\gamma})} \,\leq\, \delta, \tag{1.9}$$

where $q^0 = \mathrm{Q}[u^0]$ corresponds to ideal noise-free measurements of an exact solution to (1.3) with source $\mathrm{L}[u^0]$.

Recapitulating, in this paper we study the next ill-posed inverse source identification problem:

$$\left\{ \begin{aligned} \mathrm{L}[u] \,&=\, f\,, &&\text{in } \Omega,\\ u \,&=\, g\,, &&\text{on } \partial\Omega,\\ \mathrm{Q}[u] \,&=\, q^{\delta}, &&\text{on } \Gamma. \end{aligned} \right. \tag{1.10}$$

Namely, we *seek to specify with some selection process a suitable approximation for f from measured data q^{δ} on the compact set Γ through some observation* $\mathrm{Q}[u]$ *of the solution u*. Our analysis does not exclude the extreme cases $\Gamma = \overline{\Omega}$ (full a priori information) and $\Gamma = \emptyset$ (no a priori information), although if $\Gamma = \emptyset$ certain trivial modifications in the proofs are required which we do not discuss explicitly. The goal is a strategy to determine an "optimal" best fitting solution u^{δ} (and corresponding source $f^{\delta} := \mathrm{L}[u^{\delta}]$) to the ill-posed problem (1.10). In general, an exact solution may well not exist as (1.5) is a possibly incompatible pointwise constraint on Γ to the solution of (1.3) (due to the errors in measurements). On the other hand, it is not possible to have a uniquely determined source on the constraint-free region $\Omega \setminus \Gamma$, see e.g. [8]. In particular, if

$$\mathrm{L} = \Delta,\ \mathrm{Q}[u] = n \cdot \mathrm{D}u = \frac{\partial u}{\partial n},\ n \text{ the outer normal vector on } \partial\Omega, \Gamma = \partial\Omega,$$

then *any* f of the form $f_0 + \Delta f_1$ with $\Delta f_0 = 0$ solves the inverse problem (1.10). This happens because the boundary data $u = g$ on $\partial\Omega$ and $\partial u / \partial n = q^{\delta}$ on $\partial\Omega$

can only determine a unique biharmonic function u in Ω with $\Delta^2 u = 0$. Another popular choice in the literature for the observation operator Q consist of one of the terms in the separation of variables formula (when $\mathrm{L} = \Delta$ on rectangular domains), as e.g. in [36]. To the best of our knowledge, (1.10) has not been studied before in this generality.

Herein we follow an approach based on recent advances in Calculus of Variations in the space L^∞ (see [22, 23, 24, 25]) developed recently for functionals involving higher order derivatives. The field has been initiated in the 1960s by Gunnar Aronsson (see e.g. [3, 4, 5, 6, 7]) and is still a very active area of research; for a review of the by-now classical theory involving scalar first order functionals we refer to [21]. To this end, we provide a *regularisation strategy* inspired by the classical Tykhonov regularisation strategy in L^2 (see e.g. [27, 30]), but for the next L^∞ "error" functional:

$$\mathrm{E}_\infty(u) := \big\|\mathrm{Q}[u] - q^\delta\big\|_{L^\infty(\Gamma,\mathcal{H}^\gamma)} + \alpha\big\|\mathrm{L}[u]\big\|_{L^\infty(\Omega)}, \qquad u \in \mathcal{W}^{2,\infty}_g(\Omega), \tag{1.11}$$

where $\alpha > 0$ is a fixed regularisation parameter for the penalisation term $|\mathrm{L}[u]|$. In the variational language, it serves to make the functional coercive in the space. The benefit of finding a best fitting solution in L^∞ is apparent: we can keep the error term $|\mathrm{Q}[u] - q^\delta|$ due to the noise effects *uniformly small, not merely small on average*, which would happen if one chose to minimise the integral of the error instead of the supremum.

As it is well known to the experts of Calculus of Variations in L^∞, mere (global) minimisers of supremal functionals, albeit typically easy to obtain with standard direct minimisation methods ([13, 16]), they are not truly optimal and they do not share the nice "local" minimality properties of minimisers of their integral counterparts ([10, 32]). A popular method is to use minimisers of L^p approximating functionals as $p \to \infty$ and prove appropriate convergence of such L^p minimisers to a limiting L^∞ minimiser. This method is fairly standard nowadays and provides a selection principle of L^∞ minimisers with additional favourable properties (see e.g. [9, 11, 12, 17, 22, 23]). This idea is inspired by the simple measure-theoretic fact that the L^p norm (of a fixed $L^1 \cap L^\infty$ function) converges to the L^∞ norm as $p \to \infty$. Accordingly, we will obtain *special* minimisers of (1.11) as limits of minimisers of

$$\mathrm{E}_p(u) := \big\||\mathrm{Q}[u] - q^\delta|_{(p)}\big\|_{L^p(\Gamma,\mathcal{H}^\gamma)} + \alpha\big\||\mathrm{L}[u]|_{(p)}\big\|_{L^p(\Omega)}, \qquad u \in \big(W^{2,p} \cap W^{1,p}_g\big)(\Omega), \tag{1.12}$$

where in the above we use the normalised L^p norms

$$\|f\|_{L^p(\Gamma,\mathcal{H}^\gamma)} := \left(\fint_\Gamma |f|^p \,\mathrm{d}\mathcal{H}^\gamma\right)^{1/p}, \quad \|f\|_{L^p(\Omega)} := \left(\fint_\Omega |f|^p \,\mathrm{d}\mathcal{L}^n\right)^{1/p},$$

where the slashed integral denoting average with respect to the Hausdorff measure $\mathcal{H}^\gamma$ and the Lebesgue measure $\mathcal{L}^n$ respectively. Further, in (1.12) $|\cdot|_{(p)}$ symbolises the next p-regularisation of the absolute value away from zero:

$$|a|_{(p)} := \sqrt{|a|^2 + p^{-2}}.$$

We note also that, due to our L^p-approximation method, as an auxiliary result we also provide an L^p regularisation strategy for finite p as well, which has its own merits and could be useful in itself.

The main result in this paper is therefore the following.

Theorem 1 (L^∞ and L^p regularisations of the inverse source identification problem)**.** *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded $C^{1,1}$ domain and let also g be in $W^{2,\infty}(\Omega)$. Suppose also the operators (1.1) and (1.6) are given, satisfying the assumptions (1.2), (1.7), (1.8). Suppose further a function $q^\delta \in L^\infty(\Gamma, \mathcal{H}^\gamma)$ is given which satisfies (1.9) for $\delta > 0$. Let finally $\alpha > 0$ be fixed. Then, we have the next results in relation to the problem (1.10):*

(i) [Existence] *There exist a global minimiser $u_\infty \equiv u_\infty^{\alpha,\delta} \in \mathcal{W}_g^{2,\infty}(\Omega)$ of the functional E_∞ defined in (1.11). In particular, we have $E_\infty(u_\infty) \leq E_\infty(v)$ for all $v \in \mathcal{W}_g^{2,\infty}(\Omega)$ and*

$$f_\infty \equiv f_\infty^{\alpha,\delta} := \mathrm{L}[u_\infty^{\alpha,\delta}] \in L^\infty(\Omega).$$

In addition, there exist signed Radon measures

$$\mu_\infty \equiv \mu_\infty^{\alpha,\delta} \in \mathcal{M}(\Omega), \quad \nu_\infty \equiv \nu_\infty^{\alpha,\delta} \in \mathcal{M}(\Gamma)$$

such that the divergence PDE

$$K_r(\cdot, u_\infty, \mathrm{D}u_\infty)\nu_\infty - \mathrm{div}\big(K_p(\cdot, u_\infty, \mathrm{D}u_\infty)\nu_\infty\big) + \alpha \mathrm{L}^*[\mu_\infty] = 0, \tag{1.13}$$

is satisfied by the triplet $(u_\infty, \mu_\infty, \nu_\infty)$ in the distributional sense. In (1.13), the operator L^* is the formal adjoint of L, defined through duality, i.e.

$$\mathrm{L}^*[v] := \mathrm{div}(\mathrm{div}(Av)) - \mathrm{div}(bv) + cv$$

and K_r, K_p denote the partial derivatives of $K(x,r,p)$ with respect to $(r,p) \in \mathbb{R}\times\mathbb{R}^n$. Additionally, the error measure ν_∞ is supported in the closure of the subset of Γ of maximum noise, that is

$$\mathrm{supp}(\nu_\infty) \subseteq \Big\{ \big|Q[u_\infty] - q^\delta\big|^\bigstar = \big\|\mathrm{Q}[u_\infty] - q^\delta\big\|_{L^\infty(\Gamma,\mathcal{H}^\gamma)} \Big\}, \tag{1.14}$$

where "$(\,\cdot\,)^\bigstar$" symbolises the "essential limsup" with respect to the Radon measure $\mathcal{H}^\gamma \llcorner_\Gamma$ on Γ, see Proposition 6 that follows. If additionally the measurement function q^δ is continuous on Γ, (1.14) improves to

$$\mathrm{supp}(\nu_\infty) \subseteq \Big\{ \big|Q[u_\infty] - q^\delta\big| = \big\|\mathrm{Q}[u_\infty] - q^\delta\big\|_{L^\infty(\Gamma,\mathcal{H}^\gamma)} \Big\}. \tag{1.15}$$

(ii) [Convergence] *For any $\alpha, \delta > 0$, the minimiser u_∞ can be approximated by a family of minimisers $(u_p)_{p>n} \equiv (u_p^{\alpha,\delta})_{p>n}$ of the respective L^p functionals (1.12) and the pair of measures $(\mu_\infty, \nu_\infty) \in \mathcal{M}(\Omega) \times \mathcal{M}(\Gamma)$ can be approximated by respective absolutely continuous signed measures $(\mu_p, \nu_p)_{p>n} \equiv (\mu_p^{\alpha,\delta}, \nu_p^{\alpha,\delta})_{p>n}$, as follows:*

For any $p > n$, the functional (1.12) has a global minimiser $u_p \equiv u_p^{\alpha,\delta}$ in the space $(W^{2,p} \cap W_g^{1,p})(\Omega)$ and there exists a sequence $p_j \longrightarrow \infty$ as $j \to \infty$, such that

$$\left\{ \begin{array}{lll} u_p \longrightarrow u_\infty, & \text{in } C^{1,\kappa}(\overline{\Omega}), & \text{for any } \kappa \in (0,1), \\ \mathrm{D}^2 u_p \rightharpoonup \mathrm{D}^2 u_\infty, & \text{in } L^q(\Omega, \mathbb{R}_s^{n\times n}), & \text{for any } q \in (1,\infty), \end{array} \right. \tag{1.16}$$

as $p \to \infty$ along the sequence. Additionally, we have

$$\left\{ \begin{array}{ll} \nu_p := \dfrac{\big|\mathrm{Q}[u_p] - q^\delta\big|_{(p)}^{p-2}\big(\mathrm{Q}[u_p] - q^\delta\big)}{\mathcal{H}^\gamma(\Gamma)\,\big\||\mathrm{Q}[u_p] - q^\delta|_{(p)}\big\|_{L^p(\Gamma,\mathcal{H}^\gamma)}^{p-1}} \mathcal{H}^\gamma \llcorner_\Gamma \xrightharpoonup{*} \nu_\infty, & \text{in } \mathcal{M}(\Gamma), \\ \mu_p := \dfrac{|\mathrm{L}[u_p]|_{(p)}^{p-2}\,\mathrm{L}[u_p]}{\mathcal{L}^n(\Omega)\,\big\||\mathrm{L}[u_p]|_{(p)}\big\|_{L^p(\Omega)}^{p-1}} \mathcal{L}^n \llcorner_\Omega \xrightharpoonup{*} \mu_\infty, & \text{in } \mathcal{M}(\Omega), \end{array} \right. \tag{1.17}$$

as $p \to \infty$ along the sequence. Further, for each $p > n$, the triplet (u_p, μ_p, ν_p) solves the equation

$$K_r(\cdot, u_p, \mathrm{D}u_p)\nu_p \,-\, \mathrm{div}\big(K_p(\cdot, u_p, \mathrm{D}u_p)\nu_p\big) \,+\, \alpha \mathrm{L}^*[\mu_p] \,=\, 0, \tag{1.18}$$

in the distributional sense.

(iii) [**L^∞ error estimates**] *For any exact solution $u^0 \in \mathcal{W}^{2,\infty}_g(\Omega)$ of (1.10) (with $f = \mathrm{L}[u^0]$ and $\mathrm{Q}[u^0] = q^0$) corresponding to measurements with zero noise, we have the estimate:*

$$\Big\|\mathrm{Q}[u^{\alpha,\delta}_\infty] - \mathrm{Q}[u^0]\Big\|_{L^\infty(\Gamma,\mathcal{H}^\gamma)} \le\, 2\delta + \alpha\, \|\mathrm{L}[u^0]\|_{L^\infty(\Omega)}, \tag{1.19}$$

for any $\alpha, \delta > 0$.

(iv) [**L^p error estimates**] *For any exact solution $u^0 \in (W^{2,p} \cap W^{1,p}_g)(\Omega)$ of (1.10) (with $f = \mathrm{L}[u^0]$ and $\mathrm{Q}[u^0] = q^0$) corresponding to measurements with zero noise and for $p > n$, we have the estimate:*

$$\Big\|\mathrm{Q}[u^{\alpha,\delta}_p] - \mathrm{Q}[u^0]\Big\|_{L^p(\Gamma,\mathcal{H}^\gamma)} \le\, 2\delta + \alpha\, \|\mathrm{L}[u^0]\|_{L^p(\Omega)}, \tag{1.20}$$

for any $\alpha, \delta > 0$.

The estimate in part **(iv)** above is useful if we have merely that $\mathrm{L}[u^0] \in L^p(\Omega)$ for $p < \infty$ (namely when perhaps $\mathrm{L}[u^0] \notin L^\infty(\Omega)$).

The next two results are consequences of our main theorem.

Corollary 2 (Rates of convergence)**.** *In the setting of Theorem 1, in the case that $\mathrm{Q}[u] := u$, the estimates (1.19)-(1.20) for the L^∞ and the L^p minimisers can be improved to the linear rates of convergence*

$$\big\|u^{\alpha,\delta}_\infty - u^0\big\|_{L^\infty(\Gamma,\mathcal{H}^\gamma)} \le 2\delta + \alpha\, \|\mathrm{L}[u^0]\|_{L^\infty(\Omega)} \quad \text{as } \alpha, \delta \to 0, \tag{1.21}$$

if $\mathrm{L}[u^0] \in L^\infty(\Omega)$, and

$$\big\|u^{\alpha,\delta}_p - u^0\big\|_{L^p(\Gamma,\mathcal{H}^\gamma)} \le 2\delta + \alpha\, \|\mathrm{L}[u^0]\|_{L^p(\Omega)} \quad \text{as } \alpha, \delta \to 0, \tag{1.22}$$

if $\mathrm{L}[u^0] \in L^p(\Omega)$ for $p < \infty$.

Corollary 3. *In the setting of Theorem 1, we have*

$$L^*[\mu_p] \,=\, 0 \quad \text{in } \Omega \setminus \Gamma$$

in the distributional sense, for any $p \in (1, \infty]$. In particular, for $p < \infty$ we have

$$\mathrm{L}^*\Big(\big|\mathrm{L}[u_p]\big|^{p-2}_{(p)}\, \mathrm{L}[u_p]\Big) \,=\, 0 \quad \text{in } \Omega \setminus \Gamma,$$

in the distributional sense.

Corollary 3 expresses the fact that on the subset where we have no a priori information on the solution generating the source (and hence no constraint on the PDE), then one can select a solution whose's source is associated to a solution of the dual homogeneous problem $\mathrm{L}^*[\mu_\infty] = 0$.

Remark 4. Other possible choices for the observation operator Q which are popular in the literature, are the following:

- $\mathrm{Q}[u] := u(x,c)$, for $n=2$ and $\Omega = (a,b)\times(c,d)$ being a rectangular domain (i.e., one of the products in the separation of variables when $\mathrm{L}=\Delta$). This implies that (1.19) simplifies to

$$\left\| u_\infty^{\alpha,\delta}(\cdot,c) - u^0(\cdot,c)\right\|_{L^\infty((a,b),\mathcal{H}^1)} \leq 2\delta + \alpha\, \|\mathrm{L}[u^0]\|_{L^\infty((a,b)\times(c,d))} \quad \text{as } \alpha,\delta\to 0,$$

  and similarly for its L^p-counterpart.

- $\mathrm{Q}[u] := \mathrm{D}u\cdot n$, where n is the outer normal vector on $\partial\Omega$. In this case, (1.21) simplifies to

$$\left\| n\cdot \big(\mathrm{D}u_\infty^{\alpha,\delta} - \mathrm{D}u^0\big)\right\|_{L^\infty(\partial\Omega,\mathcal{H}^{n-1})} \leq 2\delta + \alpha\, \|\mathrm{L}[u^0]\|_{L^\infty(\Omega)} \quad \text{as } \alpha,\delta\to 0,$$

  and similarly for its L^p-counterpart.

We would like to note again that, due to the ill-posed nature of the problem, in general it is not possible to obtain an estimate on $\Omega\setminus\Gamma$.

We now provide some clarifications regarding Theorem 1.

Remark 5. **(i)** We note that in (1.13) the distributional meaning of this PDE is

$$\int_\Gamma K_r(\cdot,u_\infty,\mathrm{D}u_\infty)\,\phi\,\mathrm{d}\nu_\infty \,+\, \int_\Gamma K_p(\cdot,u_\infty,\mathrm{D}u_\infty)\cdot\mathrm{D}\phi\,\mathrm{d}\nu_\infty \,+\, \alpha\int_\Omega \mathrm{L}[\phi]\,\mathrm{d}\mu_\infty \,=\, 0,$$

for all test functions $\phi\in C^2_c(\Omega)$. Therefore, in fact the equation (1.13) is valid in the smaller space of *second order distributions*:

$$\mathcal{D}^{-2}(\Omega) \,:=\, \big(C^2_c(\Omega)\big)^*.$$

Additionally, since the measure ν_∞ is supported in the compact set Γ, by extending ν_∞ on $\Omega\setminus\Gamma$ by zero (i.e. by identifying ν_∞ with the restriction $\nu_\infty \llcorner_\Gamma$), we may rewrite (1.13) as

$$\int_\Omega \Big(K_r(\cdot,u_\infty,\mathrm{D}u_\infty)\,\phi \,+\, K_p(\cdot,u_\infty,\mathrm{D}u_\infty)\cdot\mathrm{D}\phi\Big)\,\mathrm{d}\nu_\infty \,+\, \alpha\int_\Omega \mathrm{L}[\phi]\,\mathrm{d}\mu_\infty \,=\, 0,$$

for all $\phi\in C^2_c(\Omega)$.

(ii) In index form, the definition of the formal adjoint can be written as

$$\mathrm{L}^*[v] \,=\, \sum_{i,j=1}^n \mathrm{D}^2_{ij}(A_{ij}v) \,-\, \sum_{k=1}^n \mathrm{D}_k(b_k\, v) \,+\, cv$$

and the distributional interpretation of L^* through duality is

$$\langle \mathrm{L}^*[v],\phi\rangle \,=\, \int_\Omega \bigg(\sum_{i,j=1}^n (\mathrm{D}^2_{ij}\phi)(A_{ij}v) \,+\, \sum_{k=1}^n (\mathrm{D}_k\phi)(b_k v) \,+\, \phi c v\bigg)\,\mathrm{d}\mathcal{L}^n,$$

for all $\phi\in C^2_c(\Omega)$. In a similar vein, the distributional interpretation of (1.18) is

$$\int_\Gamma K_r(\cdot,u_p,\mathrm{D}u_p)\,\phi\,\mathrm{d}\nu_p \,+\, \int_\Gamma K_p(\cdot,u_p,\mathrm{D}u_p)\cdot\mathrm{D}\phi\,\mathrm{d}\nu_p \,+\, \alpha\int_\Omega \mathrm{L}[\phi]\,\mathrm{d}\mu_p \,=\, 0,$$

for all $\phi\in C^2_c(\Omega)$. By taking into account that the measures μ_p,ν_p as given by (1.17) are in fact absolutely continuous with respect to the Lebesgue and the Hausdorff measure respectively, the above is in fact equivalent to

$$\fint_{\Gamma}\Big(K_r(\cdot,u_p,\mathrm{D}u_p)\,\phi\,+\,K_p(\cdot,u_p,\mathrm{D}u_p)\cdot\mathrm{D}\phi\Big)\frac{\big|\mathrm{Q}[u_p]-q^\delta\big|_{(p)}^{p-2}\big(\mathrm{Q}[u_p]-q^\delta\big)}{\big\||\mathrm{Q}[u_p]-q^\delta|_{(p)}\big\|_{L^p(\Gamma,\mathcal{H}^\gamma)}^{p-1}}\,\mathrm{d}\mathcal{H}^\gamma$$
$$+\,\alpha\fint_{\Omega}\mathrm{L}[\phi]\frac{|\mathrm{L}[u_p]|_{(p)}^{p-2}\,\mathrm{L}[u_p]}{\big\||\mathrm{L}[u_p]|_{(p)}\big\|_{L^p(\Omega)}^{p-1}}\,\mathrm{d}\mathcal{L}^n\;=\;0,$$

for all $\phi\in C^2_c(\Omega)$.

(iii) Since we only prescribe boundary conditions $u=g$ on $\partial\Omega$ but impose no condition on the gradient (as opposed to e.g. [22], wherein an L^∞ minimisation problem was considered by imposing $\mathrm{D}u=\mathrm{D}g$ on $\partial\Omega$ additionally to $u=g$ on $\partial\Omega$), we therefore have "natural boundary conditions" for the gradient on $\partial\Omega$. We will make no particular further use of this observation.

The following two results are of independent interest and are utilised in the proof of Theorem 1 that follows. We state and prove them in considerably greater generality than that needed herein, as they have their own merits in the Calculus of Variations in L^∞.

Proposition 6 (The essential limsup)**.** *Let $X\subseteq\mathbb{R}^n$ be a Borel set, endowed with the induced Euclidean topology and let also $\nu\in\mathcal{M}(X)$ be a positive finite Radon measure on X. For any $f\in L^\infty(X,\nu)$, we define the function $f^\bigstar\in L^\infty(X,\nu)$ by setting*

$$f^\bigstar(x)\;:=\;\lim_{\varepsilon\to0}\bigg(\nu-\operatorname*{ess\,sup}_{y\in\mathbb{B}_\varepsilon(x)}f(y)\bigg)$$

and we call $f^\star$ **the ν-essential limsup of** *f. In the above, $\mathbb{B}_\varepsilon(x)$ symbolises the open ball of radius ε centred at $x\in X$ with respect to the induced topology. Then, we have:*

(i) *It holds that $f\le f^\bigstar$, ν-a.e. on X.*

(ii) *It holds that $f^\bigstar$ is upper semicontinuous on X, namely*

$$\limsup_{X\ni y\to x}f^\bigstar(y)\;\le\;f^\bigstar(x),\qquad x\in X.$$

(iii) *$f^\bigstar$ gives a pointwise meaning to the essential supremum on X, in the sense*

$$\sup_X f^\bigstar\;=\;\nu-\operatorname*{ess\,sup}_X f.$$

Proposition 7 (L^p concentration measures)**.** *Let X be a compact metric space, endowed with a non-negative finite Borel measure ν which attaches positive values to any non-empty open set on X. Consider a sequence $(f_k)_1^\infty\subseteq L^\infty(X,\nu)$ and consider the sequence of absolutely continuous signed Radon measures $(\nu_k)_1^\infty\subseteq\mathcal{M}(X)$, given by:*

$$\nu_k\;:=\;\frac{1}{\nu(X)}\frac{\big(|f_k|_{(k)}\big)^{k-2}f_k}{\big\||f_k|_{(k)}\big\|_{L^k(X,\nu)}^{k-1}}\,\nu,\qquad k\in\mathbb{N},$$

where $|\cdot|_{(k)}=(|\cdot|^2+k^{-2})^{1/2}$. Then:

(i) *There exists a subsequence* $(k_i)_1^\infty$ *and a limit measure* $\nu_\infty \in \mathcal{M}(X)$ *such that*

$$\nu_k \xrightharpoonup{\ *\ } \nu_\infty \quad \text{in } \mathcal{M}(X),$$

as $k_i \to \infty$.

(ii) *If there exists* $f_\infty \in L^\infty(X,\nu) \setminus \{0\}$ *such that*

$$\sup_X |f_k - f_\infty| \longrightarrow 0 \quad \text{as } k \to \infty,$$

then the limit measure is supported in the set where (the ν*-essential limsup of)* $|f_\infty|$ *equals* $\|f_\infty\|_{L^\infty(X,\nu)}$*:*

$$\mathrm{supp}(\nu_\infty) \subseteq \Big\{ |f_\infty|^\bigstar = \|f_\infty\|_{L^\infty(X,\nu)} \Big\}.$$

(iii) *If additionally to the assumptions of* **(ii)** *the modulus* $|f_\infty|$ *of the uniform limit* f_∞ *is continuous on* X*, then the next stronger assertion holds true:*

$$\mathrm{supp}(\nu_\infty) \subseteq \Big\{ |f_\infty| = \|f_\infty\|_{L^\infty(X,\nu)} \Big\}.$$

2. Proofs

Herein we establish Theorem 1 and its corollaries, together with the auxiliary results Propositions 6-7. The proof of Theorem 1 consists of several lemmas. We note that some of the details might be standard to the experts of Calculus of Variations, but we do provide most of the niceties for the sake of completeness and for the convenience of the reader.

Lemma 8. *For any* $p > n$ *and fixed* $\alpha, \delta > 0$*, the functional* (1.12) *has a (global) minimiser* $u_p \in (W^{2,p} \cap W^{1,p}_g)(\Omega)$*:*

$$\mathrm{E}_p(u_p) \,=\, \inf \Big\{ \mathrm{E}_p(v) \ : \ v \in (W^{2,p} \cap W^{1,p}_g)(\Omega) \Big\}.$$

Proof. Since $g \in W^{2,\infty}(\Omega)$ (and in particular because $g, \mathrm{D}g$ are continuous on Γ and therefore $\mathcal{H}^\gamma$-measurable by identification with their precise Lebesgue representatives reconstructed through limits of average values), by the Hölder inequality and our assumption we have the a priori bound

$$\begin{aligned}
\mathrm{E}_p(g) \,&\leq \mathrm{E}_\infty(g) \\
&\leq \|q^\delta\|_{L^\infty(\Gamma,\mathcal{H}^\gamma)} \,+\, \|K(\cdot, g, \mathrm{D}g)\|_{L^\infty(\Gamma,\mathcal{H}^\gamma)} \\
&\quad + \alpha \Big(\|A\|_{L^\infty(\Omega)} + \|b\|_{L^\infty(\Omega)} + \|c\|_{L^\infty(\Omega)} \Big) \|g\|_{W^{2,\infty}(\Omega)} \\
&< \infty.
\end{aligned}$$

Hence,

$$0 \,\leq\, \inf \Big\{ \mathrm{E}_p(v) \ : \ v \in (W^{2,p} \cap W^{1,p}_g)(\Omega) \Big\} \,\leq\, \mathrm{E}_\infty(g) \,<\, \infty.$$

Further, E_p is coercive in the space $(W^{2,p} \cap W^{1,p}_g)(\Omega)$: indeed, by the L^p elliptic estimates for linear second order equations with measurable coefficients [19, Ch. 9], by our assumptions on L and the Hölder inequality we have

$$\begin{aligned}
\mathrm{E}_p(v) \,&\geq\, \alpha \|\mathrm{L}[v]\|_{L^p(\Omega)} \\
&\geq\, \frac{\alpha}{C(p,A,b,c)}\Big(\|v\|_{W^{2,p}(\Omega)} - \|g\|_{W^{2,p}(\Omega)}\Big) \\
&\geq\, \frac{\alpha}{C(p,A,b,c)}\Big(\|v\|_{W^{2,p}(\Omega)} - \|g\|_{W^{2,\infty}(\Omega)}\Big)
\end{aligned}$$

for some $C = C(p,A,b,c) > 0$ and any $v \in (W^{2,p} \cap W^{1,p}_g)(\Omega)$. Let $(u_p^m)_1^\infty$ be a minimising sequence of E_p:

$$\mathrm{E}_p(u_p^m) \longrightarrow \inf\Big\{\mathrm{E}_p(v) \ : \ v \in (W^{2,p} \cap W^{1,p}_g)(\Omega)\Big\},$$

as $m \to \infty$. Then, by the above estimates, we have the uniform bound

$$\|u_p^m\|_{W^{2,p}(\Omega)} \,\leq\, C$$

for some $C > 0$ depending on p but independent of $m \in \mathbb{N}$. By standard weak and strong compactness arguments in Sobolev spaces, there exists a subsequence $(u_p^{m_k})_1^\infty$ and a function $u_p \in (W^{2,p} \cap W^{1,p}_g)(\Omega)$ such that, along this subsequence we have

$$\left\{\begin{array}{ll}
u_p^m \longrightarrow u_p, & \text{in } L^p(\Omega), \\
\mathrm{D}u_p^m \longrightarrow \mathrm{D}u_p, & \text{in } L^p(\Omega,\mathbb{R}^n), \\
\mathrm{D}^2u_p^m \rightharpoonup \mathrm{D}^2u_p, & \text{in } L^p(\Omega,\mathbb{R}^{n\times n}_s),
\end{array}\right.$$

as $m_k \to \infty$. Additionally, since $p > n$, by the regularity of the boundary we have the compact embedding $W^{2,p}(\Omega) \Subset C^{1,k}(\overline{\Omega})$ as a consequence of the Morrey estimate. Hence,

$$u_p^m \longrightarrow u_p \quad \text{in } C^{1,\kappa}(\overline{\Omega}), \text{ for } \kappa \in \left(0, 1 - \frac{n}{p}\right),$$

as $m_k \to \infty$. The above modes of convergence and the continuity of the function K defining the operator Q imply that $\mathrm{Q}[u_p^m] \longrightarrow \mathrm{Q}[u_p]$ uniformly on Γ as $m_k \to \infty$. Therefore,

$$\big\||\mathrm{Q}[u_p^m] - q^\delta|_{(p)}\big\|_{L^p(\Gamma,\mathcal{H}^\gamma)} \longrightarrow \big\||\mathrm{Q}[u_p] - q^\delta|_{(p)}\big\|_{L^p(\Gamma,\mathcal{H}^\gamma)}$$

as $m_k \to \infty$. Additionally, by the linearity of the operator L and because its coefficients are L^∞, we have that

$$\mathrm{L}[u_p^m] \rightharpoonup \mathrm{L}[u_p] \quad \text{in } L^p(\Omega),$$

as $m_k \to \infty$. Since the functional

$$\big\||\cdot|_{(p)}\big\|_{L^p(\Omega)} \ : \ \ L^p(\Omega) \longrightarrow \mathbb{R}$$

is convex on this reflexive space and also it is strongly continuous, it is weakly lower semi-continuous and therefore

$$\big\||\mathrm{L}[u_p]|_{(p)}\big\|_{L^p(\Omega)} \,\leq\, \liminf_{k\to\infty} \big\||\mathrm{L}[u_p^{m_k}]|_{(p)}\big\|_{L^p(\Omega)}.$$

By putting all the above together, we see that

$$\mathrm{E}_p(u_p) \,\leq\, \liminf_{k\to\infty} \mathrm{E}_p(u_p^{m_k}) \,\leq\, \inf\Big\{\mathrm{E}_p(v) \ : \ v \in (W^{2,p} \cap W^{1,p}_g)(\Omega)\Big\},$$

which concludes the proof. $\square$

Lemma 9. *For any $\alpha, \delta > 0$, there exists a (global) minimiser $u_\infty \in \mathcal{W}^{2,\infty}_g(\Omega)$ and a sequence of minimisers $(u_{p_i})_1^\infty$ of the respective E_p-functionals constructed in Lemma 8, such that (1.16) holds true.*

Proof. For each $p > n$, let $u_p \in (W^{2,p} \cap W^{1,p}_g)(\Omega)$ be the minimiser of E_p given by Lemma 8. For any fixed $q \in (n, \infty)$ and $p \geq q$, the Hölder inequality and the minimality property imply the estimates

$$\mathrm{E}_q(u_p) \,\leq\, \mathrm{E}_p(u_p) \,\leq\, \mathrm{E}_p(g) \,\leq\, \mathrm{E}_\infty(g) \,<\, \infty.$$

By the coercivity of E_q in the space $(W^{2,q} \cap W^{1,q}_g)(\Omega)$, we have the estimate

$$\mathrm{E}_q(u_p) \,\geq\, \frac{\alpha}{C(q,A,b,c)}\Big(\|u_p\|_{W^{2,q}(\Omega)} - \|g\|_{W^{2,\infty}(\Omega)}\Big),$$

which implies

$$\sup_{p \geq q} \|u_p\|_{W^{2,q}(\Omega)} \,\leq\, C$$

for some $C > 0$ depending on q, the coefficient of L and α. By a standard diagonal argument, for any sequence $(p_i)_1^\infty$ with $p_i \longrightarrow \infty$ as $i \to \infty$, there exists a function

$$u_\infty \in \bigcap_{n<q<\infty} (W^{2,q} \cap W^{1,q}_g)(\Omega)$$

and a subsequence (denoted again by $(p_i)_1^\infty$) along which (1.16) holds true. It remains to show that $\mathrm{L}[u_\infty] \in L^\infty(\Omega)$ (which would guarantee membership in the space $\mathcal{W}^{2,\infty}_g(\Omega)$) and that u_∞ is in fact a minimiser of E_∞ over the same space. To this end, note that for any fixed $q \in (n, \infty)$ and $p \geq q$, we have

$$\mathrm{E}_q(u_p) \,\leq\, \mathrm{E}_p(u_p) \,\leq\, \mathrm{E}_p(v) \,\leq\, \mathrm{E}_\infty(v)$$

for any $v \in \mathcal{W}^{2,\infty}_g(\Omega)$. By the weak lower semi-continuity of E_q in the space $(W^{2,q} \cap W^{1,q}_g)(\Omega)$ demonstrated in Lemma 8, we have

$$\mathrm{E}_q(u_\infty) \,\leq\, \liminf_{i \to \infty} \mathrm{E}_q(u_{p_i}) \,\leq\, \mathrm{E}_\infty(v),$$

for any $v \in \mathcal{W}^{2,\infty}_g(\Omega)$. The particular choice $v := g$ in the above estimate gives the bound

$$\alpha \|\mathrm{L}[u_\infty]\|_{L^q(\Omega)} \,\leq\, \mathrm{E}_q(u_\infty) \,\leq\, \mathrm{E}_\infty(g).$$

By letting $q \to \infty$ in the last two estimates above, we obtain that $\mathrm{L}[u_\infty] \in L^\infty(\Omega)$ and that

$$\mathrm{E}_\infty(u_\infty) \,\leq\, \inf\Big\{\mathrm{E}_\infty(v) \,:\, v \in \mathcal{W}^{2,\infty}_g(\Omega)\Big\},$$

as desired. $\square$

Lemma 10. *For any $\alpha, \delta > 0$ and $p > n$, consider the minimiser $u_p \in (W^{2,p} \cap W^{1,p}_g)(\Omega)$ of the functional E_p constructed in Lemma 8. Consider also the signed Radon measures $\mu_p \in \mathcal{M}(\Omega)$ and $\nu_p \in \mathcal{M}(\Gamma)$, defined as in (1.17):*

$$\nu_p \,:=\, \frac{\big|\mathrm{Q}[u_p] - q^\delta\big|_{(p)}^{p-2}\big(\mathrm{Q}[u_p] - q^\delta\big)}{\mathcal{H}^\gamma(\Gamma)\,\big\||\mathrm{Q}[u_p] - q^\delta|_{(p)}\big\|^{p-1}_{L^p(\Gamma,\mathcal{H}^\gamma)}}\,\mathcal{H}^\gamma \llcorner_\Gamma,$$

$$\mu_p \,:=\, \frac{|\mathrm{L}[u_p]|_{(p)}^{p-2}\,\mathrm{L}[u_p]}{\mathcal{L}^n(\Omega)\,\big\||\mathrm{L}[u_p]|_{(p)}\big\|^{p-1}_{L^p(\Omega)}}\,\mathcal{L}^n \llcorner_\Omega.$$

Then, the triplet (u_p, μ_p, ν_p) *satisfies the PDE* (1.18) *in the distributional sense. In fact, the following stronger assertion holds: we have*

$$
\begin{aligned}
\fint_\Gamma \Big(K_r(\cdot, u_p, \mathrm{D}u_p)\,\phi \,+\, K_p(\cdot, u_p, \mathrm{D}u_p)\cdot \mathrm{D}\phi \Big) \frac{\big|\mathrm{Q}[u_p]-q^\delta\big|_{(p)}^{p-2}\big(\mathrm{Q}[u_p]-q^\delta\big)}{\big\||\mathrm{Q}[u_p]-q^\delta|_{(p)}\big\|^{p-1}_{L^p(\Gamma,\mathcal{H}^\gamma)}}\,\mathrm{d}\mathcal{H}^\gamma& \\
+\,\alpha \fint_\Omega \mathrm{L}[\phi] \frac{|\mathrm{L}[u_p]|_{(p)}^{p-2}\,\mathrm{L}[u_p]}{\big\||\mathrm{L}[u_p]|_{(p)}\big\|^{p-1}_{L^p(\Omega)}}\,\mathrm{d}\mathcal{L}^n \,=\, 0,&
\end{aligned}
$$

for all $\phi \in W^{2,p}_0(\Omega)$.

Proof. We involve a standard Gateaux differentiability argument. Let us begin by checking that μ_p, ν_p indeed define measures when $u_p \in W^{2,p}(\Omega)$. Indeed, by the Hölder inequality, we have the total variation estimates

$$
\begin{aligned}
\|\nu_p\|(\Gamma) \,&\leq\, \Big(\big\||\mathrm{Q}[u_p]-q^\delta|_{(p)}\big\|_{L^p(\Gamma,\mathcal{H}^\gamma)}\Big)^{1-p} \fint_\Gamma \big|\mathrm{Q}[u_p]-q^\delta\big|_{(p)}^{p-1}\,\mathrm{d}\mathcal{H}^\gamma \\
&\leq\, \Big(\big\||\mathrm{Q}[u_p]-q^\delta|_{(p)}\big\|_{L^p(\Gamma,\mathcal{H}^\gamma)}\Big)^{1-p} \left(\fint_\Gamma \big|\mathrm{Q}[u_p]-q^\delta\big|_{(p)}^{p}\,\mathrm{d}\mathcal{H}^\gamma \right)^{\frac{p-1}{p}} \\
&=\, 1
\end{aligned}
$$

and similarly

$$
\begin{aligned}
\|\mu_p\|(\Omega) \,&\leq\, \Big(\big\||\mathrm{L}[u_p]|_{(p)}\big\|_{L^p(\Omega)}\Big)^{1-p} \fint_\Omega \big|\mathrm{L}[u_p]\big|_{(p)}^{p-1}\,\mathrm{d}\mathcal{L}^n \\
&\leq\, \Big(\big\||\mathrm{L}[u_p]|_{(p)}\big\|_{L^p(\Omega)}\Big)^{1-p} \left(\fint_\Omega \big|\mathrm{L}[u_p]\big|_{(p)}^{p}\,\mathrm{d}\mathcal{L}^n \right)^{\frac{p-1}{p}} \\
&=\, 1.
\end{aligned}
$$

Next, fix $\phi \in C^2_c(\Omega)$. Then, by using the regularity of K, we formally compute

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} \mathrm{E}_p(u_p+\varepsilon\phi) = p\bigg(\fint_\Gamma \big|\mathrm{Q}[u_p]-q^\delta\big|_{(p)}^{p}\,\mathrm{d}\mathcal{H}^\gamma \bigg)^{\frac{1}{p}-1} \fint_\Gamma \big|\mathrm{Q}[u_p]-q^\delta\big|_{(p)}^{p-2}\big(\mathrm{Q}[u_p]-q^\delta\big)\,\cdot& \\
\cdot\,\Big[K_r(\cdot, u_p, \mathrm{D}u_p)\,\phi \,+\, K_p(\cdot, u_p, \mathrm{D}u_p)\cdot \mathrm{D}\phi\Big]\,\mathrm{d}\mathcal{H}^\gamma& \\
+\,\alpha p \left(\fint_\Omega \big|\mathrm{L}[u_p]\big|_{(p)}^{p}\,\mathrm{d}\mathcal{L}^n \right)^{\frac{1}{p}-1} \fint_\Omega \big|\mathrm{L}[u_p]\big|_{(p)}^{p-2}\,\mathrm{L}[u_p]\,\mathrm{L}[\phi]\,\mathrm{d}\mathcal{L}^n.&
\end{aligned}
$$

Since u_p is the minimiser of E_p in the space, we have that $\mathrm{E}_p(u_p) \leq \mathrm{E}_p(u_p+\varepsilon\phi)$ for all $\varepsilon \in \mathbb{R}$ and $\phi \in C^2_c(\Omega)$. Therefore, this above computation implies that the PDE (1.18) is indeed satisfied as claimed in the statement of the lemma, upon confirming that the formal computation in the integrals above is rigorous, and that therefore E_p is Gateaux differentiable at the minimiser u_p for any direction $\phi \in W^{2,p}_0(\Omega)$. This is indeed the case: since $u_p \in (C^1 \cap W^{2,p})(\Omega)$, $\mathrm{L}[u_p] \in L^p(\Omega)$ and $\mathrm{Q}[u_p]-q^\delta \in L^\infty(\Gamma,\mathcal{H}^\gamma)$, the Hölder inequality implies that

$$
\big|\mathrm{L}[u_p]\big|_{(p)}^{p-2}\,\mathrm{L}[u_p]\,\mathrm{L}[\phi] \,\in\, L^1(\Omega)
$$

and

$$
\big|\mathrm{Q}[u_p]-q^\delta\big|_{(p)}^{p-2}\big(\mathrm{Q}[u_p]-q^\delta\big)\Big[K_r(\cdot, u_p, \mathrm{D}u_p)\,\phi + K_p(\cdot, u_p, \mathrm{D}u_p)\cdot \mathrm{D}\phi\Big] \,\in\, L^1(\Gamma,\mathcal{H}^\gamma),
$$

for any $\phi \in W^{2,p}_0(\Omega) \subseteq C^1(\overline{\Omega})$, because of the continuity of $K(x,r,p)$ in x and the C^1 regularity in (r,p). $\square$

Lemma 11. *For any $\alpha, \delta > 0$, consider the minimiser u_∞ of E_∞ constructed in Lemma 9 as sequential limit of minimisers $(u_p)_{p>n}$ of the functionals $(\mathrm{E}_p)_{p>n}$ as $p_i \to \infty$. Then, there exist signed Radon measures $\mu_\infty \in \mathcal{M}(\Omega)$ and $\nu_\infty \in \mathcal{M}(\Gamma)$ such that the triplet $(u_\infty, \mu_\infty, \nu_\infty)$ satisfies the PDE (1.13) in the distributional sense, that is*

$$\int_\Omega \Big(K_r(\cdot, u_\infty, \mathrm{D}u_\infty)\,\phi \,+\, K_p(\cdot, u_\infty, \mathrm{D}u_\infty)\cdot \mathrm{D}\phi\Big)\,\mathrm{d}\nu_\infty \,+\, \alpha\int_\Omega \mathrm{L}[\phi]\,\mathrm{d}\mu_\infty \,=\, 0,$$

for all $\phi \in C^2_c(\Omega)$. Additionally, there exists a further subsequence along which the weak modes of convergence of (1.17) hold true as $p \to \infty$.*

Proof. As noted in the beginning of the proof of Lemma 10, we have the p-uniform total variation bounds $\|\mu_p\|(\Omega) \leq 1$ and $\|\nu_p\|(\Gamma) \leq 1$. Hence, by the sequential weak* compactness of the spaces of Radon measures

$$\mathcal{M}(\Omega) \,=\, \big(C^0_0(\Omega)\big)^*, \quad \mathcal{M}(\Omega) \,=\, \big(C^0(\Gamma)\big)^*,$$

there exists a further subsequence denoted again by $(p_i)_1^\infty$ such that $\mu_p \overset{*}{\rightharpoonup} \mu_\infty$ in $\mathcal{M}(\Omega)$ and $\nu_p \overset{*}{\rightharpoonup} \nu_\infty$ in $\mathcal{M}(\Gamma)$, as $p_i \to \infty$. Fix now $\phi \in C^2_c(\Omega)$. By Lemma 10, we have that the triplet (u_p, μ_p, ν_p) satisfies (1.18), that is

$$\int_\Gamma \Big(K_r(\cdot, u_p, \mathrm{D}u_p)\,\phi \,+\, K_p(\cdot, u_p, \mathrm{D}u_p)\cdot \mathrm{D}\phi\Big)\,\mathrm{d}\nu_p \,+\, \int_\Omega \mathrm{L}[\phi]\,\mathrm{d}\mu_p \,=\, 0.$$

Since

$$\mathrm{L}[\phi] \,\in\, C^0_0(\overline{\Omega}), \quad K_p(\cdot, u_\infty, \mathrm{D}u_\infty)\cdot \mathrm{D}\phi \,\in\, C^0(\Gamma)$$

and also

$$\begin{aligned} K_r(\cdot, u_p, \mathrm{D}u_p)\,\phi + K_p(\cdot, u_p, \mathrm{D}u_p)\cdot \mathrm{D}\phi &\longrightarrow \\ &K_r(\cdot, u_\infty, \mathrm{D}u_\infty)\,\phi \,+\, K_p(\cdot, u_\infty, \mathrm{D}u_\infty)\cdot \mathrm{D}\phi, \end{aligned}$$

uniformly on Γ as $p_i \to \infty$ (as a consequence of the C^1 regularity of K and the convergence $u_p \longrightarrow u_\infty$ in $C^1(\overline{\Omega})$), the weak*-strong continuity of the duality pairings between the above spaces of measures $\mathcal{M}(\Omega)$, $\mathcal{M}(\Gamma)$ and their respective predual spaces $C^0_0(\Omega)$, $C^0(\Gamma)$, allows us to conclude and obtain (1.13) by passing to the limit as $p_i \to \infty$ in (1.18). $\square$

Remark 12. By testing in the weak formulation of (1.18) against $\phi \in C^2_c(\Omega \setminus \Gamma)$ (namely for those test functions such that $\phi \equiv 0$ on Γ), we obtain $\mathrm{L}^*[\mu_\infty] = 0$ in $\Omega \setminus \Gamma$, that is

$$\mathrm{L}^*\Big(\big|\mathrm{L}[u_p]\big|^{p-2}_{(p)}\,\mathrm{L}[u_p]\Big) \,=\, 0 \quad \text{ in } \Omega \setminus \Gamma,$$

in the distributional sense. Similarly, by testing in the weak formulation of (1.13) against $\phi \in C^2_c(\Omega \setminus \Gamma)$, we obtain

$$\mathrm{L}^*[\mu_\infty] \,=\, 0 \quad \text{ in } \Omega \setminus \Gamma,$$

in the distributional sense.

Lemma 13. *For any $\alpha, \delta > 0$, $p > n$ and $u^0 \in (W^{2,p} \cap W^{1,p}_g)(\Omega)$ such that*

$$\big\|q^\delta - \mathrm{Q}[u^0]\big\|_{L^\infty(\Gamma, \mathcal{H}^\gamma)} \,\leq\, \delta,$$

the ((α,δ)-dependent) minimiser u_p of E_p (constructed in Lemmas 8-11), satisfies the error bounds (1.20), that is:

$$\Big\|\mathrm{Q}[u_p]-\mathrm{Q}[u^0]\Big\|_{L^\infty(\Gamma,\mathcal{H}^\gamma)} \le\, 2\delta+\alpha\,\|\mathrm{L}[u^0]\|_{L^p(\Omega)}.$$

If additionally $u^0\in\mathcal{W}^{2,\infty}_g(\Omega)$, then the ($(\alpha,\delta)$-dependent) minimiser u_∞ of E_∞ (constructed in Lemmas 8-11), satisfies the error bounds (1.19), that is:

$$\Big\|\mathrm{Q}[u_\infty]-\mathrm{Q}[u^0]\Big\|_{L^\infty(\Gamma,\mathcal{H}^\gamma)} \le\, 2\delta+\alpha\,\|\mathrm{L}[u^0]\|_{L^\infty(\Omega)}.$$

Proof. Let us use the symbolisation $q^0:=\mathrm{Q}[u^0]$, noting also that $q^0\in C^0(\Gamma)$ and that we have the estimate

$$\|q^\delta-q^0\|_{L^\infty(\Gamma,\mathcal{H}^\gamma)}\,\le\,\delta.$$

For any $p\in(n,\infty)$, the function u_p is a global minimiser of E_p in $(W^{2,p}\cap W^{1,p}_g)(\Omega)$. Therefore,

$$\mathrm{E}_p(u_p)\,\le\,\mathrm{E}_p(u^0).$$

This implies the estimate

$$\begin{split}\big\|\mathrm{Q}[u_p]-q^\delta\big\|_{L^p(\Gamma,\mathcal{H}^\gamma)}&+\alpha\big\|\mathrm{L}[u_p]\big\|_{L^p(\Omega)}\\ &\le\,\big\|\mathrm{Q}[u^0]-q^\delta\big\|_{L^p(\Gamma,\mathcal{H}^\gamma)}+\alpha\big\|\mathrm{L}[u^0]\big\|_{L^p(\Omega)}.\end{split}$$

The latter estimate together with the Minkowski and Hölder inequalities, in turn yield

$$\begin{split}\big\|\mathrm{Q}[u_p]-\mathrm{Q}[u^0]\big\|_{L^p(\Gamma,\mathcal{H}^\gamma)}\le\,&\big\|\mathrm{Q}[u^0]-q^\delta\big\|_{L^p(\Gamma,\mathcal{H}^\gamma)}\\ &+\,\big\|\mathrm{Q}[u^0]-q^\delta\big\|_{L^p(\Gamma,\mathcal{H}^\gamma)}+\alpha\big\|\mathrm{L}[u^0]\big\|_{L^p(\Omega)}\\ =\,&2\|q^\delta-q^0\|_{L^\infty(\Gamma,\mathcal{H}^\gamma)}+\alpha\big\|\mathrm{L}[u^0]\big\|_{L^p(\Omega)}\\ \le\,&2\delta+\alpha\,\|\mathrm{L}[u^0]\|_{L^p(\Omega)},\end{split}$$

as claimed. To obtain the corresponding estimate for u_∞ in the case that additionally $u^0\in\mathcal{W}^{2,\infty}_g(\Omega)$, we may pass to the limit as $p\to\infty$ in the last estimate above: indeed, consider the subsequence $p_i\to\infty$ along which we have the strong convergence $u_p\longrightarrow u_\infty$ in $C^1(\overline{\Omega})$ and therefore $\mathrm{Q}[u_p]\longrightarrow\mathrm{Q}[u_\infty]$ uniformly on Γ. Since by assumption $\mathrm{L}[u^0]\in L^\infty(\Omega)$, the conclusion follows by letting $i\to\infty$ in the last estimate. ☐

We now establish Proposition 6.

Proof of Proposition 6. **(i)** Let $\mathbb{B}^n_\rho(x)$ be the open ρ-ball of $\mathbb{R}^n$ centred at x. By the Lebesgue differentiation theorem (see e.g. [16]) applied to the measure $\nu\llcorner_X$ (namely to ν extended to $\mathbb{R}^n$ by zero on $\mathbb{R}^n\setminus X$) and by recalling that $\mathbb{B}_\rho(x)$ symbolises the open ball in X, we have

$$\begin{split}f(x)\,&=\,\lim_{\rho\to0}\left(⨍_{\mathbb{B}^n_\rho(x)}f\,\mathrm{d}(\nu\llcorner_X)\right)\\ &=\,\lim_{\rho\to0}\left(\frac{1}{\nu(\mathbb{B}_\rho(x))}\int_{\mathbb{B}_\rho(x)}f\,\mathrm{d}\nu\right)\end{split}$$

and therefore

$$\begin{aligned} f(x) \ &\leq\ \lim_{\rho\to 0}\left(\frac{1}{\nu(\mathbb{B}_\rho(x))}\int_{\mathbb{B}_\rho(x)} f \,\mathrm{d}\nu\right) \\ &\leq\ \lim_{\rho\to 0}\left(\nu-\underset{\mathbb{B}_\rho(x)}{\text{ess sup}}\, f\right) \\ &=\ f^\bigstar(x), \end{aligned}$$

for ν-a.e. $x \in X$.

(ii) Fix $x \in X$ and $\varepsilon > 0$. For any $\delta \in (0, \varepsilon)$ and $y \in \mathbb{B}_\delta(x)$ we have the inclusion of balls

$$\mathbb{B}_{\varepsilon-\delta}(y)\ \subseteq\ \mathbb{B}_\varepsilon(x).$$

Hence, since the limit as $\varepsilon \to 0$ in the definition of $f^\bigstar$ is in fact an infimum over all $\varepsilon > 0$, we have

$$\begin{aligned} \sup_{y\in\mathbb{B}_\delta(x)} f^\bigstar(y) \ &=\ \sup_{y\in\mathbb{B}_\delta(x)}\left[\lim_{\rho\to 0}\left(\nu-\underset{z\in\mathbb{B}_\rho(y)}{\text{ess sup}}\, f(z)\right)\right] \\ &=\ \sup_{y\in\mathbb{B}_\delta(x)}\left[\inf_{\rho>0}\left(\nu-\underset{z\in\mathbb{B}_\rho(y)}{\text{ess sup}}\, f(z)\right)\right] \\ &\leq\ \sup_{y\in\mathbb{B}_\delta(x)}\left[\nu-\underset{z\in\mathbb{B}_{\varepsilon-\delta}(y)}{\text{ess sup}}\, f(z)\right] \\ &\leq\ \sup_{y\in\mathbb{B}_\delta(x)}\left[\nu-\underset{z\in\mathbb{B}_\varepsilon(x)}{\text{ess sup}}\, f(z)\right] \\ &=\ \nu-\underset{z\in\mathbb{B}_\varepsilon(x)}{\text{ess sup}}\, f(z). \end{aligned}$$

By letting $\delta \to 0$ and $\varepsilon \to 0$, we obtain

$$\lim_{\delta\to 0}\left(\sup_{y\in\mathbb{B}_\delta(x)} f^\bigstar(y)\right)\ \leq\ \lim_{\varepsilon\to 0}\left(\nu-\underset{z\in\mathbb{B}_\varepsilon(x)}{\text{ess sup}}\, f(z)\right)\ =\ f^\bigstar(x),$$

for any $x \in X$. Hence

$$\limsup_{X\ni y\to x} f^\bigstar(y)\ \leq\ f^\bigstar(x),$$

for any $x \in X$, as desired.

(iii) We begin by noting that for any $x \in X$ and $\varepsilon > 0$ we have

$$\nu-\underset{y\in\mathbb{B}_\varepsilon(x)}{\text{ess sup}}\, f(y)\ \leq\ \nu-\underset{y\in X}{\text{ess sup}}\, f(y)$$

which readily implies

$$\sup_{x\in X} f^\bigstar(x)\ =\ \sup_{x\in X}\left(\nu-\underset{y\in\mathbb{B}_\varepsilon(x)}{\text{ess sup}}\, f(y)\right)\ \leq\ \nu-\underset{x\in X}{\text{ess sup}}\, f(x).$$

Conversely, by the definition of the essential supremum, for any $\delta > 0$, the set

$$X(\delta)\ :=\ \left\{x\in X\ :\ f(x) > \nu-\underset{y\in X}{\text{ess sup}}\, f(y)-\delta\right\}$$

satisfies

$$\nu(X(\delta))\ >\ 0.$$

By the Lebesgue-Besicovitch differentiation theorem (see e.g. [16]), ν-a.e. point $x \in X_\delta$ has density 1, namely

$$\lim_{\varepsilon\to 0} \frac{\nu\big(X(\delta)\cap \mathbb{B}^n_\varepsilon(x)\big)}{\nu(\mathbb{B}^n_\varepsilon(x))} \,=\, 1,$$

where $\mathbb{B}^n_\varepsilon(x)$ is the open ε-ball centred at x with respect to $\mathbb{R}^n$. Hence, since

$$\mathbb{B}_\varepsilon(x) \,=\, X \cap \mathbb{B}^n_\varepsilon(x),$$

for any $\delta > 0$, there exists $x_\delta \in X(\delta)$ such that

$$\nu\big(X(\delta)\cap \mathbb{B}_\varepsilon(x_\delta)\big) \,=\, \nu\big(X(\delta)\cap \mathbb{B}^n_\varepsilon(x_\delta)\big) \,>\, 0.$$

Therefore, since

$$\nu - \operatorname*{ess\,sup}_{y\in X} f(y) \,\leq\, \delta \,+\, f(x), \quad \nu - \text{a.e. } x \in X(\delta),$$

we deduce

$$\begin{aligned}\nu - \operatorname*{ess\,sup}_{y\in X} f(y) \,&\leq\, \delta \,+\, \nu - \operatorname*{ess\,sup}_{y\in \mathbb{B}_\varepsilon(x_\delta)\cap X(\delta)} f(y)\\ &\leq\, \delta \,+\, \nu - \operatorname*{ess\,sup}_{y\in \mathbb{B}_\varepsilon(x_\delta)} f(y).\end{aligned}$$

By letting $\varepsilon \to 0$ in the above inequality, we infer that

$$\begin{aligned}\nu - \operatorname*{ess\,sup}_{x\in X} f(x) \,&\leq\, \delta \,+\, \lim_{\varepsilon\to 0}\left(\nu - \operatorname*{ess\,sup}_{y\in \mathbb{B}_\varepsilon(x_\delta)} f(y)\right)\\ &=\, \delta \,+\, f^\bigstar(x_\delta)\\ &\leq\, \delta \,+\, \sup_{x\in X} f^\bigstar(x),\end{aligned}$$

for any $\delta > 0$. By letting $\delta \to 0$, we obtain

$$\nu - \operatorname*{ess\,sup}_{x\in X} f(x) \,\leq\, \sup_{x\in X} f^\bigstar(x),$$

as desired. This inequality completes the proof. $\square$

By invoking Proposition 7 whose proof follows, we readily obtain (1.14)-(1.15) by choosing

$$X = \Gamma, \quad \nu = \mathcal{H}^\gamma \llcorner_\Gamma, \quad f_k = \mathrm{Q}[u_{p_k}] - q^\delta, \quad f_\infty = \mathrm{Q}[u_\infty] - q^\delta.$$

Proof of Proposition 7. **(i)** By the definition of ν_k, we have for any continuous function $\phi \in C^0(X)$ with $|\phi| \leq 1$ that

$$\begin{aligned}\left|\int_X \phi \,\mathrm{d}\nu_k\right| \,&\leq\, \frac{1}{\big\||f_k|_{(k)}\big\|^{k-1}_{L^k(X,\nu)}} \fint_X \Big|\big(|f_k|_{(k)}\big)^{k-2} f_k\, \phi\Big| \,\mathrm{d}\nu\\ &\leq\, \frac{1}{\big\||f_k|_{(k)}\big\|^{k-1}_{L^k(X,\nu)}} \fint_X \big(|f_k|_{(k)}\big)^{k-1} \,\mathrm{d}\nu.\end{aligned}$$

Hence, by Hölder inequality, we have the total variation bound

$$\begin{aligned}\|\nu_k\|(X) \,&\leq\, \Big(\big\||f_k|_{(k)}\big\|_{L^k(X,\nu)}\Big)^{1-k} \left(\fint_X \big(|f_k|_{(k)}\big)^k \,\mathrm{d}\nu\right)^{\frac{k-1}{k}}\\ &=\, 1.\end{aligned}$$

By the sequential weak* compactness of the space $\mathcal{M}(X) = \big(C^0(X)\big)^*$, we obtain the desired subsequence $(\nu_{k_i})_1^\infty \subseteq \mathcal{M}(X)$ and the weak* sequential limit measure $\nu_\infty \in \mathcal{M}(X)$.

(ii) We begin by showing the elementary inequality

$$\big||f_k|_{(k)} - |f_\infty|\big| \,\leq\, \big|f_k - f_\infty\big| + \frac{1}{k} \quad \text{on } X.$$

Indeed, if $|f_k|_{(k)} \geq |f_\infty|$, we have

$$\begin{aligned}\big||f_k|_{(k)} - |f_\infty|\big| \,&=\, \sqrt{|f_k|^2 + k^{-2}} \,-\, |f_\infty| \\ &\leq\, |f_k| - |f_\infty| \,+\, \frac{1}{k} \\ &\leq\, \big|f_k - f_\infty\big| \,+\, \frac{1}{k}\end{aligned}$$

whilst if $|f_k|_{(k)} < |f_\infty|$, we have

$$\begin{aligned}\big||f_k|_{(k)} - |f_\infty|\big| \,&=\, |f_\infty| \,-\, \sqrt{|f_k|^2 + k^{-2}} \\ &\leq\, |f_\infty| \,-\, |f_k| \\ &\leq\, \big|f_k - f_\infty\big| \,+\, \frac{1}{k}.\end{aligned}$$

Fix now $\varepsilon > 0$. The inequality we just proved implies that if $f_k \longrightarrow f_\infty$ uniformly on X as $k \to \infty$ (note that f_k, f_∞ might be discontinuous), then $|f_k|_{(k)} \longrightarrow |f_\infty|$ uniformly on X as $k \to \infty$. Hence, there exists $k(\varepsilon) \in \mathbb{N}$ such that

$$\|f_k - f_\infty\|_{L^\infty(X,\nu)} < \frac{\varepsilon}{4}, \quad \big\||f_k|_{(k)} - |f_\infty|\big\|_{L^\infty(X,\nu)} < \frac{\varepsilon}{4},$$

for all $k \geq k(\varepsilon)$. Therefore,

$$\begin{aligned}|f_k| \,&\leq\, |f_\infty| \,+\, \frac{\varepsilon}{4}, \qquad \nu - \text{a.e. on } X, \\ |f_k|_{(k)} \,&\geq\, |f_\infty| \,-\, \frac{\varepsilon}{4}, \qquad \nu - \text{a.e. on } X.\end{aligned}$$

By integrating the latter inequality and using the Minkowski inequality, we obtain

$$\||f_k|_{(k)}\|_{L^k(X,\nu)} \,\geq\, \|f_\infty\|_{L^k(X,\nu)} \,-\, \frac{\varepsilon}{4},$$

for all $k \geq k(\varepsilon)$. Since

$$\|f_\infty\|_{L^\infty(X,\nu)} \,=\, \lim_{k\to\infty} \|f_\infty\|_{L^k(X,\nu)},$$

by choosing $k(\varepsilon)$ greater if necessary, we deduce

$$\||f_k|_{(k)}\|_{L^k(X,\nu)} \,\geq\, \|f_\infty\|_{L^\infty(X,\nu)} \,-\, \frac{\varepsilon}{2},$$

for all $k \geq k(\varepsilon)$. Let now $\mathrm{d}\nu_k/\mathrm{d}\nu$ symbolise the Radon-Nikodym derivative of ν_k with respect to ν. It follows that

$$\frac{\mathrm{d}\nu_k}{\mathrm{d}\nu} \,=\, \frac{1}{\nu(X)} \frac{\big(|f_k|_{(k)}\big)^{k-2} f_k}{\big\||f_k|_{(k)}\big\|^{k-1}_{L^k(X,\nu)}}, \quad \nu\text{-a.e. on } X.$$

By the above, for any $\varepsilon > 0$ small enough (recall that $f_\infty \not\equiv 0$) and for any $k \geq k(\varepsilon)$, we have the estimate

$$\left|\frac{\mathrm{d}\nu_k}{\mathrm{d}\nu}\right| \leq \frac{1}{\nu(X)}\left(\frac{\dfrac{1}{k} + |f_\infty| + \dfrac{\varepsilon}{4}}{\|f_\infty\|_{L^\infty(X,\nu)} - \dfrac{\varepsilon}{2}}\right)^{k-1}, \qquad \nu - \text{a.e. on } X.$$

By choosing $k(\varepsilon)$ even larger if needed, we can arrange

$$\left|\frac{\mathrm{d}\nu_k}{\mathrm{d}\nu}\right| \leq \frac{1}{\nu(X)}\left(\frac{2|f_\infty| + \varepsilon}{2\|f_\infty\|_{L^\infty(X,\nu)} - \varepsilon}\right)^{k-1}, \qquad \nu - \text{a.e. on } X.$$

Since by Proposition 6 we have $|f_\infty| \leq |f_\infty|^\bigstar$ ν-a.e. on X, we obtain

$$\left|\frac{\mathrm{d}\nu_k}{\mathrm{d}\nu}\right| \leq \frac{1}{\nu(X)}\left(\frac{2|f_\infty|^\bigstar + \varepsilon}{2\|f_\infty\|_{L^\infty(X,\nu)} - \varepsilon}\right)^{k-1}, \qquad \nu - \text{a.e. on } X.$$

Consider now for any $\varepsilon > 0$ the ν-measurable set

$$X_\varepsilon := \Big\{|f_\infty|^\bigstar < \|f_\infty\|_{L^\infty(X,\nu)} - 2\varepsilon\Big\}.$$

Notice also that X_ε is in fact open in X because $|f_\infty|^\bigstar$ is upper semicontinuous (Proposition 6). Additionally, we have the estimate

$$\left|\frac{\mathrm{d}\nu_k}{\mathrm{d}\nu}\right| \leq \frac{1}{\nu(X)}\left(\frac{2\|f_\infty\|_{L^\infty(X,\nu)} - 3\varepsilon}{2\|f_\infty\|_{L^\infty(X,\nu)} - \varepsilon}\right)^{k-1}, \qquad \nu - \text{a.e. on } X_\varepsilon.$$

The above estimate together with the Lebesgue Dominated Convergence theorem imply that for any $\varepsilon > 0$ small enough we have

$$\frac{\mathrm{d}\nu_k}{\mathrm{d}\nu} \longrightarrow 0 \quad \text{in } L^1(X_\varepsilon,\nu), \quad \text{as } k \to \infty.$$

Consider now the sequence of nonnegative total variation measures $(\|\nu_k\|)_1^\infty \subseteq \mathcal{M}(X)$. Since this sequence is also bounded in the space, there exists a nonnegative limit measure λ_∞ such that

$$\|\nu_k\| \overset{*}{\rightharpoonup} \lambda_\infty \quad \text{in } \mathcal{M}(X),$$

along perhaps a further subsequence $(k_i)_1^\infty$. Additionally, since $\nu_k \overset{*}{\rightharpoonup} \nu_\infty$ in $\mathcal{M}(X)$, we have the inequality (see e.g. [2])

$$\|\nu_\infty\| \leq \lambda_\infty.$$

Note now that for each $k \in \mathbb{N}$, by the Lebesgue-Radon-Nikodym theorem applied to $\|\nu_k\| << \nu$ we have the decomposition

$$\|\nu_k\| = \left|\frac{\mathrm{d}\nu_k}{\mathrm{d}\nu}\right|\nu.$$

Hence, we infer that

$$\|\nu_k\|(X_\varepsilon) \leq \int_{X_\varepsilon}\left|\frac{\mathrm{d}\nu_k}{\mathrm{d}\nu}\right|\mathrm{d}\nu \longrightarrow 0, \qquad \text{as } k \to \infty.$$

Therefore, since X_ε is open in X, by the weak* lower-semicontinuity of measures on open sets (see e.g. [16, 2]) and the above arguments, we have

$$\begin{aligned}
\|\nu_\infty\|(X_e) \ &\leq\ \lambda_\infty(X_\varepsilon)\\
&\leq\ \liminf_{i\to\infty}\|\nu_{k_i}\|(X_\varepsilon)\\
&\leq\ \liminf_{i\to\infty}\int_{X_\varepsilon}\left|\frac{\mathrm{d}\nu_{k_i}}{\mathrm{d}\nu}\right|\mathrm{d}\nu\\
&=\ 0.
\end{aligned}$$

Therefore, we have obtained

$$\nu_\infty\Big(\Big\{|f_\infty|^\bigstar < \|f_\infty\|_{L^\infty(X,\nu)} - 2\varepsilon\Big\}\Big)\ =\ 0, \quad \text{for any } \varepsilon>0.$$

By letting $\varepsilon\to 0$ along the sequence $\varepsilon_j := 2^{-j-1}$, the continuity of the measure ν_∞ implies

$$\begin{aligned}
\nu_\infty\Big(\Big\{|f_\infty|^\bigstar < \|f_\infty\|_{L^\infty(X,\nu)}\Big\}\Big)\ &=\ \nu_\infty\left(\bigcup_{j=1}^{\infty}\Big\{|f_\infty|^\bigstar < \|f_\infty\|_{L^\infty(X,\nu)} - 2^{-j}\Big\}\right)\\
&=\ \lim_{j\to\infty}\nu_\infty\Big(\Big\{|f_\infty|^\bigstar < \|f_\infty\|_{L^\infty(X,\nu)} - 2^{-j}\Big\}\Big)\\
&=\ 0.
\end{aligned}$$

Then, the definition of support of the measure ν_∞ and the upper semicontinuity of the function $|f_\infty|^\bigstar$ on X (by Proposition 6) yield

$$\begin{aligned}
X\setminus \mathrm{supp}(\nu_\infty)\ &=\ \bigcup\Big\{U\subseteq X \text{ open } :\ \nu_\infty(U)=0\Big\}\\
&\supseteq\ \Big\{|f_\infty|^\bigstar < \|f_\infty\|_{L^\infty(X,\nu)}\Big\}.
\end{aligned}$$

In conclusion, we infer that

$$\begin{aligned}
\mathrm{supp}(\nu_\infty)\ &\subseteq\ X\setminus\Big\{|f_\infty|^\bigstar < \|f_\infty\|_{L^\infty(X,\nu)}\Big\}\\
&=\ \Big\{|f_\infty|^\bigstar = \|f_\infty\|_{L^\infty(X,\nu)}\Big\},
\end{aligned}$$

as desired.

(iii) Suppose that $|f_\infty|$ is continuous on X and recall the properties of the essential limsup established in Proposition 6. Then, for any $x\in X$ we have

$$\begin{aligned}
\Big||f_\infty|^\bigstar(x) - |f_\infty|(x)\Big|\ &=\ \left|\lim_{\varepsilon\to 0}\left(\nu - \operatorname*{ess\,sup}_{\mathbb{B}_\varepsilon(x)}|f_\infty|\right) - |f^\infty|(x)\right|\\
&=\ \left|\lim_{\varepsilon\to 0}\left(\nu - \operatorname*{ess\,sup}_{\mathbb{B}_\varepsilon(x)}|f_\infty|\ -\ |f^\infty|(x)\right)\right|\\
&\leq\ \limsup_{\varepsilon\to 0}\left|\nu - \operatorname*{ess\,sup}_{\mathbb{B}_\varepsilon(x)}\Big(|f_\infty|\ -\ |f^\infty|(x)\Big)\right|\\
&\leq\ \limsup_{\varepsilon\to 0}\Big\||f_\infty|\ -\ |f_\infty|(x)\Big\|_{L^\infty(\mathbb{B}_\varepsilon(x),\nu)}\\
&=\ 0,
\end{aligned}$$

showing that $|f_\infty|^\bigstar\equiv|f_\infty|$, if it holds that $|f_\infty|$ is continuous on X. $\square$

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Department of Mathematics and Statistics, University of Reading, Whiteknights, PO Box 220, Reading RG6 6AX, United Kingdom

E-mail address: n.katzourakis@reading.ac.uk