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DEPARTMENT OF MATHEMATICS

VARIATIONAL PRINCIPLES AND THE FINITE ELEMENT METHOD FOR CHANNEL FLOWS

by

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Abstract

Hamilton's principle is used to devise a variational principle which has as its natural conditions the equations of irrotational motion of an incompressible, homogeneous, inviscid fluid with a free surface. By applying the shallow water approximation to the flow variables this variational principle is reduced to another one whose natural conditions are the shallow water equations of motion. Boundary terms are added to the functional of this variational principle so that the natural conditions now include boundary and initial conditions as well as the equations of motion. A quartet of variational principles for shallow water flows is derived by using Legendre transforms. These principles are modified to give other principles for steady shallow water flows by assuming that the flow variables do not depend on time. Variational principles are also derived for quasi one-dimensional shallow water flows — both time-dependent and time-independent — and for steady state discontinuous flows.

Approximations to continuous and discontinuous flows in channels of varying breadth and domain bed profiles are calculated using finite element approximations for a selection of the variational principles developed. Approximations to steady continuous flows are calculated on fixed grids using both the quasi one-dimensional and the two-dimensional formulations. Methods of generating adaptive grids in one dimension using the variational principles are also studied and an algorithm is given for generating approximations on an adaptive grid to steady discontinuous quasi one-dimensional flows. Approximations are also found for time-dependent quasi one-dimensional flows.

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Contents

Chapter 1

Introduction

The problem of fluid flow over an uneven topography and through constricting channels has been of interest to hydraulic engineers and meteorologists for many years. Variational methods have been widely used in other areas for even longer but have only recently begun to play a significant part in the problems of fluid mechanics. The finite element method is a relatively recent technique, which has advanced the construction of approximate solutions, particularly in relation to elliptic problems. This thesis brings together these three subjects.

More specifically the aim of this thesis is to generate numerical approximations to the solutions of the equations governing the irrotational motion of an homogeneous, incompressible, inviscid fluid over a fixed bed profile. The method implemented here depends on the derivation of variational principles which are satisfied for the solutions of these equations of motion. Approximate solutions to the equations are derived as those functions in a finite dimensional space for which the functionals of the variational principles are stationary with respect to variations in that space.

Luke (1967) showed that a variational principle in which the integrand (the Lagrangian density) is taken to be the fluid pressure, as given by Bernoulli's energy integral, has as its natural conditions the equations governing a free surface flow. The natural conditions of a variational principle are those which make the corresponding functional stationary. The natural conditions of Luke's principle are Laplace's equation, holding in the fluid domain, the no flow condition across the bed and the dynamic and kinematic free surface conditions. Luke assumes that all of the variations vanish on the other space boundaries and at the ends of the time interval.

Hamilton's variational principle in particle mechanics (see, for example, Goldstein (1980)) has, as the Lagrangian, the difference between the kinetic and potential energies of a system. The natural conditions of the variational principle are Lagrange's equations of motion. Salmon (1988) considers applications of classical Hamiltonian theory to fluid mechanics. Many of these applications, such as, for example, in Serrin (1959) and in Seliger and Whitham (1968), have been within the area of gas dynamics. However, Miles and Salmon (1985) derived equations describing the motion of weakly dispersive non-linear gravity waves using Hamilton's principle. In this thesis Hamilton's principle is adapted to give a principle whose natural conditions are the equations governing a free surface flow, and this principle can be rearranged to give Luke's principle, in the case where the variations vanish on all time and space boundaries except the free surface and the domain bed.

There is a point of contact between free surface flows and compressible gas flows if the shallow water approximation to free surface flows is invoked (Stoker

(1957)). Shallow water theory is an approximation to three-dimensional free surface flows in circumstances where the fluid depth is small compared with some characteristic length scale of the motion, such as the radius of curvature of the free surface. In this thesis shallow water theory at its lowest order is considered; this is the basic theory used in hydraulics to model flows in open channels and also gives good approximations to the motion of tides in the oceans and the breaking of waves on shallow beaches. The flow domain over which approximations to shallow water flows are considered here, is a channel of slowly varying breadth, so that, to a first approximation, the flow can be thought of as being quasi one-dimensional.

A substantial part of the thesis — Chapter 3 — is devoted to the derivation of the variational principles corresponding to three-dimensional free surface flows and to shallow water flows. Hamilton's principle and a modified version of Luke's principle are used as the starting points of the investigation. By approximating the variables of three-dimensional flows by their shallow water counterparts it is possible to derive variational principles which are satisfied for solutions of the shallow water equations of motion. It is shown that Hamilton's principle and the modified version of Luke's principle are essentially the same, as are the two variational principles for shallow water which are derived from them. Different representations of the variational principle for shallow water are available, based on the notion of a closed sequence of Legendre transforms introduced by Sewell (1987). The variational principles for shallow water flows are enhanced by the addition of boundary terms so that variations can be allowed which do not necessarily vanish on the boundaries. This is an important step since, in the practical implementation of a variational principle, if the variations are to vanish on the boundaries then it implies that the solution must be known there.

There is, however, an undesirable feature of these principles, that conditions on some of the flow variables must be given at both ends of the time interval. This problem does not arise in steady shallow water flows, which are considered in some detail. The variational principles for these flows are deduced from the principles for time-dependent flows.

Further variational principles are created by making the assumption that the shallow water flow is quasi one-dimensional, yielding variational principles for time-dependent and time-independent quasi one-dimensional flows.

A number of simpler variational principles can be derived by constraining the variations to satisfy one or more of the natural conditions. A selection of these constrained principles is presented, some of which fit with the notion of reciprocal variational principles.

The final section of Chapter 3 deals with the derivation of variational principles for steady discontinuous flows, that is, for flows which contain hydraulic jumps. The differential equations of shallow water flow are valid in regions of the domain excluding the discontinuity while at the discontinuity the equations of motion are replaced by jump conditions, which relate the values of the flow variables on either side of the discontinuity. One of the jump conditions is used in the formulation of the variational principles and the others are derived as natural conditions by making an assumption about the variations in the flow variables at the discontinuity.

The remainder of the thesis is concerned with using the variational principles to generate approximate solutions for flows in channels. The Ritz method (see

Strang and Fix (1973)) can be used to obtain approximate solutions of a variational principle by expanding the variables in terms of trial functions and using the variational principle to evaluate the parameters of the expansions. The finite element approach is implemented by choosing the trial functions to be piecewise polynomials, which are zero over most of the domain, these trial functions being known as finite element basis functions. The channel flows are approximated here by using piecewise linear basis functions, where the basis function corresponding to a particular node of a grid is linear and continuous and non-zero only in the elements surrounding the node, and piecewise constant basis functions, where the basis function corresponding to a particular element is non-zero only in that element. The basic method is then to seek the functions in a finite dimensional space, spanned by a set of finite element basis functions, for which the functional corresponding to a particular variational principle is stationary with respect to variations in that space.

The parameters of the expansions are found by solving one or more sets of equations, at least one set of which is non-linear. The non-linear equations are solved using Newton's method.

In Chapter 4, the algorithms for approximating time-independent quasi one-dimensional flows in shallow water are presented. Several versions of two particular variational principles are considered and used to generate approximations to continuous and discontinuous flows on fixed and adaptive grids. In the discontinuous case the positioning of the approximation to the discontinuity requires care. This is because, although all of the equations governing the motion are either implicit in the variational principle or derived as natural conditions, the jump conditions are only generated as natural conditions by imposing specific conditions on the variations. It is not clear how these conditions could be implemented in practice and the algorithm used here is based on generating separate approximations to the continuous parts of the solution and coupling the approximations at the discontinuity by using the jump conditions, in the process of which an approximation to the position of the discontinuity is also found.

Chapter 5 deals with finding approximations to steady two-dimensional continuous shallow water flows, by extending the algorithms of Chapter 4.

In Chapter 6 two further applications of the variational principles are investigated. In an attempt to study the accuracy of the shallow water approximation to free surface flows a version of Luke's principle for steady state flows is used to generate approximations in this case. Finally an algorithm to generate approximations to time-dependent quasi one-dimensional shallow water flows is considered.

Chapter 2

Background Fluid Dynamics

In this chapter the equations governing the three-dimensional motion of an incompressible, homogeneous fluid under a free surface are given and adapted to the various problems which will be considered later. An approximation to such a three-dimensional motion can be devised by assuming that the fluid depth is small compared with a typical horizontal length scale of the motion. This so-called shallow water approximation generates a simplified set of equations by removing the vertical motion, at lowest order. Shallow water theory is often applied in channels and this case only will be considered.

2.1 Free Surface Flows

In this section the equations governing the motion of a fluid under a free surface are given. The fluid is assumed to be incompressible and homogeneous and the motion is assumed to be irrotational.

Let x, y, z be cartesian coordinates, with z measured vertically upwards from the equilibrium position of the free surface, and let t be the time. Consider the

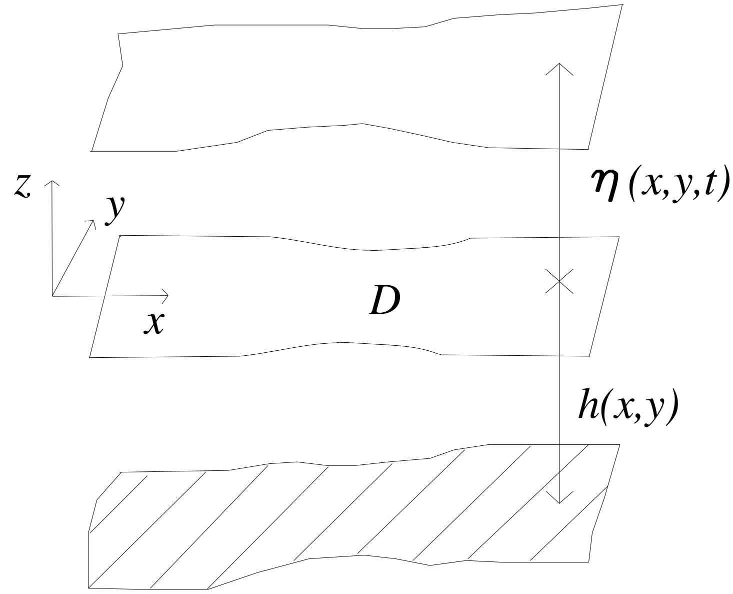

Figure 2.1: The domain of the free surface problem.

domain Ω, extending over a fixed region D in the xy plane and enclosed by the surfaces $z = -h(x, y)$ and $z = \eta(x, y, t)$, where h is the known undisturbed fluid depth and η is the unknown height of the free surface above the reference level $z = 0$, as shown in Figure 2.1.

In the domain Ω the motion is governed by the laws of conservation of mass and momentum. Let $\mathbf{u} = \mathbf{u}(x, y, z, t)$ be the Eulerian velocity, where $\mathbf{u} = (u, v, w)$. Then the conservation of mass requires

$$\tilde{\boldsymbol{\nabla}}.\mathbf{u} = 0, \tag{2.1}$$

where the operator $\tilde{\boldsymbol{\nabla}}$ is defined by

$$\tilde{\boldsymbol{\nabla}} \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right). \tag{2.2}$$

Let ρ be the density of the fluid, a constant by assumption, let g be the acceleration due to gravity, also assumed constant, and let $\tilde{p} = \tilde{p}(x, y, z, t)$ be the fluid pressure. Then the conservation of momentum equation is given by

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho}\tilde{\boldsymbol{\nabla}}\tilde{p} - \tilde{\boldsymbol{\nabla}}(gz), \tag{2.3}$$

where $\frac{D}{Dt}$ is the derivative following the motion and is defined by

$$\frac{DF}{Dt} \equiv \frac{\partial F}{\partial t} + \left(\mathbf{u}.\tilde{\boldsymbol{\nabla}}\right) F. \tag{2.4}$$

The flow is irrotational so that

$$\mathbf{u} = \tilde{\boldsymbol{\nabla}}\chi \tag{2.5}$$

for some $\chi = \chi(x, y, z, t)$, where χ is called the velocity potential. Therefore the conservation of mass equation for irrotational flow may be written as

$$\tilde{\nabla}^2\chi = 0. \tag{2.6}$$

The integrated version of the conservation of momentum equation for irrotational flow is

$$\frac{\tilde{p}}{\rho} + \chi_t + \frac{1}{2}\tilde{\boldsymbol{\nabla}}\chi.\tilde{\boldsymbol{\nabla}}\chi + gz = 0, \tag{2.7}$$

where the arbitrary function of t arising on integration has been absorbed into the χ_t term. Equation (2.7) is called the Bernoulli equation of the flow; it represents energy balance and determines $\tilde{p}$ from χ.

Equations (2.1), (2.3) and (2.5) govern the motion in the domain Ω. Further equation are needed at the boundaries of Ω.

Bernoulli's equation gives rise to the dynamic free surface condition. At the free surface, $z = \eta$, the fluid pressure $\tilde{p}$ is assumed to be a constant, set to zero for convenience. Thus Bernoulli's equation gives

$$\chi_t + \frac{1}{2}\tilde{\boldsymbol{\nabla}}\chi.\tilde{\boldsymbol{\nabla}}\chi + g\eta = 0 \qquad \text{on } z = \eta. \tag{2.8}$$

At boundaries of Ω across which there is no flow the conservation of mass equation is replaced by a boundary condition, that is, if $F(x, y, z, t) = 0$ is the equation of the boundary, at every point of this boundary the equation

$$\frac{DF}{Dt} = 0 \tag{2.9}$$

must be satisfied (Lamb (1932)).

The equation of the free surface is given by $z - \eta = 0$. Thus the kinematic free surface condition is

$$\frac{D}{Dt}(z - \eta) = 0,$$

which may be rewritten as

$$\eta_t + u\eta_x + v\eta_y - w = 0 \qquad \text{on } z = \eta. \tag{2.10}$$

The equation of the fixed bed is given by $z + h = 0$. This gives the condition of zero flow through the bed as

$$\frac{D}{Dt}(z + h) = 0,$$

or

$$uh_x + vh_y + w = 0 \qquad \text{on } z = -h. \tag{2.11}$$

Equation (2.9) also applies at any lateral boundary across which there is no flow.

Equations (2.1), (2.3), (2.5), (2.8), (2.10) and (2.11) constitute the set of equations governing three-dimensional flow in an arbitrary domain Ω.

2.2 The Shallow Water Approximation

Shallow water theory offers an approximation to free surface flows in circumstances where the water depth is much less than some characteristic length scale of the motion, such as the radius of curvature of the free surface. It is essentially an averaging process in which the fluid motion is replaced by a representative motion in the horizontal spatial coordinates. Each particle can be thought of as the aggregate of all the actual fluid particles lying in the same vertical line.

2.2.1 Derivation of the Shallow Water Equations

To lowest order, shallow water theory can be generated by assuming that the fluid pressure is hydrostatic (Stoker (1957)). That is,

$$\tilde{p}(x, y, z, t) = \rho g \left(\eta - z\right), \tag{2.12}$$

where the constant surface pressure has been set to zero for convenience.

Equation (2.12) can be used to determine a vertically averaged replacement for the pressure, $p = p(x, y, t)$, defined by

$$p = \frac{1}{\rho} \int_{-h}^{\eta} \tilde{p}\, dz,$$

from which it follows that

$$p = \frac{1}{2} g d^2, \tag{2.13}$$

where $d(x, y, t)$ is the fluid depth at location (x, y) and at time t, that is $d = h + \eta$.

Equation (2.12) implies that $\tilde{\boldsymbol{\nabla}} \tilde{p}$ is independent of z and so, from $(2.3)_{1,2}$, the acceleration of the water particles in the x and y directions is also independent of z. Thus, if the horizontal components of velocity, u and v, are independent of z at any time, they will remain independent of z throughout the motion. Substituting (2.12) into $(2.3)_3$ gives the result that, in lowest order shallow water theory, the vertical acceleration of the fluid particles is zero, that is, negligible compared with g. It is also negligible compared with u and v. These results can be summarised as

$$u_z = 0\,, \quad v_z = 0 \;\text{ and }\; w = 0. \tag{2.14}$$

The equations of shallow water motion are derived by substituting (2.14) into the equations governing the three-dimensional motion — (2.1), (2.3), (2.5), (2.8), (2.10) and (2.11).

The effect of $(2.14)_3$ on the irrotationality condition (2.5) is that the velocity potential $\chi(x, y, z, t)$ is replaced by the velocity potential $\phi(x, y, t)$ and irrotationality in shallow water is represented by

$$\mathbf{v} = \boldsymbol{\nabla}\phi, \tag{2.15}$$

where $\mathbf{v} = \mathbf{v}(x, y, t)$ is the velocity of the reduced problem, such that $\mathbf{v} = (u, v)$, and the operator $\boldsymbol{\nabla}$ is given by

$$\boldsymbol{\nabla} \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right). \tag{2.16}$$

The hypothesis (2.12) implies that the first two components of the conservation of momentum equation (2.3) can be written as

$$\mathbf{v}_t + (\mathbf{v}.\boldsymbol{\nabla})\,\mathbf{v} = -g\boldsymbol{\nabla}\eta. \tag{2.17}$$

For irrotational flow, where $\mathbf{v}$ satisfies (2.15), there exists the identity

$$(\mathbf{v}.\boldsymbol{\nabla})\,\mathbf{v} = \frac{1}{2}\boldsymbol{\nabla}\,(\mathbf{v}.\mathbf{v})\,.$$

Therefore (2.17) may be written as

$$\mathbf{v}_t + \frac{1}{2}\boldsymbol{\nabla}\,(\mathbf{v}.\mathbf{v}) = -g\boldsymbol{\nabla}\eta,$$

or, alternatively, as

$$\mathbf{v}_t + \boldsymbol{\nabla}E = g\boldsymbol{\nabla}h, \tag{2.18}$$

where E is an energy per unit mass, referred to as an energy for short, defined by

$$E = gd + \frac{1}{2}\mathbf{v}.\mathbf{v}. \tag{2.19}$$

Equation (2.18) is the equation of conservation of momentum in shallow water. The integrated version of (2.18) is more common in the variational principles which follow later and, for irrotational flow, this is given by

$$\phi_t + E = gh, \tag{2.20}$$

where an arbitrary function of t has been absorbed into ϕ_t. Equation (2.20) is the Bernoulli equation for shallow water.

For the particular case where the equilibrium depth, h, is a constant, that is, the bed is horizontal, the conservation of momentum equation is given by

$$\mathbf{v}_t + \boldsymbol{\nabla} E = 0. \tag{2.21}$$

Integrating (2.21) with respect to x and y and using (2.15) yields

$$\phi_t + E = 0, \tag{2.22}$$

where an arbitrary function of t has again been absorbed into ϕ_t. Equation (2.22) is consistent with (2.20) for the case h = constant. In (2.22) the constant term gh has been absorbed into ϕ_t which is equivalent to moving the reference level for potential energy in the coordinate system from $z = 0$ to $z = -h$.

The conservation of mass equation for free surface flow (2.1) may be written as

$$u_x + v_y + w_z = 0.$$

Integrating through the fluid depth at a point (x, y) gives

$$\int_{-h}^{\eta} (u_x + v_y + w_z)\, dz = 0. \tag{2.23}$$

Equations $(2.14)_{1,2}$ imply that u_x and v_y are independent of z so (2.23) may be rewritten as

$$(u_x + v_y)\, d + [w]_{-h}^{\eta} = 0.$$

Then, substituting for $w|_{\eta}$ and $w|_{-h}$ using the kinematic boundary conditions of zero flow through the free surface and the bed, (2.10) and (2.11), (2.23) becomes

$$d\boldsymbol{\nabla}.\mathbf{v} + \mathbf{v}.\boldsymbol{\nabla}h + \mathbf{v}.\boldsymbol{\nabla}\eta + \eta_t = 0.$$

Using $h_t \equiv 0$, the conservation of mass equation for shallow water can be written as

$$d_t + \boldsymbol{\nabla}.\mathbf{Q} = 0, \tag{2.24}$$

where

$$\mathbf{Q} = d\mathbf{v} \tag{2.25}$$

may be called the mass flow vector since d plays the part of density by analogy with gas dynamics, as will be indicated in Section 2.2.2.

Equations (2.15), (2.18) and (2.24) are the equations of motion for shallow water flow. The variables of the flow, d, $\mathbf{v}$, $\mathbf{Q}$, E and ϕ, are all functions of the time, t, and of the horizontal spatial coordinates x and y.

Let D be the domain of the reduced problem, where D extends over a fixed part of the horizontal xy plane. Let Σ be the boundary of D. Then, for consistency with conservation of mass, on any impenetrable portions of the boundary the mass flow across the boundary must be zero, that is $\mathbf{Q}.\mathbf{n} = 0$, where $\mathbf{n}$ is the outward normal to the boundary.

In the case of channel flow it is usual to define a boundary function, say $C(x, y, t)$, on Σ, such that

$$\mathbf{Q}.\mathbf{n} = C \qquad \text{on } \Sigma, \tag{2.26}$$

where $C \equiv 0$ on the fixed sides of the channel and at the inlet and outlet parts of the boundary C is an assigned mass flow.

Stoker (1957) uses the hydrostatic approximation (2.12) to derive the shallow water equations and also derives the same equations by a perturbation expansion method. In this latter case the flow variables in the exact equations, (2.1), (2.3), (2.5), (2.8), (2.10) and (2.11), are expanded in terms of a parameter which is small when the depth of fluid is much less than a typical horizontal length. The shallow water equations are obtained by equating the lowest order terms. The perturbation expansion method can also be used to generate higher order approximations to free surface flows.

2.2.2 The Gas Dynamics Analogy

The fact that the equations of motion for shallow water can be written in the same form as the equations of motion for a compressible gas flow is known as the gas dynamics analogy (Stoker (1957)).

The irrotationality condition (2.15), the conservation of momentum equation (2.18) and the conservation of mass equation (2.24) for shallow water can be rearranged to give

$$\mathbf{v} = \boldsymbol{\nabla}\phi, \tag{2.27}$$

$$\mathbf{v}_t + (\mathbf{v}.\boldsymbol{\nabla})\,\mathbf{v} = -\frac{1}{d}\boldsymbol{\nabla}p + g\boldsymbol{\nabla}h, \tag{2.28}$$

$$d_t + \nabla . (d\mathbf{v}) = 0, \tag{2.29}$$

respectively, where the pressure, p, is given by (2.13). Equations (2.27)—(2.29) may be regarded as the equations governing a two-dimensional gas flow in which d plays the part of density and the term $g\nabla h$ in (2.28) is thought of as a body force or as a heat source. For the special case where the equilibrium depth, h, is a constant, the forcing term $g\nabla h$ in (2.28) is zero and (2.28) is the usual momentum conservation equation for gas flow. Also, in gas dynamics terminology, equation (2.13), which defines p as a function of the 'density' d, is an 'adiabatic' relation (Courant and Friedrichs (1948)).

In Chapter 3 variational principles for shallow water flows are derived and subsequently used, in Chapter 4, to generate numerical approximations to channel flows. Variational principles for compressible gas flows have been developed previously by, for example, Bateman (1929), Sewell (1963) and Wixcey (1990). The analogy of shallow water theory with gas dynamics provides a connection between those principles and variational principles for shallow water.

2.3 The Quasi One-dimensional Shallow Water Approximation

For certain flow domains the motion can be approximated by making the assumption that it is dependent on one space dimension and time only.

Consider a channel which extends over the interval $[x_e, x_o]$ of the x-axis. Let $B(x)$ be the breadth of the channel, defined at each point x in $[x_e, x_o]$. Assume that the channel is of rectangular cross-section and that it is symmetric about

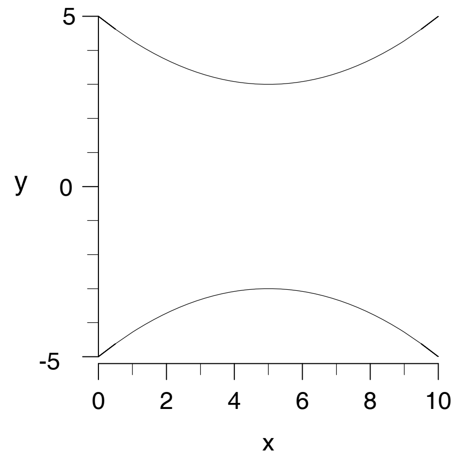

Figure 2.2: D for $x_e = 0, x_o = 10$ and $B(x) = 6 + 4\left(\frac{x}{5} - 1\right)^2$.

the x-axis so that the domain, D, of the problem is given by

$$D = \left\{(x, y) : x \in [x_e, x_o]\,; y \in \left[-\frac{B(x)}{2}, \frac{B(x)}{2}\right]\right\}. \tag{2.30}$$

Then, provided that the breadth is a slowly varying function of x, the flow is quasi one-dimensional in the x-direction, to a first approximation.

Figure 2.2 shows D for the example $x_e = 0, x_o = 10$ and $B(x) = 6+4\left(\frac{x}{5} - 1\right)^2$.

The equations of quasi one-dimensional shallow water motion can be derived from the full shallow water equations of Section 2.2.1 by assuming that the flow variables are functions of x and t only. Let the flow variables be of the form $d = d(x,t)$, $\phi = \phi(x,t)$, $E = E(x,t)$, $Q = Q(x,t)$ and $v = v(x,t)$, where Q is the one-dimensional mass flow and v is the velocity in the x-direction, the other variables being depth, velocity potential and energy, as before. Although the same symbols are used for depth, velocity potential and energy in one and two dimensions, the context will always make clear whether the flow being studied is quasi one-dimensional or two-dimensional. The operator $\boldsymbol{\nabla}$ in (2.15) and (2.18) is replaced by $\frac{\partial}{\partial x}$. The term $\boldsymbol{\nabla}.\mathbf{Q}$ in (2.24) is replaced by $\frac{1}{B}\frac{\partial}{\partial x}(QB)$.

Thus, the quasi one-dimensional shallow water equations of motion are given by

$$
\begin{aligned}
v &= \phi_x && \text{irrotationality condition,} \quad (2.31)\\
v_t + E_x &= gh_x && \text{conservation of momentum,} \quad (2.32)\\
d_t + \frac{1}{B}(BQ)_x &= 0 && \text{conservation of mass,} \quad (2.33)
\end{aligned}
$$

where the mass flow, Q, is given by

$$Q = dv, \tag{2.34}$$

and the energy, E, is given by

$$E = gd + \frac{1}{2}v^2. \tag{2.35}$$

The integrated version of the conservation of momentum equation (2.32) is

$$\phi_t + E = gh, \tag{2.36}$$

where ϕ is related to v by (2.31).

Let the equilibrium depth, h, be constant. Then the conservation of momentum equation is given by

$$v_t + E_x = 0. \tag{2.37}$$

Integrating (2.37) with respect to x gives

$$\phi_t + E = 0, \tag{2.38}$$

where an arbitrary function of t has been absorbed into ϕ_t. Equation (2.38) is consistent with (2.36) when the reference level for potential energy in the vertical is moved from $z = 0$ to $z = -h$ by redefining the velocity potential to be

$$\phi := \phi - ght. \tag{2.39}$$

The boundary conditions for quasi one-dimensional flow are given, for example, by

$$Q = C_e \qquad \text{at } x = x_e, \tag{2.40}$$

$$Q = C_o \qquad \text{at } x = x_o, \tag{2.41}$$

for known functions $C_e(t)$ and $C_o(t)$. Equations (2.40) and (2.41) can be derived from the two-dimensional shallow water boundary condition (2.26) using the fact that there is zero flow through the channel sides and that Q varies only with x and t and is constant across the channel breadth by assumption.

Equations (2.31)—(2.33) govern the motion of quasi one-dimensional shallow water. Notice that the irrotationality condition (2.31) has become essentially redundant.

2.4 Equations for Steady State Shallow Water Flows

The equations of motion for steady state shallow water can be derived from the time-dependent equations of Sections 2.2 and 2.3. In this section the steady state equations are derived by assuming that all of the flow variables are independent of time. The velocity potential ϕ, however, is not a physical flow variable and cannot be assumed time-independent, although its form can be deduced using the equations of motion.

2.4.1 Two-dimensional Flows

First assume that the flow variables do not vary with time, that is, $d = d(x, y)$ and $\mathbf{v} = \mathbf{v}(x, y)$. Then $E = E(x, y)$ and $\mathbf{Q} = \mathbf{Q}(x, y)$ by the definitions (2.19) and (2.25).

The steady state versions of the conservation of momentum equation (2.18) and the conservation of mass equation (2.24) can be immediately written as

$$\boldsymbol{\nabla} E = g \boldsymbol{\nabla} h \qquad (2.42)$$

$$\text{and} \qquad \boldsymbol{\nabla} . \mathbf{Q} = 0, \qquad (2.43)$$

respectively. The dependence of the velocity potential on t can be deduced using the irrotationality condition (2.15) and the integrated conservation of momentum equation (2.20). Differentiating equations (2.15) and (2.20) with respect to time gives

$$\mathbf{v}_t = \boldsymbol{\nabla} \phi_t \qquad (2.44)$$

$$\text{and} \qquad \phi_{tt} + E_t = 0, \qquad (2.45)$$

respectively. By the steady state assumption $\mathbf{v}_t \equiv 0$ and $E_t \equiv 0$. Thus, from equations (2.44) and (2.45), ϕ satisfies

$$\boldsymbol{\nabla} \phi_t = \mathbf{0} \;\text{ and }\; \phi_{tt} = 0. \qquad (2.46)$$

Therefore ϕ must be of the form

$$\phi(x, y, t) = f(x, y)t + \tilde{\phi}(x, y), \qquad (2.47)$$

where $\tilde{\phi}(x, y)$ is an arbitrary function and $f(x, y)$ is such that $\boldsymbol{\nabla} f = \mathbf{0}$.

Using (2.47) to substitute for ϕ in (2.15) gives

$$\mathbf{v} = \boldsymbol{\nabla} \tilde{\phi}, \qquad (2.48)$$

which is the irrotationality condition for steady flow.

Similarly, the integrated conservation of momentum equation for steady flow,

$$f + E = gh, \tag{2.49}$$

is obtained by substituting (2.47) into (2.20). Equation (2.49) implies that the function f must satisfy $f = -E + gh$, where $\boldsymbol{\nabla}(-E + gh) = 0$ from (2.42). Thus, in order to satisfy conservation of momentum, the potential must be of the form

$$\phi(x, y, t) = (-E(x, y) + gh(x, y))\, t + \tilde{\phi}(x, y), \tag{2.50}$$

where $\tilde{\phi}(x, y)$ may be identified as the velocity potential for steady flow.

Let the equilibrium depth h be constant. Then the conservation of momentum equation for steady flow is given by

$$\boldsymbol{\nabla} E = \mathbf{0}. \tag{2.51}$$

This implies that the energy E is in fact a constant, $E = \hat{E}$ say, throughout the whole domain. The integrated version (2.22) becomes

$$\phi_t + \hat{E} = 0. \tag{2.52}$$

Thus in order to satisfy conservation of momentum, the velocity potential ϕ must be of the form

$$\phi(x, y, t) = -\hat{E}t + \bar{\phi}(x, y). \tag{2.53}$$

The irrotationality condition is then

$$\mathbf{v} = \boldsymbol{\nabla}\bar{\phi} \tag{2.54}$$

and $\bar{\phi}(x, y)$ is identified as the velocity potential for steady flow in this case.

The boundary condition for steady flow is given by

$$\mathbf{n}.\mathbf{Q} = C \qquad \text{on } \Sigma,$$

where Σ is the boundary of the domain D and $C(x, y)$ is defined on Σ. For consistency with conservation of mass the function C must satisfy

$$\int_{\Sigma} C \, d\Sigma = 0.$$

Equations (2.42), (2.43) and (2.48) are the equations of motion for time-independent shallow water flows. The equations for steady state quasi one-dimensional flows can be derived from the corresponding time-dependent equations in a similar manner.

2.4.2 Quasi One-dimensional Flows

Following the derivation of the steady state equations in two dimensions, assume that the flow variables are independent of time, that is, $d = d(x)$ and $v = v(x)$. Then $E = E(x)$ and $Q = Q(x)$ by the definitions (2.34) and (2.35). As in Section 2.4.1, the dependence of the velocity potential on t must be deduced using the equations of motion.

Using $d_t \equiv 0$ and $v_t \equiv 0$, the steady state forms of the conservation of momentum equation and the conservation of mass equation can be written as

$$E' = gh' \qquad (2.55)$$

and

$$(BQ)' = 0, \qquad (2.56)$$

respectively, where $'$ represents the x derivative.

Differentiating with respect to t the irrotationality condition (2.31) and the integrated conservation of momentum equation (2.36) gives

$$v_t = \phi_{xt} \tag{2.57}$$

and
$$\phi_{tt} + E_t = 0 \tag{2.58}$$

respectively. Thus, using $v_t \equiv 0$ and $E_t \equiv 0$, the velocity potential must satisfy

$$\phi_{xt} = 0 \text{ and } \phi_{tt} = 0.$$

Therefore ϕ must be of the form

$$\phi(x,t) = f(x)t + \tilde{\phi}(x), \tag{2.59}$$

where $\tilde{\phi}$ is an arbitrary function and $f' = 0$.

Thus, for steady quasi one-dimensional flow,

$$v = \tilde{\phi}'. \tag{2.60}$$

The value of the constant function f in (2.59) can be deduced using the integrated conservation of momentum equation (2.36). Substituting for ϕ in (2.36) using (2.59) gives

$$f = -E + gh,$$

where $-E + gh$ =constant from (2.55). Thus ϕ is given by

$$\phi(x,t) = (-E(x) + gh(x))\, t + \tilde{\phi}(x),$$

where $\tilde{\phi}(x)$ is identified as the velocity potential for one-dimensional steady flow.

Let the equilibrium depth, h, be constant. Then the conservation of momentum equation for steady quasi one-dimensional motion is given by

$$E' = 0$$

which has the solution $E = \hat{E}$, where $\hat{E}$ is an arbitrary constant. The integrated version of the conservation of momentum equation is

$$\phi_t + \hat{E} = 0.$$

Thus the velocity potential for this case must satisfy

$$\phi(x,t) = -\hat{E}t + \bar{\phi}(x)$$

and

$$v = \bar{\phi}',$$

where $\bar{\phi}(x)$ is now the velocity potential for these circumstances.

The boundary conditions for steady flow are given by

$$\begin{aligned} Q &= C_e \quad \text{at } x = x_e, \\ Q &= C_o \quad \text{at } x = x_o, \end{aligned}$$

where C_e and C_o are given constants. In order to be consistent with the conservation of mass equation (2.56), C_e and C_o must satisfy

$$C_e B_e = C_o B_o,$$

where $B_e = B(x_e)$ and $B_o = B(x_o)$.

Equations (2.55), (2.56) and (2.60) are the equations of steady quasi one-dimensional motion for shallow water.

2.5 The Flow Variable Graphs

In this section graphs are used to relate the flow variables — depth and velocity — to the variations in energy and mass flow. The graphs are also used to illustrate the notion of critical flow.

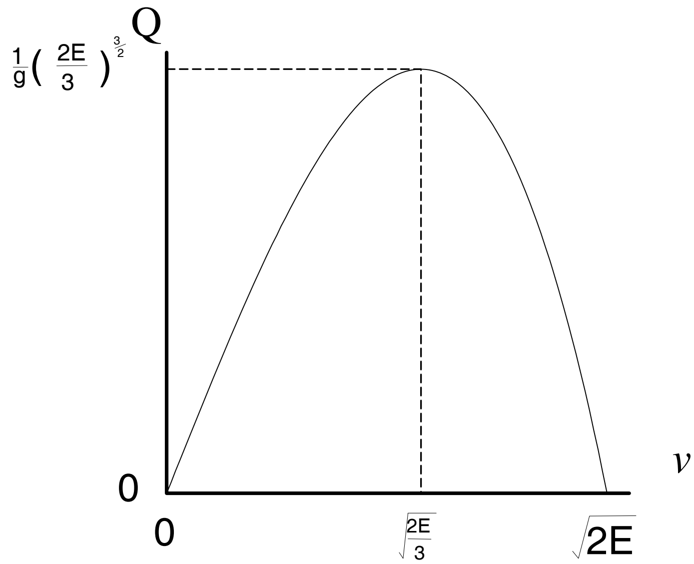

Figure 2.3: Q as a function of v for constant E.

The definitions of mass flow, Q, and energy, E, (2.34) and (2.35) can be used to express each of the variables Q, E, d and v as functions of one or more of the other variables. Furthermore, taking $Q = |\mathbf{Q}|$ and $v = |\mathbf{v}|$, the relationships also hold for two-dimensional flows.

Rearranging the definition of mass flow gives $d = \frac{Q}{v}$. Substituting this into the definition of energy and rearranging gives

$$Q = \frac{v}{g}\left(E - \frac{1}{2}v^2\right). \tag{2.61}$$

The graph of the variation of Q with v for a constant E is given in Figure 2.3. Only the portion of the curve described by (2.61) which lies in the sector $Q \geq 0$ and $v \geq 0$ is considered relevant since the motion is assumed to be always in the positive x direction.

Notice that the velocity v has the range $0 \leq v \leq \sqrt{2E}$. If v exceeded $\sqrt{2E}$ then, from (2.61), the mass flow, Q, would be negative. This would contradict the assumption of positive flow and correspond to a non-physical negative depth as can be seen directly from (2.35). The value

$$v_L = \sqrt{2E}$$

is known as the limit velocity and is the maximum velocity attainable by a flow with energy E.

Notice also that, for each E, the mass flow lies in the range $0 \leq Q \leq \frac{1}{g}\left(\frac{2E}{3}\right)^{\frac{3}{2}}$. The value

$$Q_* = \frac{1}{g}\left(\frac{2E}{3}\right)^{\frac{3}{2}} \tag{2.62}$$

of Q is known as the critical mass flow. The value of the velocity at which the critical mass flow occurs is the critical velocity and is given by

$$c_* = \sqrt{\frac{2E}{3}}. \tag{2.63}$$

A flow is termed subcritical or supercritical depending on whether v is less than or greater than the critical velocity.

A similar graph is constructed by substituting $v = \frac{Q}{d}$ into the definition of E, (2.35), to give an expression relating Q, E and d. This may be rearranged to give

$$Q = (2(E - gd))^{\frac{1}{2}}\, d. \tag{2.64}$$

The graph of the variation of Q with d for a fixed E is given in Figure 2.4. The portion of the line shown is such that $Q \geq 0$ and $d \geq 0$, the remainder of the line having no physical meaning.

As in Figure 2.3, the mass flow in Figure 2.4 lies in the range $0 \leq Q \leq \frac{1}{g}\left(\frac{2E}{3}\right)^{\frac{3}{2}}$. The depth of flow is always in the range $0 \leq d \leq \frac{E}{g}$. If d exceeds $\frac{E}{g}$ then, from (2.64), Q is undefined. The value

$$d_L = \frac{E}{g}$$

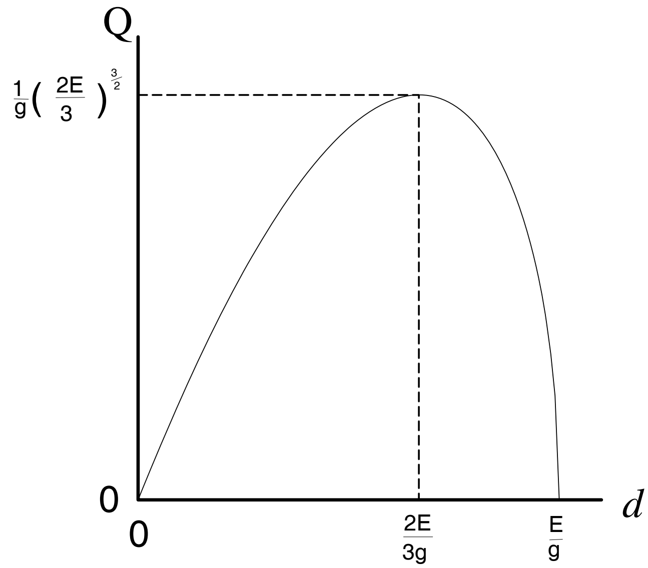

Figure 2.4: Q as a function of d for constant E.

is known as the limit depth and is the maximum attainable depth for a flow with energy E. The value of the depth which corresponds to the critical mass flow Q_*, defined by (2.62), is called the critical depth and is given by

$$d_* = \frac{2E}{3g}. \tag{2.65}$$

The critical values of depth and velocity satisfy the relationship

$$gd_* = \frac{2E}{3} = c_*^2. \tag{2.66}$$

In fact, a flow is said to be critical if the velocity, v, and the depth, d, satisfy

$$v = \sqrt{gd}. \tag{2.67}$$

Substituting (2.67) into (2.35) yields the definitions of critical velocity and critical depth, as given by (2.63) and (2.65).

From (2.35) a depth in the range $d_* \leq d \leq d_L$ corresponds to a subcritical flow; otherwise, if the depth lies in the range $0 \leq d \leq d_*$, the flow is supercritical.

From Figures 2.3 and 2.4, when the mass flow, Q, has attained its critical value there is only one possible depth and one velocity — those which correspond to critical flow. If Q is in the range $0 \le Q < Q_*$ there are two possible values of v and d — one corresponding to a supercritical flow and one to a subcritical flow.

In the case of steady quasi one-dimensional shallow water motion the flow variable graphs, Figures 2.3 and 2.4, can be used to deduce information about the variations of depth and velocity in a channel of slowly varying breadth.

The equations of motion for steady quasi one-dimensional flow are given by

$$\begin{aligned} E' &= gh', \\ (BQ)' &= 0, \end{aligned}$$

which are equations (2.55) and (2.56), that is, conservation of momentum and conservation of mass. These equations may be integrated to give

$$\begin{aligned} E - gh &= \hat{E} \\ \text{and} \qquad BQ &= CB_e, \end{aligned}$$

where $\hat{E}$ and C are constants, to be defined, and $B_e = B(x_e)$.

Thus, if $B(x)$ and $h(x)$ are given for x in $[x_e, x_o]$, the values of energy E and mass flow Q are known at each point in the interval $[x_e, x_o]$. That is,

$$E = \hat{E} + gh \tag{2.68}$$

$$Q = \frac{CB_e}{B}. \tag{2.69}$$

Solution values for the velocity lie on the surface of which Figure 2.3 is a cross-section for constant E and for the depth lie on the surface of which Figure 2.4 is a cross-section for constant E. Thus as Q and E vary with x, the variations of both velocity and depth can be deduced from these surfaces.

If the equilibrium depth of the fluid is constant the energy E given by (2.68) is also constant and the solutions of v and d lie on the curves in Figures 2.3 and 2.4 for fixed E. Consider a channel whose breadth decreases to a minimum, B_{min} say, such that $B(x) = B_{\text{min}}$ for some $x \in (x_e, x_o)$. An example of such a channel is given in Figure 2.2. Moving along the channel, from the inlet at $x = x_e$, as B decreases Q increases (using (2.69)) and so, from Figure 2.3, a subcritical v will increase and a supercritical v will decrease in value. Once the point of minimum breadth has been passed, Q decreases as B increases so that the subcritical v decreases and the supercritical v increases in value. Similarly, using Figure 2.4, moving along the channel from $x = x_e$ a subcritical d will decrease then increase and a supercritical d will increase then decrease, in step with the increase then decrease of mass flow Q.

When $h' \not\equiv 0$, less information about the flow can be obtained from the curves given by (2.61) and (2.64). Consider the curve in Figure 2.3 to be a cross-section for constant E, through the surface created by taking Q to be function of v and E in equation (2.61). The solution lies on the surface and, as Q and E vary in accordance with equations (2.68) and (2.69), the values that v takes during the motion can be traced on the surface. A similar surface representing Q as a function of d and E, as defined by equation (2.64), enables the variation of d to be traced as Q and E vary during the motion. As Q varies in response to a changing channel breadth it is not possible, in general, to determine whether the velocity and depth of flow will increase or decrease, since this depends also on the change in E as determined by the variation in h.

For the energy $E = \hat{E} + gh$, assumed known, the mass flow in a channel of breadth $B(x)$ cannot exceed the value of the critical mass flow Q_*, given by (2.62). This bound on the maximum possible value of the mass flow imposes a lower bound on the minimum breadth of the channel. From (2.69) the minimum breadth, $\hat{B}(x)$ at each point x, for which a continuous flow is possible is

$$\hat{B}(x) = \frac{CB_e}{Q_*} = CB_e g \left(\frac{2\left(\hat{E} + gh(x)\right)}{3} \right)^{-\frac{3}{2}}.$$

If $B(x) < \hat{B}(x)$ for any x in $[x_e, x_o]$ then the flow becomes blocked. If $B(x) > \hat{B}(x)$ for all x in $[x_e, x_o]$ then the flow remains wholly subcritical or wholly supercritical throughout the channel. If $B(x) = \hat{B}(x)$ at a particular point in $[x_e, x_o]$ then the flow is critical at that point and there is the possibility of transitional flow.

It can be shown, using (2.68) and (2.69), that a flow with constant energy E may be critical at a point, x_c say, only if the breadth at that point is stationary with respect to x, that is $B'(x_c) = 0$. Using the definition of mass flow (2.34) and the conservation of mass equation (2.33) it is possible to obtain an expression for v' in terms of d' and v. This is given by

$$v' = -\frac{B'}{B}v - \frac{d'}{d}v. \tag{2.70}$$

Using (2.35) to substitute for E in (2.68) and differentiating with respect to x gives

$$gd' + vv' = gh'.$$

Then substituting for v', using (2.70), and rearranging yields

$$gd'\left(1 - \frac{v^2}{gd}\right) = gh' + \frac{B'}{B}v^2. \tag{2.71}$$

Equation (2.71) implies that when a flow is critical, that is, equation (2.67) is satisfied, the breadth and equilibrium depth must be such that

$$gh' + \frac{B'}{B}v^2 = 0.$$

Thus for a domain with $h' \equiv 0$ critical flow occurs only at a point where $B' = 0$. Conversely, if $h' = 0$ and $B' = 0$ at a point then either $v^2 = gd$ and the flow is critical or $d' = 0$, that is, the fluid depth has reached either a minimum or a maximum at that point. For a domain where $h' \not\equiv 0$ equation (2.71) provides less information; for example, it is not possible in general to determine without knowing the solution in advance where stationary points of the solution might lie or whether, given appropriate conditions, the flow becomes critical.

One further flow variable graph is considered here. This involves a quantity which is of particular use when considering discontinuous motions, namely, the flow stress P. The reason for this utility is that the value of flow stress varies continuously even when the flow variables are discontinuous (in the sense of hydraulic jumps) — as is described in Section 2.6. The flow stress is defined by

$$P = p + dv^2 \tag{2.72}$$

for quasi one-dimensional flow, where p is the pressure given by (2.13). Using the definition of energy E, (2.35), to substitute for $d = \frac{1}{g}\left(E - \frac{1}{2}v^2\right)$ and rearranging gives

$$P = \frac{1}{2g}\left(E - \frac{1}{2}v^2\right)\left(E + \frac{3}{2}v^2\right). \tag{2.73}$$

Equation (2.73) can be used to draw a flow variable graph of P as a function of E and v, but a more interesting relationship is that between P, Q and E.

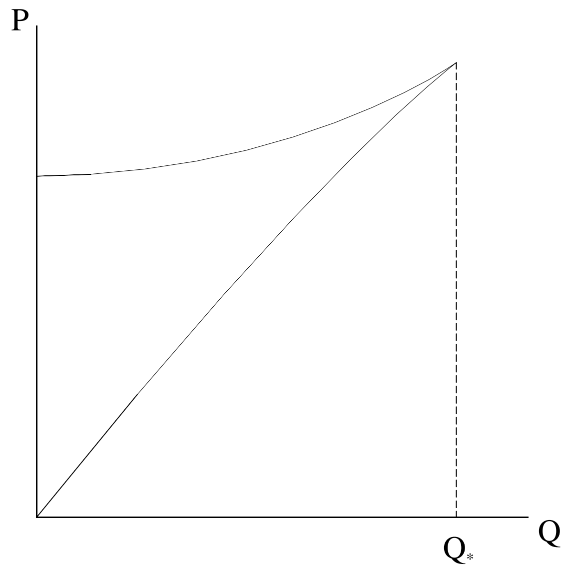

Figure 2.5: P as a function of Q for constant E.

The flow variable graph in Figure 2.5 is created by regarding v as a parameter in equations (2.61) and (2.73). In practice it is plotted by taking, for each fixed E, $2n-1$ values of v in the permitted range, that is,

$$
\begin{aligned}
v_i &= \frac{i-1}{n-1}\sqrt{\frac{2E}{3}} && i = 1,\ldots,n, \\
v_i &= \frac{i-n}{n-1}\left(\sqrt{2E}-\sqrt{\frac{2E}{3}}\right)+\sqrt{\frac{2E}{3}} && i = n+1,\ldots,2n-1.
\end{aligned}
$$

Then the $2n-1$ points (Q_i, P_i), given by

$$
\begin{aligned}
Q_i &= \frac{v_i}{g}\left(E-\frac{1}{2}v_i^2\right), \\
P_i &= \frac{1}{2g}\left(E-\frac{1}{2}v_i^2\right)\left(E+\frac{3}{2}v_i^2\right),
\end{aligned}
$$

for $i = 1,\ldots,2n-1$, trace the curve for P as a function of Q for E fixed. The cusp of the graph in Figure 2.5 is at the critical point, for each value of E, and marks the division between the subcritical (upper) branch and the supercritical (lower) branch of each curve.

The connection with discontinuous flow is discussed in Section 2.6.

2.6 Discontinuous Flows

This chapter has, so far, dealt with the equations of motion for continuous flows. In this section equations of motion for discontinuous flows in shallow water are considered.

It is possible to control the depth and velocity, and therefore also the mass flow and energy, of a flow at the inlet and outlet positions of a channel using, for example, weirs or sluice gates. Thus a situation might occur where the imposed inlet and outlet conditions cannot be achieved by a continuous flow in the channel. In such circumstances a discontinuity may occur.

The differential equations of Sections 2.2, 2.3 and 2.4, which model the flow in shallow water, are only valid for continuous solutions. At points of discontinuity the differential equations no longer apply and other equations are needed to govern the motion. These equations, known as jump conditions, relate the values of the flow variables on one side of the discontinuity to their values on the other side.

In this thesis only time-independent discontinuous flows are considered, the stationary discontinuity being known as a hydraulic jump. The jump conditions for quasi one-dimensional and two-dimensional flows are given in Sections 2.6.1 and 2.6.2.

2.6.1 Discontinuous Flows in One Dimension

In quasi one-dimensional motion a hydraulic jump consists of a point (a value of x) where the depth and velocity of the shallow water flow are discontinuous.

Consider the channel which extends over the interval $[x_e, x_o]$ of the x-axis and has slowly varying breadth $B(x)$. Let $x_s \in (x_e, x_o)$ be the position of the hydraulic jump. Then the equations of motion for continuous flow hold in the two intervals (x_e, x_s) and (x_s, x_o). Thus in $(x_e, x_s) \cup (x_s, x_o)$ the flow variables satisfy

$$E' = gh' \qquad \text{conservation of momentum,} \tag{2.74}$$

$$(BQ)' = 0 \qquad \text{conservation of mass,} \tag{2.75}$$

where Q and E are defined by (2.34) and (2.35), as before.

At the position of the hydraulic jump, x_s, the flow variables must satisfy the jump conditions which are alternative statements of conservation of mass and momentum, valid at a discontinuity. The jump conditions are given by

$$[P]_{x_s} = 0 \tag{2.76}$$

$$\text{and} \qquad [BQ]_{x_s} = 0, \tag{2.77}$$

from Stoker (1957), where P is the flow stress defined by (2.72). The brackets $[\,\cdot\,]_{x_s}$ denote the jump in the value of the quantity at the point x_s. That is, for example, $[P]_{x_s} = P|_{x_s+} - P|_{x_s-}$, where $+$ denotes the x_o side of x_s and $-$ the x_e side of x_s. The third jump condition is given by

$$[E]_{x_s} \neq 0,$$

which states that the energy E is not conserved at a jump. Discounting the possibility that there is an energy source at x_s gives the inequality

$$[E]_{x_s} < 0, \tag{2.78}$$

which is justified by the fact that, in reality, mechanical energy may be converted into heat energy through turbulence at the jump.

Equations (2.74)—(2.78) govern the motion of a quasi one-dimensional shallow water flow with a discontinuity at the point x_s. In certain cases the jump conditions (2.76)—(2.78) can be used to uniquely determine the position of the hydraulic jump. This may be illustrated using the graph in Figure 2.5 which relates flow stress, mass flow and energy.

The usefulness of the flow variable graph lies in the fact that equations (2.74) and (2.75) can be solved for E and Q, given a particular domain. Applying the jump conditions (2.77) and (2.78) to the solutions of (2.74) and (2.75) gives the variations of Q and E throughout the channel, as follows.

Using equations (2.75) and (2.77) gives the variation of Q in the channel as

$$Q(x) = \frac{CB_e}{B(x)} \qquad x \in [x_e, x_o] . \tag{2.79}$$

From equation (2.74)

$$E(x) - gh(x) = \text{ constant} \qquad x \in (x_e, x_s) \cup (x_s, x_o).$$

Assuming that $[h]_{x_s} = 0$ equation (2.78) gives

$$[E - gh]_{x_s} < 0.$$

Thus let

$$\begin{aligned} E - gh &= E_e \qquad x \in [x_e, x_s) \\ \text{and} \qquad E - gh &= E_o \qquad x \in (x_s, x_o] , \end{aligned}$$

where E_e and E_o are constants such that $E_e > E_o$.

In Figure 2.6, which shows the variation of P with Q for two distinct values of E, let $E_1 = E_e + gh(x_s)$ and $E_2 = E_o + gh(x_s)$. Then the point of intersection,

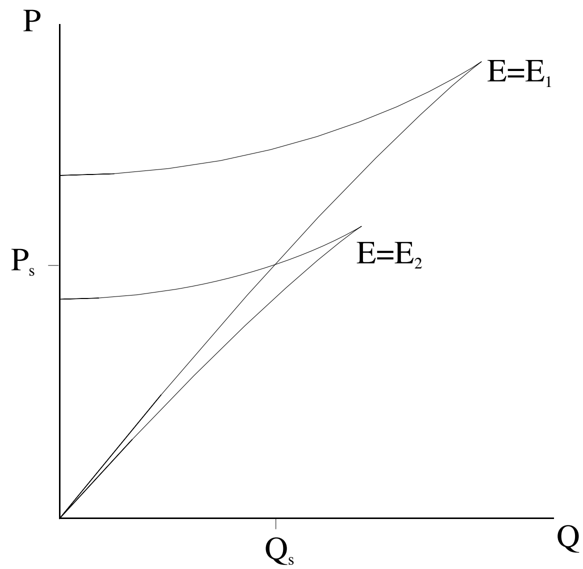

Figure 2.6: P as a function of Q for two distinct values of E.

where $P = P_s$ and $Q = Q_s$, is the point of discontinuity of a flow with energy $E = E_e + gh(x_e)$ at inlet and energy $E = E_o + gh(x_o)$ at outlet. Notice that the point of intersection occurs on the supercritical branch of the line corresponding to E_1 and on the subcritical branch of the line corresponding to E_2. The jump condition (2.78) ensures that this is always true, that is, the flow on the inlet side of a discontinuity is always supercritical and the flow immediately on the other side of the discontinuity is always subcritical.

Let the undisturbed fluid depth, h, be constant. Then the discontinuous flow, determined by $E = E_e + gh$ at inlet and $E = E_o + gh$ at outlet, can be traced on the curves corresponding to $E_1 = E_e + gh$ and $E_2 = E_o + gh$ in Figure 2.6. In particular, given just E_e and E_o, the mass flow at the discontinuity, Q_s, can be deduced. In this way the position of the discontinuity in the interval $[x_e, x_o]$ may be found. From (2.79) the position of the discontinuity, x_s, satisfies

$$B(x_s) = \frac{CB_e}{Q_s} \tag{2.80}$$

and, since the breadth function $B(x)$ is known, the value of x_s can be calculated.

There are three possible situations arising.

1. x_s in (x_e, x_o) is uniquely determined by inverting equation (2.80).

2. There is no value of x in (x_e, x_o) which satisfies (2.80).

3. There is more than one value of x in (x_e, x_o) which satisfies (2.80).

Case 1 yields the position of the hydraulic jump. In Case 2 there is no solution containing a hydraulic jump to the problem with $E = E_e + gh$ at inlet and $E = E_o + gh$ at outlet. In Case 3 there is more than one point in the channel which satisfies the jump conditions (2.76)—(2.78). To avoid the possibility of ambiguity conditions are sought under which Case 3 does not occur.

A unique solution could be achieved by defining B so that (2.80) is uniquely invertible on (x_e, x_o). For the purpose of seeking numerical approximations to discontinuous flows in this thesis converging/diverging channels, similar to that of Figure 2.2, are used with boundary conditions which cause the flow to be critical at the point of minimum breadth — the channel throat. In this circumstance there may still be two distinct points in the channel, x_1 and x_2 say, such that x_1 and x_2 satisfy (2.80). The shape of the channel ensures that one of these points lies on the inlet side of the channel throat and the other on the outlet side. The condition that the flow is critical at the channel throat forces the discontinuity to lie in the diverging section of the channel. Otherwise, because of condition (2.78), the flow would become blocked.

The flow path for this type of discontinuous critical flow is shown by arrows in Figure 2.7. At inlet the mass flow is given by Q_e. For an initially subcritical flow the solution moves along the E_1 curve, in the direction shown, as far as the

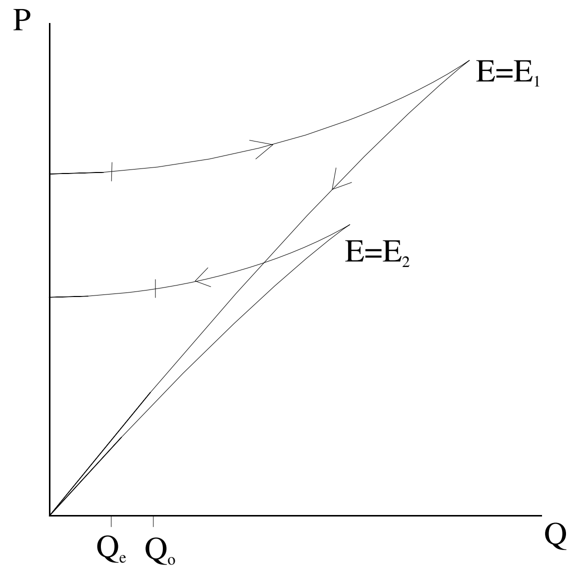

Figure 2.7: Flow path for a discontinuous critical flow.

critical point where $Q = Q_*$. The flow must become supercritical here for the discontinuity to occur so the solution tracks along the supercritical branch until it reaches the position of the discontinuity. The solution point then switches to the subcritical branch of the E_2 curve until the mass flow equals the mass flow Q_o at the outlet of the channel.

For the example with constant undisturbed fluid depth Figure 2.7 can be used to define a range of possible outlet conditions, given an inlet condition for the flow. For the case of a critical flow a hydraulic jump may occur anywhere in the range $(\hat{x}, x_o)$, where $\hat{x}$ is the position of the channel throat. Let E_1 be the energy of the flow at inlet and let E_2 be the energy at outlet. A discontinuity at the channel throat requires that the curves for E_1 and E_2 intersect at the cusp of the E_1 curve. This requires $E_2 = E_1$ so that the discontinuity is of zero strength. The value $E = E_1$ is the maximum energy at outlet that a discontinuous flow can achieve. The minimum value of E at outlet is obtained when the discontinuity lies right at the channel outlet. In Figure 2.7 let the curve corresponding to

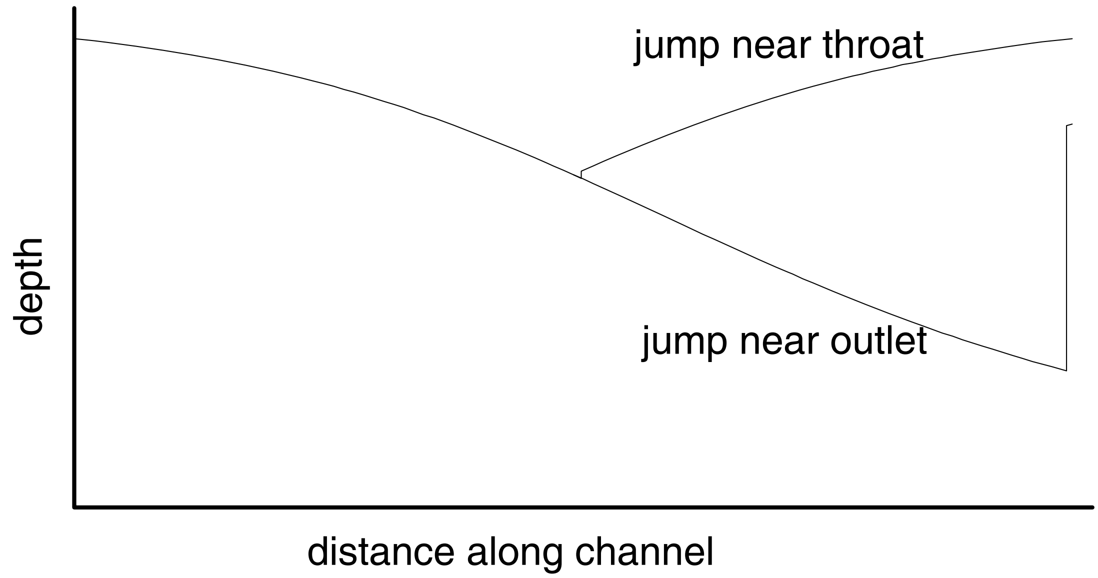

Figure 2.8: Range of possible outlet depths.

$E = E_2$ intersect with the curve corresponding to $E = E_1$ at the point $Q = Q_o$. Then the energy $E = E_2$ is the minimum energy at outlet which can be achieved by a discontinuous flow. From (2.64), using the fact that the depth at outlet is subcritical, the minimum and maximum values of E give minimum and maximum achievable outlet depths. Figure 2.8 shows an example of minimum and maximum outlet depths for the channel shown in Figure 2.2.

Some similar properties can be deduced for specific domains with non-constant equilibrium depths. Results are particular to each case since the solutions no longer lie on the lines of constant energy E. Discontinuous motions in such domains are not considered in this thesis.

2.6.2 Discontinuous Flows in Two Dimensions

In two dimensions a hydraulic jump is a curve in the xy plane which marks a discontinuity in the depth and velocity of the flow. Although hydraulic jumps which terminate in mid-channel do exist, for example in supercritical flow at a concave bend, only hydraulic jumps which extend across the whole width of the channel are studied here.

Consider a steady discontinuous shallow water flow in a channel D. Let Σ_s be the line where the velocity and depth of flow are discontinuous, that is, the hydraulic jump. Then, in $D\backslash\Sigma_s$, the flow variables satisfy the differential equations

$$\mathbf{v} = \boldsymbol{\nabla}\phi \qquad \text{irrotationality,} \tag{2.81}$$

$$\boldsymbol{\nabla}E = g\boldsymbol{\nabla}h \qquad \text{conservation of momentum,} \tag{2.82}$$

$$\boldsymbol{\nabla}.\mathbf{Q} = 0 \qquad \text{conservation of mass,} \tag{2.83}$$

where the energy and mass flow are defined by (2.19) and (2.25), as before.

At the curve Σ_s the flow variables are related by the two-dimensional jump conditions. Let $\mathbf{n}$ be the unit normal vector to the line Σ_s and let $\boldsymbol{\tau}$ be the unit tangential vector. Then the jump conditions

$$[P]_{\Sigma_s} = 0, \tag{2.84}$$

$$[\mathbf{Q}.\mathbf{n}]_{\Sigma_s} = 0, \tag{2.85}$$

$$[\mathbf{v}.\boldsymbol{\tau}]_{\Sigma_s} = 0, \tag{2.86}$$

$$[E]_{\Sigma_s} < 0, \tag{2.87}$$

may be deduced from Chadwick (1976), where P is the two-dimensional flow stress given by

$$P = p + d\left(\mathbf{v}.\mathbf{n}\right)^2, \tag{2.88}$$

p being the pressure defined by (2.13). The symbol $[\,\cdot\,]_{\Sigma_s}$ denotes the change in the value of the quantity on crossing the line Σ_s, that is, for example $[P]_{\Sigma_s} = P|_{\Sigma_{s+}} - P|_{\Sigma_{s-}}$, where $+$ denotes the downstream side of the discontinuity and $-$ denotes the upstream side.

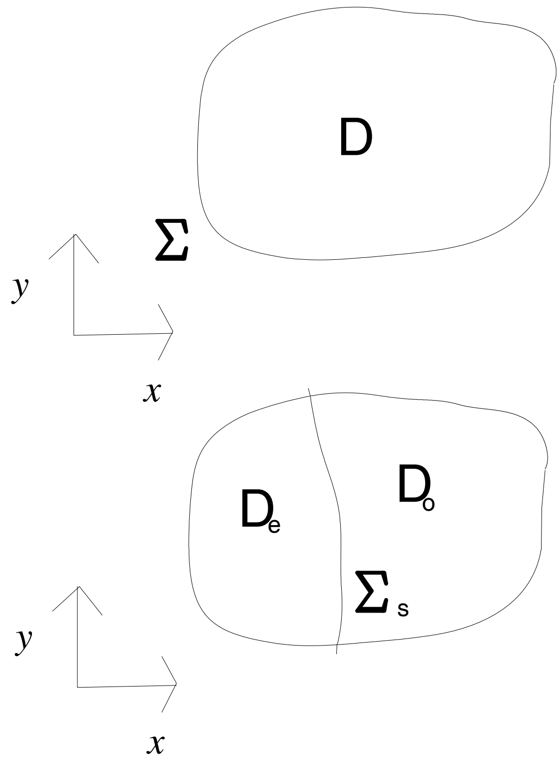

Figure 2.9: Domain for two-dimensional discontinuous flows.

The first three of these conditions (2.84)–(2.86) state that the value of the flow stress, the component of mass flow normal to Σ_s and the component of velocity tangential to Σ_s are conserved on crossing the jump. Equations (2.84) and (2.85) are the two-dimensional counterparts of the one-dimensional jump conditions (2.76) and (2.77). Equation (2.86) is a 'no shear' condition.

The final jump condition (2.87), the two-dimensional counterpart of (2.78), states that the energy E is not conserved on crossing the hydraulic jump.

Consider the domain D divided into two parts by the curve Σ_s, as in Figure 2.9. Let D_e be the subdomain of D on the inlet side of Σ_s and let D_o be the subdomain of D on the outlet side. Then the equation of conservation of momentum in D_e is satisfied by

$$E = E_e + gh,$$

where E_e is a constant. The equation of conservation of momentum in D_o is satisfied by

$$E = E_o + gh,$$

where E_o is a constant such that $E_o < E_e$.

As in the one-dimensional case the flow is supercritical before the hydraulic jump, that is in D_e, and subcritical after the hydraulic jump, that is in D_o.

Chapter 3

Variational Principles

The purpose of this chapter is to present a collection of functionals which are stationary for solutions of the equations of motion for free surface flows, particularly those flows approximated using shallow water theory. The chapter starts by establishing variational principles for three-dimensional free surface flows which are then used to derive principles for shallow water flows. Time-dependent and time-independent motions are considered as is quasi one-dimensional shallow water flow. In the final section variational principles corresponding to discontinuous shallow water flows are derived for the time-independent case.

3.1 Variational Principles for Free Surface Flows

In this section the equations of irrotational motion of an inviscid, incompressible, homogeneous fluid with a free surface are shown to be the natural conditions of two variational principles which, although derived from different viewpoints, are, in fact, closely related.

Let x, y, z be cartesian coordinates, defined as in Chapter 2, and let t be the time. Let $\Omega = D \times (-h, \eta)$ be the spatial domain to be considered, where D is a fixed region of the xy plane, h is the undisturbed fluid depth and η is the height of the free surface above the equilibrium position, as shown in Figure 2.1.

3.1.1 Luke's Principle

The Bernoulli equation (2.7) for free surface flows gives an expression for the fluid pressure $\tilde{p}$ as a function of the velocity potential χ, that is,

$$\tilde{p} = -\rho \left(\chi_t + gz + \frac{1}{2} \tilde{\boldsymbol{\nabla}} \chi . \tilde{\boldsymbol{\nabla}} \chi \right), \tag{3.1}$$

where $\tilde{\boldsymbol{\nabla}}$ is defined by (2.2).

Luke (1967) uses the expression (3.1), for $\tilde{p}$, as the Lagrangian density (the integrand of the Lagrangian) in a variational principle. Luke's principle was stated for a constant equilibrium depth but can be generalised to allow for a non-constant depth and, for the given three-dimensional domain Ω, the modified variational principle is

$$\delta \tilde{J}_1(\eta, \chi) = \delta \left\{ \int_{t_1}^{t_2} \iint_D \int_{-h}^{\eta} -\rho \left(\chi_t + gz + \frac{1}{2} \tilde{\boldsymbol{\nabla}} \chi . \tilde{\boldsymbol{\nabla}} \chi \right) dz\, dx\, dy\, dt \right\} = 0. \tag{3.2}$$

Let the variations in χ and η be such that $\delta\chi = 0$ and $\delta\eta = 0$ on the lateral boundaries of Ω (that is, on the boundary of D for all $z \in [-h, \eta]$) for each constant $t \in [t_1, t_2]$ and at the times t_1 and t_2 everywhere in Ω. Then, using the First Mean Value Theorem for Definite Integrals (Johnson and Reiss (1982)) to identify the $\delta\eta$ contribution,

$$
\begin{aligned}
\delta \tilde{J}_1 \;=\; & -\rho \int_{t_1}^{t_2} \iint_D \left\{ \delta\eta \left(\chi_t + gz + \frac{1}{2}\tilde{\boldsymbol{\nabla}}\chi.\tilde{\boldsymbol{\nabla}}\chi \right)\Big|_{z=\eta} \right. \\
& - \left(\delta\chi\left(\eta_t + \chi_x\eta_x + \chi_y\eta_y - \chi_z\right)\right)\big|_{z=\eta} - \left(\delta\chi\left(\chi_x h_x + \chi_y h_y + \chi_z\right)\right)\big|_{z=-h} \\
& \left. - \int_{-h}^{\eta} \delta\chi \tilde{\nabla}^2\chi \, dz \right\} dx\, dy\, dt = 0,
\end{aligned}
$$

which yields the natural conditions

$$\tilde{\nabla}^2\chi = 0 \quad \text{in } \Omega, \tag{3.3}$$

$$\chi_t + gz + \frac{1}{2}\tilde{\boldsymbol{\nabla}}\chi.\tilde{\boldsymbol{\nabla}}\chi = 0 \quad \text{on } z = \eta, \tag{3.4}$$

$$\eta_t + \chi_x\eta_x + \chi_y\eta_y - \chi_z = 0 \quad \text{on } z = \eta, \tag{3.5}$$

$$\chi_x h_x + \chi_y h_y + \chi_z = 0 \quad \text{on } z = -h, \tag{3.6}$$

for $t \in (t_1, t_2)$. Equation (3.3) is the equation of conservation of mass for irrotational flow, that is, (2.6). Equations (3.4)—(3.6) are equivalent to (2.8), (2.10) and (2.11), where the irrotationality condition (2.5) has been assumed, and are therefore the dynamic free surface condition, the kinematic free surface condition and the condition of zero flow through the bed, respectively. Thus the variational principle (3.2) generates the governing equations of a free surface flow. No attempt is made at this stage to include boundary or initial conditions as natural conditions.

Notice that using (3.1) to define the pressure in terms of the velocity potential assumes conservation of momentum and that the flow is irrotational. In other words, irrotationality and conservation of momentum, in the form of the energy integral (3.1), are implicit constraints of the free variational principle (3.2), by which is meant that they do not have to be applied as explicit constraints on (3.2) nor do they belong to the set of natural conditions of (3.2).

Luke's principle can be extended to deliver irrotationality as a natural condition. The revised variational principle which achieves this is

$$\delta\left\{\int_{t_1}^{t_2}\iint_D\int_{-h}^{\eta}\left(-\rho\left(\chi_t+gz+\frac{1}{2}\mathbf{u}.\mathbf{u}\right)+\tilde{\mathbf{Q}}.\left(\mathbf{u}-\tilde{\boldsymbol{\nabla}}\chi\right)\right)dz\,dx\,dy\,dt\right\}=0, \quad (3.7)$$

where $\tilde{\mathbf{Q}}=(\tilde{Q}_1,\tilde{Q}_2,\tilde{Q}_3)$ is a Lagrange multiplier and $\mathbf{u}=(u,v,w)$ is the fluid velocity. The functional in (3.7) depends on η, χ, $\mathbf{u}$ and $\tilde{\mathbf{Q}}$. The natural conditions of (3.7) are

$$\left.\begin{aligned}\tilde{\boldsymbol{\nabla}}.\tilde{\mathbf{Q}} &= 0\\ \tilde{\mathbf{Q}} &= \rho\mathbf{u}\\ \mathbf{u} &= \tilde{\boldsymbol{\nabla}}\chi\end{aligned}\right\}\quad \text{in } \Omega, \qquad (3.8)$$

$$-\rho\left(\chi_t+gz+\frac{1}{2}\mathbf{u}.\mathbf{u}\right)+\tilde{\mathbf{Q}}.\left(\mathbf{u}-\tilde{\boldsymbol{\nabla}}\chi\right) = 0 \quad \text{on } z=\eta, \qquad (3.9)$$

$$\rho\eta_t+\tilde{Q}_1\eta_x+\tilde{Q}_2\eta_y-\tilde{Q}_3 = 0 \quad \text{on } z=\eta, \qquad (3.10)$$

$$\tilde{Q}_1h_x+\tilde{Q}_2h_y+\tilde{Q}_3 = 0 \quad \text{on } z=-h, \qquad (3.11)$$

for $t\in(t_1,t_2)$, obtained using the same method by which (3.3)—(3.6) were derived from (3.2). Equations (3.8) together are equivalent to (3.3); the multiplier $\tilde{\mathbf{Q}}$ is identified by $(3.8)_2$ as the three-dimensional mass flow vector. Using $(3.8)_2$ and $(3.8)_3$, (3.9)—(3.11) can be recognised as the dynamic free surface condition, the kinematic free surface condition and the condition of no flow through the bed, respectively.

3.1.2 Hamilton's Principle

Hamilton's principle, in its classical form, is given by

$$\delta\left\{\int_{t_1}^{t_2}L\,dt\right\}=0, \qquad (3.12)$$

where the Lagrangian $L=T-V$, T and V respectively denoting the kinetic and potential energies of the mechanical system being considered. Thus, if the i th particle of an n-particle system has mass m_i and position $\mathbf{x}_i(t)$ at a time t,

$$T = \frac{1}{2}\sum_{i=1}^{n} m_i \dot{\mathbf{x}}_i.\dot{\mathbf{x}}_i,$$

$V = V(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ is a given function and therefore $L = L(\mathbf{x}_1, \ldots, \mathbf{x}_n, \dot{\mathbf{x}}_1, \ldots, \dot{\mathbf{x}}_n)$. The principle (3.12) produces the usual Lagrange equations of motion.

A direct application of Hamilton's principle to fluid flow requires the use of Lagrangian coordinates, in which the position of a fluid particle at time t is denoted by $\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$, where $\mathbf{X}$ is the initial location of that particle. The label $\mathbf{X}$ effectively replaces the label i in the point mass system above, and the summation is correspondingly replaced by an integration over the domain initially occupied by the fluid.

For compressible flows, Seliger and Whitham (1968) have shown that this way of applying Hamilton's principle to a continuum is correct, in that it produces the momentum balance equations in Lagrangian form.

From a practical point of view, the Eulerian framework is more useful than the Lagrangian system. Salmon (1988) discusses the translation of Hamilton's principle from one framework to the other. Here, a direct way of using Hamilton's principle in the Eulerian context is sought. The key point is that conservation of mass is implicit in the Lagrangian setting because integration is carried out over all the mass in the system. In Eulerian coordinates, however, where the flow through a domain fixed in space (by lateral boundaries and the bed for free surface flows) is considered, conservation of mass is not automatic and must be enforced. One way of doing this is to apply conservation of mass constraints to the continuum version of the variational principle (3.12).

For the free surface problem, the difference in kinetic energy and potential energy for the flow in the given domain is

$$\iint_D \int_{-h}^{\eta} \left(\frac{1}{2}\rho \mathbf{u}.\mathbf{u} - \rho g z\right) dz\, dx\, dy. \tag{3.13}$$

The functional (3.13) is evidently the Lagrangian L in this case. Luke (1967) refers to this form of the Lagrangian and mentions that the difference between his variational principle and Hamilton's principle is related to conservation of mass.

The variational principle for free surface flow, based on Hamilton's principle, uses (3.13) as the Lagrangian, subject to the constraint that conservation of mass is satisfied. The constraint may be incorporated into the functional by using Lagrange multipliers.

In the fixed domain Ω conservation of mass is given by $\tilde{\boldsymbol{\nabla}}.\mathbf{u} = 0$ (equation (2.1)). The kinematic free surface condition in the form

$$(\eta_t + u\eta_x + v\eta_y - w)|_{z=\eta} = 0, \tag{3.14}$$

guarantees that there is no flow across the free surface and the condition of no flow through the bed is

$$(uh_x + vh_y + w)|_{z=-h} = 0. \tag{3.15}$$

These conditions must also be enforced in the variational principle being constructed.

The conservation of mass requirements are met by adding (2.1), (3.14) and (3.15) into the functional as constraints using the Lagrange multipliers $\nu = \nu(x, y, z, t)$, $\lambda = \lambda(x, y, t)$ and $\mu = \mu(x, y, t)$. The extra terms to be added to the functional (3.13) are

$$\nu \tilde{\boldsymbol{\nabla}}.\mathbf{u},$$

integrated over Ω and (t_1, t_2), and

$$\lambda\left(\eta_t + u\eta_x + v\eta_y - w\right)\big|_{z=\eta} + \mu\left(uh_x + vh_y + w\right)\big|_{z=-h},$$

integrated over D and (t_1, t_2). The variational principle based on Hamilton's principle is therefore

$$\delta J_2(\eta, \mathbf{u}, \lambda, \mu, \nu) = \delta\left\{\int_{t_1}^{t_2}\iint_D\left(\int_{-h}^{\eta}\left(\rho\left(\frac{1}{2}\mathbf{u}.\mathbf{u} - gz\right) + \nu\tilde{\boldsymbol{\nabla}}.\mathbf{u}\right)dz\right.\right. \tag{3.16}$$
$$\left.\left. + \lambda\left(\eta_t + u\eta_x + v\eta_y - w\right)\big|_{z=\eta} + \mu\left(uh_x + vh_y + w\right)\big|_{z=-h}\right)dx\,dy\,dt\right\} = 0.$$

As in the case of Luke's principle (3.2) the variations are assumed to vanish on the lateral space boundaries and on the time boundaries. The natural conditions of (3.16) are given by

$$\left.\begin{aligned}\rho\mathbf{u} - \tilde{\boldsymbol{\nabla}}\nu &= \mathbf{0}\\ \tilde{\boldsymbol{\nabla}}.\mathbf{u} &= 0\end{aligned}\right\}\quad \text{in } \Omega, \tag{3.17}$$

$$\lambda_t + \tilde{\boldsymbol{\nabla}}.\left(\lambda\mathbf{u}\right) - \rho\left(\frac{1}{2}\mathbf{u}.\mathbf{u} - gz\right) - \nu\tilde{\boldsymbol{\nabla}}.\mathbf{u} = 0 \quad \text{on } z = \eta, \tag{3.18}$$

$$\eta_t + u\eta_x + v\eta_y - w = 0 \quad \text{on } z = \eta, \tag{3.19}$$

$$(\nu - \lambda)\,\tilde{\boldsymbol{\nabla}}\left(z - \eta\right) = 0 \quad \text{on } z = \eta, \tag{3.20}$$

$$uh_x + vh_y + w = 0 \quad \text{on } z = -h, \tag{3.21}$$

$$(\nu - \mu)\,\tilde{\boldsymbol{\nabla}}\left(z + h\right) = 0 \quad \text{on } z = -h, \tag{3.22}$$

for $t \in (t_1, t_2)$. The fluid is homogeneous by hypothesis. Therefore, identifying $\frac{\nu}{\rho}$ as the velocity potential, equations (3.17)—(3.22) together are equivalent to (3.8)—(3.11) and hence to (3.3)— (3.6). For consistency of notation the Lagrange multiplier ν is relabelled as $\nu = \rho\chi$, so that $(3.17)_1$ gives the usual irrotationality condition (2.5). Then, using (3.20) and (3.22), the Lagrange multipliers λ and μ may be identified as $\lambda = \rho\chi|_{z=\eta}$ and $\mu = \rho\chi|_{z=-h}$.

Equation (3.16), with λ, μ and ν as defined above, can be shown to be the same as equation (3.7), except for terms on the lateral boundary of Ω and at times t_1 and t_2. The variational principle (3.16), with λ, μ and ν as stated, can be rearranged to give

$$\delta\left\{\int_{t_1}^{t_2}\iint_D\left(\int_{-h}^{\eta}\rho\left(gz+\frac{1}{2}\mathbf{u}.\mathbf{u}+\mathbf{u}.\left(\tilde{\boldsymbol{\nabla}}\chi-\mathbf{u}\right)-\tilde{\boldsymbol{\nabla}}.\left(\chi\mathbf{u}\right)\right)dz\right.\right.$$
$$\left.\left.-\left(\rho\chi\left(\eta_t+u\eta_x+v\eta_y-w\right)\right)\Big|_{z=\eta}-\left(\rho\chi\left(uh_x+vh_y+w\right)\right)\Big|_{z=-h}\right)dx\,dy\,dt\right\}=0.$$

Using the divergence theorem,

$$I=\int_{t_1}^{t_2}\iint_D\int_{-h}^{\eta}\rho\tilde{\boldsymbol{\nabla}}.\left(\chi\mathbf{u}\right)dz\,dx\,dy\,dt=\int_{t_1}^{t_2}\iint_\sigma\rho\chi\mathbf{u}.\mathbf{n}\,d\sigma\,dt,$$

where σ is the whole boundary of Ω. The parts of σ of interest here are the surfaces $z=\eta$ and $z=-h$, which contribute to I the terms

$$\int_{t_1}^{t_2}\iint_D\left(-\left(\rho\chi\left(u\eta_x+v\eta_y-w\right)\right)\Big|_{z=\eta}-\left(\rho\chi\left(uh_x+vh_y+w\right)\right)\Big|_{z=-h}\right)dx\,dy\,dt.$$

It follows that if Σ denotes the lateral boundary of Ω and $\tilde{\mathbf{Q}}$ is defined by $(3.8)_2$ then (3.16) becomes

$$\delta\left\{\int_{t_1}^{t_2}\left(\iint_D\int_{-h}^{\eta}\left(-\rho\left(\chi_t+gz+\frac{1}{2}\mathbf{u}.\mathbf{u}\right)+\tilde{\mathbf{Q}}.\left(\mathbf{u}-\tilde{\boldsymbol{\nabla}}\chi\right)\right)dz\,dx\,dy\right.\right.$$
$$\left.\left.+\iint_\Sigma\rho\chi\mathbf{u}.\mathbf{n}\,d\sigma\right)dt+\iint_D\left[\int_{-h}^{\eta}\rho\chi\,dz\right]_{t_1}^{t_2}dx\,dy\right\}=0,$$

which is just (3.7) with added boundary terms. The variations are assumed to vanish on Σ and at times t_1 and t_2 so the boundary terms may be neglected. Moreover, if (3.16) is constrained to satisfy the irrotationality condition by substituting $\mathbf{u}=\tilde{\boldsymbol{\nabla}}\chi$ into the integrand, the resulting principle can be shown to be the same as Luke's principle (3.2), to within boundary terms which may be neglected, as before.

Thus Hamilton's principle can be adapted to give (3.16) which, on relabelling $\nu = \rho\chi$ and using (3.20) and (3.22) to identify the Lagrange multipliers λ and μ, has as its natural conditions the irrotationality condition and the conservation of mass equation in the domain Ω and boundary conditions on the free surface and the bed.

The adaptation of Hamilton's principle to fluid flow is given by Seliger and Whitham (1968) for compressible flows. In that case the conservation of mass and two other constraints on the principle are necessary, the other constraints being related to energy balance, in the form of entropy conservation for a particle, and conservation of particle identities. In the current problem of irrotational free surface flows, the entropy does not appear and conservation of particle labels is apparently not required.

3.2 Shallow Water Flows

Variational principles for shallow water flows can be considered from two points of view. The principles in Section 3.1 for free surface flows can be modified, by applying the shallow water approximation to the variables, or Hamilton's principle can be applied directly to the variables of shallow water theory, using the gas dynamics analogy of Section 2.2.2. This section deals with the first method.

3.2.1 Shallow Water Principles from Free Surface Principles

Consider the two variational principles for free surface flows – the 'pressure' principle (3.7) and the 'Hamilton' principle (3.16). The effect of applying the shallow water approximation to these principles is to replace the variables by their two-dimensional counterparts. The velocity potential $\chi = \chi(x, y, z, t)$ reduces to a function $\phi = \phi(x, y, t)$, $\mathbf{u} = \tilde{\boldsymbol{\nabla}}\chi$ is replaced by $\mathbf{v} = \boldsymbol{\nabla}\phi$, where $\boldsymbol{\nabla}$ is defined by (2.16), and the Lagrange multiplier $\tilde{\mathbf{Q}}$ is replaced by $\mathbf{Q} = (Q_1, Q_2)$, where $\mathbf{Q} = \mathbf{Q}(x, y, t)$.

Making these substitutions in the functional of the 'pressure' principle (3.7) and evaluating the integral over z yields the functional $J_1(\mathbf{Q}, d, \mathbf{v}, \phi)$ defined by

$$J_1 = \int_{t_1}^{t_2} \iint_D \rho \left(\frac{1}{2} g d^2 - d \left(\phi_t + \frac{1}{2}\mathbf{v}.\mathbf{v} + gd - gh \right) + \mathbf{Q}.\left(\mathbf{v} - \boldsymbol{\nabla}\phi\right) \right) dx\, dy\, dt, \tag{3.23}$$

where $d = h + \eta$ is the total fluid depth.

Assuming that variations vanish on the boundary of D and at t_1 and t_2, the natural conditions of $\delta J_1 = 0$ are

$$\left.\begin{aligned} \phi_t + \tfrac{1}{2}\mathbf{v}.\mathbf{v} + g\eta &= 0 \\ \mathbf{v} - \boldsymbol{\nabla}\phi &= \mathbf{0} \\ \mathbf{Q} - d\mathbf{v} &= \mathbf{0} \\ d_t + \boldsymbol{\nabla}.\mathbf{Q} &= 0 \end{aligned}\right\} \quad \text{in } \hat{D}, \tag{3.24}$$

where $\hat{D} = D \times (t_1, t_2)$. These equations are respectively the integrated conservation of momentum equation (2.20), the irrotationality condition (2.15), the definition of mass flow (2.25) and the conservation of mass equation (2.24) for shallow water flow.

Now consider (3.16) – the 'Hamilton' principle. Under the conditions of the shallow water approximation the integral over z can be carried out and the terms evaluated at $z = -h$ and $z = \eta$ can be combined since, from (2.14), u and v take the same values at these levels. The result is the functional $J_2 = J_2(d, \mathbf{v}, \phi)$ given by

$$J_2 = \int_{t_1}^{t_2} \iint_D \rho \left(\frac{1}{2}d\mathbf{v}.\mathbf{v} - \frac{1}{2}gd^2 + gdh + \phi\left(d_t + \boldsymbol{\nabla}.\left(d\mathbf{v}\right)\right)\right) dx\, dy\, dt. \qquad (3.25)$$

Assuming that the variations vanish on the space and time boundaries, the natural conditions of $\delta J_2 = 0$ are

$$\left.\begin{aligned} \phi_t + g\eta + \mathbf{v}.\boldsymbol{\nabla}\phi - \tfrac{1}{2}\mathbf{v}.\mathbf{v} &= 0 \\ \mathbf{v} - \boldsymbol{\nabla}\phi &= \mathbf{0} \\ d_t + \boldsymbol{\nabla}.\left(d\mathbf{v}\right) &= 0 \end{aligned}\right\} \quad \text{in } \hat{D}, \qquad (3.26)$$

which together are equivalent to (3.24).

Thus the 'pressure' and 'Hamilton' free surface principles reduce to 'shallow water' principles when the variables are approximated using shallow water theory.

By using the divergence theorem and integration by parts, the functional (3.23) can be rearranged to give

$$\begin{aligned} J_1 = \int_{t_1}^{t_2} \iint_D \rho \left(\left(\mathbf{Q} - \frac{1}{2}d\mathbf{v}\right).\mathbf{v} - \frac{1}{2}gd^2 + gdh + \phi\left(d_t + \boldsymbol{\nabla}.\mathbf{Q}\right)\right) dx\, dy\, dt \\ - \int_{t_1}^{t_2} \int_\Sigma \rho\phi\mathbf{Q}.\mathbf{n}\, d\Sigma - \iint_D \left[\rho d\phi\right]_{t_1}^{t_2} dx\, dy, \end{aligned}$$

that is, for $\mathbf{Q} = d\mathbf{v}$ the two functionals (3.23) and (3.25) are the same, to within boundary terms.

3.2.2 Further Functionals

Before considering variational principles with boundary conditions as natural conditions the 'pressure' and 'Hamilton' functionals for unsteady shallow water motion are rewritten in different variables. The variational principles with boundary terms, generated using the modified functionals, can then be related to two further variational principles.

First consider (3.23), which was derived from the free surface principle based on an expression for the pressure. As the fluid is assumed homogeneous, the density ρ can be set equal to unity without losing generality. As a further simplification of notation, the term

$$\phi_t + \frac{1}{2}\mathbf{v}.\mathbf{v} + gd - gh$$

may be written as

$$\phi_t + E - gh.$$

This suggests the use of E as a new variable. From the definition of E, (2.19), we have

$$d = \frac{1}{g}\left(E - \frac{1}{2}\mathbf{v}.\mathbf{v}\right),$$

which allows for the definition of a new function $p(\mathbf{v}, E)$ obtained by substituting for d in the 'pressure' $\frac{1}{2}gd^2$. Thus

$$p(\mathbf{v}, E) = \frac{1}{2g}\left(E - \frac{1}{2}\mathbf{v}.\mathbf{v}\right)^2. \tag{3.27}$$

The integrand of the functional being constructed is now

$$p(\mathbf{v}, E) - d\left(\phi_t + E - gh\right) + \mathbf{Q}.\left(\mathbf{v} - \boldsymbol{\nabla}\phi\right), \tag{3.28}$$

which is the integrand of (3.23) after making the substitutions outlined above. A functional with integrand (3.28) will be referred to as a 'p' functional for unsteady shallow water flow.

The 'Hamilton' functional (3.25) is rewritten similarly. The density ρ is taken to be unity as before, the change of variable made is from $\mathbf{v}$ to $\mathbf{Q}$ using a re-arrangement of (2.25), that is $\mathbf{v} = \frac{\mathbf{Q}}{d}$, and the function $r(\mathbf{Q}, d)$ is defined to be

$$r(\mathbf{Q}, d) = \frac{1}{2}\frac{\mathbf{Q}.\mathbf{Q}}{d} - \frac{1}{2}gd^2. \tag{3.29}$$

Making these substitutions in the integrand of (3.25) yields the expression

$$r(\mathbf{Q}, d) + gdh + \phi\,(d_t + \boldsymbol{\nabla}.\mathbf{Q})\,. \tag{3.30}$$

A functional with integrand (3.30) will be referred to as an 'r' functional for unsteady shallow water flow.

The structures of the integrands of the 'p' and 'r' functionals are similar in that they may both be expressed in the form

$$\text{function} + \text{multiplier} \times \text{conservation law.}$$

For the 'p' functional (3.28) is

$$p + \text{multiplier} \times \begin{matrix}\text{conservation of} \\ \\ \text{momentum}\end{matrix} + \text{multiplier} \times \begin{matrix}\text{irrotationality} \\ \\ \text{condition}\end{matrix},$$

and for the 'r' functional (3.30) is

$$r + gdh + \text{multiplier} \times \text{conservation of mass.}$$

These forms for the integrands of the 'p' and 'r' functionals suggest obvious ways of constraining the variational principles based on these functionals. For example, if a ‘p’ variational principle is constrained to satisfy the conservation of momentum equation, by setting $E = gh - \phi_t$, and irrotationality, by setting $\mathbf{v} = \boldsymbol{\nabla}\phi$, the expression (3.28) becomes

$$p\left(\boldsymbol{\nabla}\phi, gh - \phi_t\right),$$

which depends on the variable ϕ alone. Constrained variational principles are dealt with in more detail in Sections 3.5, 3.6 and 3.7.

In Section 3.5 boundary terms will be added to the ‘p’ and ‘r’ functionals so that variations which do not necessarily vanish at the space and time boundaries are allowed. Two further functionals, whose integrands are related to (3.28) and (3.30) by a closed quartet of Legendre transforms will also be constructed.

It is evident that there exists a number of constrained and unconstrained variational principles related to time-dependent shallow water flows. To avoid deriving separately the natural conditions of each variational principle a functional of the general form of the time-dependent shallow water functional can be used to generate the natural conditions of a general shallow water variational principle. Then the natural conditions for each different case can be obtained immediately. This general variational principle, along with those for steady state and quasi one-dimensional shallow water flows, is considered in Section 3.3. The natural conditions of general variational principles for time-independent discontinuous flows are derived in Section 3.4.

3.3 Continuous Variables

In this section the natural conditions of a general form of variational principle which includes the cases of shallow water flows are derived. All of the variables are assumed to be continuous.

The general variational principles are referred to as being for one or two dimensions, by which it is meant that the corresponding principles for shallow water theory are for flows in one or two space dimensions. In both cases an extra coordinate is added to allow for time-dependent flows.

3.3.1 General Variational Principles for One-dimensional Flow

Let the domain of integration of the problem be $\{(x,t) : x \in [x_0, x_1]; t \in [t_0, t_1]\}$. In shallow water theory the x coordinate will represent a single space dimension and t will represent the time.

Let $u_1(x,t), \ldots, u_m(x,t)$ be a set of functions which will subsequently be identified with the variables of shallow water theory. The u_i are assumed to be differentiable functions of x and t, since this is all the smoothness required for the shallow water principles.

Consider a general functional of the form

$$\begin{aligned} I_1(u_1, \ldots, u_m) = & \int_{t_0}^{t_1} \int_{x_0}^{x_1} f(x, t, u_1, \ldots, u_m, u_{1x}, \ldots, u_{mx}, u_{1t}, \ldots, u_{mt}) dx\, dt \\ & + \int_{t_0}^{t_1} [g(x, t, u_1, \ldots, u_m)]_{x_0}^{x_1} dt + \int_{x_0}^{x_1} [h(x, t, u_1, \ldots, u_m)]_{t_0}^{t_1} dx, \quad (3.31) \end{aligned}$$

where $u_{ix} \equiv \frac{\partial u_i}{\partial x}$ and $u_{it} \equiv \frac{\partial u_i}{\partial t}$ for $i = 1, \ldots, m$. Let the functions g and h be differentiable and f be twice differentiable with respect to their arguments.

Using Taylor series and integrating by parts, the first variation of (3.31) may be written as

$$\delta I_1 = \int_{t_0}^{t_1} \int_{x_0}^{x_1} \left\{ \sum_{i=1}^{m} \left(f_{u_i} - (f_{u_{it}})_t - (f_{u_{ix}})_x \right) \delta u_i \right\} dx\, dt$$
$$+ \int_{t_0}^{t_1} \left[\sum_{i=1}^{m} (g_{u_i} + f_{u_{ix}})\, \delta u_i \right]_{x_0}^{x_1} dt + \int_{x_0}^{x_1} \left[\sum_{i=1}^{m} (h_{u_i} + f_{u_{it}})\, \delta u_i \right]_{t_0}^{t_1} dx.$$

Thus the natural conditions of $\delta I_1 = 0$ are

$$\delta u_i : \quad f_{u_i} - (f_{u_{it}})_t - (f_{u_{ix}})_x = 0 \qquad x \in (x_0, x_1)\,;\, t \in (t_0, t_1), \tag{3.32}$$

$$\delta u_i\big|_{x_j} : \quad (g_{u_i} + f_{u_{ix}})\big|_{x_j} = 0 \qquad j = 0, 1\,;\, t \in (t_0, t_1), \tag{3.33}$$

$$\delta u_i\big|_{t_j} : \quad (h_{u_i} + f_{u_{it}})\big|_{t_j} = 0 \qquad j = 0, 1\,;\, x \in (x_0, x_1), \tag{3.34}$$

for $i = 1, \ldots, m$. Equations (3.32) are the Euler equations of the variational principle and equations (3.33) and (3.34) are natural boundary conditions.

The functions $u_i(x,t)$ which cause the functional (3.31) to be stationary with respect to variations in the u_i also satisfy equations (3.32) in the domain $\{(x,t) : x \in (x_0, x_1); t \in (t_0, t_1)\}$ and the boundary conditions (3.33) and (3.34). In Section 3.7 variational principles are derived which have as their natural conditions, deduced using (3.32)–(3.34), the equations of time-dependent quasi one-dimensional motion in shallow water.

The general form of a functional for steady state quasi one-dimensional shallow water flow is

$$I_2(u_1, \ldots, u_m) = \int_{x_0}^{x_1} f(x, u_1, \ldots, u_m, u_1', \ldots, u_m')\, dx + [g(x, u_1, \ldots, u_m)]_{x_0}^{x_1}, \tag{3.35}$$

where the general functions u_i now depend on x alone and $u_i' \equiv \frac{du_i}{dx}$.

The first variation of (3.35) may be written as

$$\delta I_2 = \int_{x_0}^{x_1} \left\{ \sum_{i=1}^{m} \left(f_{u_i} - \frac{d}{dx} f_{u_i'} \right) \delta u_i \right\} dx + \left[\sum_{i=1}^{m} \left(g_{u_i} + f_{u_i'} \right) \delta u_i \right]_{x_0}^{x_1},$$

using Taylor series and integration by parts. The natural conditions of $\delta I_2 = 0$ are therefore

$$\delta u_i : \quad f_{u_i} - \tfrac{d}{dx} f_{u_i'} = 0 \qquad x \in (x_0, x_1), \tag{3.36}$$

$$\delta u_i|_{x_j} : \quad \left(g_{u_i} + f_{u_i'} \right)\Big|_{x_j} = 0 \qquad j = 0, 1, \tag{3.37}$$

for $i = 1, \ldots, m$.

3.3.2 General Variational Principles for Two-dimensional Flow

The domain D in two-dimensional flow is a simply connected open set in the xy plane. For the general time-dependent case a third coordinate t is added, where t lies in the interval $[t_0, t_1]$. Let $u_1(x, y, t), \ldots, u_m(x, y, t)$ be m scalar variables and let $\mathbf{v}_1(x, y, t), \ldots, \mathbf{v}_n(x, y, t)$ be n vector variables, where $\mathbf{v}_i = (v_{i1}, v_{i2})$ for $i = 1, \ldots, n$. The variables u_i and $\mathbf{v}_i$ are assumed to be differentiable functions of x, y and t.

Consider a functional of the form

$$\begin{aligned} I_3(u_i, \mathbf{v}_k) = &\int_{t_0}^{t_1} \iint_D F(x, y, t, u_i, \boldsymbol{\nabla} u_i, u_{it}, \mathbf{v}_k, \boldsymbol{\nabla}.\mathbf{v}_k, \mathbf{v}_{kt})\, dx\, dy\, dt \\ &+ \int_{t_0}^{t_1} \int_\Sigma G(x, y, t, u_i, \mathbf{v}_k)\, d\Sigma\, dt + \iint_D \left[H(x, y, t, u_i, \mathbf{v}_k) \right]_{t_0}^{t_1} dx\, dy, \end{aligned} \tag{3.38}$$

where $i = 1, \ldots, m$, $k = 1, \ldots, n$ and Σ is the boundary of D. Let the functions G and H be differentiable with respect to their arguments and let F be twice differentiable with respect to its arguments.

Using Taylor series, the divergence theorem and integration by parts the first variation of the functional (3.38) may be written as

$$
\begin{aligned}
\delta I_3 = \int_{t_0}^{t_1} \iint_D \Bigg\{ & \sum_{i=1}^{m} \left(F_{u_i} - \boldsymbol{\nabla}.F_{\boldsymbol{\nabla} u_i} - (F_{u_{it}})_t \right) \delta u_i \\
& + \sum_{k=1}^{n} \left(F_{\mathbf{v}_k} - \boldsymbol{\nabla} F_{\boldsymbol{\nabla}.\mathbf{v}_k} - \left(F_{\mathbf{v}_{kt}} \right)_t \right) .\delta \mathbf{v}_k \Bigg\} \, dx\, dy\, dt \\
+ \int_{t_0}^{t_1} \int_{\Sigma} \Bigg\{ & \sum_{i=1}^{m} \left(G_{u_i} + F_{\boldsymbol{\nabla} u_i}.\mathbf{n} \right) \delta u_i + \sum_{k=1}^{n} \left(G_{\mathbf{v}_k} + F_{\boldsymbol{\nabla}.\mathbf{v}_k} \mathbf{n} \right) .\delta \mathbf{v}_k \Bigg\} \, d\Sigma\, dt \\
+ \iint_D \Bigg[& \sum_{i=1}^{m} (H_{u_i} + F_{u_{it}}) \delta u_i + \sum_{k=1}^{n} \left(H_{\mathbf{v}_k} + F_{\mathbf{v}_{kt}} \right) .\delta \mathbf{v}_k \Bigg]_{t_0}^{t_1} dx\, dy,
\end{aligned}
$$

where the unit vector $\mathbf{n}$ is the outward normal to the boundary Σ and the following notation is used.

$$
F_{u_i} \equiv \frac{\partial F}{\partial u_i}, \qquad F_{u_{it}} \equiv \frac{\partial F}{\partial u_{it}}, \qquad F_{\boldsymbol{\nabla} u_i} \equiv \left(\frac{\partial F}{\partial u_{ix}}, \frac{\partial F}{\partial u_{iy}} \right).
$$

$$
F_{\mathbf{v}_k} \equiv \left(\frac{\partial F}{\partial v_{k1}}, \frac{\partial F}{\partial v_{k2}} \right), \quad F_{\mathbf{v}_{kt}} \equiv \left(\frac{\partial F}{\partial v_{k1t}}, \frac{\partial F}{\partial v_{k2t}} \right), \quad F_{\boldsymbol{\nabla}.\mathbf{v}_k} \equiv \frac{\partial F}{\partial \left(\boldsymbol{\nabla}.\mathbf{v}_k \right)}.
$$

The natural conditions of $\delta I_3 = 0$ are given by

$$
\delta u_i : \quad F_{u_i} - \boldsymbol{\nabla}.F_{\boldsymbol{\nabla} u_i} - (F_{u_{it}})_t = 0 \quad (x,y) \in D\,;\, t \in (t_0, t_1), \tag{3.39}
$$

$$
\delta \mathbf{v}_k : \quad F_{\mathbf{v}_k} - \boldsymbol{\nabla} F_{\boldsymbol{\nabla}.\mathbf{v}_k} - \left(F_{\mathbf{v}_{kt}} \right)_t = \mathbf{0} \quad (x,y) \in D\,;\, t \in (t_0, t_1), \tag{3.40}
$$

$$
\delta u_i|_{\Sigma} : \quad G_{u_i} + F_{\boldsymbol{\nabla} u_i}.\mathbf{n} = 0 \quad (x,y) \in \Sigma\,;\, t \in (t_0, t_1), \tag{3.41}
$$

$$
\delta \mathbf{v}_k|_{\Sigma} : \quad G_{\mathbf{v}_k} + F_{\boldsymbol{\nabla}.\mathbf{v}_k} \mathbf{n} = \mathbf{0} \quad (x,y) \in \Sigma\,;\, t \in (t_0, t_1), \tag{3.42}
$$

$$
\delta u_i|_{t_j} : \quad (H_{u_i} + F_{u_{it}})|_{t_j} = 0 \quad j = 0, 1\,;\, (x,y) \in D, \tag{3.43}
$$

$$
\delta \mathbf{v}_k|_{t_j} : \quad \left(H_{\mathbf{v}_k} + F_{\mathbf{v}_{kt}} \right)\Big|_{t_j} = \mathbf{0} \quad j = 0, 1\,;\, (x,y) \in D, \tag{3.44}
$$

for $i = 1, \ldots, m$ and $k = 1, \ldots, n$. Equations (3.39) and (3.40) are the Euler equations of the variational principle and equations (3.41)–(3.44) are natural boundary conditions.

The functions u_i for $i = 1, \ldots, m$ and $\mathbf{v}_k$ for $k = 1, \ldots, n$ which cause I_3 to be stationary with respect to variations in its arguments also satisfy equations

(3.39)–(3.44). In Sections 3.5.1 and 3.5.2 variational principles are derived whose natural conditions, given by (3.39)–(3.44), are the equations of motion for time-dependent shallow water flows.

The general form of a functional for steady state shallow water flow is

$$I_4(u_i, \mathbf{v}_k) = \iint_D F(x, y, u_i, \boldsymbol{\nabla} u_i, \mathbf{v}_k, \boldsymbol{\nabla}.\mathbf{v}_k)\, dx\, dy + \int_\Sigma G(x, y, u_i, \mathbf{v}_k)\, d\Sigma, \quad (3.45)$$

where the general variables u_i for $i = 1, \ldots, m$ and $\mathbf{v}_k$ for $k = 1, \ldots, n$ are functions of x and y only.

Using Taylor series and the divergence theorem the first variation of (3.45) may be written as

$$\delta I_4 = \iint_D \left\{ \sum_{i=1}^{m} \left(F_{u_i} - \boldsymbol{\nabla}.F_{\boldsymbol{\nabla} u_i} \right) \delta u_i + \sum_{k=1}^{n} \left(F_{\mathbf{v}_k} - \boldsymbol{\nabla} F_{\boldsymbol{\nabla}.\mathbf{v}_k} \right) .\delta \mathbf{v}_k \right\} dx\, dy$$
$$+ \int_\Sigma \left\{ \sum_{i=1}^{m} \left(G_{u_i} + F_{\boldsymbol{\nabla} u_i}.\mathbf{n} \right) \delta u_i + \sum_{k=1}^{n} \left(G_{\mathbf{v}_k} + F_{\boldsymbol{\nabla}.\mathbf{v}_k} \mathbf{n} \right) .\delta \mathbf{v}_k \right\} d\Sigma,$$

which yields the natural conditions

$$\delta u_i : \quad F_{u_i} - \boldsymbol{\nabla}.F_{\boldsymbol{\nabla} u_i} = 0 \quad (x, y) \in D, \quad (3.46)$$

$$\delta \mathbf{v}_k : \quad F_{\mathbf{v}_k} - \boldsymbol{\nabla} F_{\boldsymbol{\nabla}.\mathbf{v}_k} = \mathbf{0} \quad (x, y) \in D, \quad (3.47)$$

$$\delta u_i|_\Sigma : \quad G_{u_i} + F_{\boldsymbol{\nabla} u_i}.\mathbf{n} = 0 \quad (x, y) \in \Sigma, \quad (3.48)$$

$$\delta \mathbf{v}_k|_\Sigma : \quad G_{\mathbf{v}_k} + F_{\boldsymbol{\nabla}.\mathbf{v}_k} \mathbf{n} = \mathbf{0} \quad (x, y) \in \Sigma, \quad (3.49)$$

for $i = 1, \ldots, m$ and $k = 1, \ldots, n$.

Using the results of this section the natural conditions of any variational principle for continuous shallow water motion can be written down immediately.

3.4 Discontinuous Variables

In this section general versions of the shallow water variational principles allowing for discontinuous variables are studied. Only time-independent discontinuous flows will be considered so the extra coordinate of Section 3.3, which is identified with the time, is no longer used.

3.4.1 General Variational Principles for One-dimensional Flow

Let the interval $[x_0, x_1]$ of the x-axis be the domain over which the integrand of the general functional is integrated and let x_s be a point in the interior of this interval. Let $u_i(x)$ $i = 1, \ldots, m$ be a set of functions defined on $[x_0, x_1]$, as before. Assume that all of the u_i are continuous in $[x_0, x_s) \cup (x_s, x_1]$. This allows for one or more of the u_i to be discontinuous at the point x_s.

Consider a functional of the form

$$\hat{I}_2(u_1, \ldots, u_m, x_s) = \left(\int_{x_0}^{x_s} + \int_{x_s}^{x_1} \right) f(x, u_1, \ldots, u_m, u_1', \ldots, u_m')\, dx + [g(x, u_1, \ldots, u_m)]_{x_0}^{x_1}. \qquad (3.50)$$

Using Taylor series the first variation of $\hat{I}_2$ is given by

$$\delta \hat{I}_2 = \left(\int_{x_0}^{x_s} + \int_{x_s}^{x_1} \right) \left\{ \sum_{i=1}^{m} f_{u_i} \delta u_i + f_{u_i'} \delta u_i' \right\} dx + \left[\sum_{i=1}^{m} g_{u_i} \delta u_i \right]_{x_0}^{x_1} + \delta x_s\, f|_{x_s-} - \delta x_s\, f|_{x_s+}, \qquad (3.51)$$

where the superscript $-$ denotes the x_0 side of x_s and $+$ denotes the x_1 side of x_s. The first three terms are due to the variations of the u_i and the last two are due to the variation of the position of the discontinuity, x_s.

Applying integration by parts to the $\delta u_i'$ terms of (3.51) yields

$$\delta \hat{I}_2 = \left(\int_{x_0}^{x_s} + \int_{x_s}^{x_1}\right)\left\{\sum_{i=1}^{m}\left(f_{u_i} - \frac{d}{dx} f_{u_i'}\right)\delta u_i\right\} dx$$
$$+ \left[\sum_{i=1}^{m}\left(g_{u_i} + f_{u_i'}\right)\delta u_i\right]_{x_0}^{x_1} + \left[\sum_{i=1}^{m} f_{u_i'}\delta u_i\right]_{x_s+}^{x_s-} + \delta x_s\,[f]_{x_s+}^{x_s-}. \qquad (3.52)$$

The total variation $\hat{\delta u_i}$ of the variable u_i at the point x_s is given by

$$\left.\hat{\delta u_i}\right|_{x_s\pm} = \left.\delta u_i\right|_{x_s\pm} + u_i'(x_s\pm)\delta x_s,$$

which is the first order term in the expansion of

$$\left.(u_i + \delta u_i)\right|_{x_s\pm+\delta x_s}.$$

The only variations considered are those for which the total variations on either side of the discontinuity are equal for each variable, that is,

$$\left.\hat{\delta u_i}\right|_{x_s+} = \left.\hat{\delta u_i}\right|_{x_s-} = \left.\hat{\delta u_i}\right|_{x_s},$$

say, and, in the case of shallow water, the coefficients of these terms in the variational principle give rise to the required jump conditions. Substituting for $\left.\delta u_i\right|_{x_s\pm}$ in (3.52) yields

$$\delta \hat{I}_2 = \left(\int_{x_0}^{x_s} + \int_{x_s}^{x_1}\right)\left\{\sum_{i=1}^{m}\left(f_{u_i} - \frac{d}{dx} f_{u_i'}\right)\delta u_i\right\} dx$$
$$+ \left[\sum_{i=1}^{m}\left(g_{u_i} + f_{u_i'}\right)\delta u_i\right]_{x_0}^{x_1} + \sum_{i=1}^{m}\left[f_{u_i'}\right]_{x_s}\left.\hat{\delta u_i}\right|_{x_s} + \delta x_s\left[f - \sum_{i=1}^{m} f_{u_i'}u_i'\right]_{x_s}, \qquad (3.53)$$

where the brackets $[\cdot]_{x_s}$ denote the change in the argument on crossing x_s from the x_0 side to the x_1 side. That is, for example, $[f]_{x_s} = f|_{x_s-} - f|_{x_s+}$, where $-$ denotes the the x_0 side of x_s and $+$ the x_1 side, as before.

Thus, from (3.53), the natural conditions of $\delta \hat{I}_2 = 0$ are

$$\delta u_i : \qquad f_{u_i} - \tfrac{d}{dx} f_{u_i'} = 0 \qquad\qquad x \in (x_0, x_s) \cup (x_s, x_1), \qquad (3.54)$$

$$\delta u_i\big|_{x_j}: \qquad \left(g_{u_i} + f_{u_i'}\right)\Big|_{x_j} = 0 \qquad j = 0, 1, \tag{3.55}$$

$$\hat{\delta u_i}\Big|_{x_s}: \qquad \left[f_{u_i'}\right]_{x_s} = 0, \tag{3.56}$$

$$\delta x_s: \quad \left[f - \sum_{i=1}^{m} f_{u_i'} u_i'\right]_{x_s} = 0, \tag{3.57}$$

for $i = 1, \ldots, m$. Equations (3.54) are the Euler equations and (3.55) are boundary conditions at x_0 and x_1. Equations (3.54) and (3.55) are identical to the corresponding natural conditions for continuous variables, derived in Section 3.3. Equations (3.56) and (3.57) are due to the discontinuities in the u_i at the point x_s. It is possible that there is a number of such points of discontinuity in the interval (x_0, x_1). If so, then there are equations of the form (3.56) and (3.57) corresponding to each of these points. In this case the Euler equations (3.54) are derived for the interval (x_0, x_1) excluding all points of discontinuity.

3.4.2 General Variational Principles for Two-dimensional Flow

The natural conditions of a variational principle with discontinuous variables, defined in two dimensions, are derived in this section. The method is similar to that of the one-dimensional case in Section 3.4.1. The difference is that in two dimensions the variables are discontinuous across a curve in the domain of integration instead of at an isolated point.

Let D be the domain in the xy plane over which the integrand of the functional is to be integrated and let Σ be the boundary of D, as before. Let Σ_s be a smooth, non self-intersecting curve which divides D into two distinct regions, D^- and D^+, as in Figure 3.1.

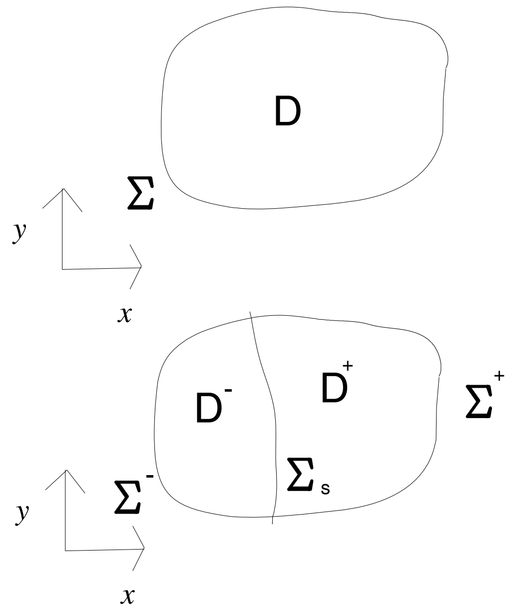

Figure 3.1: The two-dimensional domain.

Let $u_i = u_i(x,y)$ for $i = 1,\ldots,m$ and $\mathbf{v}_i = (v_{i1}(x,y), v_{i2}(x,y))$ for $i = 1,\ldots,n$ be the scalar and vector variables. The u_i and $\mathbf{v}_i$ are assumed to be continuous in the domains D^- and D^+ but may be discontinuous across the curve Σ_s.

Consider a functional of the form

$$\begin{aligned} \hat{I}_4(u_i, \mathbf{v}_k, \Sigma_s) = {} & \left(\iint_{D^-} + \iint_{D^+} \right) F(x, y, u_i, \boldsymbol{\nabla} u_i, \mathbf{v}_k, \boldsymbol{\nabla}.\mathbf{v}_k)\, dx\, dy \\ & + \left(\int_{\Sigma^-} + \int_{\Sigma^+} \right) G(x, y, u_i, \mathbf{v}_k)\, d\Sigma, \end{aligned} \tag{3.58}$$

where $i = 1,\ldots,m$, $k = 1,\ldots,n$, Σ^- is the part of Σ which bounds D^- and Σ^+ is the part which bounds D^+.

The first variation of $\hat{I}_4$ is given by

$$\begin{aligned} \delta \hat{I}_4 = {} & \left(\iint_{D^-} + \iint_{D^+} \right) \left\{ \sum_{i=1}^{m} \left(F_{u_i} \delta u_i + F_{\boldsymbol{\nabla} u_i} . \boldsymbol{\nabla} \left(\delta u_i \right) \right) \right. \\ & \qquad \left. + \sum_{k=1}^{n} \left(F_{\mathbf{v}_k} . \delta \mathbf{v}_k + F_{\boldsymbol{\nabla} . \mathbf{v}_k} \boldsymbol{\nabla} . \left(\delta \mathbf{v}_k \right) \right) \right\} dx\, dy \\ & + \left(\int_{\Sigma^-} + \int_{\Sigma^+} \right) \left\{ \sum_{i=1}^{m} G_{u_i} \delta u_i + \sum_{k=1}^{n} G_{\mathbf{v}_k} . \delta \mathbf{v}_k \right\} d\Sigma \\ & + \iint_{\delta D} F^-\, dx\, dy - \iint_{\delta D} F^+\, dx\, dy + \int_{\delta \Sigma} G^-\, d\Sigma - \int_{\delta \Sigma} G^+\, d\Sigma, \end{aligned} \tag{3.59}$$

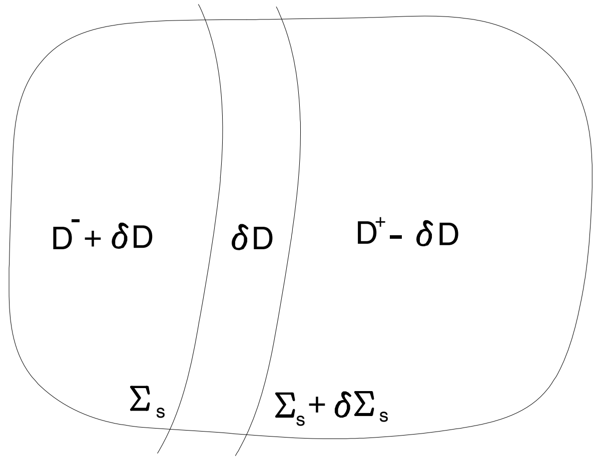

Figure 3.2: Variation of Σ_s.

where δD is the region of D enclosed by Σ_s and its variation $\Sigma_s + \delta\Sigma_s$ — see Figure 3.2 — and $\delta\Sigma$ is the change in the length of Σ^- caused by the variation of Σ_s. The superscripts $-$ and $+$ indicate that the functions are evaluated on the D^- or D^+ sides of Σ_s, respectively.

Applying the divergence theorem to the $\boldsymbol{\nabla}(\delta u_i)$ and $\boldsymbol{\nabla}.(\delta \mathbf{v}_k)$ terms in (3.59) yields

$$\begin{aligned}
\delta \hat{I}_4 = \left(\iint_{D^-} + \iint_{D^+}\right) & \left\{ \sum_{i=1}^{m} \left(F_{u_i} - \boldsymbol{\nabla}.F_{\boldsymbol{\nabla} u_i}\right) \delta u_i \right. \\
& \left. + \sum_{k=1}^{n} \left(F_{\mathbf{v}_k} - \boldsymbol{\nabla} F_{\boldsymbol{\nabla}.\mathbf{v}_k}\right) .\delta \mathbf{v}_k \right\} dx\,dy \\
& + \left(\int_{\Sigma^-} + \int_{\Sigma^+}\right) \left\{ \sum_{i=1}^{m} \left(G_{u_i} + F_{\boldsymbol{\nabla} u_i}.\mathbf{n}\right) \delta u_i + \sum_{k=1}^{n} \left(G_{\mathbf{v}_k} + F_{\boldsymbol{\nabla}.\mathbf{v}_k}\mathbf{n}\right) .\delta \mathbf{v}_k \right\} d\Sigma \\
& + \iint_{\delta D} F^-\,dx\,dy - \iint_{\delta D} F^+\,dx\,dy + \int_{\delta\Sigma} G^-\,d\Sigma - \int_{\delta\Sigma} G^+\,d\Sigma \\
& + \int_{\Sigma_s} \left\{ \sum_{i=1}^{m} F^-_{\boldsymbol{\nabla} u_i}.\mathbf{n}\,\delta u_i + \sum_{k=1}^{n} F^-_{\boldsymbol{\nabla}.\mathbf{v}_k}\mathbf{n}.\delta \mathbf{v}_k \right. \\
& \left. - \sum_{i=1}^{m} F^+_{\boldsymbol{\nabla} u_i}.\mathbf{n}\,\delta u_i - \sum_{k=1}^{n} F^+_{\boldsymbol{\nabla}.\mathbf{v}_k}\mathbf{n}.\delta \mathbf{v}_k \right\} d\Sigma.
\end{aligned}$$

Consider a point on the curve Σ_s. Let δn be the displacement of this point, under the variation, in the direction of $\mathbf{n}$, the unit normal on the surface Σ_s with direction out of the subdomain D^-. Then the integral over the domain δD can be written in terms of an integral along Σ_s and the variation δn, that is,

$$\iint_{\delta D} F\, dx\, dy = \int_{\Sigma_s} F\, \delta n\, d\Sigma.$$

As in the one-dimensional case equations relating the values of the variables on either side of the discontinuity are obtained by using the total variations. Let $\delta\tau$ be the displacement of a point on Σ_s, under the variation, in the direction of $\boldsymbol{\tau}$, the unit tangent vector at the point. Then the total variations of the flow variables on the curve Σ_s are given by

$$\begin{aligned} \hat{\delta u}_i^{\pm}\Big|_{\Sigma_s} &= \delta u_i^{\pm}\Big|_{\Sigma_s} + \frac{\partial u_i}{\partial n}^{\pm}\Big|_{\Sigma_s} \delta n + \frac{\partial u_i}{\partial \tau}^{\pm}\Big|_{\Sigma_s} \delta\tau \quad i = 1, \ldots, m, \\ \hat{\delta \mathbf{v}}_k^{\pm}\Big|_{\Sigma_s} &= \delta \mathbf{v}_k^{\pm}\Big|_{\Sigma_s} + \frac{\partial \mathbf{v}_k}{\partial n}^{\pm}\Big|_{\Sigma_s} \delta n + \frac{\partial \mathbf{v}_k}{\partial \tau}^{\pm}\Big|_{\Sigma_s} \delta\tau \quad k = 1, \ldots, n, \end{aligned}$$

where $\frac{\partial}{\partial n}$ denotes differentiation in the direction of $\mathbf{n}$ and $\frac{\partial}{\partial \tau}$ differentiation in the direction of $\boldsymbol{\tau}$. As in the one-dimensional case, only variations whose total variations are equal at the discontinuity are considered, that is,

$$\begin{aligned} \hat{\delta u}_i^{+}\Big|_{\Sigma_s} &= \hat{\delta u}_i^{-}\Big|_{\Sigma_s} = \hat{\delta u}_i\Big|_{\Sigma_s}, \\ \hat{\delta \mathbf{v}}_k^{+}\Big|_{\Sigma_s} &= \hat{\delta \mathbf{v}}_k^{-}\Big|_{\Sigma_s} = \hat{\delta \mathbf{v}}_k\Big|_{\Sigma_s}. \end{aligned}$$

Thus the first variation of I_4 can be written as

$$\begin{aligned} \delta \hat{I}_4 = \left(\iint_{D^-} + \iint_{D^+}\right) & \left\{\sum_{i=1}^{m} \left(F_{u_i} - \boldsymbol{\nabla}.F_{\boldsymbol{\nabla} u_i}\right) \delta u_i \right. \\ & \left. + \sum_{k=1}^{n} \left(F_{\mathbf{v}_k} - \boldsymbol{\nabla} F_{\boldsymbol{\nabla}.\mathbf{v}_k}\right).\delta \mathbf{v}_k \right\} dx\, dy \\ & + \left(\int_{\Sigma^-} + \int_{\Sigma^+}\right) \left\{\sum_{i=1}^{m} \left(G_{u_i} + F_{\boldsymbol{\nabla} u_i}.\mathbf{n}\right) \delta u_i + \sum_{k=1}^{n} \left(G_{\mathbf{v}_k} + F_{\boldsymbol{\nabla}.\mathbf{v}_k}\mathbf{n}\right).\delta \mathbf{v}_k\right\} d\Sigma \\ & + \int_{\delta\Sigma} \left(G^- - G^+\right) d\Sigma \\ & + \int_{\Sigma_s} \left\{\left(F^- - \sum_{i=1}^{m} F^-_{\boldsymbol{\nabla} u_i}.\mathbf{n}\frac{\partial u_i}{\partial n}^- - \sum_{k=1}^{n} F^-_{\boldsymbol{\nabla}.\mathbf{v}_k}\frac{\partial}{\partial n}\left(\mathbf{v}_k^-.\mathbf{n}\right)\right)\delta n \right. \\ & \quad - \left(F^+ - \sum_{i=1}^{m} F^+_{\boldsymbol{\nabla} u_i}.\mathbf{n}\frac{\partial u_i}{\partial n}^+ - \sum_{k=1}^{n} F^+_{\boldsymbol{\nabla}.\mathbf{v}_k}\frac{\partial}{\partial n}\left(\mathbf{v}_k^+.\mathbf{n}\right)\right)\delta n \end{aligned}$$

$$
\begin{aligned}
&-\left(\sum_{i=1}^{m} F^{-}_{\boldsymbol{\nabla} u_i}.\mathbf{n}\frac{\partial u_i}{\partial \tau}^{-} + \sum_{k=1}^{n} F^{-}_{\boldsymbol{\nabla}.\mathbf{v}_k}\frac{\partial}{\partial \tau}\left(\mathbf{v}_k^{-}.\mathbf{n}\right)\right)\delta\tau \\
&+\left(\sum_{i=1}^{m} F^{+}_{\boldsymbol{\nabla} u_i}.\mathbf{n}\frac{\partial u_i}{\partial \tau}^{+} + \sum_{k=1}^{n} F^{+}_{\boldsymbol{\nabla}.\mathbf{v}_k}\frac{\partial}{\partial \tau}\left(\mathbf{v}_k^{+}.\mathbf{n}\right)\right)\delta\tau \\
&+\sum_{i=1}^{m}\left(F^{-}_{\boldsymbol{\nabla} u_i}.\mathbf{n} - F^{+}_{\boldsymbol{\nabla} u_i}.\mathbf{n}\right)\hat{\delta u_i} + \sum_{k=1}^{n}\left(F^{-}_{\boldsymbol{\nabla}.\mathbf{v}_k} - F^{+}_{\boldsymbol{\nabla}.\mathbf{v}_k}\right)\mathbf{n}.\hat{\delta \mathbf{v}_k}\Bigg\}\, d\Sigma.
\end{aligned}
$$

The natural conditions of $\delta\hat{I}_4 = 0$ are

$$\delta u_i: \quad F_{u_i} - \boldsymbol{\nabla}.F_{\boldsymbol{\nabla} u_i} = 0 \qquad (x,y) \in D^- \cup D^+, \tag{3.60}$$

$$\delta \mathbf{v}_k: \quad F_{\mathbf{v}_k} - \boldsymbol{\nabla} F_{\boldsymbol{\nabla}.\mathbf{v}_k} = \mathbf{0} \qquad (x,y) \in D^- \cup D^+, \tag{3.61}$$

$$\delta u_i: \quad G_{u_i} + F_{\boldsymbol{\nabla} u_i}.\mathbf{n} = 0 \qquad (x,y) \in \Sigma^- \cup \Sigma^+, \tag{3.62}$$

$$\delta \mathbf{v}_k: \quad G_{\mathbf{v}_k} + F_{\boldsymbol{\nabla}.\mathbf{v}_k}\mathbf{n} = \mathbf{0} \qquad (x,y) \in \Sigma^- \cup \Sigma^+, \tag{3.63}$$

$$\delta n: \quad \left[F - \sum_{i=1}^{m} F_{\boldsymbol{\nabla} u_i}.\mathbf{n}\frac{\partial u_i}{\partial n} - \sum_{k=1}^{n} F_{\boldsymbol{\nabla}.\mathbf{v}_k}\frac{\partial}{\partial n}\left(\mathbf{v}_k.\mathbf{n}\right)\right]_{\Sigma_s} = 0, \tag{3.64}$$

$$\delta \tau: \quad \left[\sum_{i=1}^{m} F_{\boldsymbol{\nabla} u_i}.\mathbf{n}\frac{\partial u_i}{\partial \tau} + \sum_{k=1}^{n} F_{\boldsymbol{\nabla}.\mathbf{v}_k}\frac{\partial}{\partial \tau}\left(\mathbf{v}_k.\mathbf{n}\right)\right]_{\Sigma_s} = 0, \tag{3.65}$$

$$\hat{\delta u_i}: \quad \left[F_{\boldsymbol{\nabla} u_i}.\mathbf{n}\right]_{\Sigma_s} = 0, \tag{3.66}$$

$$\hat{\delta \mathbf{v}_k}: \quad \left[F_{\boldsymbol{\nabla}.\mathbf{v}_k}\right]_{\Sigma_s} = 0, \tag{3.67}$$

for $i = 1, \ldots, m$ and $k = 1, \ldots, n$, where the brackets $[\cdot]_{\Sigma_s}$ denote the change in the quantity enclosed on crossing Σ_s from D^- to D^+. The sets of equations (3.60)–(3.63) are the same as the natural conditions for continuous variables, as derived in Section 3.3.2 although, instead of being valid on the whole of D or Σ, (3.60) and (3.61) are valid on D^- and D^+ separately and (3.62) and (3.63) are valid on Σ^- and Σ^+ separately. Equations (3.64) and (3.65) arise from the variation in the position of the line of discontinuity Σ_s and equations (3.66) and (3.67) are the result of matching the coefficients of the total variations in the variables at this line.

3.5 Time-dependent Shallow Water Flows

The previous two sections have dealt with generating general expressions for the natural conditions of certain types of variational principles. These are used in this section and the remaining sections of this chapter to deduce the natural conditions of variational principles for shallow water flows.

The derivation of variational principles for time-dependent motion in shallow water is continued here using the functionals created in Section 3.2.2.

3.5.1 Boundary Conditions

The variational principles for shallow water flows considered so far have all been such that the variations vanish at the boundaries of the domains of integration. In this section boundary terms are added to the functionals of Section 3.2.2 and variations which are not necessarily zero on the space and time boundaries are allowed. In this way boundary conditions for shallow water flows are derived.

First consider the functional with the integrand (3.28) — the 'p' functional. If we examine the associated variational principle and allow variations which do not vanish at the boundaries, the variables $\mathbf{Q}$ and ϕ will appear in terms integrated around the space boundary of the domain and d and ϕ will appear in terms evaluated at the initial and final times t_1 and t_2. This can be seen using the natural conditions (3.41)–(3.44) of the general two-dimensional variational principle in Section 3.3, where the function F is taken to be the integrand of the 'p' functional and G and H are, as yet, unspecified boundary functions. With this motivation the boundary Σ of domain D is divided into two parts, say $\Sigma = \Sigma_\phi + \Sigma_Q$, where boundary conditions for ϕ are sought on Σ_ϕ and for $\mathbf{Q}$ on Σ_Q

for $t \in (t_1, t_2)$. Similarly the domain is divided into two parts, say $D = D_d + D_\phi$, and conditions at the time boundaries t_1 and t_2 are sought for d in D_d and for ϕ in D_ϕ.

In this way it is possible to construct a 'p' functional with boundary terms. Let the functional $I_1(E, \mathbf{Q}, d, \mathbf{v}, \phi)$ be given by

$$\begin{aligned} I_1 = & \int_{t_1}^{t_2} \iint_D \left(p(\mathbf{v}, E) - d\,(\phi_t + E - gh) + \mathbf{Q}.\left(\mathbf{v} - \boldsymbol{\nabla}\phi\right)\right)\, dx\, dy\, dt \\ & + \int_{t_1}^{t_2} \int_{\Sigma_Q} C\phi\, d\Sigma\, dt + \int_{t_1}^{t_2} \int_{\Sigma_\phi} (\phi - f)\, \mathbf{Q}.\mathbf{n}\, d\Sigma\, dt \\ & + \iint_{D_\phi} \left(\left(d\,(\phi - h_2)\right)\big|_{t_2} - \left(d\,(\phi - h_1)\right)\big|_{t_1} \right)\, dx\, dy \\ & + \iint_{D_d} \left(\phi\big|_{t_2} g_2 - \phi\big|_{t_1} g_1 \right)\, dx\, dy, \end{aligned} \tag{3.68}$$

where $f = f(x, y, t)$ and $C = C(x, y, t)$ are given functions on Σ_ϕ and Σ_Q respectively and $g_i = g_i(x, y)$ and $h_i = h_i(x, y)$ for $i = 1, 2$ are given functions in D_d and D_ϕ respectively.

The natural conditions of the revised 'p' principle $\delta I_1 = 0$, deduced using equations (3.39)–(3.44), are

$$\left.\begin{aligned} p_{\mathbf{v}} + \mathbf{Q} &= \mathbf{0} \\ p_E - d &= 0 \\ d_t + \boldsymbol{\nabla}.\mathbf{Q} &= 0 \\ \phi_t + E - gh &= 0 \\ \mathbf{v} - \boldsymbol{\nabla}\phi &= \mathbf{0} \end{aligned}\right\} \quad \text{in } \hat{D},$$

$$\begin{aligned} C - \mathbf{Q}.\mathbf{n} &= 0 && \text{on } \Sigma_Q \text{ for } t \in (t_1, t_2), \\ f - \phi &= 0 && \text{on } \Sigma_\phi \text{ for } t \in (t_1, t_2), \\ d\big|_{t_i} - g_i &= 0 && \text{in } D_d \text{ for } i = 1, 2, \\ \phi\big|_{t_i} - h_i &= 0 && \text{in } D_\phi \text{ for } i = 1, 2, \end{aligned}$$

where $\hat{D} = D \times (t_1, t_2)$,

$$p_{\mathbf{v}} \equiv \frac{\partial p}{\partial \mathbf{v}} = -\frac{\mathbf{v}}{g}\left(E - \frac{1}{2}\mathbf{v}.\mathbf{v}\right) \quad \text{and} \quad p_E \equiv \frac{\partial p}{\partial E} = \frac{1}{g}\left(E - \frac{1}{2}\mathbf{v}.\mathbf{v}\right).$$

Thus the first two conditions in $\hat{D}$ are

$$-\frac{\mathbf{v}}{g}\left(E - \frac{1}{2}\mathbf{v}.\mathbf{v}\right) + \mathbf{Q} = \mathbf{0} \quad \text{and} \quad \frac{1}{g}\left(E - \frac{1}{2}\mathbf{v}.\mathbf{v}\right) - d = 0,$$

which together give

$$\mathbf{Q} = d\mathbf{v} \quad \text{and} \quad E = gd + \frac{1}{2}\mathbf{v}.\mathbf{v},$$

so that the last three natural conditions in $\hat{D}$ are the conservation laws and the irrotationality condition. The last four natural conditions are space and time boundary conditions. The first of these is a condition on the normal component of mass flow on Σ_Q and the second is a condition on the velocity potential on $\Sigma\phi$, both for $t \in (t_1, t_2)$. The remaining conditions are for depth and velocity potential, in D_d and D_ϕ respectively, at the initial and final times. These last conditions are not desirable in a practical sense since they require knowledge of the solution at the final time.

Consider now the 'r' functional — that with integrand (3.30). The domain and domain boundary are again divided into two, as for the 'p' principle, to provide a choice of boundary conditions. Using the same functions, $f = f(x, y, t)$, $C = C(x, y, t)$, $g_i = g_i(x, y)$ and $h_i = h_i(x, y)$ for $i = 1, 2$, a second functional $I_2(\mathbf{Q}, d, \phi)$ can be derived, namely,

$$\begin{aligned} I_2 = & \int_{t_1}^{t_2} \iint_D \left(r(\mathbf{Q}, d) + gdh + \phi\left(d_t + \boldsymbol{\nabla}.\mathbf{Q}\right)\right) \, dx\, dy\, dt \\ & + \int_{t_1}^{t_2} \int_{\Sigma_Q} \phi\left(C - \mathbf{Q}.\mathbf{n}\right) \, d\Sigma\, dt - \int_{t_1}^{t_2} \int_{\Sigma_\phi} f\mathbf{Q}.\mathbf{n} \, d\Sigma\, dt \\ & - \iint_{D_d} \left(\left(\phi\left(d - g_2\right)\right)\big|_{t_2} - \left(\phi\left(d - g_1\right)\right)\big|_{t_1} \right) \, dx\, dy \\ & - \iint_{D_\phi} \left(d\big|_{t_2} h_2 - d\big|_{t_1} h_1 \right) \, dx\, dy. \end{aligned} \tag{3.69}$$

The natural conditions of the 'r' principle $\delta I_2 = 0$ may be deduced using (3.39)–(3.44) and are given by

$$\left.\begin{array}{rcl} r_d + gh - \phi_t & = & 0 \\ r_{\mathbf{Q}} - \boldsymbol{\nabla}\phi & = & \mathbf{0} \\ d_t + \boldsymbol{\nabla}.\mathbf{Q} & = & 0 \end{array}\right\} \qquad \text{in } \hat{D},$$

$$\begin{array}{rcll} C - \mathbf{Q}.\mathbf{n} & = & 0 & \text{on } \Sigma_Q \text{ for } t \in (t_1, t_2), \\ f - \phi & = & 0 & \text{on } \Sigma_\phi \text{ for } t \in (t_1, t_2), \\ d|_{t_i} - g_i & = & 0 & \text{in } D_d \text{ for } i = 1, 2, \\ \phi|_{t_i} - h_i & = & 0 & \text{in } D_\phi \text{ for } i = 1, 2. \end{array}$$

The first two natural conditions in the domain may be rewritten as

$$-\frac{1}{2}\frac{\mathbf{Q}.\mathbf{Q}}{d^2} - gd + gh - \phi_t = 0 \ , \ \ \frac{\mathbf{Q}}{d} - \boldsymbol{\nabla}\phi = \mathbf{0}$$

so that, using (2.25), the equations of motion and the irrotationality condition have again been derived. The boundary conditions are identical to those of the 'p' principle.

Thus there exist two functionals, (3.68) and (3.69), whose natural conditions of the first variation are the equations of motion in the domain of the problem together with prescribed conditions on mass flow and velocity potential on the boundary of the domain, and conditions on the depth and velocity potential over regions of the domain at the initial and final times.

3.5.2 A Quartet of Functionals

A sequence of Legendre transforms can be used to generate a quartet of functionals which have as natural conditions of their first variations the equations of time-dependent motion in shallow water. Two such functionals — based on the p and r functions — have already been described and were independently derived from the 'pressure' and 'Hamilton' functionals for three-dimensional free surface flows. Two further functionals are now sought.

By applying the divergence theorem and integration by parts, the 'p' functional (3.68) can be expressed in the form

$$\begin{aligned} I_1 = & \int_{t_1}^{t_2} \iint_D \left(p(\mathbf{v}, E) - Ed + \mathbf{Q}.\mathbf{v} + gdh + \phi \left(d_t + \boldsymbol{\nabla}.\mathbf{Q} \right) \right) \, dx \, dy \, dt \\ & + \int_{t_1}^{t_2} \int_{\Sigma_Q} \phi \left(C - \mathbf{Q}.\mathbf{n} \right) \, d\Sigma \, dt - \int_{t_1}^{t_2} \int_{\Sigma_\phi} f \mathbf{Q}.\mathbf{n} \, d\Sigma \, dt \\ & - \iint_{D_d} \left(\left(\phi \left(d - g_2 \right) \right) \big|_{t_2} - \left(\phi \left(d - g_1 \right) \right) \big|_{t_1} \right) \, dx \, dy \\ & - \iint_{D_\phi} \left(d \big|_{t_2} h_2 - d \big|_{t_1} h_1 \right) \, dx \, dy. \end{aligned} \tag{3.70}$$

Comparing this with I_2, as given by (3.69), suggests that there is a relationship between the two functions $p(\mathbf{v}, E)$ and $r(\mathbf{Q}, d)$ such that

$$r(\mathbf{Q}, d) = p(\mathbf{v}, E) - Ed + \mathbf{Q}.\mathbf{v}$$

in value, which can be confirmed directly using (2.19) and (2.25). The relation is in fact a Legendre transformation as is now shown.

The Legendre transform $R(\mathbf{v}, d)$ of $p(\mathbf{v}, E)$, with E and d as dual active variables and $\mathbf{v}$ passive, is defined by

$$R(\mathbf{v}, d) = Ed - p(\mathbf{v}, E) \tag{3.71}$$

and is such that

$$R_{\mathbf{v}} = -p_{\mathbf{v}} \quad , \quad R_d = E.$$

Using (2.19) R can be constructed from (3.71) and is

$$R(\mathbf{v}, d) = \frac{1}{2} g d^2 + \frac{1}{2} d \mathbf{v}.\mathbf{v}. \tag{3.72}$$

Notice that R is equal to the total energy of a fluid particle.

The function R is also a Legendre transform of $r(\mathbf{Q}, d)$, with $\mathbf{Q}$ active and d passive, in that, using (2.25), we may write

$$R(\mathbf{v}, d) = \mathbf{Q}.\mathbf{v} - r(\mathbf{Q}, d) \tag{3.73}$$

with the first derivatives

$$R_d = -r_d \ , \ R_{\mathbf{v}} = \mathbf{Q}.$$

Equations (3.71) and (3.73) imply the required connection, that

$$r(\mathbf{Q}, d) = \mathbf{Q}.\mathbf{v} - R(\mathbf{v}, d) = p(\mathbf{v}, E) - Ed + \mathbf{Q}.\mathbf{v}$$

in value.

The intermediate function R can be bypassed and p and r may be connected directly by a Legendre transform. Since $p_{\mathbf{v}} = -\mathbf{Q}$ and $p_E = d$, then if $\mathbf{v}$ and E are both active variables, the transformation of p is

$$r(\mathbf{Q}, d) = \mathbf{Q}.\mathbf{v} - Ed + p(\mathbf{v}, E) \tag{3.74}$$

and

$$r_{\mathbf{Q}} = \mathbf{v} \ , \ r_d = -E.$$

A fourth function $P(\mathbf{Q}, E)$ completes a closed quartet of functions related by Legendre transforms and is derivable from p, r and R by using appropriate active variables. P cannot be given explicitly, but is defined by eliminating $\mathbf{v}$ and d from

$$P(\mathbf{Q}, E) = \frac{1}{2}gd^2 + d\mathbf{v}.\mathbf{v} \ , \ \mathbf{Q} = d\mathbf{v} \ , \ E = gd + \frac{1}{2}\mathbf{v}.\mathbf{v}, \tag{3.75}$$

and has the values of flow stress. The function P is related to p and r by

$$p(\mathbf{v}, E) - P(\mathbf{Q}, E) \ = \ -\mathbf{Q}.\mathbf{v} \tag{3.76}$$

$$r(\mathbf{Q}, d) - P(\mathbf{Q}, E) \quad = \quad -Ed. \tag{3.77}$$

The functions P and R can be used as bases for functionals, the natural conditions of the first variations of which include the equations of motion in shallow water. The functionals may be generated by substituting for p and r in the integrands of (3.68) and (3.69). The process is to use (3.76) to substitute for p in the integrand of (3.68) and (3.73) to substitute for r in the integrand of (3.69) by what is essentially a change of variables using the definitions of E and $\mathbf{Q}$, (2.19) and (2.25). Although (3.71) could be used to substitute for p in (3.68) and (3.77) could be used to substitute for r in (3.69) it would not change the nature of the functionals being generated. For instance integration by parts and the divergence theorem can be used on the 'P' functional generated by substituting (3.77) into (3.69) to give the functional formed by substituting (3.76) into (3.68).

The 'P' Principle

Let the functional $I_3(E, \mathbf{Q}, d, \phi)$ be defined by

$$\begin{aligned} I_3 = & \int_{t_1}^{t_2} \iint_D \left(P(\mathbf{Q}, E) - \mathbf{Q}.\boldsymbol{\nabla}\phi - d\left(\phi_t + E - gh\right)\right)\, dx\, dy\, dt \\ & + \int_{t_1}^{t_2} \int_{\Sigma_Q} C\phi\, d\Sigma\, dt + \int_{t_1}^{t_2} \int_{\Sigma_\phi} (\phi - f)\, \mathbf{Q}.\mathbf{n}\, d\Sigma\, dt \\ & + \iint_{D_\phi} \Big(\left(d\left(\phi - h_2\right)\right)\big|_{t_2} - \left(d\left(\phi - h_1\right)\right)\big|_{t_1} \Big)\, dx\, dy \\ & + \iint_{D_d} \Big(\phi|_{t_2}\, g_2 - \phi|_{t_1}\, g_1 \Big)\, dx\, dy. \end{aligned} \tag{3.78}$$

Then the natural conditions of the 'P' principle $\delta I_3 = 0$ are

$$\left.\begin{aligned} P_{\mathbf{Q}} - \boldsymbol{\nabla}\phi &= \mathbf{0} \\ P_E - d &= 0 \\ \phi_t + E - gh &= 0 \\ d_t + \boldsymbol{\nabla}.\mathbf{Q} &= 0 \end{aligned}\right\} \quad \text{in } \hat{D},$$

$$
\begin{aligned}
C - \mathbf{Q}.\mathbf{n} &= 0 && \text{on } \Sigma_Q \text{ for } t \in (t_1, t_2), \\
f - \phi &= 0 && \text{on } \Sigma_\phi \text{ for } t \in (t_1, t_2), \\
d|_{t_i} - g_i &= 0 && \text{in } D_d \text{ for } i = 1, 2, \\
\phi|_{t_i} - h_i &= 0 && \text{in } D_\phi \text{ for } i = 1, 2.
\end{aligned}
$$

The first condition in $\hat{D}$ is

$$\mathbf{v} - \boldsymbol{\nabla}\phi = \mathbf{0}.$$

Thus if equations (2.19) and (2.25) are assumed, the 'P' principle yields the conservation laws and the irrotationality condition as natural conditions in $\hat{D}$, and gives the same boundary conditions on ϕ and $\mathbf{Q}$ at space boundaries and on d and ϕ at time boundaries as are obtained from the 'p' and 'r' principles.

The 'R' Principle

Now consider a principle based on the function R. Let the functional $I_4(\mathbf{Q}, d, \mathbf{v}, \phi)$ be given by

$$
\begin{aligned}
I_4 = &\int_{t_1}^{t_2} \iint_D \left(-R(\mathbf{v}, d) + \mathbf{Q}.\mathbf{v} + gdh + \phi\,(d_t + \boldsymbol{\nabla}.\mathbf{Q})\right)\, dx\, dy\, dt \\
&+ \int_{t_1}^{t_2} \int_{\Sigma_Q} \phi\,(C - \mathbf{Q}.\mathbf{n})\; d\Sigma\, dt - \int_{t_1}^{t_2} \int_{\Sigma_\phi} f\mathbf{Q}.\mathbf{n}\, d\Sigma\, dt \\
&- \iint_{D_d} \left((\phi\,(d - g_2))|_{t_2} - (\phi\,(d - g_1))|_{t_1} \right)\, dx\, dy \\
&- \iint_{D_\phi} \left(d|_{t_2}\, h_2 - d|_{t_1}\, h_1 \right)\, dx\, dy. \qquad (3.79)
\end{aligned}
$$

The natural conditions of the 'R' principle $\delta I_4 = 0$ are

$$
\left.
\begin{aligned}
-R_{\mathbf{v}} + \mathbf{Q} &= \mathbf{0} \\
-R_d + gh - \phi_t &= 0 \\
\mathbf{v} - \boldsymbol{\nabla}\phi &= \mathbf{0} \\
d_t + \boldsymbol{\nabla}.\mathbf{Q} &= 0
\end{aligned}
\right\} \quad \text{in } \hat{D},
$$

$$\begin{aligned}
C - \mathbf{Q}.\mathbf{n} &= 0 \qquad \text{on } \Sigma_Q \text{ for } t \in (t_1, t_2), \\
f - \phi &= 0 \qquad \text{on } \Sigma_\phi \text{ for } t \in (t_1, t_2), \\
d|_{t_i} - g_i &= 0 \qquad \text{in } D_d \text{ for } i = 1, 2, \\
\phi|_{t_i} - h_i &= 0 \qquad \text{in } D_\phi \text{ for } i = 1, 2.
\end{aligned}$$

The first two conditions in $\hat{D}$ may be written as

$$-d\mathbf{v} + \mathbf{Q} = \mathbf{0} \;\text{ and }\; -gd - \frac{1}{2}\mathbf{v}.\mathbf{v} + gh - \phi_t = 0.$$

Thus the natural conditions of the 'R' principle include the equations of motion in shallow water and the same boundary conditions as obtained previously from the 'p', 'r' and 'P' principles.

So there exists a quartet of functionals (3.68), (3.69), (3.78) and (3.79), based on the four functions p, r, P and R, from which the shallow water equations can be derived as the natural conditions of the first variations. Figure 3.3 shows the relations between the p, r, P and R functions.

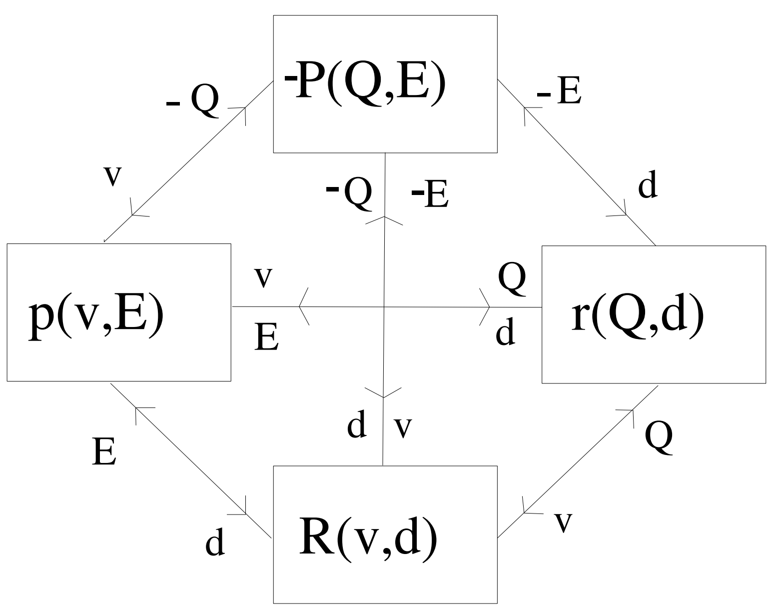

Figure 3.3: A quartet of Legendre transforms.

3.5.3 Constrained and Reciprocal Principles

Variational principles can be constrained by allowing only variations which satisfy one or more of the natural conditions. The principles constrained in this way will have the remaining natural conditions as natural conditions (Courant and Hilbert (1953)). There are several ways in which the variations of the 'p', 'r', 'P' and 'R' principles of Sections 3.5.1 and 3.5.2 can be constrained; only those constraints which produce variational principles which are reciprocal, in a sense to be defined shortly, are studied here.

Reciprocal 'p' and 'r' Principles

The functional (3.68), used in the 'p' principle, has an integrand which contains the integrated conservation of momentum equation and the irrotationality condition explicitly. It seems natural to constrain the 'p' principle to satisfy these two conditions. This can be done by specifying

$$\left.\begin{array}{rcl} E &=& -\phi_t + gh \\ \mathbf{v} &=& \nabla\phi \end{array}\right\}, \tag{3.80}$$

which results in the functional I_1 reducing to a functional $I_1^c(\mathbf{Q}, d, \phi)$. The constrained principle is given by

$$\begin{aligned} \delta I_1^c = \delta \Bigg\{ &\int_{t_1}^{t_2} \iint_D \hat{p}(\phi)\, dx\, dy\, dt + \int_{t_1}^{t_2} \int_{\Sigma_Q} C\phi\, d\Sigma\, dt \\ &+ \int_{t_1}^{t_2} \int_{\Sigma_\phi} (\phi - f)\, \mathbf{Q}.\mathbf{n}\, d\Sigma\, dt + \iint_{D_\phi} \Big((d\,(\phi - h_2))\big|_{t_2} - (d\,(\phi - h_1))\big|_{t_1} \Big)\, dx\, dy \\ &+ \iint_{D_d} \Big(\phi|_{t_2}\, g_2 - \phi|_{t_1}\, g_1 \Big)\, dx\, dy \Bigg\} = 0, \end{aligned} \tag{3.81}$$

where

$$\hat{p}(\phi) = p(\nabla\phi, -\phi_t + gh) = \frac{1}{2g}\left(\phi_t - gh + \frac{1}{2}\nabla\phi.\nabla\phi\right)^2.$$

The natural conditions are

$$\left(-\frac{1}{g}\left(\phi_t - gh + \frac{1}{2}\boldsymbol{\nabla}\phi.\boldsymbol{\nabla}\phi\right)\right)_t + \boldsymbol{\nabla}.\left(-\frac{1}{g}\left(\phi_t - gh + \frac{1}{2}\boldsymbol{\nabla}\phi.\boldsymbol{\nabla}\phi\right)\boldsymbol{\nabla}\phi\right) = 0 \quad \text{in } \hat{D},$$

$$\begin{aligned}
\frac{1}{g}\left(\phi_t - gh + \frac{1}{2}\boldsymbol{\nabla}\phi.\boldsymbol{\nabla}\phi\right)\boldsymbol{\nabla}\phi.\mathbf{n} + C &= 0 && \text{on } \Sigma_Q \text{ for } t \in (t_1, t_2),\\
\frac{1}{g}\left(\phi_t - gh + \frac{1}{2}\boldsymbol{\nabla}\phi.\boldsymbol{\nabla}\phi\right)\boldsymbol{\nabla}\phi.\mathbf{n} + \mathbf{Q}.\mathbf{n} &= 0 && \text{on } \Sigma_\phi \text{ for } t \in (t_1, t_2),\\
\phi - f &= 0 && \text{on } \Sigma_\phi \text{ for } t \in (t_1, t_2),\\
\left(\frac{1}{g}\left(\phi_t - gh + \frac{1}{2}\boldsymbol{\nabla}\phi.\boldsymbol{\nabla}\phi\right) + d\right)\Bigg|_{t_i} &= 0 && \text{in } D_\phi \text{ for } i = 1, 2,\\
\phi|_{t_i} - h_i &= 0 && \text{in } D_\phi \text{ for } i = 1, 2,\\
\frac{1}{g}\left(\phi_t - gh + \frac{1}{2}\boldsymbol{\nabla}\phi.\boldsymbol{\nabla}\phi\right)\Bigg|_{t_i} + g_i &= 0 && \text{in } D_d \text{ for } i = 1, 2,
\end{aligned}$$

the first of which may be recognised as conservation of mass, written in terms of ϕ, in the domain. Boundary conditions are given for ϕ, d and $\mathbf{Q}$.

If $\Sigma_Q = \Sigma$ and $D_d = D$ then (3.81) becomes

$$\delta\left\{\int_{t_1}^{t_2}\iint_D \hat{p}(\phi)\,dx\,dy\,dt + \int_{t_1}^{t_2}\int_\Sigma C\phi\,d\Sigma\,dt + \iint_D \left(\phi|_{t_2} g_2 - \phi|_{t_1} g_1\right)\,dx\,dy\right\} = 0 \tag{3.82}$$

in which the functional depends on the variable ϕ alone.

A constrained 'r' principle can be constructed to satisfy conservation of mass by specifying

$$\left.\begin{aligned} d &= \boldsymbol{\nabla}.\boldsymbol{\psi} \\ \mathbf{Q} &= -\boldsymbol{\psi}_t \end{aligned}\right\}, \tag{3.83}$$

for some vector $\boldsymbol{\psi} = \boldsymbol{\psi}(x, y, t)$, and substituting into equation (3.69). The resulting functional depends on $\boldsymbol{\psi}$ and ϕ and the variational principle is given by

$$\delta\left\{\int_{t_1}^{t_2}\iint_D \left(\hat{r}(\boldsymbol{\psi}) + g\boldsymbol{\nabla}.\boldsymbol{\psi} h\right) dx\,dy\,dt + \int_{t_1}^{t_2}\int_{\Sigma_Q} \phi\left(C + \boldsymbol{\psi}_t.\mathbf{n}\right)\, d\Sigma\, dt \right.$$
$$+ \int_{t_1}^{t_2}\int_{\Sigma_\phi} f\boldsymbol{\psi}_t.\mathbf{n}\, d\Sigma\, dt - \iint_{D_d}\left(\left(\phi\left(\boldsymbol{\nabla}.\boldsymbol{\psi} - g_2\right)\right)\big|_{t_2} - \left(\phi\left(\boldsymbol{\nabla}.\boldsymbol{\psi} - g_1\right)\right)\big|_{t_1}\right) dx\,dy$$
$$\left. - \iint_{D_\phi}\left(\boldsymbol{\nabla}.\boldsymbol{\psi}|_{t_2} h_2 - \boldsymbol{\nabla}.\boldsymbol{\psi}|_{t_1} h_1\right) dx\,dy\right\} = 0, \tag{3.84}$$

where

$$\hat{r}(\boldsymbol{\psi}) = r(-\boldsymbol{\psi}_t, \boldsymbol{\nabla}.\boldsymbol{\psi}) = \frac{1}{2}\frac{\boldsymbol{\psi}_t.\boldsymbol{\psi}_t}{\boldsymbol{\nabla}.\boldsymbol{\psi}} - \frac{1}{2}g\left(\boldsymbol{\nabla}.\boldsymbol{\psi}\right)^2.$$

The natural conditions are

$$-\boldsymbol{\Psi}_t + \boldsymbol{\nabla}\left(g\boldsymbol{\nabla}.\boldsymbol{\psi} - gh + \frac{1}{2}\boldsymbol{\Psi}.\boldsymbol{\Psi}\right) = \mathbf{0} \qquad \text{in } \hat{D},$$

$$\begin{aligned}
C + \boldsymbol{\psi}_t.\mathbf{n} &= 0 && \text{on } \Sigma_Q \text{ for } t \in (t_1, t_2),\\
-\phi_t - \frac{1}{2}\boldsymbol{\Psi}.\boldsymbol{\Psi} - g\boldsymbol{\nabla}.\boldsymbol{\psi} + gh &= 0 && \text{on } \Sigma_Q \text{ for } t \in (t_1, t_2),\\
-f_t - \frac{1}{2}\boldsymbol{\Psi}.\boldsymbol{\Psi} - g\boldsymbol{\nabla}.\boldsymbol{\psi} + gh &= 0 && \text{on } \Sigma_\phi \text{ for } t \in (t_1, t_2),\\
\boldsymbol{\nabla}.\boldsymbol{\psi}|_{t_i} - g_i &= 0 && \text{in } D_d \text{ for } i = 1, 2,\\
\boldsymbol{\nabla}\phi|_{t_i} + \boldsymbol{\Psi}|_{t_i} &= \mathbf{0} && \text{in } D_d \text{ for } i = 1, 2,\\
\boldsymbol{\nabla}h_i + \boldsymbol{\Psi}|_{t_i} &= \mathbf{0} && \text{in } D_\phi \text{ for } i = 1, 2,\\
\phi|_{t_i} - h_i &= 0 && \text{on } \Sigma_Q \cap \Sigma_{\hat{\phi}} \text{ for } i = 1, 2,\\
\phi|_{t_i} - \phi|_{t_i} &= 0 && \text{on } \Sigma_Q \cap \Sigma_d \text{ for } i = 1, 2,\\
f|_{t_i} - h_i &= 0 && \text{on } \Sigma_\phi \cap \Sigma_{\hat{\phi}} \text{ for } i = 1, 2,\\
f|_{t_i} - \phi|_{t_i} &= 0 && \text{on } \Sigma_\phi \cap \Sigma_d \text{ for } i = 1, 2,
\end{aligned}$$

where $\Sigma_{\hat{\phi}}$ is the boundary of D_ϕ and Σ_d is the boundary of D_d and, for neatness, $\boldsymbol{\Psi}$ represents the term $\boldsymbol{\psi}_t/\boldsymbol{\nabla}.\boldsymbol{\psi}$.

The natural condition in $\hat{D}$ is recognisable as the equation of conservation of momentum — not the integrated form usually generated. The irrotationality condition can be derived as a consequence of the conservation of momentum and the boundary conditions which specify that the flow is irrotational at $t = t_1$.

If $\Sigma_\phi = \Sigma$ and $D_\phi = D$, (3.84) reduces to a variational principle involving a functional of $\boldsymbol{\psi}$ alone, namely,

$$\delta \left\{ \int_{t_1}^{t_2} \iint_D \left(\hat{r}(\boldsymbol{\psi}) + g \boldsymbol{\nabla}.\boldsymbol{\psi} h \right) dx\, dy\, dt + \int_{t_1}^{t_2} \int_\Sigma f \boldsymbol{\psi}_t.\mathbf{n}\, d\Sigma\, dt \right.$$
$$\left. - \iint_D \left(\boldsymbol{\nabla}.\boldsymbol{\psi}|_{t_2} h_2 - \boldsymbol{\nabla}.\boldsymbol{\psi}|_{t_1} h_1 \right) dx\, dy \right\} = 0. \qquad (3.85)$$

The variational principles (3.82) and (3.85) can be described as reciprocal. We use this term to mean that the constraints satisfied in the domain by the variations in one principle are the natural conditions, in the domain, of the other principle. The boundary conditions also exhibit reciprocity in that the natural boundary conditions of (3.82) are given for mass flow, as a function of ϕ, on Σ and depth, as a function of ϕ, in D for $t = t_1\,,\, t_2$ whereas in (3.85) conditions are for the energy, as a function of $\boldsymbol{\psi}$, on Σ and velocity, as a function of $\boldsymbol{\psi}$, in D for $t = t_1\,,\, t_2$.

Reciprocal 'P' and 'R' Principles

Now consider the other two variational principles — based on P and R. The integrands of the 'P' and 'R' functionals are not expressible in either of the forms

$$\text{function of } (\mathbf{Q}, d) + \text{multiplier} \times \text{conservation of mass} \qquad \text{or}$$

$$\text{function of } (\mathbf{v}, E) + \text{multiplier} \times \begin{matrix}\text{conservation of}\\ \text{momentum}\end{matrix} + \text{multiplier} \times \begin{matrix}\text{irrotationality}\\ \text{condition}\end{matrix},$$

which are the structures that allow the 'p' and 'r' variational principles to be constrained to depend on only one variable. There is no corresponding way of constraining the 'P' and 'R' principles and the functionals cannot be reduced to depend on one variable. However, the following structure can be deduced.

Consider the 'P' functional (3.78). Let $\Sigma_Q = \Sigma$ and $D_d = D$, and constrain the variables to satisfy conservation of momentum using the first of (3.80). Then the variational principle becomes

$$\delta \left\{ \int_{t_1}^{t_2} \iint_D (P(\mathbf{Q}, -\phi_t + gh) - \mathbf{Q}.\nabla\phi)\, dx\, dy\, dt + \int_{t_1}^{t_2} \int_\Sigma C\phi\, d\Sigma\, dt \right. \\ \left. + \iint_D \left(\phi|_{t_2} g_2 - \phi|_{t_1} g_1 \right) dx\, dy \right\} = 0, \tag{3.86}$$

where the variables are $\mathbf{Q}$ and ϕ. The natural conditions are given by

$$\left. \begin{aligned} P_{\mathbf{Q}} - \nabla\phi &= \mathbf{0} \\ (P_{-\phi_t+gh})_t + \nabla.\mathbf{Q} &= 0 \end{aligned} \right\} \quad \text{in } \hat{D},$$

$$\begin{aligned} C - \mathbf{Q}.\mathbf{n} &= 0 \qquad \text{on } \Sigma \text{ for } t \in (t_1, t_2), \\ g_i - P_{-\phi_t+gh}|_{t_i} &= 0 \qquad \text{in } D \text{ for } i = 1, 2. \end{aligned}$$

The first two conditions may be rewritten as

$$\left. \begin{aligned} \mathbf{v} - \nabla\phi &= \mathbf{0} \\ d_t + \nabla.\mathbf{Q} &= 0 \end{aligned} \right\} \quad \text{in } \hat{D},$$

which are the irrotationality condition and the conservation of mass equation.

In the 'R' functional (3.79) let $\Sigma_\phi = \Sigma$ and $D_\phi = D$ and constrain the variations to satisfy conservation of mass by imposing (3.83). Then the variational principle becomes

$$\delta \left\{ \int_{t_1}^{t_2} \iint_D (-R(\mathbf{v}, \nabla.\boldsymbol{\psi}) - \boldsymbol{\psi}_t.\mathbf{v} + g\nabla.\boldsymbol{\psi} h)\, dx\, dy\, dt + \int_{t_1}^{t_2} \int_\Sigma f\boldsymbol{\psi}_t.\mathbf{n}\, d\Sigma\, dt \right. \\ \left. - \iint_D \left(\nabla.\boldsymbol{\psi}|_{t_2} h_2 - \nabla.\boldsymbol{\psi}|_{t_1} h_1 \right) dx\, dy \right\} = 0, \tag{3.87}$$

which involves a functional of $\mathbf{v}$ and $\boldsymbol{\psi}$. The natural conditions are given by

$$\left.\begin{aligned} -R_{\mathbf{v}} - \boldsymbol{\psi}_t &= \mathbf{0} \\ \boldsymbol{\nabla}\left(R_{\boldsymbol{\nabla}.\boldsymbol{\psi}} - gh\right) + \mathbf{v}_t &= \mathbf{0} \end{aligned}\right\} \quad \text{in } \hat{D},$$

$$\begin{aligned} -R_{\boldsymbol{\nabla}.\boldsymbol{\psi}} + gh - f_t &= 0 && \text{on } \Sigma \text{ for } t \in (t_1, t_2), \\ \boldsymbol{\nabla} h_i - \mathbf{v}|_{t_i} &= \mathbf{0} && \text{in } D \text{ for } i = 1, 2, \\ f|_{t_i} - h_i &= 0 && \text{on } \Sigma \text{ for } i = 1, 2. \end{aligned}$$

The first two equations are

$$\left.\begin{aligned} -\left(\boldsymbol{\nabla}.\boldsymbol{\psi}\right)\mathbf{v} - \boldsymbol{\psi}_t &= \mathbf{0} \\ \mathbf{v}_t + \boldsymbol{\nabla}\left(g\boldsymbol{\nabla}.\boldsymbol{\psi} - gh + \tfrac{1}{2}\mathbf{v}.\mathbf{v}\right) &= \mathbf{0} \end{aligned}\right\} \quad \text{in } \hat{D},$$

the second of which is conservation of momentum. This, together with the natural condition in D for t_1, implies the irrotationality condition in D for $t \in [t_1, t_2]$.

The constrained 'P' and 'R' principles (3.86) and (3.87) are reciprocal since the constraint of conservation of momentum in (3.86) is a natural condition of (3.87) and the conservation of mass constraint in (3.87) is a natural condition of (3.86). The irrotationality condition is a natural condition of both principles.

3.6 Time-independent Shallow Water Flows

The discussion so far has concerned derivation of variational principles whose natural conditions include the time-dependent shallow water equations of motion. We now seek to apply these principles to steady state conditions.

3.6.1 Steady Principles from Unsteady Principles

In Chapter 2 the equations of motion for time-independent shallow water flows are deduced from the equations for time-dependent motion by making the assumption that all of the flow variables — mass flow, energy, depth and velocity — are independent of time. The potential ϕ is not a flow variable and is not assumed to be independent of time although its general form is deduced as

$$\phi(x, y, t) = -\tilde{E}t + \tilde{\phi}(x, y) \tag{3.88}$$

(equation (2.50)), where the constant energy $\tilde{E}$ is the steady counterpart of $E - gh$.

The variational principles for time-independent motion can be derived from the variational principles for time-dependent motion of Section 3.5 in a similar way. That is, by assuming that the flow variables are independent of time and that ϕ is of the form (3.88). With appropriately modified boundary functions the integrals with respect to time in the principles for time-dependent flow can be evaluated.

Consider the four functionals (3.68), (3.69), (3.78) and (3.79). The boundary functions $C(x, y, t)$, $f(x, y, t)$, $g_i(x, y)$ and $h_i(x, y)$ for $i = 1, 2$ must be treated with care in transforming from unsteady to steady motion. In the natural conditions the function C will be used to provide a boundary condition for the mass flow on Σ_Q. As mass flow is now assumed independent of time, C must be replaced by a function $\hat{C}(x, y)$. The function f will be used to provide a boundary condition for ϕ on Σ_ϕ. The variation of ϕ with time is known from (3.88) and so, for consistency, f must be replaced by $\tilde{f}(x, y, t) = -\tilde{E}t + \tilde{f}_1(x, y)$. The functions g_1 and g_2 are the time boundary conditions on the depth in domain D_d but since the depth does not vary with time we must have $g_1 = g_2 = \hat{g}(x, y)$. The functions

h_1 and h_2 give time boundary conditions on ϕ in D_ϕ, and from (3.88) we know that

$$\phi|_{t_2} - \phi|_{t_1} = -\tilde{E}(t_2 - t_1) = -\tilde{E}T,$$

where $T = t_2 - t_1$. Therefore h_1 and h_2 must be specified so that $h_2 - h_1 = -\tilde{E}T$ for consistency.

First consider the 'p' principle. The functional I_1, given by (3.68), under steady state conditions becomes

$$I_1^s = \iint_D T\left(p(\mathbf{v}, E) + \mathbf{Q}.\left(\mathbf{v} - \boldsymbol{\nabla}\tilde{\phi}\right)\right) dx\,dy + \int_{\Sigma_Q} \hat{C}\left(\int_{t_1}^{t_2} \phi\,dt\right) d\Sigma$$
$$+ \int_{\Sigma_\phi} \left(\int_{t_1}^{t_2} \left(\phi - \tilde{f}\right) dt\right) \mathbf{Q}.\mathbf{n}\,d\Sigma - \iint_{D_d} T\hat{g}\tilde{E}\,dx\,dy,$$

where $I_1^s = I_1^s(\mathbf{Q}, \mathbf{v}, \phi)$. To simplify this define

$$\hat{f}(x, y) = \tilde{f}_1(x, y) - \frac{1}{2}\tilde{E}(t_1 + t_2)$$

so that

$$\int_{t_1}^{t_2} \tilde{f}(x, y, t)\,dt = \left[\tilde{f}_1(x, y)t - \frac{1}{2}\tilde{E}t^2\right]_{t_1}^{t_2} = T\left(\tilde{f}_1(x, y) - \frac{1}{2}\tilde{E}(t_1 + t_2)\right) = T\hat{f}(x, y).$$

Also, let

$$\hat{\phi}(x, y) = \tilde{\phi}(x, y) - \frac{1}{2}\tilde{E}(t_1 + t_2)$$

so that

$$\int_{t_1}^{t_2} \phi(x, y, t)\,dt = \left[\tilde{\phi}(x, y)t - \frac{1}{2}\tilde{E}t^2\right]_{t_1}^{t_2} = T\left(\tilde{\phi}(x, y) - \frac{1}{2}\tilde{E}(t_1 + t_2)\right) = T\hat{\phi}(x, y)$$

and $\boldsymbol{\nabla}\hat{\phi} = \boldsymbol{\nabla}\tilde{\phi}$. Then

$$I_1^s = \iint_D T\left(p(\mathbf{v}, E) + \mathbf{Q}.\left(\mathbf{v} - \boldsymbol{\nabla}\hat{\phi}\right)\right) dx\,dy + \int_{\Sigma_Q} T\hat{C}\hat{\phi}\,d\Sigma$$
$$+ \int_{\Sigma_\phi} T\left(\hat{\phi} - \hat{f}\right) \mathbf{Q}.\mathbf{n}\,d\Sigma - \iint_{D_d} T\hat{g}\tilde{E}\,dx\,dy. \qquad (3.89)$$

Notice that the final term in (3.89) is a constant and so it may be discarded. Also, throughout the functional there is a constant non-zero multiplier T which may be set equal to unity. Finally, for neatness, the ˆ notation is suppressed and the 'p' functional for use in the steady state variational principle is written as

$$L_1 = \iint_D \left(p(\mathbf{v}, E) + \mathbf{Q}.\left(\mathbf{v} - \boldsymbol{\nabla}\phi\right)\right) dx\, dy + \int_{\Sigma_Q} C\phi\, d\Sigma + \int_{\Sigma_\phi} (\phi - f)\, \mathbf{Q}.\mathbf{n}\, d\Sigma, \tag{3.90}$$

where $L_1 = L_1(\mathbf{Q}, \mathbf{v}, \phi)$ and E is the known function $E = \tilde{E} + gh$, for consistency with conservation of momentum (2.42).

By the same process steady state forms of (3.69), (3.78) and (3.79) can be generated, and using the method by which (3.90) was deduced from (3.89) the steady state 'r', 'P' and 'R' functionals may be written

$$\begin{aligned} L_2 &= \iint_D \left(r(\mathbf{Q}, d) + Ed + \phi\boldsymbol{\nabla}.\mathbf{Q}\right) dx\, dy \\ &\qquad + \int_{\Sigma_Q} \phi\,(C - \mathbf{Q}.\mathbf{n})\, d\Sigma - \int_{\Sigma_\phi} f\mathbf{Q}.\mathbf{n}\, d\Sigma, & (3.91) \\ L_3 &= \iint_D \left(P(\mathbf{Q}, E) + \phi\boldsymbol{\nabla}.\mathbf{Q}\right) dx\, dy \\ &\qquad + \int_{\Sigma_Q} \phi\,(C - \mathbf{Q}.\mathbf{n})\, d\Sigma - \int_{\Sigma_\phi} f\mathbf{Q}.\mathbf{n}\, d\Sigma, & (3.92) \\ L_4 &= \iint_D \left(-R(\mathbf{v}, d) + \mathbf{Q}.\mathbf{v} + Ed + \phi\boldsymbol{\nabla}.\mathbf{Q}\right) dx\, dy \\ &\qquad + \int_{\Sigma_Q} \phi\,(C - \mathbf{Q}.\mathbf{n})\, d\Sigma - \int_{\Sigma_\phi} f\mathbf{Q}.\mathbf{n}\, d\Sigma, & (3.93) \end{aligned}$$

where $L_2 = L_2(\mathbf{Q}, d, \phi)$, $L_3 = L_3(\mathbf{Q}, \phi)$ and $L_4 = L_4(\mathbf{Q}, d, \mathbf{v}, \phi)$.

The natural conditions of the steady state variational principles $\delta L_1 = 0$, $\delta L_2 = 0$, $\delta L_3 = 0$ and $\delta L_4 = 0$ are expected to include the shallow water equations of motion (2.43) and (2.48) and possibly (2.25) or (2.19). Equation (2.42) is satisfied exactly since the energy E is regarded as a known function $E = \tilde{E} + gh$, where $\tilde{E}$ is a given constant.

The natural conditions of $\delta L_1 = 0$, the 'p' principle, are

$$\left.\begin{aligned} p_{\mathbf{v}} + \mathbf{Q} &= \mathbf{0} \\ \mathbf{v} - \nabla\phi &= \mathbf{0} \\ \nabla.\mathbf{Q} &= 0 \end{aligned}\right\} \qquad \text{in } D,$$

$$\begin{aligned} C - \mathbf{Q}.\mathbf{n} &= 0 \qquad \text{on } \Sigma_Q, \\ f - \phi &= 0 \qquad \text{on } \Sigma_\phi, \end{aligned}$$

the first equation being

$$-\frac{\mathbf{v}}{g}\left(E - \frac{1}{2}\mathbf{v}.\mathbf{v}\right) + \mathbf{Q} = \mathbf{0} \qquad \text{in } D.$$

The natural conditions of $\delta L_2 = 0$, the 'r' principle, are

$$\left.\begin{aligned} r_{\mathbf{Q}} - \nabla\phi &= \mathbf{0} \\ r_d + E &= 0 \\ \nabla.\mathbf{Q} &= 0 \end{aligned}\right\} \qquad \text{in } D,$$

$$\begin{aligned} C - \mathbf{Q}.\mathbf{n} &= 0 \qquad \text{on } \Sigma_Q, \\ f - \phi &= 0 \qquad \text{on } \Sigma_\phi, \end{aligned}$$

the first two equations being

$$\left.\begin{aligned} \tfrac{\mathbf{Q}}{d} - \nabla\phi &= \mathbf{0} \\ -\tfrac{1}{2}\tfrac{\mathbf{Q}.\mathbf{Q}}{d^2} - gd + E &= 0 \end{aligned}\right\} \qquad \text{in } D.$$

The natural conditions of $\delta L_3 = 0$, the 'P' principle, are

$$\left.\begin{aligned} P_{\mathbf{Q}} - \nabla\phi &= \mathbf{0} \\ \nabla.\mathbf{Q} &= 0 \end{aligned}\right\} \qquad \text{in } D,$$

$$\begin{aligned} C - \mathbf{Q}.\mathbf{n} &= 0 \qquad \text{on } \Sigma_Q, \\ f - \phi &= 0 \qquad \text{on } \Sigma_\phi, \end{aligned}$$

the first equation being

$$\mathbf{v} - \boldsymbol{\nabla}\phi = \mathbf{0} \qquad \text{in } D,$$

where $\mathbf{v}$ is a function of $\mathbf{Q}$ and E using (2.25) and (2.19).

The natural conditions of $\delta L_4 = 0$, the 'R' principle, are

$$\left.\begin{aligned} -R_{\mathbf{v}} + \mathbf{Q} &= \mathbf{0} \\ -R_d + E &= 0 \\ \mathbf{v} - \boldsymbol{\nabla}\phi &= \mathbf{0} \\ \boldsymbol{\nabla}.\mathbf{Q} &= 0 \end{aligned}\right\} \qquad \text{in } D,$$

$$\begin{aligned} C - \mathbf{Q}.\mathbf{n} &= 0 \qquad \text{on } \Sigma_Q, \\ f - \phi &= 0 \qquad \text{on } \Sigma_\phi, \end{aligned}$$

the first two equations being

$$\left.\begin{aligned} -d\mathbf{v} + \mathbf{Q} &= \mathbf{0} \\ -gd - \tfrac{1}{2}\mathbf{v}.\mathbf{v} + E &= 0 \end{aligned}\right\} \qquad \text{in } D.$$

Thus the natural conditions of the variational principles for steady state motion, derived from the principles for unsteady motion, include the steady state equations in shallow water — (2.43) and (2.48). In order that the equations are expressed in the forms of (2.43) and (2.48) it is necessary to assume in the 'p' principle that $d = \frac{1}{g}(E - \frac{1}{2}\mathbf{v}.\mathbf{v})$, in the 'r' principle that $\mathbf{v} = \frac{\mathbf{Q}}{d}$ and in the 'P' principle that $E = gd + \frac{1}{2}\mathbf{v}.\mathbf{v}$ and $\mathbf{Q} = d\mathbf{v}$.

Incidentally the same 'p' and 'r' functionals (3.90) and (3.91) can be derived from the 'pressure' and 'Hamilton' free surface functionals by a different route. Instead of applying the shallow water approximation and then considering steady flow the assumption of steady state conditions can be made first. The method is dependent on the addition of appropriate boundary terms to the free surface functionals for unsteady flow in the same way that boundary terms were added to the functionals for unsteady flow in shallow water in Section 3.5.

3.6.2 Constrained and Reciprocal Principles

The variational principles for steady motion can be constrained in the same way as the ones for unsteady motion were in Section 3.5. The variational principles for time-dependent motion have three natural conditions which can be used as constraints — singly or in pairs — conservation of mass, conservation of momentum and the irrotationality condition. For the variational principles for steady flow there are just two — conservation of mass and the irrotationality condition — since conservation of momentum is satisfied implicitly.

Reciprocal 'p' and 'P' Principles

Consider the integrands of the functionals (3.90)–(3.93). In Section 3.5 emphasis was placed on the structure of the integrands of the 'p' and 'r' functionals — they were expressed as a function of $(\mathbf{Q}, d)$ or $(\mathbf{v}, E)$ plus a multiple of a conservation law or the irrotationality condition. For steady flows the 'p' and 'P' functionals exhibit a similar property, that is, the integrands can be expressed as functions of $\mathbf{Q}$ or $\mathbf{v}$ plus a multiple of conservation of mass or the irrotationality condition, since E is a known function. Thus the 'p' principle will be constrained by irrotationality and the 'P' principle by conservation of mass.

Let $\Sigma_Q = \Sigma$. Then the 'p' principle constrained by irrotationality is a functional of ϕ alone and is given by

$$\delta \left\{ \iint_D p(\boldsymbol{\nabla}\phi, E)\, dx\, dy + \int_\Sigma C\phi\, d\Sigma \right\} = 0, \tag{3.94}$$

with the natural conditions

$$\begin{aligned} \boldsymbol{\nabla}.\left(\frac{1}{g}\left(E-\frac{1}{2}\boldsymbol{\nabla}\phi.\boldsymbol{\nabla}\phi\right)\boldsymbol{\nabla}\phi\right) &= 0 \qquad \text{in } D, \\ \frac{1}{g}\left(E-\frac{1}{2}\boldsymbol{\nabla}\phi.\boldsymbol{\nabla}\phi\right)\boldsymbol{\nabla}\phi.\mathbf{n}-C &= 0 \qquad \text{on } \Sigma, \end{aligned}$$

the first of which is conservation of mass in D.

Let $\Sigma_Q = \Sigma$ and $\mathbf{Q}.\mathbf{n} = C$ on Σ. Then the constrained 'P' principle is a functional of $\mathbf{Q}$ alone and is given by

$$\delta\left\{\iint_D P(\mathbf{Q}, E)\,dx\,dy\right\} = 0, \tag{3.95}$$

where $\mathbf{Q}$ is constrained by $\boldsymbol{\nabla}.\mathbf{Q} = 0$ in D and by $\mathbf{Q}.\mathbf{n} = C$ on Σ. The variational principle (3.95) has as its only natural condition the irrotationality condition.

Thus the 'p' and 'P' steady principles display the same relationship as the 'p' and 'r' principles for unsteady flow (3.82) and (3.85) — they are both functionals of one variable and are reciprocal in the sense defined earlier.

The particular relationship between the 'p' and 'r' principles (3.82) and (3.85) for unsteady flow has not survived the transition to principles for steady flow. The 'p' and 'r' functionals (3.90) and (3.91) cannot be constrained so that they each depend on just one function and have reciprocal constraints and natural conditions. The relationship of the 'p' and 'r' principles in unsteady motion is a result of the integrands being expressible in the form

function of $(\mathbf{Q}, d)$ or $(\mathbf{v}, E)$ + multiplier $\times$ conservation law

and once the variables are constrained to satisfy the relevant conservation law and, in the case of the 'p' principle, irrotationality, the functions of $(\mathbf{Q}, d)$ or $(\mathbf{v}, E)$ can each be written in terms of one variable. In the steady motion functional

(3.90) the pressure function p is still expressed as a function of $\mathbf{v}$ and E but, since E is a known function and no longer a variable of the problem, p is in fact a function of $\mathbf{v}$ alone. The flow stress P is also a function of one variable so that the constrained 'p' and 'P' principles for steady motion exhibit the same relationship as the constrained 'p' and 'r' principles for unsteady motion in terms of being reciprocal and depending on just one variable.

For the case of constant equilibrium depth h, where the energy E is a constant, the constrained 'p' and 'P' principles (3.94) and (3.95) are examples of Bateman's functions (Bateman (1929)), using the gas dynamics analogy.

Reciprocal 'r' and 'R' Principles

The function r depends on $\mathbf{Q}$ and d and cannot be written as a function of one variable by requiring the irrotationality condition or conservation of mass to be satisfied. The function R also depends on two variables and cannot be reduced to a function of one variable. However the 'r' and 'R' principles for steady motion can be constrained to give reciprocal principles.

Let $\Sigma_Q = \Sigma$. Then constraining the 'R' principle to satisfy the irrotationality condition gives

$$\delta\left\{\iint_D (-R(\boldsymbol{\nabla}\phi, d) + Ed)\, dx\, dy + \int_\Sigma C\phi\, d\Sigma\right\} = 0 \tag{3.96}$$

which depends on ϕ and d. The variational principle (3.96) has natural conditions

$$\left.\begin{aligned} -R_d + E &= 0 \\ \boldsymbol{\nabla}.\left(R_{\boldsymbol{\nabla}\phi}\right) &= 0 \end{aligned}\right\} \qquad \text{in } D,$$

$$R_{\boldsymbol{\nabla}\phi}.\mathbf{n} - C = 0 \qquad \text{on } \Sigma,$$

the second of which is conservation of mass in the form

$$\boldsymbol{\nabla}.\left(d\boldsymbol{\nabla}\phi\right)=0.$$

Let $\Sigma_Q = \Sigma$ and $\mathbf{Q}.\mathbf{n} = C$ on Σ. Then constraining the 'r' principle to satisfy conservation of mass gives

$$\delta\left\{\iint_D \left(r(\mathbf{Q},d)+Ed\right)dx\,dy\right\}=0, \tag{3.97}$$

where $\boldsymbol{\nabla}.\mathbf{Q}=0$, which has as natural conditions

$$r_d+E=0$$

and the irrotationality condition in D.

The 'r' and 'R' principles are reciprocal since the constraint of one principle is a natural condition of the other. Unlike the constrained 'p' and 'P' principles though, the 'r' and 'R' principles are functionals of two variables which yield a second natural condition for each.

3.7 Quasi One-dimensional Shallow Water Flows

In the same way that variational principles for time-dependent shallow water flows were derived from principles for free surface flows in Section 3.2, by making the shallow water approximation, the quasi one-dimensional approximation can be applied to (3.68), (3.69), (3.78), (3.79) and (3.90)–(3.93) to give functionals whose corresponding variational principles have as their natural conditions the equations of unsteady and steady quasi one-dimensional motion.

The domain of the problem is the channel

$$D = \left\{ (x,y) : x \in [x_e, x_o]; y \in \left[-\frac{B(x)}{2}, \frac{B(x)}{2} \right] \right\},$$

as in Section 2.3.

3.7.1 Time-dependent Flows

Consider the functionals (3.68), (3.69), (3.78) and (3.79) for time-dependent shallow water flows. Corresponding functionals for quasi one-dimensional flows may be derived from these by assuming that all of the flow variables are independent of y, that is, by substituting $E = E(x,t)$, $Q = Q(x,t)$, $d = d(x,t)$, $v = v(x,t)$ and $\phi = \phi(x,t)$ for their two-dimensional counterparts. Then, after replacing the operators by their one-dimensional versions, that is, replacing $\boldsymbol{\nabla}.\mathbf{Q}$ by $\frac{1}{B}(BQ)_x$ in (3.69) and (3.79) and $\boldsymbol{\nabla}\phi$ by ϕ_x in (3.68) and (3.78), the integrals with respect to y can be evaluated.

The boundary functions must also be independent of the y coordinate. In the two-dimensional functionals $C = C(x,y,t)$ and $f = f(x,y,t)$ are defined on Σ_Q and Σ_ϕ respectively, where $\Sigma = \Sigma_Q + \Sigma_\phi$ is the boundary of D. In the natural conditions of the corresponding variational principles the function C is identified as the normal component of mass flow on Σ_Q, thus C must be zero on any portion of Σ_Q which lies on the channel sides across which there is no flow. Therefore the boundary conditions are much simplified by letting Σ_Q include all parts of the boundary across which there is no flow, in fact we take $\Sigma_Q = \Sigma$. The boundary function C must be of the form $C(x_e,t) = C_e(t)$ at inlet, $C(x_o,t) = C_o(t)$ at outlet and $C(x,t) = 0$ elsewhere.

For the boundary terms evaluated at t_1 and t_2 in the functionals (3.68), (3.69), (3.78) and (3.79) the boundary functions are $g_i = g_i(x, y)$ and $h_i = h_i(x, y)$ for $i = 1, 2$ which, in the natural conditions of the variational principles, yield boundary conditions for d and ϕ respectively at t_1 and t_2. Let $D = D_d$ so that in the one-dimensional case only the boundary function for d, given by $g_i = g_i(x)$ in D for $i = 1, 2$, exists.

Making the above substitutions into (3.68), (3.69), (3.78) and (3.79) and integrating with respect to y yields the one-dimensional functionals $K_1(E, Q, d, v, \phi)$, $K_2(Q, d, \phi)$, $K_3(E, Q, d, \phi)$ and $K_4(Q, d, v, \phi)$ given by

$$\begin{aligned} K_1 = & \int_{t_1}^{t_2} \int_{x_e}^{x_o} \left(p(v, E) - d\left(\phi_t + E - gh\right) + Q\left(v - \phi_x\right)\right) B\, dx\, dt \\ & + \int_{t_1}^{t_2} \Big(C_o \left.(B\phi)\right|_{x_o} - C_e \left.(B\phi)\right|_{x_e}\Big)\, dt + \int_{x_e}^{x_o} \Big(\left.\phi\right|_{t_2} g_2 - \left.\phi\right|_{t_1} g_1\Big) B\, dx, \end{aligned} \quad (3.98)$$

$$\begin{aligned} K_2 = & \int_{t_1}^{t_2} \int_{x_e}^{x_o} \left(r(Q, d) + gdh + \phi\left(d_t + \frac{1}{B}\left(BQ\right)_x\right)\right) B\, dx\, dt \\ & + \int_{t_1}^{t_2} \Big(\left.\left(B\phi\left(C_o - Q\right)\right)\right|_{x_o} - \left.\left(B\phi\left(C_e - Q\right)\right)\right|_{x_e}\Big)\, dt \\ & - \int_{x_e}^{x_o} \Big(\left.\left(\phi\left(d - g_2\right)\right)\right|_{t_2} - \left.\left(\phi\left(d - g_1\right)\right)\right|_{t_1}\Big) B\, dx, \end{aligned} \quad (3.99)$$

$$\begin{aligned} K_3 = & \int_{t_1}^{t_2} \int_{x_e}^{x_o} \left(P(Q, E) - Q\phi_x - d\left(\phi_t + E - gh\right)\right) B\, dx\, dt \\ & + \int_{t_1}^{t_2} \Big(C_o \left.(B\phi)\right|_{x_o} - C_e \left.(B\phi)\right|_{x_e}\Big)\, dt + \int_{x_e}^{x_o} \Big(\left.\phi\right|_{t_2} g_2 - \left.\phi\right|_{t_1} g_1\Big) B\, dx, \end{aligned} \quad (3.100)$$

$$\begin{aligned} K_4 = & \int_{t_1}^{t_2} \int_{x_e}^{x_o} \left(-R(v, d) + gdh + Qv + \phi\left(d_t + \frac{1}{B}\left(BQ\right)_x\right)\right) B\, dx\, dt \\ & + \int_{t_1}^{t_2} \Big(\left.\left(B\phi\left(C_o - Q\right)\right)\right|_{x_o} - \left.\left(B\phi\left(C_e - Q\right)\right)\right|_{x_e}\Big)\, dt \\ & - \int_{x_e}^{x_o} \Big(\left.\left(\phi\left(d - g_2\right)\right)\right|_{t_2} - \left.\left(\phi\left(d - g_1\right)\right)\right|_{t_1}\Big) B\, dx. \end{aligned} \quad (3.101)$$

The functions $p(v, E)$, $r(Q, d)$, $P(Q, E)$ and $R(v, d)$ are the one-dimensional counterparts of $p(\mathbf{v}, E)$, $r(\mathbf{Q}, d)$, $P(\mathbf{Q}, E)$ and $R(\mathbf{v}, d)$ (defined by (3.27), (3.29), (3.75) and (3.72)), namely,

$$p(v, E) = \frac{1}{2g}\left(E - \frac{1}{2}v^2\right)^2, \quad (3.102)$$

$$r(Q,d) = \frac{1}{2}\frac{Q^2}{d} - \frac{1}{2}gd^2, \tag{3.103}$$

$$R(v,d) = \frac{1}{2}gd^2 + \frac{1}{2}v^2 d \tag{3.104}$$

and $P(Q,E)$ is defined by eliminating v and d from

$$P = \frac{1}{2}gd^2 + dv^2\,,\ Q = dv\,,\ E = gd + \frac{1}{2}v^2. \tag{3.105}$$

The natural conditions of $\delta K_1 = 0$, $\delta K_2 = 0$, $\delta K_3 = 0$ and $\delta K_4 = 0$ may be deduced from the general formulae (3.32)–(3.34) and are as follows.

For the 'p' principle $\delta K_1 = 0$,

$$\left.\begin{aligned} p_v + Q &= 0 \\ P_E - d &= 0 \\ d_t + \tfrac{1}{B}(BQ)_x &= 0 \\ \phi_t + E - gh &= 0 \\ v - \phi_x &= 0 \end{aligned}\right\} \quad \text{in } (x_e, x_o) \text{ for } t \in (t_1, t_2),$$

$$\begin{aligned} C_o - Q|_{x_o} &= 0 \quad \text{for } t \in (t_1,t_2), \\ C_e - Q|_{x_e} &= 0 \quad \text{for } t \in (t_1,t_2), \\ g_i - d|_{t_i} &= 0 \quad \text{in } (x_e,x_o) \text{ for } i = 1,2. \end{aligned}$$

For the 'r' principle $\delta K_2 = 0$,

$$\left.\begin{aligned} r_d - \phi_t + gh &= 0 \\ r_Q - \phi_x &= 0 \\ d_t + \tfrac{1}{B}(BQ)_x &= 0 \end{aligned}\right\} \quad \text{in } (x_e, x_o) \text{ for } t \in (t_1, t_2),$$

$$\begin{aligned} C_o - Q|_{x_o} &= 0 \quad \text{for } t \in (t_1,t_2), \\ C_e - Q|_{x_e} &= 0 \quad \text{for } t \in (t_1,t_2), \\ g_i - d|_{t_i} &= 0 \quad \text{in } (x_e,x_o) \text{ for } i = 1,2. \end{aligned}$$

For the 'P' principle $\delta K_3 = 0$,

$$\left.\begin{aligned} P_Q - \phi_x &= 0 \\ P_E - d &= 0 \\ \phi_t + E - gh &= 0 \\ d_t + \tfrac{1}{B}\left(BQ\right)_x &= 0 \end{aligned}\right\} \quad \text{in } (x_e, x_o) \text{ for } t \in (t_1, t_2),$$

$$\begin{aligned} C_o - Q|_{x_o} &= 0 \quad \text{for } t \in (t_1, t_2), \\ C_e - Q|_{x_e} &= 0 \quad \text{for } t \in (t_1, t_2), \\ g_i - d|_{t_i} &= 0 \quad \text{in } (x_e, x_o) \text{ for } i = 1, 2. \end{aligned}$$

For the 'R' principle $\delta K_4 = 0$,

$$\left.\begin{aligned} -R_v + Q &= 0 \\ -R_d - \phi_t + gh &= 0 \\ v - \phi_x &= 0 \\ d_t + \tfrac{1}{B}\left(BQ\right)_x &= 0 \end{aligned}\right\} \quad \text{in } (x_e, x_o) \text{ for } t \in (t_1, t_2),$$

$$\begin{aligned} C_o - Q|_{x_o} &= 0 \quad \text{for } t \in (t_1, t_2), \\ C_e - Q|_{x_e} &= 0 \quad \text{for } t \in (t_1, t_2), \\ g_i - d|_{t_i} &= 0 \quad \text{in } (x_e, x_o) \text{ for } i = 1, 2. \end{aligned}$$

Thus each set of natural conditions includes the equations of motion for time-dependent quasi one-dimensional shallow water flow. The conservation of mass equation (2.33) is explicit in each set. The conservation of momentum equation, in its integrated form, is explicit in the natural conditions of the 'p' and 'P' principles but for the 'r' principle the relations $Q = dv$ and $E = gd + \frac{1}{2}v^2$ are needed and for the 'R' principle the relation $E = gd + \frac{1}{2}v^2$ is required. The boundary conditions in each case are the same, that is, on the mass flow at inlet and outlet, for $t \in (t_1, t_2)$, and on the depth in (x_e, x_o) at $t = t_1,\ t_2$.

3.7.2 Constrained and Reciprocal Principles

The variational principles for unsteady quasi one-dimensional flows can be constrained in the same ways as the corresponding variational principles for two-dimensional flows. In particular, constrained 'p' and 'r' principles which are reciprocal to one another and constrained 'P' and 'R' principles which are reciprocal to one another can be derived.

Reciprocal 'p' and 'r' Principles

The 'p' principle corresponding to the functional (3.98) can be reduced to a principle which depends on just one variable by constraining the variations to satisfy the conservation of momentum equation, in its integrated form, and the one-dimensional version of the irrotationality condition, that is, by specifying

$$\left.\begin{aligned} E &= -\phi_t + gh \\ v &= \phi_x \end{aligned}\right\}. \tag{3.106}$$

The constrained 'p' principle is given by

$$\begin{aligned} \delta K_1^c = \delta \Big\{ &\int_{t_1}^{t_2} \int_{x_e}^{x_o} \hat{p}(\phi) B \, dx \, dt + \int_{t_1}^{t_2} \Big(C_o \left. (B\phi) \right|_{x_o} - C_e \left. (B\phi) \right|_{x_e} \Big) \, dt \\ &+ \int_{x_e}^{x_o} \Big(\left. \phi \right|_{t_2} g_2 - \left. \phi \right|_{t_1} g_1 \Big) \, B \, dx \Big\} = 0, \end{aligned} \tag{3.107}$$

where $K_1^c = K_1^c(\phi)$ and

$$\hat{p}(\phi) = \frac{1}{2g} \left(\phi_t - gh + \frac{1}{2} \phi_x^2 \right)^2 .$$

The natural conditions of $\delta K_1^c = 0$ are

$$\left(-\frac{1}{g}\left(\phi_t - gh + \frac{1}{2}\phi_x^2\right)\right)_t + \frac{1}{B}\left(-\frac{1}{g}\left(\phi_t - gh + \frac{1}{2}\phi_x^2\right)\phi_x B\right)_x = 0 \quad \text{in } (x_e, x_o) \text{ for } t \in (t_1, t_2),$$

$$\begin{aligned}
C_o + \left(\frac{1}{g}\left(\phi_t - gh + \frac{1}{2}\phi_x^2\right)\phi_x\right)\Bigg|_{x_o} &= 0 \quad \text{for } t \in (t_1, t_2),\\
C_e + \left(\frac{1}{g}\left(\phi_t - gh + \frac{1}{2}\phi_x^2\right)\phi_x\right)\Bigg|_{x_e} &= 0 \quad \text{for } t \in (t_1, t_2),\\
g_i + \frac{1}{g}\left(\phi_t - gh + \frac{1}{2}\phi_x^2\right)\Bigg|_{t_i} &= 0 \quad \text{in } (x_e, x_o) \text{ for } i = 1, 2,
\end{aligned}$$

which correspond to conservation of mass in the domain and boundary conditions on ϕ.

The 'r' principle, corresponding to the functional (3.99), may be constrained to satisfy conservation of mass by specifying

$$\left.\begin{aligned} d &= \tfrac{\psi_x}{B} \\ Q &= -\tfrac{\psi_t}{B} \end{aligned}\right\}, \tag{3.108}$$

for some function $\psi(x,t)$. The constrained 'r' principle depends on ψ and ϕ and is given by

$$\begin{aligned}
\delta K_2^c = \delta \Bigg\{ &\int_{t_1}^{t_2}\int_{x_e}^{x_o}\left(\hat{r}(\psi) + g\frac{\psi_x}{B}h\right) B\,dx\,dt \\
&+ \int_{t_1}^{t_2}\left(\left(B\phi\left(C_o + \frac{\psi_t}{B}\right)\right)\Bigg|_{x_o} - \left(B\phi\left(C_e + \frac{\psi_t}{B}\right)\right)\Bigg|_{x_e}\right)dt \\
&- \int_{x_e}^{x_o}\left(\left(\phi\left(\frac{\psi_x}{B} - g_2\right)\right)\Bigg|_{t_2} - \left(\phi\left(\frac{\psi_x}{B} - g_1\right)\right)\Bigg|_{t_1}\right)B\,dx\Bigg\} = 0,
\end{aligned} \tag{3.109}$$

where

$$\hat{r}(\psi) = \frac{1}{2}\left(\frac{\psi_t^2}{\psi_x B} - g\frac{\psi_x^2}{B^2}\right).$$

The natural conditions of (3.109) are given by

$$-\Psi_t + \left(\frac{1}{2}\Psi^2 + g\frac{\psi_x}{B} - gh\right)_x = 0 \quad \text{in } (x_e, x_o) \text{ for } t \in (t_1, t_2),$$

$$
\begin{aligned}
C_o + \left.\frac{\psi_t}{B}\right|_{x_o} &= 0 \quad \text{for } t \in (t_1, t_2), \\
C_e + \left.\frac{\psi_t}{B}\right|_{x_e} &= 0 \quad \text{for } t \in (t_1, t_2), \\
\left.\left(-\phi_t - \frac{1}{2}\Psi^2 - g\frac{\psi_x}{B} + gh\right)\right|_{x_o} &= 0 \quad \text{for } t \in (t_1, t_2), \\
\left.\left(-\phi_t - \frac{1}{2}\Psi^2 - g\frac{\psi_x}{B} + gh\right)\right|_{x_e} &= 0 \quad \text{for } t \in (t_1, t_2), \\
g_i - \left.\frac{\psi_x}{B}\right|_{t_i} &= 0 \quad \text{in } (x_e, x_o) \text{ for } i = 1, 2, \\
\left.(\phi_x + \Psi)\right|_{t_i} &= 0 \quad \text{in } (x_e, x_o) \text{ for } i = 1, 2,
\end{aligned}
$$

where $\Psi = \frac{\psi_t}{\psi_x}$. The first of these conditions is the conservation of momentum equation in the domain. The equation $v = \phi_x$ is not derived but is essentially redundant anyway in quasi one-dimensional flow.

The variational principles (3.107) and (3.109) are reciprocal in that the natural condition in D of (3.107) is the constraint of (3.109) and the natural condition in D of (3.109) is the constraint of (3.107).

Reciprocal 'P' and 'R' Principles

The 'P' principle constrained to satisfy conservation of momentum by specifying $(3.106)_1$ is

$$
\delta K_3^c = \delta \left\{ \int_{t_1}^{t_2} \int_{x_e}^{x_o} \left(P(Q, -\phi_t + gh) - Q\phi_x\right) B\, dx\, dt \right.
$$
$$
\left. + \int_{t_1}^{t_2} \left(C_o \left.(B\phi)\right|_{x_o} - C_e \left.(B\phi)\right|_{x_e}\right) dt + \int_{x_e}^{x_o} \left(\left.\phi\right|_{t_2} g_2 - \left.\phi\right|_{t_1} g_1\right) B\, dx \right\} = 0, \quad (3.110)
$$

which has the natural conditions

$$
\left.
\begin{aligned}
P_Q - \phi_x &= 0 \\
(P_{-\phi_t + gh})_t + \tfrac{1}{B}(BQ)_x &= 0
\end{aligned}
\right\} \quad \text{in } (x_e, x_o) \text{ for } t \in (t_1, t_2),
$$

$$
C_o - \left.Q\right|_{x_o} = 0 \quad \text{for } t \in (t_1, t_2),
$$

$$C_e - Q|_{x_e} = 0 \quad \text{for } t \in (t_1, t_2),$$

$$g_i - P_{-\phi_t + gh}|_{t_i} = 0 \quad \text{in } (x_e, x_o) \text{ for } i = 1, 2,$$

the second of which is the equation of conservation of mass.

The 'R' principle constrained to satisfy conservation of mass by specifying (3.108) is

$$\begin{aligned} \delta K_4^c = \delta \Bigg\{ & \int_{t_1}^{t_2} \left(-R(v, \frac{\psi_x}{B}) - \frac{\psi_t}{B} v + g \frac{\psi_x}{B} h \right) B \, dx \, dt \\ & + \int_{t_1}^{t_2} \left(\left(B\phi \left(C_o + \frac{\psi_t}{B} \right) \right) \Big|_{x_o} - \left(B\phi \left(C_e + \frac{\psi_t}{B} \right) \right) \Big|_{x_e} \right) dt \\ & - \int_{x_e}^{x_o} \left(\left(\phi \left(\frac{\psi_x}{B} - g_2 \right) \right) \Big|_{t_2} - \left(\phi \left(\frac{\psi_x}{B} - g_1 \right) \right) \Big|_{t_1} \right) B \, dx \Bigg\} = 0, \quad (3.111) \end{aligned}$$

which has the natural conditions

$$\left. \begin{aligned} -R_v - \tfrac{\psi_t}{B} &= 0 \\ \left(R_{\frac{\psi_x}{B}} - gh \right)_x + v_t &= 0 \end{aligned} \right\} \quad \text{in } (x_e, x_o) \text{ for } t \in (t_1, t_2),$$

$$\begin{aligned} C_o + \frac{\psi_t}{B} \Big|_{x_o} &= 0 \quad \text{for } t \in (t_1, t_2), \\ C_e + \frac{\psi_t}{B} \Big|_{x_e} &= 0 \quad \text{for } t \in (t_1, t_2), \\ \left(-\phi_t - R_{\frac{\psi_x}{B}} + gh \right) \Big|_{x_o} &= 0 \quad \text{for } t \in (t_1, t_2), \\ \left(-\phi_t - R_{\frac{\psi_x}{B}} + gh \right) \Big|_{x_e} &= 0 \quad \text{for } t \in (t_1, t_2), \\ g_i - \frac{\psi_x}{B} \Big|_{t_i} &= 0 \quad \text{in } (x_e, x_o) \text{ for } i = 1, 2, \\ (\phi_x - v)|_{t_i} &= 0 \quad \text{in } (x_e, x_o) \text{ for } i = 1, 2, \end{aligned}$$

the second of which is the equation of conservation of momentum.

Thus the constraint on (3.110) is one of the natural conditions of (3.111) and the constraint on (3.111) is one of the natural conditions of (3.110), that is, these constrained 'P' and 'R' principles are reciprocal in the sense defined earlier.

3.7.3 Time-independent Flows

Functionals for time-independent quasi one-dimensional flows can be derived either from the functionals for time-dependent one-dimensional flows (3.98)–(3.101) or from the functionals for steady two-dimensional flows (3.90)–(3.93). In both cases the procedure is to assume that the physical flow variables are functions of x alone and in the time-dependent one-dimensional case to deduce the dependence of ϕ on x and t, as was done for the two-dimensional case. The integrals with respect to t in (3.98)–(3.101) and with respect to y in (3.90)–(3.93) can then be evaluated.

The functionals for time-independent quasi one-dimensional shallow water flows, derived in either of these ways, may be written as

$$M_1 = \int_{x_e}^{x_o} (p(v,E) + Q\,(v - \phi'))\, B\, dx + C B_e\,(\phi(x_o) - \phi(x_e))\,, \quad (3.112)$$

$$M_2 = \int_{x_e}^{x_o} (r(Q,d) + Ed - Q\phi')\, B\, dx + C B_e\,(\phi(x_o) - \phi(x_e))\,, \quad (3.113)$$

$$M_3 = \int_{x_e}^{x_o} (P(Q,E) - Q\phi')\, B\, dx + C B_e\,(\phi(x_o) - \phi(x_e))\,, \quad (3.114)$$

$$M_4 = \int_{x_e}^{x_o} (-R(v,d) + Q(v - \phi') + Ed)\, B\, dx + C B_e\,(\phi(x_o) - \phi(x_e))\,, \quad (3.115)$$

where $M_1 = M_1(v,Q,\phi)$, $M_2 = M_2(d,Q,\phi)$, $M_3 = M_3(Q,\phi)$ and $M_4 = M_4(d,v,Q,\phi)$.

The boundary function $C = C(x,y,t)$ is now defined to have the values

$$\begin{aligned} C(x,y,t) &= C && \text{at } x = x_e, \\ C(x,y,t) &= \tfrac{CB_e}{B_o} && \text{at } x = x_o, \\ \text{and} \qquad C(x,y,t) &= 0 && \text{otherwise,} \end{aligned}$$

for consistency with conservation of mass, where $B_e = B(x_e)$ and $B_o = B(x_o)$. The energy E is the known function $E = \tilde{E} + gh$, where $\tilde{E}$ is a given constant, in order to satisfy conservation of momentum.

The natural conditions of $\delta M_1 = 0$, $\delta M_2 = 0$, $\delta M_3 = 0$ and $\delta M_4 = 0$ may be deduced from the general formulae (3.36) and (3.37) and are as follows.

For the 'p' principle $\delta M_1 = 0$,

$$\left.\begin{aligned} p_v + Q &= 0 \\ v - \phi' &= 0 \\ (BQ)' &= 0 \end{aligned}\right\} \quad \text{in } (x_e, x_o),$$

$$\begin{aligned} C - Q(x_e) &= 0, \\ CB_e - Q(x_o)B_o &= 0. \end{aligned}$$

For the 'r' principle $\delta M_2 = 0$,

$$\left.\begin{aligned} r_Q - \phi' &= 0 \\ r_d + E &= 0 \\ (BQ)' &= 0 \end{aligned}\right\} \quad \text{in } (x_e, x_o),$$

$$\begin{aligned} C - Q(x_e) &= 0, \\ CB_e - Q(x_o)B_o &= 0. \end{aligned}$$

For the 'P' principle $\delta M_3 = 0$,

$$\left.\begin{aligned} P_Q - \phi' &= 0 \\ (BQ)' &= 0 \end{aligned}\right\} \quad \text{in } (x_e, x_o),$$

$$\begin{aligned} C - Q(x_e) &= 0, \\ CB_e - Q(x_o)B_o &= 0. \end{aligned}$$

For the 'R' principle $\delta M_4 = 0$,

$$\left.\begin{aligned} -R_v + Q &= 0 \\ -R_d + E &= 0 \\ v - \phi' &= 0 \\ (BQ)' &= 0 \end{aligned}\right\} \quad \text{in } (x_e, x_o),$$

$$\begin{aligned} C - Q(x_e) &= 0, \\ CB_e - Q(x_o)B_o &= 0. \end{aligned}$$

Thus each set of natural conditions contains the conservation of mass equation, the formula $v = \phi'$, expressed in different variables in the cases of the 'p' and 'r' principles, and also the boundary conditions $Q(x_e) = C$ and $Q(x_o) = \frac{CB_e}{B_o}$, that is, the equations of motion, as required. Conservation of momentum is implicit from using $E = \tilde{E} + gh$.

3.7.4 Constrained and Reciprocal Principles

The variational principles for steady quasi one-dimensional flows can be constrained in the same way as the two-dimensional versions.

Constrained 'p' and 'P' Principles

The 'p' principle based on the functional (3.112) can be constrained to depend on only one variable by substituting $v = \phi'$ into the integrand. The constrained 'p' principle is given by

$$\delta M_1^c = \delta \left\{ \int_{x_e}^{x_o} p(\phi', E) B \, dx + CB_e \left(\phi(x_o) - \phi(x_e) \right) \right\} = 0, \qquad (3.116)$$

where $M_1^c = M_1^c(\phi)$, which has the natural conditions

$$\begin{aligned} \left(\frac{1}{g}\left(E - \frac{1}{2}\phi'^2\right)\phi' B\right)' &= 0 \quad \text{in } (x_e, x_o), \\ C - \left(\frac{1}{g}\left(E - \frac{1}{2}\phi'^2\right)\phi'\right)\Bigg|_{x_e} &= 0, \\ CB_e - \left(\frac{1}{g}\left(E - \frac{1}{2}\phi'^2\right)\phi' B\right)\Bigg|_{x_o} &= 0, \end{aligned}$$

the first of which is the conservation of mass equation.

The 'P' principle may be constrained to satisfy conservation of mass by specifying $QB = CB_e$ in $[x_e, x_o]$. The constrained 'P' principle is given by

$$\delta M_3^c = \delta\left\{\int_{x_e}^{x_o} P(Q, E) B \, dx\right\} = 0,$$

where $Q = \frac{CB_e}{B}$, which has no natural conditions since both Q and E are known functions.

Constrained 'r' and 'R' Principles

Following the derivation of the two-dimensional reciprocal principles, the 'r' principle can be constrained to satisfy conservation of mass by specifying $Q = \frac{CB_e}{B}$ and the 'R' principle can be constrained by substituting $v = \phi'$ into the integrand of the functional. The constrained principles are given by

$$\delta M_2^c = \delta\left\{\int_{x_e}^{x_o} (r(Q, d) + Ed)\, B \, dx\right\} = 0, \tag{3.117}$$

where $Q = \frac{CB_e}{B}$, which has the natural condition

$$r_d + E = 0 \quad \text{in } (x_e, x_o),$$

defining E as a function of Q and d, and

$$\delta M_4^c = \delta\left\{\int_{x_e}^{x_o} (-R(\phi', d) + Ed)\, B \, dx + CB_e\left(\phi(x_o) - \phi(x_e)\right)\right\} = 0,$$

which has the natural conditions

$$\left.\begin{aligned} (BR_{\phi'})' = 0 \\ -R_d + E = 0 \end{aligned}\right\} \quad \text{in } (x_e, x_o),$$

$$\begin{aligned} CB_e - (BR_{\phi'})\big|_{x_e} &= 0, \\ CB_e - (BR_{\phi'})\big|_{x_o} &= 0, \end{aligned}$$

that is, conservation of mass and the definition of E as a function of ϕ and d in the domain and boundary conditions on the mass flow.

The reciprocity of the constrained variational principles that occurred in two-dimensional flows and time-dependent one-dimensional flows has not survived to the steady one-dimensional case since there is essentially only the one equation (the conservation of mass equation) which can be used as a constraint.

3.8 Discontinuous Flows

The variational principles considered so far are only valid for continuous shallow water flows since, in deriving the natural conditions using integration by parts and the divergence theorem, the variables have been assumed to be differentiable. In this section variational principles for time-independent discontinuous flows in one and two dimensions are derived.

In the variational principles of Sections 3.6 and 3.7 for steady state shallow water flows the conservation of momentum equation is satisfied implicitly by specifying $E = \tilde{E} + gh$ in D, where $\tilde{E}$ is a constant and E is the energy defined by either (2.35) or (2.19), depending on whether the flow being considered is one-dimensional or two-dimensional. For discontinuous flows there is a jump in the value of E on crossing the discontinuity and this property is used in generating the variational principles for discontinuous flows.

Let the domain be of the form

$$D = \left\{ (x, y) : x \in [x_e, x_o]; y \in \left[-\frac{B(x)}{2}, \frac{B(x)}{2} \right] \right\},$$

that is, a channel, where there is flow into the channel at $x = x_e$ and out of the channel at $x = x_o$.

Consider a discontinuous flow in D which has energy $E = E_e$ at inlet and $E = E_o$ at outlet, where E_e and E_o are constants such that $E_e > E_o$. Let the channel bed be horizontal so that the undisturbed fluid depth h is a constant for $x \in [x_e, x_o]$. Then, for a time-independent flow, $E = E_e$ everywhere on the inlet side of the discontinuity and $E = E_o$ everywhere on the outlet side. Substituting these values for the energy into the functionals (3.112)–(3.115) and (3.90)–(3.93) and allowing the variables to be discontinuous yields functionals whose corresponding variational principles have as natural conditions the equations of discontinuous motion in one and two dimensions.

3.8.1 Two-dimensional Flows

Let Σ_s be the line in D across which the flow is discontinuous. Assume that it divides D into the two regions D_e and D_o, where D_e borders the inlet boundary and D_o the outlet boundary.

The functionals for discontinuous flows in two dimensions, derived from (3.90)–(3.93), are

$$N_1 = \iint_{D_e} (p(\mathbf{v}, E_e) + \mathbf{Q}.(\mathbf{v} - \boldsymbol{\nabla}\phi))\, dx\, dy$$
$$+ \iint_{D_o} (p(\mathbf{v}, E_o) + \mathbf{Q}.(\mathbf{v} - \boldsymbol{\nabla}\phi))\, dx\, dy$$
$$+ \int_{\Sigma_e} C\phi\, d\Sigma + \int_{\Sigma_o} C\phi\, d\Sigma, \tag{3.118}$$
$$N_2 = \iint_{D_e} (r(\mathbf{Q}, d) + E_e d - \mathbf{Q}.\boldsymbol{\nabla}\phi)\, dx\, dy$$
$$+ \iint_{D_o} (r(\mathbf{Q}, d) + E_o d - \mathbf{Q}.\boldsymbol{\nabla}\phi)\, dx\, dy$$
$$+ \int_{\Sigma_e} C\phi\, d\Sigma + \int_{\Sigma_o} C\phi\, d\Sigma, \tag{3.119}$$
$$N_3 = \iint_{D_e} (P(\mathbf{Q}, E_e) - \mathbf{Q}.\boldsymbol{\nabla}\phi)\, dx\, dy$$
$$+ \iint_{D_o} (P(\mathbf{Q}, E_o) - \mathbf{Q}.\boldsymbol{\nabla}\phi)\, dx\, dy$$
$$+ \int_{\Sigma_e} C\phi\, d\Sigma + \int_{\Sigma_o} C\phi\, d\Sigma, \tag{3.120}$$
$$N_4 = \iint_{D_e} (-R(\mathbf{v}, d) + \mathbf{Q}.(\mathbf{v} - \boldsymbol{\nabla}\phi) + E_e d)\, dx\, dy$$
$$+ \iint_{D_o} (-R(\mathbf{v}, d) + \mathbf{Q}.(\mathbf{v} - \boldsymbol{\nabla}\phi) + E_o d)\, dx\, dy$$
$$+ \int_{\Sigma_e} C\phi\, d\Sigma + \int_{\Sigma_o} C\phi\, d\Sigma, \tag{3.121}$$

where $N_1 = N_1(\mathbf{v}, \mathbf{Q}, \phi, \Sigma_s)$, $N_2 = N_2(d, \mathbf{Q}, \phi, \Sigma_s)$, $N_3 = N_3(\mathbf{Q}, \phi, \Sigma_s)$ and $N_4 = N_4(d, \mathbf{v}, \mathbf{Q}, \phi, \Sigma_s)$. The section of the boundary Σ_Q in the functionals (3.90)–(3.93) is taken to be Σ, the whole boundary of D, and the boundary function C is set to zero on the parts of the boundary across which there is no flow; Σ_e is the inlet boundary and Σ_o is the outlet boundary.

The natural conditions of $\delta N_1 = 0$, $\delta N_2 = 0$, $\delta N_3 = 0$ and $\delta N_4 = 0$ may be deduced from the general formulae (3.60)–(3.67).

The natural conditions of the 'p' principle $\delta N_1 = 0$ are

$$\left.\begin{aligned} p_{\mathbf{v}} + \mathbf{Q} &= \mathbf{0} \\ \mathbf{v} - \boldsymbol{\nabla}\phi &= \mathbf{0} \\ \boldsymbol{\nabla}.\mathbf{Q} &= 0 \end{aligned}\right\} \quad \text{in } D_e \cup D_o,$$

$$
\begin{aligned}
C - \mathbf{Q}.\mathbf{n} &= 0 \quad \text{on } \Sigma_e \cup \Sigma_o, \\
\left[p + \mathbf{Q}.\mathbf{v} - \mathbf{Q}.\boldsymbol{\tau} \frac{\partial \phi}{\partial \tau} \right]_{\Sigma_s} &= 0, \\
\left[\mathbf{Q}.\mathbf{n} \frac{\partial \phi}{\partial \tau} \right]_{\Sigma_s} &= 0, \\
[\mathbf{Q}.\mathbf{n}]_{\Sigma_s} &= 0,
\end{aligned}
$$

where the brackets $[\cdot]_{\Sigma_s} = \cdot|_{\Sigma_{s+}} - \cdot|_{\Sigma_{s-}}$, $+$ denotes the downstream side of Σ_s and $-$ the upstream side. The first four of these conditions are the same as for the continuous case but are valid separately in D_e and D_o and on Σ_e and Σ_o. Using the $\mathbf{v} = \boldsymbol{\nabla}\phi$ natural condition the last three natural conditions can be rewritten as

$$
\begin{aligned}
[p + (\mathbf{Q}.\mathbf{n})(\mathbf{v}.\mathbf{n})]_{\Sigma_s} &= 0, \\
[\mathbf{v}.\boldsymbol{\tau}]_{\Sigma_s} &= 0, \\
[\mathbf{Q}.\mathbf{n}]_{\Sigma_s} &= 0,
\end{aligned}
$$

which are the required jump conditions (2.84)–(2.86) for discontinuous shallow water flows.

The natural conditions of the 'r' principle $\delta N_2 = 0$ are

$$
\left.
\begin{aligned}
r_{\mathbf{Q}} - \boldsymbol{\nabla}\phi = \mathbf{0} \\
r_d + E = 0 \\
\boldsymbol{\nabla}.\mathbf{Q} = 0
\end{aligned}
\right\} \quad \text{in } D_e \cup D_o,
$$

$$
\begin{aligned}
C - \mathbf{Q}.\mathbf{n} &= 0 \quad \text{on } \Sigma_e \cup \Sigma_o, \\
\left[r + Ed - \mathbf{Q}.\boldsymbol{\nabla}\phi + \mathbf{Q}.\mathbf{n} \frac{\partial \phi}{\partial n} \right]_{\Sigma_s} &= 0, \\
\left[\mathbf{Q}.\mathbf{n} \frac{\partial \phi}{\partial \tau} \right]_{\Sigma_s} &= 0, \\
[\mathbf{Q}.\mathbf{n}]_{\Sigma_s} &= 0.
\end{aligned}
$$

The natural conditions of the 'P' principle $\delta N_3 = 0$ are

$$\left.\begin{aligned} P_{\mathbf{Q}} - \boldsymbol{\nabla}\phi = \mathbf{0} \\ \boldsymbol{\nabla}.\mathbf{Q} = 0 \end{aligned}\right\} \quad \text{in } D_e \cup D_o,$$

$$\begin{aligned} C - \mathbf{Q}.\mathbf{n} &= 0 \quad \text{on } \Sigma_e \cup \Sigma_o, \\ \left[P - \mathbf{Q}.\boldsymbol{\nabla}\phi + \mathbf{Q}.\mathbf{n}\frac{\partial\phi}{\partial n}\right]_{\Sigma_s} &= 0, \\ \left[\mathbf{Q}.\mathbf{n}\frac{\partial\phi}{\partial\tau}\right]_{\Sigma_s} &= 0, \\ [\mathbf{Q}.\mathbf{n}]_{\Sigma_s} &= 0. \end{aligned}$$

The natural conditions of the 'R' principle $\delta N_4 = 0$ are

$$\left.\begin{aligned} -R_{\mathbf{v}} + \mathbf{Q} = \mathbf{0} \\ -R_d + E = 0 \\ \mathbf{v} - \boldsymbol{\nabla}\phi = \mathbf{0} \\ \boldsymbol{\nabla}.\mathbf{Q} = 0 \end{aligned}\right\} \quad \text{in } D_e \cup D_o,$$

$$\begin{aligned} C - \mathbf{Q}.\mathbf{n} &= 0 \quad \text{on } \Sigma_e \cup \Sigma_o, \\ \left[-R + \mathbf{Q}.\left(\mathbf{v} - \boldsymbol{\nabla}\phi\right) + Ed + \mathbf{Q}.\mathbf{n}\frac{\partial\phi}{\partial n}\right]_{\Sigma_s} &= 0, \\ \left[\mathbf{Q}.\mathbf{n}\frac{\partial\phi}{\partial\tau}\right]_{\Sigma_s} &= 0, \\ [\mathbf{Q}.\mathbf{n}]_{\Sigma_s} &= 0. \end{aligned}$$

Using the relationships between p and r, P and R, (3.74), (3.76) and (3.71), the last three natural conditions of $\delta N_2 = 0$, $\delta N_3 = 0$ and $\delta N_4 = 0$ can be seen to be the same as the last three natural conditions of $\delta N_1 = 0$. Thus the variational principles based on the functionals (3.118)–(3.121) have as their natural conditions the equations of shallow water motion, including the jump conditions at the discontinuity.

3.8.2 One-dimensional Flows

Let x_s be the position of the discontinuity in (x_e, x_o). Then the functionals for discontinuous flow in one dimension, derived from (3.112)–(3.115), are

$$\begin{aligned} S_1 = & \int_{x_e}^{x_s} \left(p(v, E_e) + Q\,(v - \phi')\right) B\,dx \\ & + \int_{x_s}^{x_o} \left(p(v, E_o) + Q\,(v - \phi')\right) B\,dx \\ & + CB_e\left(\phi(x_o) - \phi(x_e)\right), \end{aligned} \tag{3.122}$$

$$\begin{aligned} S_2 = & \int_{x_e}^{x_s} \left(r(Q, d) + E_e d - Q\phi'\right) B\,dx \\ & + \int_{x_s}^{x_o} \left(r(Q, d) + E_o d - Q\phi'\right) B\,dx \\ & + CB_e\left(\phi(x_o) - \phi(x_e)\right), \end{aligned} \tag{3.123}$$

$$\begin{aligned} S_3 = & \int_{x_e}^{x_s} \left(P(Q, E_e) - Q\phi'\right) B\,dx \\ & + \int_{x_s}^{x_o} \left(P(Q, E_o) - Q\phi'\right) B\,dx \\ & + CB_e\left(\phi(x_o) - \phi(x_e)\right), \end{aligned} \tag{3.124}$$

$$\begin{aligned} S_4 = & \int_{x_e}^{x_s} \left(-R(v, d) + Q\,(v - \phi') + E_e d\right) B\,dx \\ & + \int_{x_s}^{x_o} \left(-R(v, d) + Q\,(v - \phi') + E_o d\right) B\,dx \\ & + CB_e\left(\phi(x_o) - \phi(x_e)\right), \end{aligned} \tag{3.125}$$

where $S_1 = S_1(v, Q, \phi, x_s)$, $S_2 = S_2(d, Q, \phi, x_s)$, $S_3 = S_3(Q, \phi, x_s)$ and $S_4 = S_4(d, v, Q, \phi, x_s)$.

The natural conditions of $\delta S_1 = 0$, $\delta S_2 = 0$, $\delta S_3 = 0$ and $\delta S_4 = 0$ may be deduced using (3.54)–(3.57) and are as follows.

For the 'p' principle $\delta S_1 = 0$,

$$\left.\begin{aligned} p_v + Q &= 0 \\ v - \phi' &= 0 \\ (BQ)' &= 0 \end{aligned}\right\} \quad \text{in } (x_e, x_s) \cup (x_s, x_o),$$

$$
\begin{aligned}
C - Q(x_e) &= 0, \\
CB_e - Q(x_o)B_o &= 0, \\
[BQ]_{x_s} &= 0, \\
[(p + Qv)\,B]_{x_s} &= 0,
\end{aligned}
$$

where $[\cdot]_{x_s} = \cdot|_{x_s+} - \cdot|_{x_s-}$, the $+$ denotes the x_o side of x_s and the $-$ the x_e side; E is given the value E_e in $[x_e, x_s)$ and E_o in $(x_s, x_o]$.

For the 'r' principle $\delta S_2 = 0$,

$$
\left.
\begin{aligned}
r_Q - \phi' = 0 \\
r_d + E = 0 \\
(BQ)' = 0
\end{aligned}
\right\} \quad \text{in } (x_e, x_s) \cup (x_s, x_o),
$$

$$
\begin{aligned}
C - Q(x_e) &= 0, \\
CB_e - Q(x_o)B_o &= 0, \\
[BQ]_{x_s} &= 0, \\
[(r + Ed)\,B]_{x_s} &= 0.
\end{aligned}
$$

For the 'P' principle $\delta S_3 = 0$,

$$
\left.
\begin{aligned}
P_Q - \phi' = 0 \\
(BQ)' = 0
\end{aligned}
\right\} \quad \text{in } (x_e, x_s) \cup (x_s, x_o),
$$

$$
\begin{aligned}
C - Q(x_e) &= 0, \\
CB_e - Q(x_o)B_o &= 0, \\
[BQ]_{x_s} &= 0, \\
[BP]_{x_s} &= 0.
\end{aligned}
$$

For the ‘R’ principle $\delta S_4 = 0$,

$$\left.\begin{array}{r} -R_v + Q = 0 \\ -R_d + E = 0 \\ v - \phi' = 0 \\ (BQ)' = 0 \end{array}\right\} \quad \text{in } (x_e, x_s) \cup (x_s, x_o),$$

$$\begin{aligned} C - Q(x_e) &= 0, \\ CB_e - Q(x_o)B_o &= 0, \\ [BQ]_{x_s} &= 0, \\ [(-R + Qv + Ed)\, B]_{x_s} &= 0. \end{aligned}$$

Thus the natural conditions include the equations of shallow water motion in (x_e, x_s) and (x_s, x_o) and boundary conditions on the mass flow at x_e and x_o. The first jump condition in each case is the condition of no jump in the mass flow (2.77). Using one-dimensional versions of the equations (3.74), (3.76) and (3.71), which relate p to r, P and R, the second jump condition in each case can be recognised as the condition of no jump in the value of the flow stress, defined by (2.72), on crossing the point of discontinuity x_s, that is condition (2.76).

Chapter 4

Approximations to Quasi One-dimensional Shallow Water Flows

The variational principles of Section 3.7 have as natural conditions the equations of steady state quasi one-dimensional motion in shallow water. This chapter is concerned with using some of these variational principles to generate approximations to the variables of shallow water flows in channels.

The equations of motion for quasi one-dimensional flow are satisfied by functions for which the functionals of the variational principles of Section 3.7 are stationary. Solutions of the shallow water equations can be approximated by finding the functions for which the functionals are stationary with respect to variations in a finite dimensional space. In this chapter the variables of shallow water are expanded in terms of finite element basis functions, defined on a grid of points extending over the domain of the problem.

The particular variational principles used here are the 'p' and 'r' principles based on the functionals (3.112) and (3.113). The constrained versions of these principles, (3.116) and (3.117), both depend on only one variable and are used to develop algorithms for generating approximations, defined on fixed grids, to the velocity and depth of flow in a channel. The constrained 'p' principle is also used to generate approximations on adaptive grids.

The final section of this chapter is concerned with deriving approximations to discontinuous flows in channels. The constrained 'r' principle is used to generate approximations to the depth on a grid with one moving node which is placed at the position of the discontinuity. This algorithm is extended to give a method for approximating discontinuous depth profiles on adaptive grids.

The domains of the problems to be considered are channels with breadth $B(x)$ for $x \in [x_e, x_o]$ and undisturbed fluid depth $h(x)$ for $x \in [x_e, x_o]$, where B and h are functions to be defined later.

4.1 The Constrained 'r' Principle

The 'r' principle based on the functional (3.113), with variations constrained to satisfy the conservation of mass equation, can be used to generate approximations to the depth of fluid for shallow water flow in a channel.

The functional of the constrained principle (3.117) is given by

$$M_2^c(d) = \int_{x_e}^{x_o} (r(Q,d) + Ed)\, B\, dx, \tag{4.1}$$

where Q and E are known functions of x, namely,

$$Q(x) = \frac{CB_e}{B(x)} \qquad \text{for } x \in [x_e, x_o], \tag{4.2}$$

from the conservation of mass constraint, and

$$E(x) = \tilde{E} + gh(x) \qquad \text{for } x \in [x_e, x_o], \tag{4.3}$$

corresponding to conservation of momentum. In practice the constants C and $\tilde{E}$ are calculated from given values of two of the three variables mass flow, depth and velocity at the inlet boundary. Given the values of two of these variables at $x = x_e$ the value of the third can be deduced from (2.34). Then, using (2.35),

$$\begin{aligned} C &= Q(x_e) \\ \text{and} \qquad \tilde{E} &= gd(x_e) + \frac{1}{2}v(x_e)^2 - gh(x_e). \end{aligned}$$

For a continuous flow to exist notice that C and $\tilde{E}$ must satisfy

$$CB_e \leq \frac{1}{g}\left(\frac{2\left(\tilde{E} + gh(x)\right)}{3}\right)^{\frac{3}{2}} B(x) \qquad \text{for } x \in [x_e, x_o], \tag{4.4}$$

using (2.62) and (2.56).

The function d which satisfies $\delta M_2^c = 0$ is the depth of fluid in the channel. The nature of the stationary value of M_2^c can be determined by considering the second derivative

$$\frac{d^2 M_2^c}{dd^2} = \int_{x_e}^{x_o} r_{dd} B \, dx.$$

From the definition of r, (3.103),

$$r_{dd} = \frac{Q^2}{d^3} - g, \tag{4.5}$$

which is positive if $\frac{Q^2}{d^2} > gd$, that is (using (2.34)), if $v^2 > gd$ and the flow is supercritical and negative if $\frac{Q^2}{d^2} < gd$, that is, if the flow is subcritical. Thus, if the flow is supercritical in the whole of $[x_e, x_o]$, the function d satisfying $\delta M_2^c = 0$ minimises M_2^c and, if the flow is subcritical in the whole of $[x_e, x_o]$, the stationary function d maximises M_2^c. If the flow is critical at isolated points in the channel then these statements still hold but, if both subcritical and supercritical flows exist in the channel, it is not possible to say whether the stationary function minimises or maximises M_2^c.

4.1.1 The Algorithm

The method used here for generating approximations to d is to substitute into the functional (4.1) finite element expansions for d and to find the parameters of the expansions for which M_2^c is stationary with respect to variations in the parameters.

Let the interval $[x_e, x_o]$ be divided into $n-1$ regular intervals by the points $x_1, \ldots, x_n$ given by

$$x_i = \frac{(i-1)}{(n-1)}(x_o - x_e) + x_e \qquad i = 1, \ldots, n. \tag{4.6}$$

Let $\alpha_1(x), \ldots, \alpha_n(x)$ be finite element basis functions, defined on the grid given by (4.6), and let

$$d^h(x) = \sum_{i=1}^{n} d_i \alpha_i(x) \tag{4.7}$$

be the approximation to d, where the d_i $(i = 1, \ldots, n)$ are parameters of the solution, to be determined.

Consider the finite dimensional version of the functional (4.1),

$$L(\mathbf{d}) = \int_{x_1}^{x_n} \left(r(Q, d^h) + E d^h \right) B \, dx,$$

where $\mathbf{d} = (d_1, \ldots, d_n)^T$ and Q and E are given by (4.2) and (4.3).

The parameters $\mathbf{d}$ for which (4.7) is an approximation to d are such that L is stationary with respect to variations in $\mathbf{d}$. They are found by solving the non-linear set of equations

$$F_i(\mathbf{d}) = \frac{\partial L}{\partial d_i} = \int_{x_1}^{x_n} (r_{d^h} + E)\, \alpha_i B \, dx = 0 \qquad i = 1, \ldots, n. \tag{4.8}$$

There is more than one solution of the set of equations (4.8). One possible solution involves negative values of d_i and is not considered since it has no physical meaning. In the case of approximations to non-critical flows there are two other solutions — one which approximates subcritical flow and one which approximates supercritical flow. In the case of flows which become critical at a point in the domain there is a further possibility, that is, an approximation to transitional flow.

In the present work (4.8) is solved using Newton's method. The Jacobian J is given by

$$J(\mathbf{d}) = \{J_{ij}\} = \left\{ \frac{\partial F_i}{\partial d_j} \right\} = \left\{ \frac{\partial^2 L}{\partial d_j \, \partial d_i} \right\} = \left\{ \int_{x_1}^{x_n} r_{d^h d^h} \alpha_i \alpha_j B \, dx \right\}, \tag{4.9}$$

and is the Hessian of L.

Given an approximation $\mathbf{d}^k$ to the solution $\mathbf{d}$, Newton's method provides an updated approximation

$$\mathbf{d}^{k+1} = \mathbf{d}^k + \delta \mathbf{d}^k, \tag{4.10}$$

where

$$J(\mathbf{d}^k)\, \delta \mathbf{d}^k = -\mathbf{F}(\mathbf{d}^k). \tag{4.11}$$

This yields a sequence of approximations to $\mathbf{d}$. The process is repeated until

$$\max_i \left| \delta d_i^k \right| < \text{tolerance}. \tag{4.12}$$

Then $d_i = d_i^k$ for $i = 1, \ldots, n$ are the values of the parameters in the approximation (4.7) which make $L(\mathbf{d})$ stationary. The Jacobian J and the vector $\mathbf{F}$ are calculated using five point Gaussian quadrature, where it is assumed that the error introduced by the numerical integration is sufficiently small that the finite element solution, for a particular tolerance in (4.12), is unaffected.

Let $\hat{\mathbf{d}}$ satisfy $\mathbf{F}(\hat{\mathbf{d}}) = \mathbf{0}$ and let $\gamma > 0$ be such that the domain $S = \left\{\mathbf{d} : \left\|\mathbf{d} - \hat{\mathbf{d}}\right\| < \gamma\right\} \subset \Re^n$ contains the point $\hat{\mathbf{d}}$, where $\|\cdot\|$ is an appropriate norm. Assume that the first derivatives of J are continuous in S and that J is non-singular in S. Then, there exists $\epsilon > 0$ such that Newton's method is quadratically convergent whenever $\left\|\mathbf{d}^0 - \hat{\mathbf{d}}\right\| < \epsilon$ (Johnson and Riess (1982)).

From (4.9) J has the form of a weighted mass matrix, where $Br_{d^h d^h}$ is the weight function. Using (4.5) it can be seen that, if the approximate solution in $[x_1, x_n]$ is subcritical throughout the Newton iteration, J is negative definite and the solution of (4.8) maximises L. Alternatively, if the approximate solution is supercritical in $[x_1, x_n]$ for all iterations, J is positive definite and the solution of (4.8) minimises L.

Thus, given values for $\tilde{E}$ and C, it is possible, using Newton's method, to generate finite element approximations to the depth of flow in a channel for continuous flows which are either supercritical in the whole domain or subcritical in the whole domain. The success of the method relies on choosing the initial approximation $\mathbf{d}^0$ to $\mathbf{d}$ such that the approximations $\mathbf{d}^k$, calculated from (4.10) and (4.11), have either all subcritical components or all supercritical components. For each set of conditions, $\tilde{E}$ and C, two approximations will be considered — one corresponding to subcritical flow and the other to supercritical flow; the choice of $\mathbf{d}^0$ determines which solution is found by the algorithm.

If the flow for which an approximation is being sought includes both subcritical and supercritical motion and if an approximation, to the approximate solution, at an iteration step has both subcritical and supercritical values the Jacobian is indefinite and Newton's method may fail to converge to the solution.

The algorithm is implemented on the equi-spaced grid given by (4.6), with $x_e = 0$, $x_o = 10$ and $n = 21$. Two sets of basis functions are considered; the first, α_i^l for $i = 1, \ldots, n$, leads to continuous piecewise linear approximations and the second, α_i^c for $i = 1, \ldots, n-1$, gives discontinuous piecewise constant approximations. The basis functions are defined by

$$\begin{aligned}
\alpha_1^l(x) &= \begin{cases} \dfrac{x_2 - x}{x_2 - x_1} & x \in [x_1, x_2] \\[2ex] 0 & x \notin [x_1, x_2] \end{cases}, \\
\alpha_i^l(x) &= \begin{cases} \dfrac{x - x_{i-1}}{x_i - x_{i-1}} & x \in [x_{i-1}, x_i] \\ \dfrac{x_{i+1} - x}{x_{i+1} - x_i} & x \in [x_i, x_{i+1}] \\[2ex] 0 & x \notin [x_{i-1}, x_{i+1}] \end{cases} \qquad i = 2, \ldots, n-1, \\
\alpha_n^l(x) &= \begin{cases} \dfrac{x - x_{n-1}}{x_n - x_{n-1}} & x \in [x_{n-1}, x_n] \\[2ex] 0 & x \notin [x_{n-1}, x_n] \end{cases}
\end{aligned} \tag{4.13}$$

and

$$\alpha_i^c(x) = \begin{cases} 1 & x \in (x_i, x_{i+1}) \\[2ex] 0 & x \notin (x_i, x_{i+1}) \end{cases} \qquad i = 1, \ldots, n-1, \tag{4.14}$$

and are shown in Figure 4.1.

For the basis functions defined by (4.13), J is tridiagonal and (4.11) is solved quickly for $\delta\mathbf{d}^k$ using Gaussian elimination and back substitution. For the basis functions defined by (4.14), J is diagonal and (4.11) is easily solved.

The method is used to find approximations to flows in a number of different

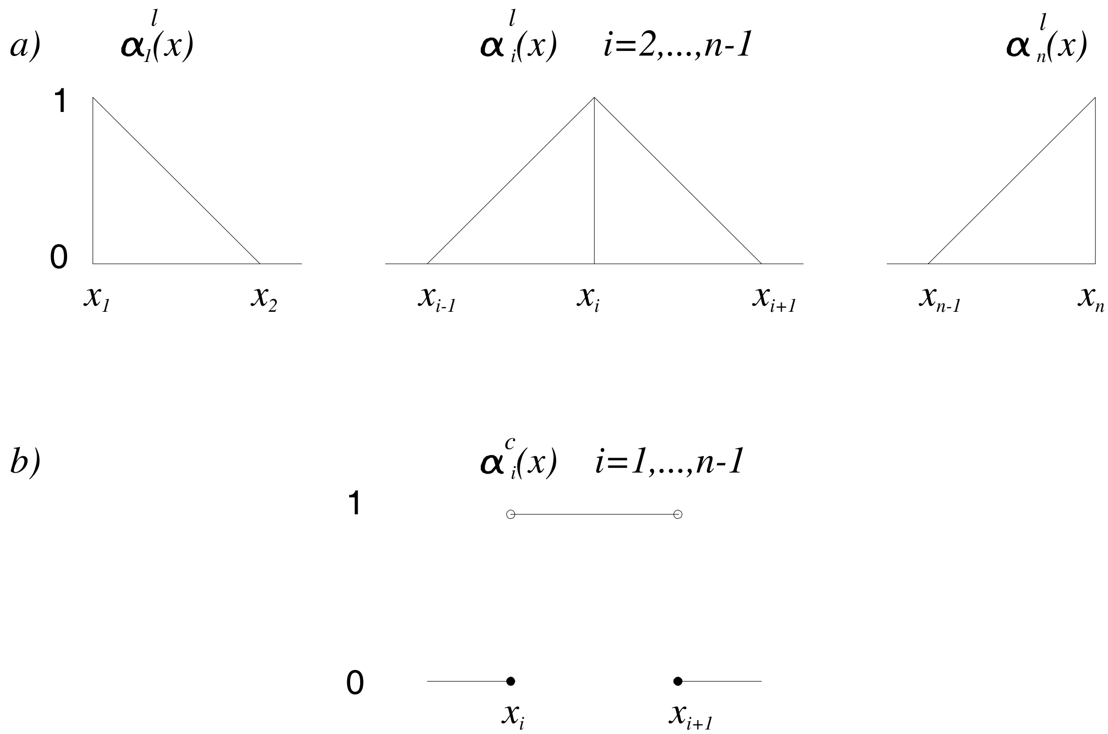

Figure 4.1: One-dimensional basis functions *a*) piecewise linear and *b*) piecewise constant.

channels. Several breadth functions are considered. These are

$$B_{1,k}(x) = 6 + 4\left(1 - 2\frac{x - x_e}{x_o - x_e}\right)^k \quad \text{in } [x_e, x_o], \text{ for } k = 2, 4, 6, 8, \tag{4.15}$$

$$B_{2,\sigma}(x) = \begin{cases} 6 + 4\left(1 - \left(\frac{x - x_e}{\nu - x_e}\right)^\sigma\right) & \text{in } [x_e, \nu] \\ 6 + 4\left(1 - \left(\frac{x_o - x}{x_o - \nu}\right)^\sigma\right) & \text{in } [\nu, x_o] \end{cases}, \text{ for } \sigma = \frac{1}{2}, 1, \frac{3}{2}, 2. \tag{4.16}$$

Moving the reference level for potential energy from $z = 0$ to $z = -h(x_e)$ is equivalent to redefining the equilibrium depth to be $h(x) := h(x) - h(x_e)$, so that $h(x_e) = 0$, and the constant $\tilde{E}$ to be $\tilde{E} := \tilde{E} + gh(x_e)$. This is now assumed to be the case. The equilibrium depth functions considered here are

$$h_1(x) = 0 \quad \text{in } [x_e, x_o], \tag{4.17}$$

$$h_2(x) = H\frac{x - x_e}{x_o - x_e} \quad \text{in } [x_e, x_o], \tag{4.18}$$

$$h_3(x) = 4H\frac{(x - x_e)(x_o - x)}{(x_o + x_e)^2} \quad \text{in } [x_e, x_o]. \tag{4.19}$$

The energy $\tilde{E}$ is given the value 50. In order to guarantee that a continuous solution exists the value of mass flow at inlet C must satisfy

$$C \leq \frac{1}{g}\left(\frac{2\left(\tilde{E}+gh(x)\right)}{3}\right)^{\frac{3}{2}}\frac{B(x)}{B_e} \qquad \text{in } [x_e, x_o],$$

from (4.4). For the case $h = h_1$ this is just

$$C \leq \frac{1}{g}\left(\frac{2\tilde{E}}{3}\right)^{\frac{3}{2}}\frac{B_{\min}}{B_e},$$

where

$$B_{\min} = \min_{x\in[x_e,x_o]} B(x).$$

Thus, for the given breadth functions (4.15) and (4.16), C must have a value such that $C \leq C_*$, where

$$C_* = \frac{20}{\sqrt{3}}. \tag{4.20}$$

A value of $C = C_*$ yields flows which are critical at the point of minimum breadth. A value of $C = 10$ is used to give examples of non-critical flows.

The initial approximation $\mathbf{d}^0$ to the solution $\mathbf{d}$ determines whether the finite element solution is an approximation to subcritical or to supercritical flow. In practice subcritical approximations are obtained by specifying $d_i^0 > d_*$ for $i = 1, \ldots, n$, where d_* is the critical depth, given by (2.65). In this case, for $h = h_1$, $d_* = \frac{100}{30} \approx 3.33$. Supercritical approximations are obtained by specifying $d_i^0 < d_*$ for $i = 1, \ldots, n$. Transitional flows are not considered because of problems with the convergence of Newton's method with an indefinite Jacobian.

Let the tolerance on the Newton iteration be 10^{-3}. Consider the channel with breadth $B = B_{1,6}$. Using the piecewise linear basis functions (4.13) Newton's method converges to the supercritical approximation from the initial approximation $d_i^0 = 1$ for $i = 1, \ldots, n$ in 15 iterations for critical flow and 7 iterations

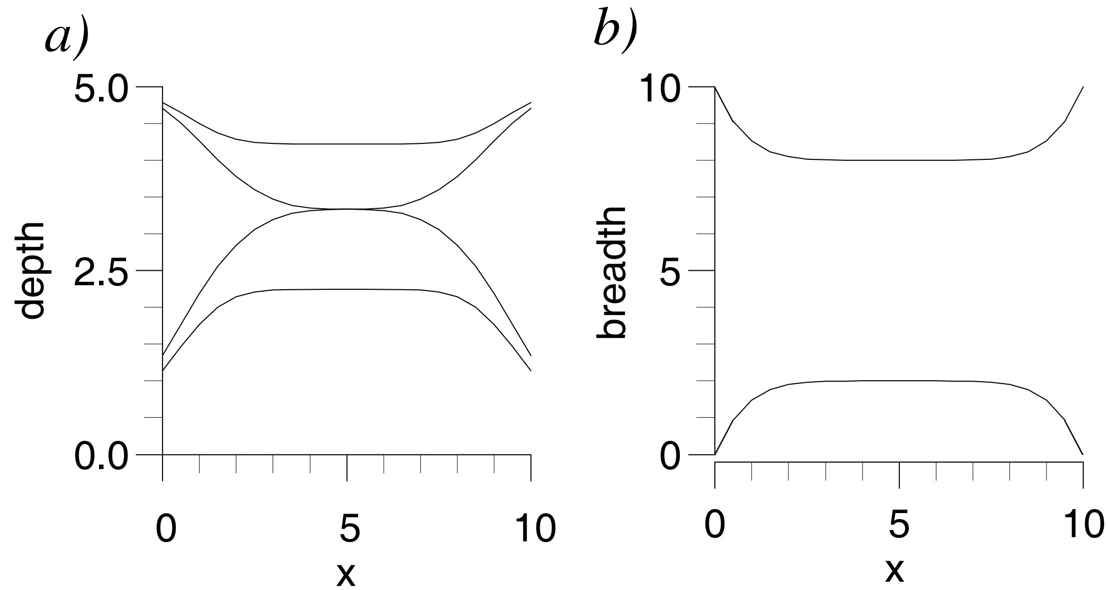

Figure 4.2: *a*) Piecewise linear depth approximations on a fixed grid and *b*) $B_{1,6}(x)$.

for non-critical flow. Subcritical approximations are obtained, using $d_i^0 = 4$ for $i = 1, \ldots, n$, in 10 iterations for critical flow and 3 iterations for non-critical flow. Figure 4.2*a* shows the finite element approximations to the depth for the critical and non-critical flows generated under these conditions. The top two lines are the approximations to the subcritical flows and the other two approximate the supercritical flows, for the two values of mass flow at inlet $C = 10$ and $C = C_*$, where C_* is defined by (4.20). Figure 4.2*b* shows a linear interpolation to the breadth function using the 21 grid points given by (4.6). The sides of the channel are almost parallel for part of its length at the narrowest part so the depths of the two critical approximations are close to the critical depth value for some distance around the point $x = 5$.

Figure 4.3 shows corresponding results for $B = B_{2,2}$ with $\nu = 7.5$.

Using the piecewise constant basis functions (4.14), in the channel with $B =$

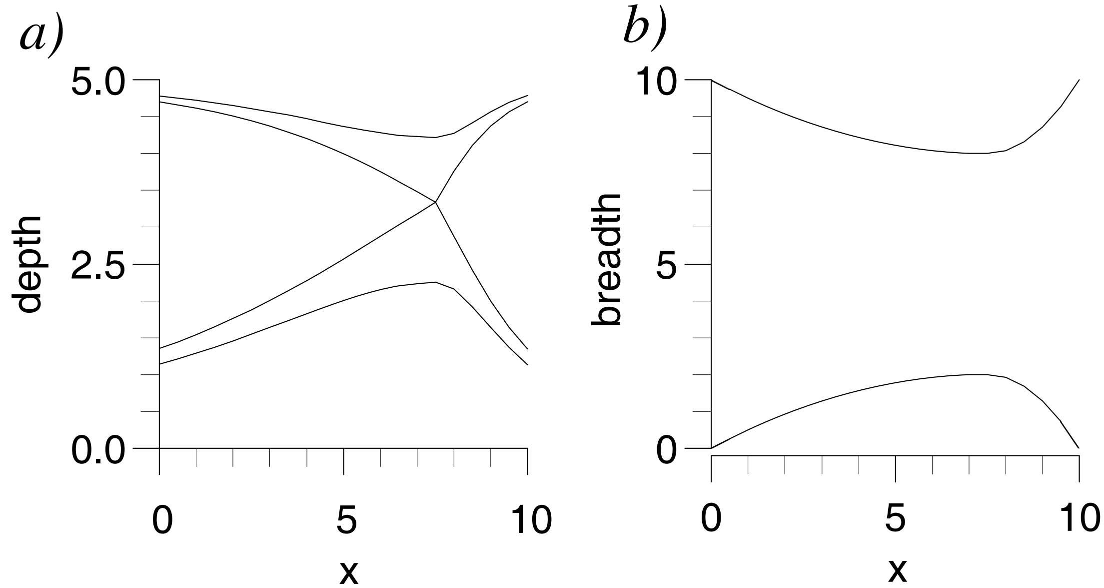

Figure 4.3: a) Piecewise linear depth approximations on a fixed grid and b) $B_{2,2}(x)$ with $\nu = 7.5$.

$B_{1,6}$ and $h = h_1$, the supercritical approximation, using $d_i^0 = 1$ for $i = 1, \ldots, n$, is found after 14 iterations for critical flow and 7 iterations for non-critical flow while the subcritical approximation, using $d_i^0 = 4$ for $i = 1, \ldots, n$, is found after 10 iterations for critical flow and 3 iterations for non-critical flow. Figure 4.4a shows these approximations and Figure 4.4b is the channel breadth.

Consider the channel with $B = B_{1,2}$ and $h = h_2$ for $H = 0.2$. Figure 4.5a shows the subcritical and supercritical piecewise linear approximations for $C = 10$ and $C = 7.7$. The dashed line shows the position of the channel bed. Notice that the depth profiles are no longer symmetric about the line $x = 5$. The breadth $B_{1,2}$ is shown in Figure 4.5b. With $d_i^0 = 1$ for $i = 1, \ldots, n$ the supercritical approximations are found after 6 iterations in the $C = 10$ case and 5 iterations in the $C = 7.7$ case. Using $d_i^0 = 4$ for $i = 1, \ldots, n$ the subcritical approximations are found after 3 iterations in the $C = 10$ case and 3 iterations in the $C = 7.7$ case.

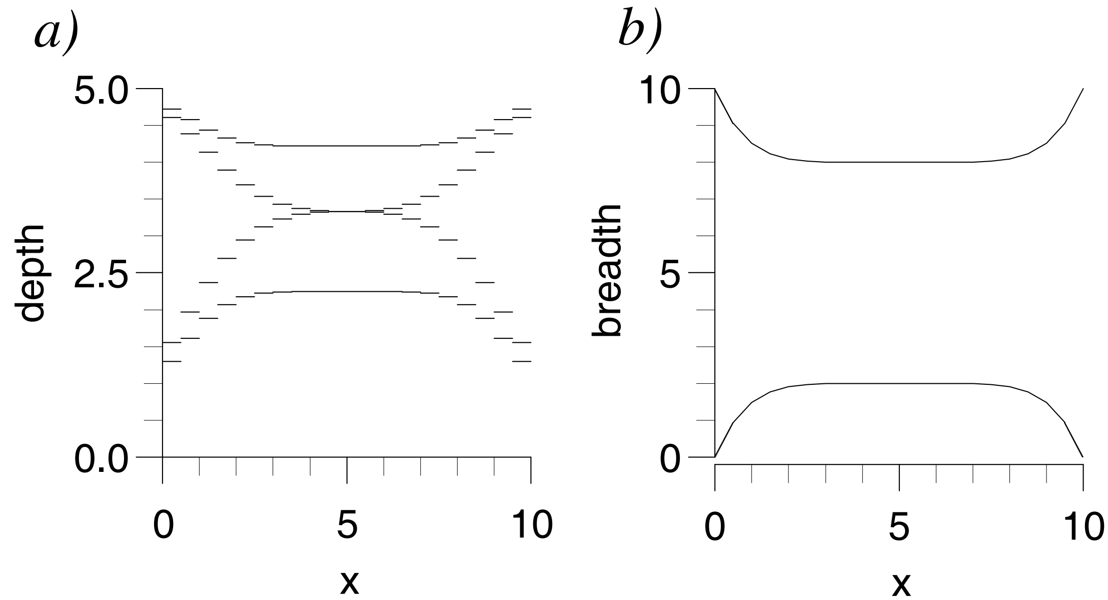

Figure 4.4: *a*) Piecewise constant depth approximations on a fixed grid and *b*) $B_{1,6}(x)$.

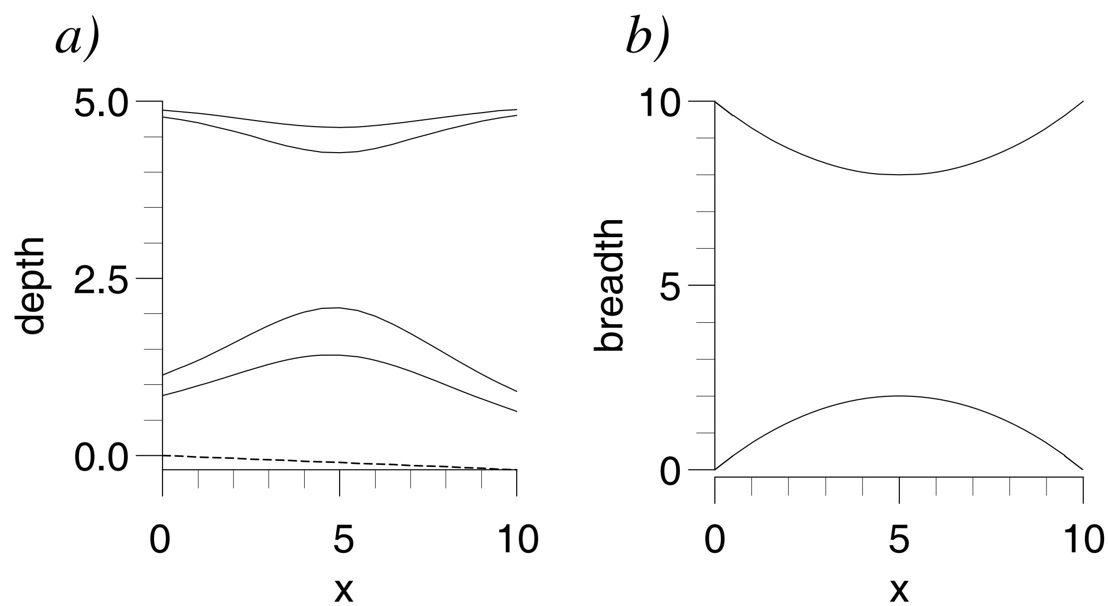

Figure 4.5: *a*) Piecewise linear depth approximations for $B = B_{1,2}$ and $h = h_2$ with $H = 0.2$ and *b*) $B_{1,2}(x)$.

4.1.2 Error Bounds

An error bound for the piecewise constant approximations to d can be calculated.

Proposition The piecewise constant approximations, defined using (4.7) and (4.14) and generated from (4.8), converge linearly to the shallow water depth for wholly subcritical or wholly supercritical flows.

Proof The parameters of the approximation d^h are defined as those which satisfy (4.8), that is,

$$\int_{x_1}^{x_n} (r_{d^h} + E)\, \alpha_i^c B\, dx = 0 \qquad i = 1, \ldots, n-1. \tag{4.21}$$

The exact depth d satisfies the equation

$$r_d + E = 0,$$

from the definitions of r (3.103), mass flow (2.34) and energy (2.35). Thus

$$\int_{x_1}^{x_n} (r_d + E)\, \alpha_i^c B\, dx = 0 \qquad i = 1, \ldots, n-1. \tag{4.22}$$

Subtracting (4.22) from (4.21) gives

$$\int_{x_1}^{x_n} (r_{d^h} - r_d)\, \alpha_i^c B\, dx = 0 \qquad i = 1, \ldots, n-1,$$

and so

$$\int_{x_i}^{x_{i+1}} (r_{d^h} - r_d)\, B\, dx = 0 \qquad i = 1, \ldots, n-1, \tag{4.23}$$

using (4.14).

Both d and d^h are differentiable on each interval $[x_i, x_{i+1}]$ and thus, using the Mean Value Theorem,

$$r_{d^h} - r_d = \left(d^h - d\right) r_{\psi\psi}(Q, \psi)\big|_{\psi=\theta}\,, \tag{4.24}$$

for $\theta(x)$ between $d^h(x)$ and $d(x)$, where $r_{\psi\psi} = \frac{Q^2}{\psi^3} - g$, from (3.103). Thus if d^h and d are completely supercritical $r_{\psi\psi} > 0$ and if d^h and d are completely subcritical $r_{\psi\psi} < 0$ in $[x_i, x_{i+1}]$. Therefore, substituting (4.24) into (4.23) to give

$$\int_{x_i}^{x_{i+1}} \left(d^h - d\right) r_{\psi\psi}(Q,\psi)\big|_{\psi=\theta} \, B \, dx = 0,$$

implies that $d^h - d = 0$ at at least one point (say $x = \hat{x}$) in (x_i, x_{i+1}) for completely subcritical or supercritical flows, since $B > 0$.

Now d^h is constant on $[x_i, x_{i+1}]$ so, for $x \in [x_i, x_{i+1}]$,

$$\int_{\hat{x}}^{x} d'(\sigma)\, d\sigma = \int_{\hat{x}}^{x} \left(d'(\sigma) - d^{h\prime}(\sigma)\right) d\sigma = \left[d(\sigma) - d^h(\sigma)\right]_{\hat{x}}^{x} = d(x) - d^h(x).$$

Thus

$$\begin{aligned}
\int_{x_i}^{x_{i+1}} \left(d(x) - d^h(x)\right)^2 dx &= \int_{x_i}^{x_{i+1}} \left(\int_{\hat{x}}^{x} d'(\sigma)\, d\sigma\right)^2 dx \\
&\leq \int_{x_i}^{x_{i+1}} \left((x_{i+1} - x_i) \max_{I_i} |d'|\right)^2 dx \\
&= (x_{i+1} - x_i)^3 \max_{I_i} |d'|^2 ,
\end{aligned}$$

where $\max_{I_i} |d'|^2 = \max_{x \in [x_i, x_{i+1}]} |d'(x)|^2$.

Therefore the square of the L_2 error is

$$\begin{aligned}
\left|\left|d - d^h\right|\right|^2 &= \int_{x_e}^{x_o} \left(d - d^h\right)^2 dx \qquad (4.25)\\
&= \sum_{i=1}^{n-1} \int_{x_i}^{x_{i+1}} \left(d - d^h\right)^2 dx \\
&\leq \sum_{i=1}^{n-1} (x_{i+1} - x_i)^3 \max_{I_i} |d'|^2 \\
&\leq \max_i (x_{i+1} - x_i)^2 \max_{x \in [x_e, x_o]} |d'|^2 \sum_{i=1}^{n-1} (x_{i+1} - x_i) \\
&= \max_i (x_{i+1} - x_i)^2 \max_{x \in [x_e, x_o]} |d'|^2 (x_o - x_e),
\end{aligned}$$

that is,

$$\left|\left|d - d^h\right|\right| \leq \Delta x \max_{x \in [x_e, x_o]} |d'| \, (x_o - x_e)^{\frac{1}{2}},$$

where, for the equi-spaced grid (4.6), $\Delta x = \frac{x_e - x_o}{n-1}$.

n	Δx	critical flows		non-critical flows	
		subcritical	supercritical	subcritical	supercritical
3	$\frac{10}{2}$	6.028×10^{-1}	8.608×10^{-1}	2.687×10^{-1}	5.097×10^{-1}
5	$\frac{10}{2^2}$	1.870×10^{-1}	3.521×10^{-1}	1.188×10^{-1}	2.654×10^{-1}
9	$\frac{10}{2^3}$	7.249×10^{-2}	1.550×10^{-1}	4.882×10^{-2}	1.218×10^{-1}
17	$\frac{10}{2^4}$	3.198×10^{-2}	7.249×10^{-2}	2.192×10^{-2}	5.772×10^{-2}
33	$\frac{10}{2^5}$	1.504×10^{-2}	3.504×10^{-2}	1.038×10^{-2}	2.805×10^{-2}
65	$\frac{10}{2^6}$	7.293×10^{-3}	1.722×10^{-2}	5.050×10^{-3}	1.382×10^{-2}
129	$\frac{10}{2^7}$	3.592×10^{-3}	8.539×10^{-3}	2.491×10^{-3}	6.860×10^{-3}
257	$\frac{10}{2^8}$	1.782×10^{-3}	4.252×10^{-3}	1.237×10^{-3}	3.417×10^{-3}
513	$\frac{10}{2^9}$	8.878×10^{-4}	2.121×10^{-3}	6.164×10^{-4}	1.705×10^{-3}
1025	$\frac{10}{2^{10}}$	4.431×10^{-4}	1.060×10^{-3}	3.077×10^{-4}	8.520×10^{-4}

Table 4.1: L_2 errors for piecewise constant depth approximations.

Thus the piecewise constant depth approximation converges linearly with n to the solution d.

The L_2 error is calculated for piecewise constant approximations on grids with different numbers of nodes for the example $B = B_{1,2}$, defined by (4.15), and $h = h_1$, defined by (4.17). The energy $\tilde{E} = 50$ and both $C = C_*$, defined by (4.20), and $C = 10$ are considered. The results are given in Table 4.1, from which it can be seen, more especially for larger n, that as the interval length Δx halves the L_2 error also halves.

The L_2 errors for the corresponding piecewise linear approximations are given in Table 4.2. It can be seen that the convergence is almost quadratic.

n	Δx	critical flows		non-critical flows	
		subcritical	supercritical	subcritical	supercritical
3	$\frac{10}{2}$	1.178×10^{-1}	9.217×10^{-2}	1.087×10^{-2}	3.975×10^{-2}
5	$\frac{10}{2^2}$	2.087×10^{-2}	2.084×10^{-2}	6.606×10^{-3}	9.668×10^{-3}
9	$\frac{10}{2^3}$	4.395×10^{-3}	5.122×10^{-3}	1.155×10^{-3}	1.769×10^{-3}
17	$\frac{10}{2^4}$	9.825×10^{-4}	1.235×10^{-3}	2.842×10^{-4}	4.714×10^{-4}
33	$\frac{10}{2^5}$	2.304×10^{-4}	3.135×10^{-4}	6.858×10^{-5}	1.249×10^{-4}
65	$\frac{10}{2^6}$	5.595×10^{-5}	7.657×10^{-5}	1.651×10^{-5}	3.976×10^{-5}
129	$\frac{10}{2^7}$	1.280×10^{-5}	2.234×10^{-5}	4.401×10^{-6}	1.453×10^{-5}

Table 4.2: L_2 errors for piecewise linear depth approximations.

4.2 The Unconstrained ‘r’ Principle

Finite element expansions for the mass flow and the velocity potential, as well as for the fluid depth, can be obtained using the unconstrained ‘r’ principle, based on the functional (3.113). The method used here is a simple extension of the algorithm in Section 4.1.

Consider the grid defined by the points (4.6), with $x_e = 0$, $x_o = 10$ and $n = 21$. Let

$$Q^h(x) = \sum_{i=1}^{n} Q_i \alpha_i(x)\,, \ d^h(x) = \sum_{i=1}^{n} d_i \alpha_i(x)\,, \ \phi^h(x) = \sum_{i=1}^{n} \phi_i \alpha_i(x) \tag{4.26}$$

be approximations to the mass flow, depth and velocity potential, respectively, where the α_i $(i = 1, \ldots, n)$ are finite element basis functions. Substituting (4.26) into the functional (3.113) yields the finite dimensional version

$$L(\mathbf{Q},\mathbf{d},\boldsymbol{\phi}) = \int_{x_1}^{x_n} \left(r(Q^h,d^h) + Ed^h - \phi^{h\prime}Q^h\right) B\,dx + CB_e\left(\phi^h(x_n) - \phi^h(x_1)\right), \tag{4.27}$$

where $\mathbf{Q} = (Q_1,\ldots,Q_n)^T$, $\mathbf{d} = (d_1,\ldots,d_n)^T$, $\boldsymbol{\phi}=(\phi_1,\ldots,\phi_n)^T$ and $E(x) = \tilde{E} + gh(x)$. The parameters $\mathbf{Q}$, $\mathbf{d}$ and $\boldsymbol{\phi}$ are calculated by solving

$$\frac{\partial L}{\partial Q_i} = 0\,,\ \frac{\partial L}{\partial d_i} = 0\,,\ \frac{\partial L}{\partial \phi_i} = 0 \quad \text{for } i = 1,\ldots,n. \tag{4.28}$$

Let the α_i be the piecewise linear basis functions defined by (4.13). Then equations $(4.28)_3$ yield

$$-\int_{x_1}^{x_n} \alpha_i' Q^h B\,dx + CB_e\left(\alpha_i(x_n) - \alpha_i(x_1)\right) = 0 \quad i = 1,\ldots,n,$$

which may be rewritten as

$$\begin{aligned}
\sum_{j=1}^{2} Q_j \int_{x_1}^{x_2} \alpha_1'\alpha_j B\,dx &= -CB_e,\\
\sum_{j=i-1}^{i+1} Q_j \int_{x_{i-1}}^{x_{i+1}} \alpha_i'\alpha_j B\,dx &= 0 \qquad i = 2,\ldots,n-1,\\
\sum_{j=n-1}^{n} Q_j \int_{x_{n-1}}^{x_n} \alpha_n'\alpha_j B\,dx &= CB_e,
\end{aligned}$$

or as,

$$A_Q\mathbf{Q} = C_Q, \tag{4.29}$$

where A_Q is a constant $n\times n$ matrix and C_Q is a constant $n\times 1$ vector with only first and last entries non-zero. The matrix A_Q is of rank $n-1$ and is singular but, using the boundary condition $Q_1 = C$, the solution of (4.29) is unique. A_Q is tridiagonal and $\mathbf{Q}$ is calculated using Gaussian elimination and back substitution.

Equations $(4.28)_2$ yield

$$\int_{x_1}^{x_n} \left(r_{d^h} + E\right)\alpha_i B\,dx = 0 \quad i = 1,\ldots,n,$$

which, once Q^h is known, can be solved for d^h by the method of Section 4.1.

Equations $(4.28)_1$ give

$$\int_{x_1}^{x_n} \left(r_{Q^h} - \phi^{h\prime} \right) \alpha_i B \, dx = 0 \qquad i = 1, \ldots, n,$$

which may be written as

$$\begin{aligned}
\sum_{j=1}^{2} \phi_j \int_{x_1}^{x_2} \alpha_1 \alpha_j' B \, dx &= \int_{x_1}^{x_2} r_{Q^h} \alpha_1 B \, dx, \\
\sum_{j=i-1}^{i+1} \phi_j \int_{x_{i-1}}^{x_{i+1}} \alpha_i \alpha_j' B \, dx &= \int_{x_{i-1}}^{x_{i+1}} r_{Q^h} \alpha_i B \, dx \qquad i = 2, \ldots, n-1, \\
\sum_{j=n-1}^{n} \phi_j \int_{x_{n-1}}^{x_n} \alpha_n \alpha_j' B \, dx &= \int_{x_{n-1}}^{x_n} r_{Q^h} \alpha_n B \, dx,
\end{aligned}$$

or as

$$A_\phi \boldsymbol{\phi} = C_\phi, \tag{4.30}$$

where A_ϕ is an $n \times n$ matrix and C_ϕ is an $n \times 1$ vector. Once Q^h and d^h are known $\boldsymbol{\phi}$ can be calculated directly. The matrix A_ϕ is of rank $n-1$ and singular but ϕ is a potential function and the important quantity is its gradient so one of the values, say ϕ_1, is specified arbitrarily. This procedure is equivalent to setting the arbitrary constant in ϕ by assigning its value at the boundary.

Results for critical flow in a channel with $B = B_{1,4}$, defined by (4.15), and $h = h_1$, defined by (4.17), are shown in Figure 4.6. The energy $\tilde{E}$ is taken to be 50. The piecewise linear approximation to the mass flow is shown in Figure 4.6*a*. The piecewise linear approximations to the velocity potential and depth for a supercritical flow are given in Figures 4.6*b* and 4.6*c*, respectively. Figure 4.6*d* shows the piecewise constant approximation to the supercritical velocity derived by taking the gradient of the piecewise linear velocity potential approximation in each interval $[x_i, x_{i+1}]$ for $i = 1, \ldots, n-1$. The Newton iteration to find d^h

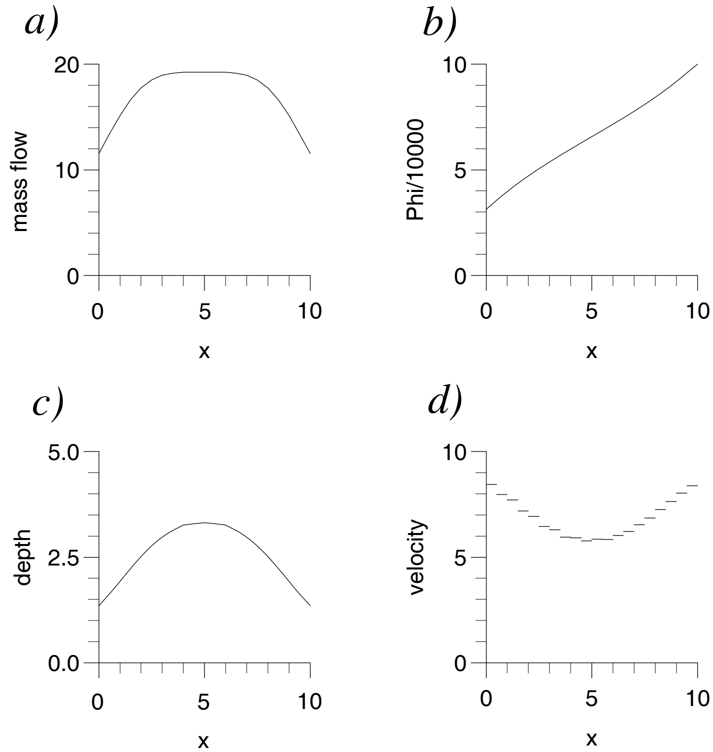

Figure 4.6: a) Mass flow, b) velocity potential, c) depth and d) velocity approximations for $B = B_{1,4}$ — supercritical case.

converges after 13 iterations, using $d_i^0 = 1$ for $i = 1, \ldots, n$, with a tolerance of 10^{-3}.

Corresponding results for the subcritical flow are given in Figure 4.7. The Newton iteration converges from $d_i^0 = 4$ for $i = 1, \ldots, n$ in 8 iterations.

Notice that the velocity approximation is not quite symmetric about the line $x = 5$, even though the breadth and equilibrium fluid depth functions are. This is probably a consequence of using approximations to mass flow and depth in (4.30). By increasing the number of grid points the approximations can be improved — Figure 4.8 shows the supercritical solutions for $n = 61$.

Thus, although approximations to all of the variables can be generated using the unconstrained 'r' principle, in the case of the velocity potential (and therefore

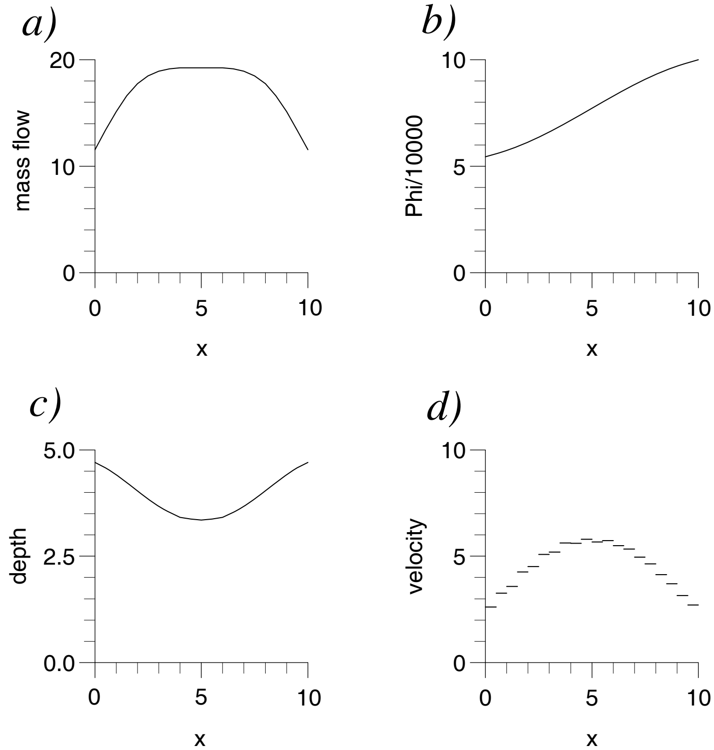

Figure 4.7: a) Mass flow, b) velocity potential, c) depth and d) velocity approximations for $B = B_{1,4}$ — subcritical case.

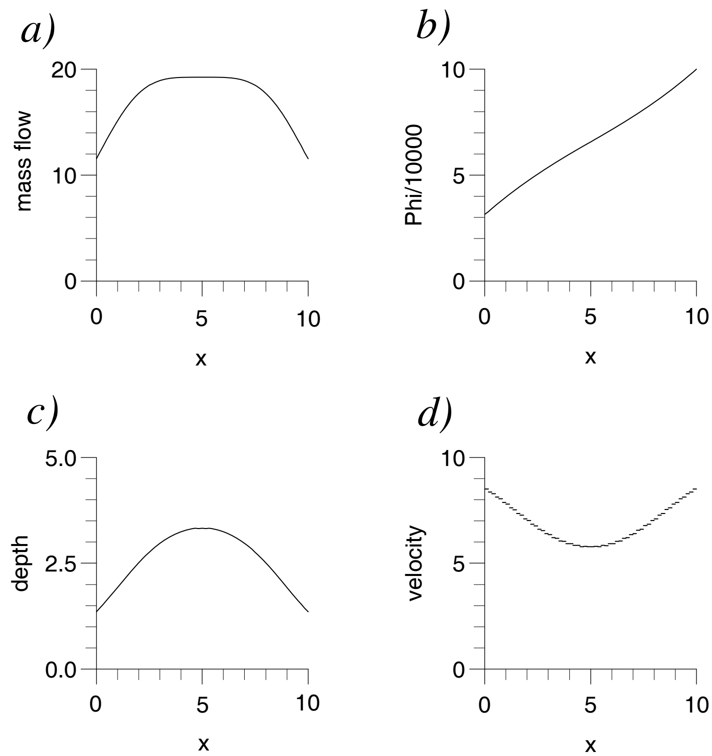

Figure 4.8: As Figure 4.6 — $n = 61$.

the velocity) it is not ideal. However a variational principle exists which depends on the velocity potential alone, that is, the 'p' principle, based on the functional (3.112), constrained by $v = \phi'$. In using this constrained principle to seek an approximation to ϕ (and therefore v) no other approximations are made and more accurate results might be expected.

4.3 The Constrained 'p' Principle

The functional of the constrained 'p' principle (3.116) is given by

$$M_1^c(\phi) = \int_{x_e}^{x_o} p(\phi', E)B\, dx + CB_e\left(\phi(x_o) - \phi(x_e)\right), \tag{4.31}$$

where $E(x) = \tilde{E} + gh(x)$ and the constants $\tilde{E}$ and C are prescribed.

The velocity potential of a shallow water flow is the function ϕ which satisfies $\delta M_1^c = 0$. The nature of the stationary value of M_1^c can be deduced by considering

$$\frac{d^2 M_1^c}{d\phi'^2} = \int_{x_e}^{x_o} p_{\phi'\phi'} B\, dx.$$

From the definition of p, (3.102),

$$p_{\phi'\phi'} = \frac{1}{g}\left(\frac{3}{2}\phi'^2 - E\right). \tag{4.32}$$

Thus, from (4.32), if the flow is supercritical in the whole of $[x_e, x_o]$ then the solution ϕ of $\delta M_1^c = 0$ minimises M_1^c and if the flow is subcritical in the whole of $[x_e, x_o]$ the solution maximises M_1^c.

4.3.1 The Algorithm

The algorithm for generating an approximation to the velocity potential using (4.31) is similar to that of Section 4.1.

Let the x_i $(i = 1, \ldots, n)$, given by (4.6), define the grid. Let the finite element approximation to the velocity potential be given by

$$\phi^h(x) = \sum_{i=1}^{n} \phi_i \alpha_i(x),$$

where the α_i are the piecewise linear basis functions (4.13) and the ϕ_i are parameters of the solution. Thus the finite dimensional version of the functional of the constrained 'p' principle is given by

$$L(\boldsymbol{\phi}) = \int_{x_1}^{x_n} p({\phi^h}', E) B \, dx + C B_e \left(\phi^h(x_n) - \phi^h(x_1) \right),$$

where $E(x) = \tilde{E} + gh(x)$ and $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_n)^T$. The approximation to the velocity potential is determined by the $\boldsymbol{\phi}$ which causes L to be stationary, that is, the $\boldsymbol{\phi}$ which satisfies

$$F_i(\boldsymbol{\phi}) = \frac{\partial L}{\partial \phi_i} = \int_{x_1}^{x_n} p_{{\phi^h}'} \alpha_i' B \, dx + C B_e \left(\alpha_i(x_n) - \alpha_i(x_1) \right) = 0 \qquad i = 1, \ldots, n. \tag{4.33}$$

The solution of the non-linear set of equations (4.33) is found using Newton's method. The Jacobian is given by

$$J(\boldsymbol{\phi}) = \{J_{ij}\} = \left\{ \frac{\partial F_i}{\partial \phi_j} \right\} = \left\{ \frac{\partial^2 L}{\partial \phi_j \partial \phi_i} \right\} = \left\{ \int_{x_1}^{x_n} p_{{\phi^h}'{\phi^h}'} \alpha_i' \alpha_j' B \, dx \right\},$$

which is the Hessian of L and has the form of a weighted mass matrix, with weight $p_{{\phi^h}'{\phi^h}'} B$. From (4.32) J is negative definite for wholly subcritical flows and positive definite for wholly supercritical flows.

Given an initial approximation $\boldsymbol{\phi}^0$ to the solution $\boldsymbol{\phi}$ Newton's method produces a sequence of approximations $\boldsymbol{\phi}^k$ from

$$\boldsymbol{\phi}^{k+1} = \boldsymbol{\phi}^k + \delta \boldsymbol{\phi}^k, \tag{4.34}$$

where

$$J(\boldsymbol{\phi}^k)\,\delta\boldsymbol{\phi}^k = -\mathbf{F}(\boldsymbol{\phi}^k). \tag{4.35}$$

The sequence ends when

$$\max_i \left|\delta\phi_i^k\right| < \text{tolerance}. \tag{4.36}$$

The Jacobian and the vector $\mathbf{F}$ are integrated exactly. The Jacobian is tridiagonal and (4.35) is solved by Gaussian elimination and back substitution. The initial approximation $\boldsymbol{\phi}^0$ is given by

$$\phi_i^0 = (i-1)\,v^0 \qquad i = 1, \ldots, n,$$

where v^0 is assigned a value which determines whether the approximation being calculated is an approximation to subcritical or to supercritical flow. Let $c_*^{\min} = \min_{x\in[x_1,x_n]} c_*$, where c_* is defined by (2.63). Then, if $v^0 < \frac{x_n - x_1}{n-1} c_*^{\min}$, the approximation will be subcritical. Let $c_*^{\max} = \max_{x\in[x_1,x_n]} c_*$. Then, if $v^0 > \frac{x_n-x_1}{n-1} c_*^{\max}$, the approximation will be supercritical.

The algorithm is implemented on the grid (4.6), with $x_e = 0$, $x_o = 10$ and $n = 21$. The energy $\tilde{E}$ is again taken to be 50. Approximations to flows in channels with breadths given by (4.15) and (4.16) and fluid depths below the level $z = 0$ given by (4.17) and (4.18) are considered.

For $h = h_1$ the value of mass flow at inlet $C = C_*$, where C_* is given by (4.20), is used to give examples of critical flows and $C = 10$ is used to give examples of non-critical flows.

Consider the channel with breadth $B = B_{1,6}$ and let the tolerance in (4.36) be 10^{-3}. The method converges to the subcritical approximation in 4 iterations, using $v^0 = 1$, and to the supercritical approximation in 5 iterations, using $v^0 = 5$,

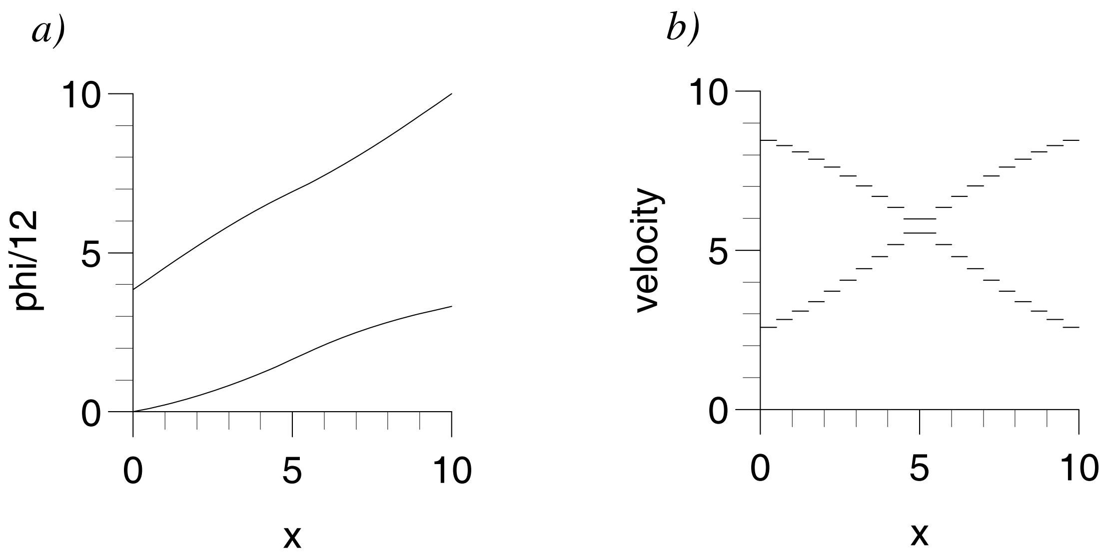

Figure 4.9: *a*) Velocity potential and *b*) velocity approximations for $B = B_{1,6}$ and $h = h_1$.

for non-critical flows. For critical flows the method converges to the subcritical approximation in 7 iterations, using $v^0 = 1$, and to the supercritical approximation in 8 iterations, using $v^0 = 5$. Results for the critical flows are shown in Figure 4.9. Figure 4.9*a* shows the piecewise linear velocity potential approximations, the top line corresponding to supercritical flow and the bottom line to subcritical flow. Figure 4.9*b* shows the piecewise constant velocity approximations derived from the gradients of the velocity potential approximations in each element. Notice that the velocity approximations are approximately symmetric about the line $x = 5$ as is expected for flows in a channel whose breadth and equilibrium depth functions are symmetric about this line.

For $B = B_{1,2}$ and $h = h_2$ with $H = 0.2$ Figure 4.10*a* shows the piecewise linear approximations to the velocity potential for subcritical and supercritical

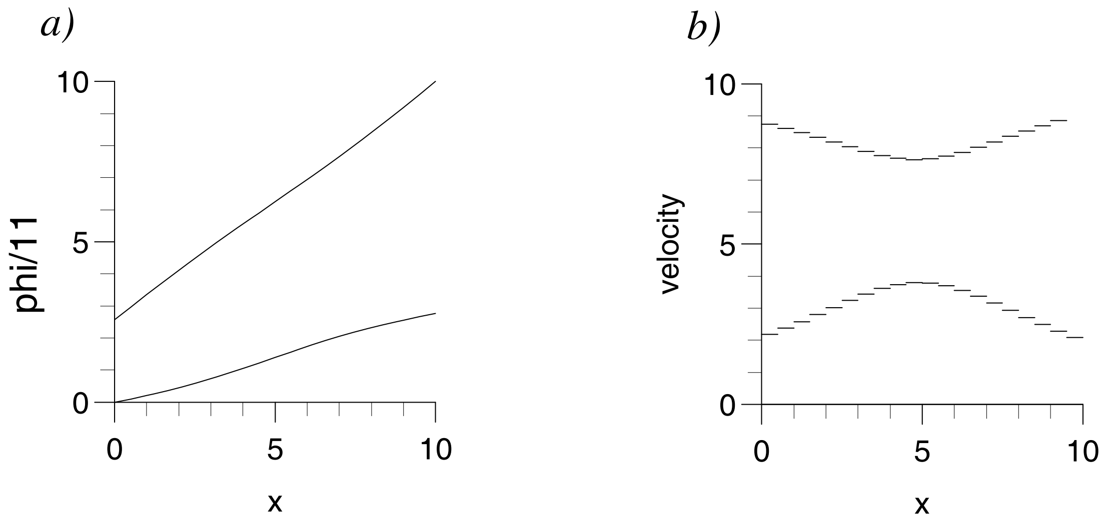

Figure 4.10: *a*) Velocity potential and *b*) velocity approximations for $B = B_{1,2}$ and $h = h_2$ with $H = 0.2$.

flows with $C = 10$. The corresponding piecewise constant approximations to the velocity are given in Figure 4.10*b*.

4.3.2 Errors

The L_2 error of the approximations to the velocity, derived in Section 4.3.1, is defined by

$$\left\| v - \phi^{h\prime} \right\| = \left(\int_{x_e}^{x_o} \left(v - \phi^{h\prime} \right)^2 dx \right)^{\frac{1}{2}}.$$

Table 4.3 shows the L_2 errors for piecewise constant velocity approximations in the channel with $B = B_{1,2}$ and $h = h_1$. The energy $\tilde{E}$ is given the value 50, $C = C_*$, defined by (4.20), is used to derive the critical approximations and $C = 10$ produces the non-critical approximations. It can be seen that, as the grid is refined, the convergence of $\phi^{h\prime}$ to v is linear.

n	Δx	critical flows		non-critical flows	
		subcritical	supercritical	subcritical	supercritical
3	$\frac{10}{2}$	3.266×10^{0}	2.744×10^{0}	1.933×10^{0}	1.444×10^{0}
5	$\frac{10}{2^2}$	1.582×10^{0}	1.352×10^{0}	8.914×10^{-1}	6.549×10^{-1}
9	$\frac{10}{2^3}$	7.782×10^{-1}	6.653×10^{-1}	4.449×10^{-1}	3.261×10^{-1}
17	$\frac{10}{2^4}$	3.861×10^{-1}	3.296×10^{-1}	2.227×10^{-1}	1.632×10^{-1}
33	$\frac{10}{2^5}$	1.923×10^{-1}	1.640×10^{-1}	1.114×10^{-1}	8.164×10^{-2}
65	$\frac{10}{2^6}$	9.597×10^{-2}	8.179×10^{-2}	5.569×10^{-2}	4.082×10^{-2}
129	$\frac{10}{2^7}$	4.794×10^{-2}	4.084×10^{-2}	2.784×10^{-2}	2.041×10^{-2}
257	$\frac{10}{2^8}$	2.396×10^{-2}	2.041×10^{-2}	1.392×10^{-2}	1.021×10^{-2}
513	$\frac{10}{2^9}$	1.198×10^{-2}	1.020×10^{-2}	6.961×10^{-3}	5.103×10^{-3}
1025	$\frac{10}{2^{10}}$	5.987×10^{-3}	5.100×10^{-3}	3.481×10^{-3}	2.552×10^{-3}

Table 4.3: L_2 errors for piecewise constant velocity approximations.

4.4 The Constrained 'p' Principle — Adaptive Grid

The approximations derived so far in this chapter have all been defined on the fixed regular grid given by the points (4.6). In this section a method of generating irregular grids using the constrained 'p' principle (3.116) is investigated.

The method of generating irregular grids and the corresponding approximations to the velocity potential using (3.116) is similar to the method of Section 4.3 in that a finite element expansion for the velocity potential is substituted into the functional (4.31) and the values of the parameters of the expansion are found such that the functional is stationary with respect to variations. The difference here is that the positions of the internal grid points are also allowed to vary.

Let the domain of integration $[x_e, x_o]$ be divided initially into $n-1$ regular intervals $[x_i, x_{i+1}]$ $(i = 1, \ldots, n-1)$ by the points x_i $(i = 1, \ldots, n)$ defined by (4.6). Then let

$$\phi^h(x, \mathbf{x}) = \phi_1 \alpha_1(x, x_1, x_2) + \sum_{i=2}^{n-1} \phi_i \alpha_i(x, x_{i-1}, x_i, x_{i+1}) + \phi_n \alpha_n(x, x_{n-1}, x_n)$$

be the finite element expansion of the velocity potential ϕ, defined on this grid. The α_i are the piecewise linear basis functions given by

$$\alpha_1(x, x_1, x_2) = \begin{cases} \dfrac{x_2 - x}{x_2 - x_1} & x \in [x_1, x_2] \\ 0 & x \notin [x_1, x_2] \end{cases},$$

$$\alpha_i(x, x_{i-1}, x_i, x_{i+1}) = \begin{cases} \dfrac{x - x_{i-1}}{x_i - x_{i-1}} & x \in [x_{i-1}, x_i] \\ \dfrac{x_{i+1} - x}{x_{i+1} - x_i} & x \in [x_i, x_{i+1}] \\ 0 & x \notin [x_{i-1}, x_{i+1}] \end{cases} \qquad i = 2, \ldots, n-1,$$

$$\alpha_n(x, x_{n-1}, x_n) = \begin{cases} \dfrac{x - x_{n-1}}{x_n - x_{n-1}} & x \in [x_{n-1}, x_n] \\ 0 & x \notin [x_{n-1}, x_n] \end{cases},$$

the ϕ_i are the values of the approximation at the grid points and $\mathbf{x} = (x_1, \ldots, x_n)^T$ is the vector of grid points.

The discrete version of the functional of the constrained principle, in this case, is

$$L(\boldsymbol{\phi}, \mathbf{x}) = \left(\int_{x_1}^{x_2} + \cdots + \int_{x_{n-1}}^{x_n} \right) p(\phi^{h\prime}, E) B \, dx + C B_e \left(\phi^h(x_n, \mathbf{x}) - \phi^h(x_1, \mathbf{x}) \right),$$

where $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_n)^T$, $\phi^{h\prime} \equiv \frac{\partial \phi^h}{\partial x}$ and $E = \tilde{E} + gh$. The initial finite element solution for the velocity potential is found by solving

$$F_i(\boldsymbol{\phi}, \mathbf{x}) = \frac{\partial L}{\partial \phi_i} = \left(\int_{x_1}^{x_2} + \cdots + \int_{x_{n-1}}^{x_n} \right) p_{\phi^{h\prime}} \alpha_i' B \, dx + C B_e \left[\alpha_i \right]_{x_1}^{x_n} = 0 \quad i = 1, \ldots, n \tag{4.37}$$

for $\boldsymbol{\phi}$ with the x_i fixed and given by (4.6). This is done using Newton's method, as in Section 4.3.

New positions for the internal grid points are then found by solving

$$\begin{aligned} G_i(\boldsymbol{\phi}, \mathbf{x}) &= \frac{\partial L}{\partial x_i} \\ &= -\left[pB \right]_{x_i} + \left(\int_{x_1}^{x_2} + \cdots + \int_{x_{n-1}}^{x_n} \right) p_{\phi^{h\prime}} \frac{\partial \phi^{h\prime}}{\partial x_i} B \, dx + C B_e \left[\frac{\partial \phi^h}{\partial x_i} \right]_{x_1}^{x_n} = 0 \\ & \qquad i = 2, \ldots, n-1, \end{aligned} \tag{4.38}$$

for x_i $(i = 2, \ldots, n-1)$, by Newton's method. The Jacobian is the $(n-2) \times (n-2)$ matrix given by

$$\begin{aligned} J(\boldsymbol{\phi}, \mathbf{x}) &= \{J_{ij}\} = \left\{ \frac{\partial L}{\partial x_j \partial x_i} \right\} \\ &= \Bigg\{ - \left[p_{\phi^{h\prime}} \frac{\partial \phi^{h\prime}}{\partial x_j} B \right]_{x_i} - \left[p_{\phi^{h\prime}} \frac{\partial \phi^{h\prime}}{\partial x_i} B \right]_{x_j} \\ &\quad + \left(\int_{x_1}^{x_2} + \cdots + \int_{x_{n-1}}^{x_n} \right) \left(p_{\phi^{h\prime}\phi^{h\prime}} \frac{\partial \phi^{h\prime}}{\partial x_j} \frac{\partial \phi^{h\prime}}{\partial x_i} + p_{\phi^{h\prime}} \frac{\partial^2 \phi^{h\prime}}{\partial x_j \partial x_i} \right) B \, dx \\ &\quad + C B_e \left[\frac{\partial^2 \phi^h}{\partial x_j \partial x_i} \right]_{x_1}^{x_n} \Bigg\}, \end{aligned}$$

which is tridiagonal so that the equation

$$J(\boldsymbol{\phi}, \mathbf{x}^k) \, \delta \mathbf{x}^k = -\mathbf{G}(\boldsymbol{\phi}, \mathbf{x}^k)$$

is solved for $\delta \mathbf{x}^k$ by Gaussian elimination and back substitution. A sequence of approximations to $\mathbf{x}$ is generated using

$$\mathbf{x}^{k+1} = \mathbf{x}^k + \delta \mathbf{x}^k.$$

The process is repeated until

$$\max_i \left|\delta x_i^k\right| < \text{tolerance}. \tag{4.39}$$

The procedure is to return to (4.37) and find $\boldsymbol{\phi}$ on the new grid, using the solution for $\boldsymbol{\phi}$ on the previous grid as the initial approximation $\boldsymbol{\phi}^0$. Equations (4.38) are then solved again to modify the grid further, the initial approximation $\mathbf{x}^0$ being given the values of the solution $\mathbf{x}$ at the previous iteration.

This process is repeated until

$$\max_i \left(|F_i|, |G_i|\right) \tag{4.40}$$

changes by less than some percentage between successive iterations on the positions of the grid nodes.

The energy $\tilde{E}$ is assigned the value 50. The two values of mass flow at inlet $C = C_*$, given by (4.20), which generates critical flows, and $C = 10$, which generates examples of non-critical flows, are considered. The criterion for convergence using (4.40) is that (4.40) changes by less than 5% between two successive iterations.

Results are given for the channel with $x_e = 0$, $x_o = 10$ and breadth $B = B_3$, where

$$B_3(x) = 8 + 2\cos\left(\frac{\pi x}{5}\right),$$

which is shown in Figure 4.11. The fluid depth below the reference level $z = 0$ is $h = h_1$ (equation (4.17)).

The tolerance on the Newton iteration for $\boldsymbol{\phi}$ is taken to be 10^{-7} and on (4.39) to be $\frac{x_o - x_e}{n-1}10^{-4}$, where n is the number of grid points. The approximations to subcritical and supercritical velocities for $C = C_*$, derived as the gradients of the piecewise linear approximations to the velocity potential, for $n = 5$, 7 and 11 are

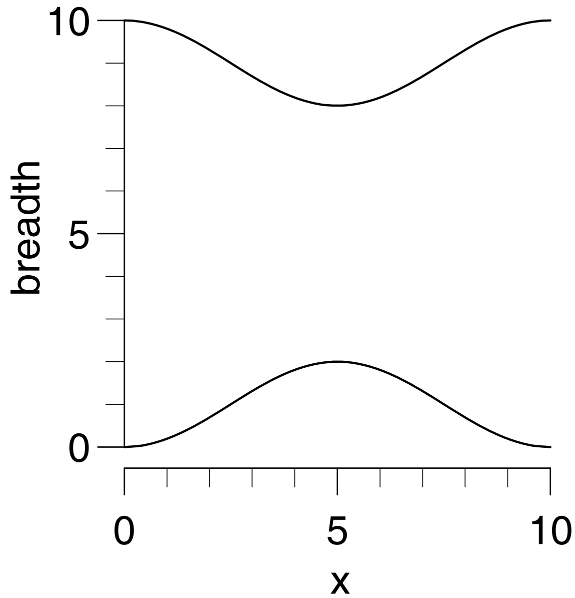

Figure 4.11: The breadth function $B_3(x)$.

shown in Figure 4.12. The dots represent the final positions of the grid points. The subcritical approximation with $n = 5$, Figure 4.12a, requires 28 sets of iterations to converge and the grid points have moved from their initial equi-spaced positions. The subcritical approximations with $n = 7$ and 11, Figures 4.12c and 4.12e respectively, both converge after one set of iterations, the grid points have moved slightly towards the line $x = 5$, although this is not obvious from the figure. The supercritical approximations with $n = 5$, 7 and 11, Figures 4.12b, 4.12d and 4.12f respectively, all converge after one set of iterations; there is no discernible motion of the grid points in these cases.

Table 4.4 gives the L_2 errors of the approximate solutions for various n, in the same channel and with the same conditions as above. For comparison, the corresponding L_2 errors for approximations generated on fixed, equi-spaced grids are also given. For the supercritical approximations there is a slight improvement in the L_2 error with $n = 5$, 7 and 9. For the subcritical approximations the

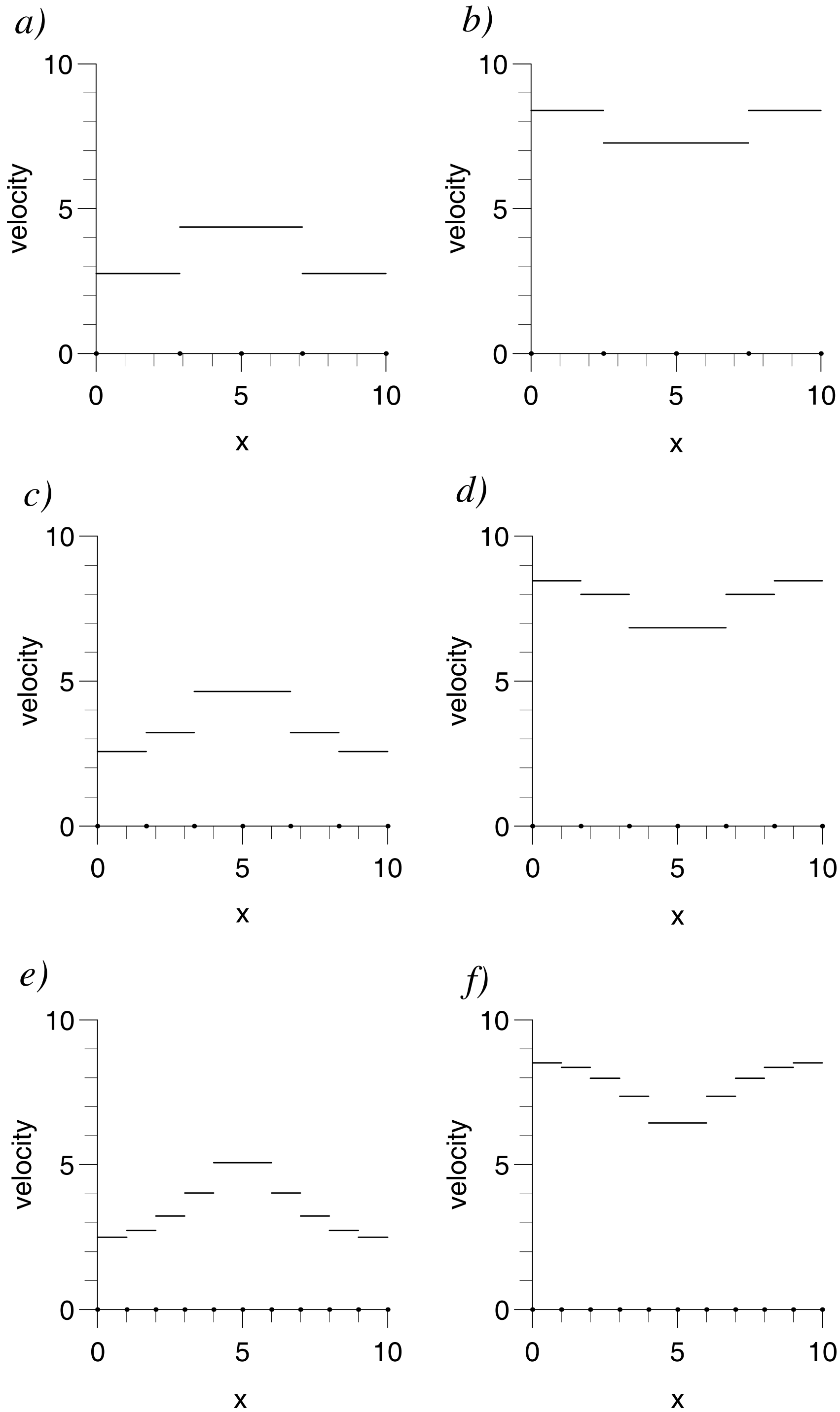

Figure 4.12: Velocity approximations for a) and b) $n = 5$, c) and d) $n = 7$ and e) and f) $n = 11$

n	Δx	fixed grids		adaptive grids	
		subcritical	supercritical	subcritical	supercritical
3	$\frac{10}{2}$	3.322×10^{0}	2.730×10^{0}	3.322×10^{0}	2.730×10^{0}
5	$\frac{10}{4}$	1.826×10^{0}	1.574×10^{0}	1.608×10^{0}	1.572×10^{0}
7	$\frac{10}{6}$	1.212×10^{0}	1.054×10^{0}	1.208×10^{0}	1.053×10^{0}
9	$\frac{10}{8}$	9.045×10^{-1}	7.876×10^{-1}	9.030×10^{-1}	7.875×10^{-1}
11	$\frac{10}{10}$	7.211×10^{-1}	6.280×10^{-1}	7.204×10^{-1}	6.280×10^{-1}
21	$\frac{10}{20}$	3.576×10^{-1}	3.111×10^{-1}	3.575×10^{-1}	3.111×10^{-1}
31	$\frac{10}{30}$	2.377×10^{-1}	2.066×10^{-1}	2.376×10^{-1}	2.066×10^{-1}

Table 4.4: Comparison of L_2 errors for piecewise constant velocity approximations.

improvement is more pronounced with $n = 5$, 7 and 9 but is only slight for $n = 11$ and negligible for $n = 21$ and 31.

4.5 Discontinuous Flows — The Constrained 'r' Principle

In this section the 'r' principle, based on the functional (3.113), constrained to satisfy the conservation of mass equation is used to generate approximations to the depths in discontinuous shallow water flows. In order to achieve an accurate finite element approximation to the depth one of the grid nodes must be positioned at the point of discontinuity; this requires the use of irregular grids.

The functional of the 'r' principle for discontinuous flow (3.123), constrained to satisfy conservation of mass, is

$$S_2^c(d, x_s) = \int_{x_e}^{x_s} (r(Q, d) + E_e d)\, B\, dx + \int_{x_s}^{x_o} (r(Q, d) + E_o d)\, B\, dx, \qquad (4.41)$$

where $Q(x) = \frac{CB_e}{B(x)}$. The equilibrium fluid depth h is assumed constant so that the energy E, defined by (2.35), has the constant value E_e in $[x_e, x_s)$ and the constant value E_o in $(x_s, x_o]$. The values of E_e and E_o are deduced from boundary conditions and, from (2.78), are such that $E_e > E_o$. The natural conditions of the first variation of S_2^c are

$$\begin{aligned} r_d + E_e &= 0 \quad \text{in } (x_e, x_s), \\ r_d + E_o &= 0 \quad \text{in } (x_s, x_o), \\ [r + Ed]_{x_s} &= 0, \end{aligned} \qquad (4.42)$$

where the coefficients of the total variation of d on either side of x_s have been equated, that is, the equation

$$\delta d|_{x_{s+}} + d'|_{x_{s+}} \delta x_s = \delta d|_{x_{s-}} + d'|_{x_{s-}} \delta x_s \qquad (4.43)$$

is assumed satisfied. It is not obvious how, in practice, it might be possible to construct variations that satisfy (4.43). It is the assumption that (4.43) is true which gives rise to the natural jump condition $(4.42)_3$. So, if variations satisfying (4.43) cannot be found then, in order to generate approximations to the depth in discontinuous shallow water flows, $(4.42)_3$ must be enforced in some way.

The method of finding approximations is based on that of Section 4.1 in that finite element expansions for d in the regions of the domain before and after the discontinuity are substituted into a finite dimensional version of (4.41). Then the node which separates the pre- and post-discontinuity approximations must be repositioned in order to satisfy $(4.42)_3$. An algorithm based on this is given in Section 4.5.1. The method is then extended in Section 4.5.2 to give an algorithm generating approximations on grids where all of the internal grid nodes are positioned using $(4.42)_3$.

4.5.1 Grid with One Moving Node

Let the domain of the problem $[x_e, x_o]$ be divided into $n-1$ adjacent regular intervals $[x_i, x_{i+1}]$ by the points x_i $(i = 1, \ldots, n)$ defined by (4.6). One of these nodes must be chosen as being the initial approximation to the position of the discontinuity and the number of the node nearest to the actual position of the hydraulic jump needs to be deduced. Let x_N be the initial guess for the jump position.

The method requires that approximations to the flow in front of and behind the jump are generated separately and coupled, by means of a discontinuity, at the position of the hydraulic jump.

Let the approximation to the depth in the pre-jump region $[x_1, x_N]$ be

$$d^e(x) = \sum_{i=1}^{N} d_i^e \alpha_i^e(x),$$

where

$$\alpha_1^e(x) = \begin{cases} \dfrac{x_2 - x}{x_2 - x_1} & x \in [x_1, x_2] \\ 0 & x \notin [x_1, x_2] \end{cases},$$

$$\alpha_i^e(x) = \begin{cases} \dfrac{x - x_{i-1}}{x_i - x_{i-1}} & x \in [x_{i-1}, x_i] \\ \dfrac{x_{i+1} - x}{x_{i+1} - x_i} & x \in [x_i, x_{i+1}] \\ 0 & x \notin [x_{i-1}, x_{i+1}] \end{cases} \qquad i = 2, \ldots, N-1,$$

$$\alpha_N^e(x) = \begin{cases} \dfrac{x - x_{N-1}}{x_N - x_{N-1}} & x \in [x_{N-1}, x_N] \\ 0 & x \notin [x_{N-1}, x_N] \end{cases} .$$

Let the approximation to the depth in the post-jump region $[x_N, x_n]$ be

$$d^o(x) = \sum_{i=N}^{n} d_i^o \alpha_i^o(x),$$

where

$$\alpha_N^o(x) = \begin{cases} \dfrac{x_{N+1} - x}{x_{N+1} - x_N} & x \in [x_N, x_{N+1}] \\ 0 & x \notin [x_N, x_{N+1}] \end{cases} ,$$

$$\alpha_i^o(x) = \begin{cases} \dfrac{x - x_{i-1}}{x_i - x_{i-1}} & x \in [x_{i-1}, x_i] \\ \dfrac{x_{i+1} - x}{x_{i+1} - x_i} & x \in [x_i, x_{i+1}] \\ 0 & x \notin [x_{i-1}, x_{i+1}] \end{cases} \qquad i = N+1, \ldots, n-1,$$

$$\alpha_n^o(x) = \begin{cases} \dfrac{x - x_{n-1}}{x_n - x_{n-1}} & x \in [x_{n-1}, x_n] \\ 0 & x \notin [x_{n-1}, x_n] \end{cases} .$$

The algorithm is in two parts. Firstly the two finite element approximations d^e and d^o are derived by finding the values of $\mathbf{d}^e = (d_1^e, \ldots, d_N^e)^T$ and $\mathbf{d}^o = (d_N^o, \ldots, d_n^o)^T$ such that

$$L(\mathbf{d}^e, \mathbf{d}^o) = \int_{x_1}^{x_N} \left(r(Q, d^e) + E_e d^e\right) B \, dx + \int_{x_N}^{x_n} \left(r(Q, d^o) + E_o d^o\right) B \, dx$$

is stationary with respect to variations in $\mathbf{d}^e$ and $\mathbf{d}^o$. This requires solving the two sets of equations

$$\frac{\partial L}{\partial d_i^e} = 0 \;\; i = 1, \ldots, N \quad \text{and} \quad \frac{\partial L}{\partial d_i^o} = 0 \;\; i = N, \ldots, n,$$

using Newton's method, as described in Section 4.1. The initial approximation to $\mathbf{d}^e$ must be supercritical in order that the supercritical flow in the region before the jump is approximated and the initial approximation to $\mathbf{d}^o$ must be subcritical.

The second stage of the algorithm is to alter the position of x_N by employing the jump condition $(4.42)_3$. If x_s is the exact position of the jump and d is the exact solution then, from $(4.42)_3$,

$$(r(Q,d) + E_e d)|_{x_{s-}} - (r(Q,d) + E_o d)|_{x_{s+}} = 0.$$

If the approximation satisfies

$$\left|(r(Q,d^e) + E_e d^e)|_{x_N} - (r(Q,d^o) + E_o d^o)|_{x_N}\right| < \text{tolerance}, \tag{4.44}$$

for some specified tolerance, then the approximate solution has been found and x_N is the approximate position of the hydraulic jump. If (4.44) is not satisfied then a new approximation to the jump position is found using the jump condition, as follows.

The equation

$$r(Q_s, d^e_N) + E_e d^e_N - r(Q_s, d^o_N) - E_o d^o_N = 0 \tag{4.45}$$

is solved for Q_s, the value of the mass flow which would occur at the jump if d^e_N and d^o_N were the actual depths of the flow before and after the jump. The conservation of mass constraint gives

$$Q(x)B(x) = CB_e \qquad x \in [x_1, x_n]$$

and, since $B(x)$ and C are to be specified, this can be used to find the point x_s^N in the channel where the mass flow is Q_s. From the discussion in Chapter 2 only flows which are critical at the channel throat will be considered so that

$$B(x_s^N) = \frac{CB_e}{Q_s} \tag{4.46}$$

can be solved, by bisection, to give a unique value for x_s^N.

The process which occurs on solving (4.45) is explained more fully in the following pages and then the algorithm for positioning a node at the point of discontinuity is completed.

Let $\hat{x}_s$ be an approximation to the exact position x_s of the hydraulic jump. From the conservation of mass equation the mass flow at $\hat{x}_s$ can be calculated to be $\hat{Q}_s = \frac{CB_e}{B(\hat{x}_s)}$. The flow on the inlet side of the jump must be supercritical and on the outlet side must be subcritical. Assume this to be the case here. Then the values of the depth at points immediately either side of $\hat{x}_s$ can be calculated using the definitions of mass flow (2.34) and energy (2.35).

Let d^- be the supercritical solution of

$$E_e = gd^- + \frac{1}{2}\left(\frac{\hat{Q}_s}{d^-}\right)^2,$$

that is, the root which lies between 0 and $\frac{2E_e}{3g}$. Let d^+ be the subcritical solution of

$$E_o = gd^+ + \frac{1}{2}\left(\frac{\hat{Q}_s}{d^+}\right)^2,$$

that is, the root which lies between $\frac{2E_o}{3g}$ and $\frac{E_o}{g}$. In the approximation method, if $x_N = \hat{x}_s$, d_N^e is an approximation to d^- and d_N^o is an approximation to d^+.

In Figure 4.13 the graphs of flow stress P, defined by (2.72), against mass flow Q for $E = E_e$ and $E = E_o$, with $E_o < E_e$, are drawn in solid lines. The two dotted lines are the curves

$$P^-(Q) = r(Q, d^-) + E_e d^-,$$

which touches the supercritical branch of the E_e curve at $Q = \hat{Q}_s$, and

$$P^+(Q) = r(Q, d^+) + E_o d^+,$$

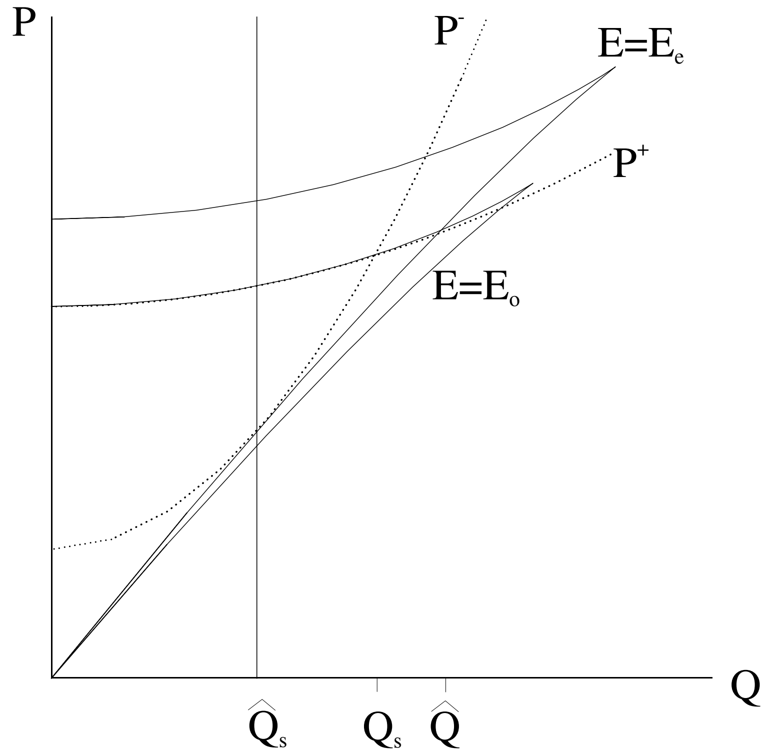

Figure 4.13: Intersection of P^+ and P^- for $\hat{Q}_s < \hat{Q}$.

which touches the subcritical branch of the E_o curve at $Q = \hat{Q}_s$. If d_N^e and d_N^o are the exact values of d^- and d^+ then, solving equation (4.45) for Q_s, is equivalent to finding the value of Q at which the two curves P^- and P^+ intersect.

For any value of $\hat{Q}_s$, lying in the range $0 \leq \hat{Q}_s \leq \frac{1}{g}\left(\frac{2E_o}{3}\right)^{\frac{3}{2}}$, it can be shown that the curve P^- lies above the supercritical branch of $P(Q)$ for $E = E_e$, except at the point $Q = \hat{Q}_s$ where the two curves are tangent. It can also be shown that P^+ lies below the subcritical branch of $P(Q)$ for $E = E_o$, except at the point $Q = \hat{Q}_s$ where the two curves are tangent to one another.

In Figure 4.13 $\hat{Q}_s$ is less than the actual value of mass flow $\hat{Q}$ at the jump and in these circumstances the value $Q = Q_s$, at the point of intersection of P^- and P^+, always lies between $\hat{Q}_s$ and $\hat{Q}$ and thus is an improvement on $\hat{Q}_s$. Repeating the process by letting $\hat{Q}_s = Q_s$, generating the corresponding curves P^- and

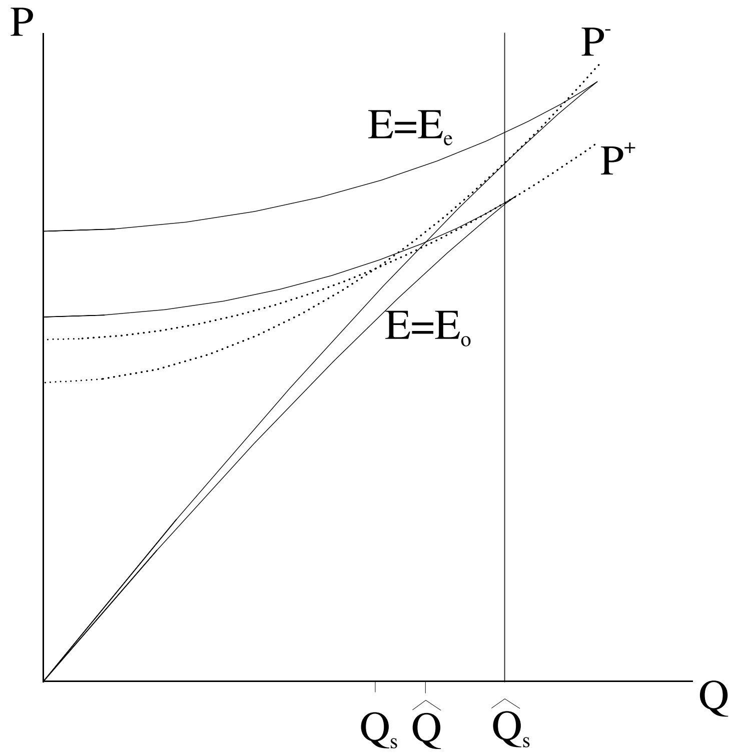

Figure 4.14: Intersection of P^+ and P^- for $\hat{Q}_s > \hat{Q}$.

P^+ and finding the point of intersection yields a sequence of values of Q_s, each lying between the previous value of Q_s and $\hat{Q}$. Thus this iteration will eventually converge to give the exact value of the mass flow at the jump, from which, using the conservation of mass equation, can be deduced the position of the jump.

Figure 4.14 gives an example of the graph for $\hat{Q}_s$ greater than $\hat{Q}$. In these circumstances if the P^- and P^+ curves intersect at a value of $Q = Q_s$ then $Q_s < \hat{Q}$. It is not possible to prove that a value of $Q_s > 0$ exists for which $P^-(Q_s) = P^+(Q_s)$ and, even if such a value does exist, the mass flow might never achieve the value Q_s in a particular flow. If such a Q_s does exist and is achieved at a point in the domain then, letting $\hat{Q}_s = Q_s$, gives the situation in Figure 4.13.

The situation is slightly different in the approximation case. Equation (2.64) gives an expression for Q as a function of d, that is,

$$Q = d\sqrt{2\,(E - gd)}.$$

Let

$$Q^e = d_N^e\sqrt{2\,(E_e - gd_N^e)} \quad \text{and} \quad Q^o = d_N^o\sqrt{2\,(E_o - gd_N^o)},$$

where d_N^e is the approximation to d on the x_e side of x_N, the approximation to the jump position, and d_N^o is the approximation on the x_o side of x_N. Let

$$P^e(Q) = r(Q, d_N^e) + E_e d_N^e$$

and

$$P^o(Q) = r(Q, d_N^o) + E_o d_N^o.$$

Figure 4.15 shows a sketch of the curves P^e and P^o on a graph of P as a function of Q for two different values of E, $E_o < E_e$. Notice that P^e touches the E_e curve at $Q = Q^e$ and P^o touches the E_o curve at $Q = Q^o$. Note also that neither Q^e nor Q^o is necessarily equal to the mass flow $Q(x_N) = \frac{CB_e}{B(x_N)}$. The point of intersection of the P^e and P^o curves gives the value of Q_s equivalent to solving (4.45). There are four possible situations arising.

1. $Q^e < \hat{Q}$ and $Q^o < \hat{Q}$ so that $\hat{Q} > Q_s > \max(Q^e, Q^o)$.

2. $Q^e < \hat{Q}$ and $Q^o > \hat{Q}$ so that $\hat{Q} > Q_s > Q^e$.

3. $Q^e > \hat{Q}$ and $Q^o > \hat{Q}$ so that $Q_s < \hat{Q}$.

4. $Q^e > \hat{Q}$ and $Q^o < \hat{Q}$ so that $Q_s < \hat{Q}$.

These properties are deduced using the facts that P^e is tangent to the E_e curve at $Q = Q^e$ and always lies on or above the supercritical branch and that P^o is tangent to the E_o curve at $Q = Q^o$ and always lies on or below the subcritical

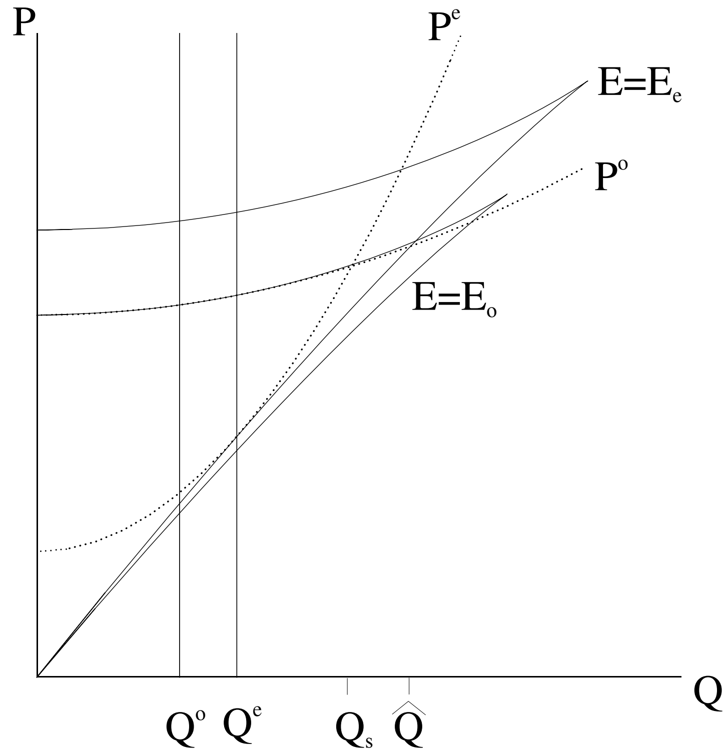

Figure 4.15: Intersection of P^e and P^o curves.

branch. In cases 1 and 2, although $Q^e < Q_s < \hat{Q}$, it does not necessarily mean that $Q(x_N) < Q_s < \hat{Q}$. The convergence of the iteration using (4.45) depends on this being true, which it will be if d_N^e is close to the supercritical solution of $E_e = gd + \frac{1}{2}\left(\frac{Q(x_N)}{d}\right)^2$ so that Q^e is close to $Q(x_N)$. In cases 3 and 4 if $Q_s > 0$ exists, and is achieved in a particular flow, solving (4.45) to give a further approximation to the mass flow at the jump may improve the approximation, although this cannot be shown. In practice the values of Q^e and Q^o are sufficiently close together so that situations 2 and 3 occur only when the approximation to the jump position is very close to the actual jump position.

The algorithm for positioning a node at the jump is in two parts. Firstly, beginning with $N = n - 1$, the corresponding value of x_s^{n-1} is found using (4.45) and (4.46). Then, stepping backwards along the channel to the $n - 2$ th node, the value of x_s^{n-2} is found. If $(x_{n-1} - x_s^{n-1})(x_{n-2} - x_s^{n-2}) < 0$ then x_s lies between x_{n-1} and x_{n-2}. Otherwise the process is repeated until the node j is found, where $(x_j - x_s^j)(x_{j-1} - x_s^{j-1}) < 0$. Then, if $|x_j - x_s^j| < |x_{j-1} - x_s^{j-1}|$, the number N of the node to be moved to the jump position is j; otherwise $N = j - 1$.

Once the number of the node to be moved to the jump position has been established in this way, x_N is moved to x_s^N. The finite element approximations d^e and d^o are recalculated on the modified grid and, if (4.44) is still not satisfied, (4.45) and (4.46) are used to reposition x_N and the process is repeated until (4.44) is satisfied. The approximate solution has then been found and x_N is an approximation to the jump position.

The algorithm is applied to a grid with $x_e = 0$, $x_o = 10$ and $n = 21$. The energy at inlet E_e is given the value 50 and the mass flow at inlet $C = C_*$, where C_* is defined by (4.20), to give a critical flow in a channel with breadth $B = B_{1,k}$, defined by (4.15). The depth at outlet d_o is given for each case and is used to deduce the value of E_o, using the definitions of mass flow (2.34) and energy (2.35). From the conservation of mass equation $Q(x_o) = \frac{CB_e}{B(x_o)}$, which yields

$$E_o = gd_o + \frac{1}{2}\left(\frac{CB_e}{B(x_o)d_o}\right)^2.$$

The piecewise linear approximation to the discontinuous depth profile with $d_o = 4.69$ and breadth $B = B_{1,6}$ is given in Figure 4.16a. For a tolerance on the Newton iteration of 10^{-3} and on the jump condition (4.44) of 10^{-3}, the method converges in 3 iterations on the position of the discontinuity, once the node to be placed at the discontinuity has been found; in this case it is node number 16. These iterations require 15, 8 and 8 Newton iterations. The initial approximation on the original regular grid is given the values $d_i^e = 1$ $(i = 1, \ldots, N)$

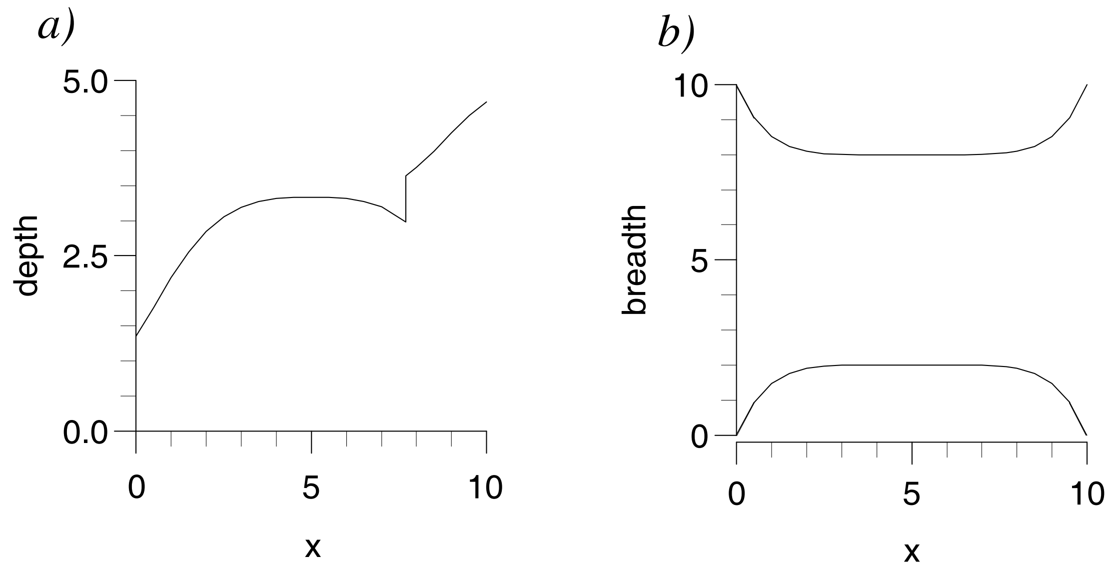

Figure 4.16: a) Piecewise linear depth approximation for $d_o = 4.69$ and b) $B_{1,6}(x)$.

and $d_i^o = 4.69$ $(i = N, \ldots, n)$. Once the number of the node to approximate the jump position is found subsequent approximations to the finite element solutions use the approximation on the previous grid as the first guess in Newton's method to find the approximation on the new grid. Figure 4.16b shows the breadth $B_{1,6}$.

The piecewise linear approximation for $d_o = 3.86$ is shown in Figure 4.17. This converges in 3 iterations on the position of node 20, which is selected by the algorithm to be moved to approximate the jump position, requiring 15, 4 and 4 Newton iterations.

The algorithm in this section generates approximations to the depth for discontinuous flows in channels, where the approximations are defined on grids in which all of the grid points except one are fixed. The one movable grid point is positioned, using the jump condition $(4.42)_3$, in such a way that $(4.42)_3$ is approximately satisfied. In Section 4.5.2 this method is extended, by allowing all of

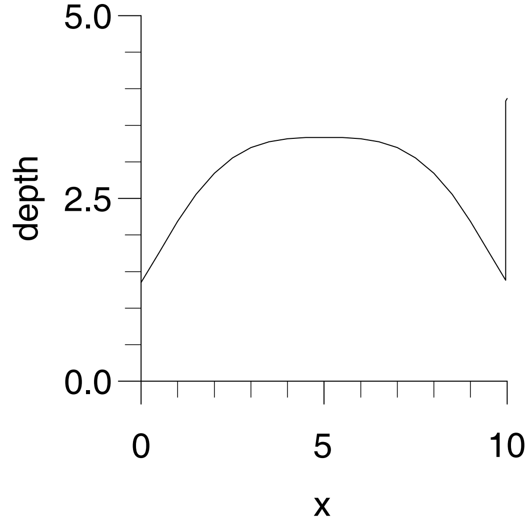

Figure 4.17: Piecewise linear depth approximation for $d_o = 3.86$.

the internal grid points to move, in order to generate irregular grids.

4.5.2 Adaptive Grids

The domain of the problem $[x_e, x_o]$ is divided into $n-1$ regular intervals by the points x_i $(i = 1, \ldots, n)$ defined by (4.6). Finite element approximations to the depth are generated separately on each interval $[x_i, x_{i+1}]$ and the jump condition $(4.42)_3$ is used at each internal node to reposition the node. Instead of having just two finite element approximations coupled at a point, as in Section 4.5.1, there will be $n-1$ approximations coupled at the $n-2$ internal grid points.

Let

$$d_i^h(x) = d_i^L \alpha_i^L(x) + d_i^R \alpha_i^R(x) \tag{4.47}$$

be the finite element approximation to d in the i th element $[x_i, x_{i+1}]$, where

$$\alpha_i^L(x) \quad = \quad \begin{cases} \dfrac{x_{i+1} - x}{x_{i+1} - x_i} & x \in [x_i, x_{i+1}] \\ 0 & x \notin [x_i, x_{i+1}] \end{cases} \qquad i = 1, \ldots, n-1,$$

$$\alpha_i^R(x) = \begin{cases} \dfrac{x - x_i}{x_{i+1} - x_i} & x \in [x_i, x_{i+1}] \\ 0 & x \notin [x_i, x_{i+1}] \end{cases} \qquad i = 1, \ldots, n-1.$$

Let N be the number of the node chosen to be the initial approximation to the position of the hydraulic jump. Then, in the element $[x_i, x_{i+1}]$,

$$E = E_e \qquad \text{if } i+1 \leq N$$

and

$$E = E_o \qquad \text{if } i \geq N,$$

where E_e is the value of the energy E at inlet and E_o is the value at outlet.

The finite element solution on each element is given by the values of $\mathbf{d}_i = (d_i^L, d_i^R)$ such that

$$\begin{aligned} L(\mathbf{d}_1, \ldots, \mathbf{d}_{n-1}) &= \sum_{i=1}^{N-1} \left(\int_{x_i}^{x_{i+1}} \left(r(Q, d_i^h) + E_e d_i^h \right) B \, dx \right) \\ &+ \sum_{i=N}^{n-1} \left(\int_{x_i}^{x_{i+1}} \left(r(Q, d_i^h) + E_o d_i^h \right) B \, dx \right), \end{aligned}$$

where $Q(x) = \frac{CB_e}{B(x)}$, is stationary with respect to variations in $\mathbf{d}_i$ $(i = 1, \ldots, n-1)$. The solutions of the $n-1$ sets of non-linear equations

$$\frac{\partial L}{\partial d_i^L} = 0 \;\; , \;\; \frac{\partial L}{\partial d_i^R} = 0 \quad i = 1, \ldots, n-1,$$

each with two unknowns, are found using Newton's method.

Once the $\mathbf{d}_i$ have been calculated on the initial grid the jump condition is applied at each internal node. If

$$\left| \left(r(Q, d_i^L) + E_1 d_i^L \right) \Big|_{x_i} - \left(r(Q, d_{i-1}^R) + E_2 d_{i-1}^R \right) \Big|_{x_i} \right| < \text{tolerance}, \tag{4.48}$$

where

$$E_1 = E_2 = E_e \qquad \text{if } i < N,$$

$$E_1 = E_o, \;\; E_2 = E_e \quad \text{if } i = N,$$

$$E_1 = E_2 = E_o \qquad \text{if } i > N,$$

for all $i = 2, \ldots, n-1$ and a specified tolerance, the required approximate solution has been found. If (4.48) is not satisfied for a particular value of i then

$$r(Q_i, d_i^L) + E_1 d_i^L - r(Q_i, d_{i-1}^R) - E_2 d_{i-1}^R = 0 \tag{4.49}$$

is solved for Q_i and the new position of the grid point x_i is found from Q_i using the conservation of mass law and bisection.

The grid point closest to the jump position in the regular grid defined by (4.6) is found in the same way as in Section 4.5.1. The approximation x_N to the jump position is initially taken to be x_{n-1}; equation (4.49) then yields the new approximation x_s^{n-1}. The process is repeated using x_{n-2} as the approximation to the jump position and then stepping backwards along the channel to each grid point in turn until $(x_{j-1} - x_s^{j-1})(x_j - x_s^j) < 0$ for some j. Then, if $|x_j - x_s^j| < |x_{j-1} - x_s^{j-1}|$, $N = j$ is the number of the node which will be used to approximate the jump position; otherwise $N = j - 1$.

With N fixed the $n - 1$ finite element approximations (4.47) are calculated on the new grid, using the solutions on the previous grid as the initial guess in Newton's method. If (4.48) is not satisfied for some i in the range $2, \ldots, n-1$, the internal grid points are repositioned using (4.49). The process is repeated until (4.48) is satisfied for all i in the range $2, \ldots, n-1$. An approximation to the depth for discontinuous shallow water flow has then been found.

The algorithm is applied in a channel where $x_e = 0$, $x_o = 10$ and $n = 21$. The breadth $B = B_{1,k}$ is given by (4.15). The energy at inlet E_e is given the value 50 and the mass flow at inlet is given the value which causes the flow to be critical at the channel throat, that is $C = C_*$, where C_* is defined by (4.20). The initial approximations to the depth used in Newton's method has the value

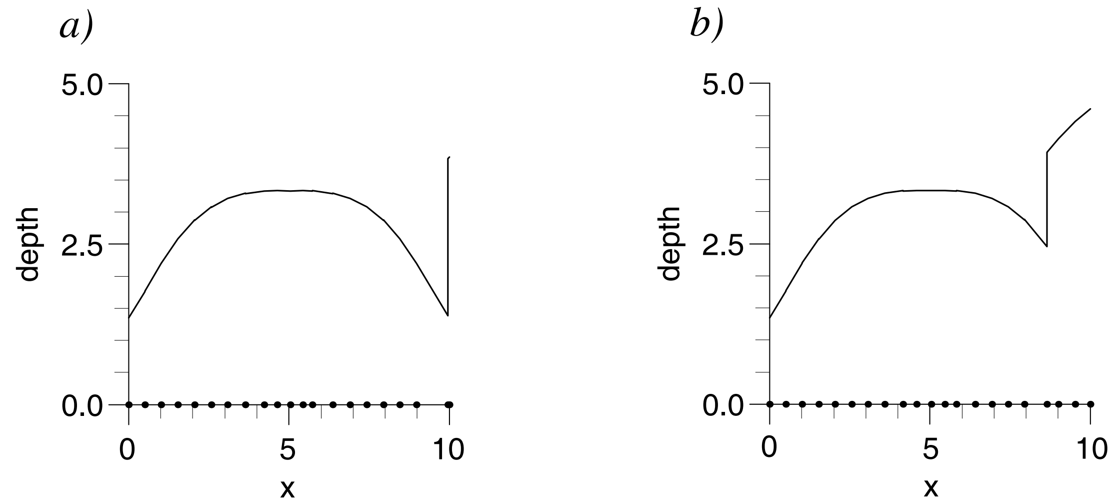

Figure 4.18: Piecewise linear depth approximations for *a*) $d_o = 3.86$ and *b*) $d_o = 4.6$.

1 at the nodes on the inlet side of x_N and the value of the outlet depth d_o at the nodes on the outlet side of x_N.

Figure 4.18*a* shows the result for $B = B_{1,6}$ and the outlet depth $d_o = 3.86$. The dots on the x-axis show the final positions of the grid points, grid point 20 approximates the jump position. For a tolerance on the Newton iteration of 10^{-5} and on the jump condition (4.48) of 10^{-3} the algorithm converges in 3 iterations on the positions of the grid nodes. Figure 4.18*b* shows the corresponding result for $d_o = 4.6$. Here node 18 approximates the jump position and, for the same tolerances, the algorithm converges in 3 iterations on the positions of the grid nodes.

Chapter 5

Approximations to Continuous Two-dimensional Shallow Water Flows

In this chapter the constrained variational principles derived in Section 3.6.2 are used to generate approximations to two-dimensional shallow water flows. The method is an extension of the method used in Chapter 4 to approximate one-dimensional flows.

The functionals of the constrained variational principles for steady state flows, (3.94), (3.97), (3.95) and (3.96), are

$$\begin{aligned}
L_1^c(\phi) &= \iint_D p(\boldsymbol{\nabla}\phi, E)\,dx\,dy + \int_\Sigma C\phi\,d\Sigma, \qquad (5.1)\\
L_2^c(\mathbf{Q}, d) &= \iint_D (r(\mathbf{Q}, d) + Ed)\,dx\,dy,\\
L_3^c(\mathbf{Q}) &= \iint_D P(\mathbf{Q}, E)\,dx\,dy,\\
L_4^c(\phi, d) &= \iint_D (-R(\boldsymbol{\nabla}\phi, d) + Ed)\,dx\,dy + \int_\Sigma C\phi\,d\Sigma, \qquad (5.2)
\end{aligned}$$

where, in L_2^c and L_3^c, the mass flow $\mathbf{Q}$ is required to satisfy $\boldsymbol{\nabla}.\mathbf{Q} = 0$ in D and $\mathbf{n}.\mathbf{Q} = C$ on Σ, for the given boundary function C, and, in L_1^c and L_4^c, the constraint $\mathbf{v} = \boldsymbol{\nabla}\phi$ has been applied. The energy E is the function $E(x, y) = \tilde{E} + gh(x, y)$, where $\tilde{E}$ is a known constant.

In the variational principles based on L_2^c and L_3^c the variations, $\delta\mathbf{Q}$, of $\mathbf{Q}$ must satisfy $\boldsymbol{\nabla}.(\mathbf{Q} + \delta\mathbf{Q}) = 0$ in D and $\mathbf{n}.(\mathbf{Q} + \delta\mathbf{Q}) = C$ on Σ, that is, $\boldsymbol{\nabla}.\,\delta\mathbf{Q} = 0$ in D and $\mathbf{n}.\,\delta\mathbf{Q} = 0$ on Σ. This can be achieved in practice by introducing a new variable $\psi = \psi(x, y)$, such that, $\mathbf{Q} = (\psi_y, -\psi_x)$. Then $\delta\mathbf{Q} = (\delta\psi_y, -\delta\psi_x)$ where $\mathbf{n}.(\delta\psi_y, -\delta\psi_x) = 0$ on Σ.

In this chapter the functional L_1^c is used to generate approximations to the velocity potential ϕ, from which approximations to the velocity $\mathbf{v}$ are deduced using $\mathbf{v} = \boldsymbol{\nabla}\phi$, and L_4^c is used to generate approximations to the depth d and to ϕ (and hence also to $\mathbf{v}$). The variational principles based on L_1^c and L_4^c do not require the variations to satisfy any boundary conditions.

Let the domain of the problem be the channel

$$D = \left\{(x, y) : x \in [x_e, x_o];\ y \in \left[-\frac{B(x)}{2}, \frac{B(x)}{2}\right]\right\},$$

where $B(x)$, the breadth of the channel, is a function to be defined. The channel has an axis of symmetry along the line $y = 0$ and so only the half of the channel

$$\tilde{D} = \left\{(x, y) : x \in [x_e, x_o];\ y \in \left[0, \frac{B(x)}{2}\right]\right\}$$

is considered, the flow over the region $D \backslash \tilde{D}$ being deduced using the symmetry property. The function $h(x, y)$ is the depth of the fluid below the level $z = 0$, in line with the definition of the one-dimensional version of h.

The boundary function C is taken to be zero on the lateral sides of $\tilde{D}$ and is

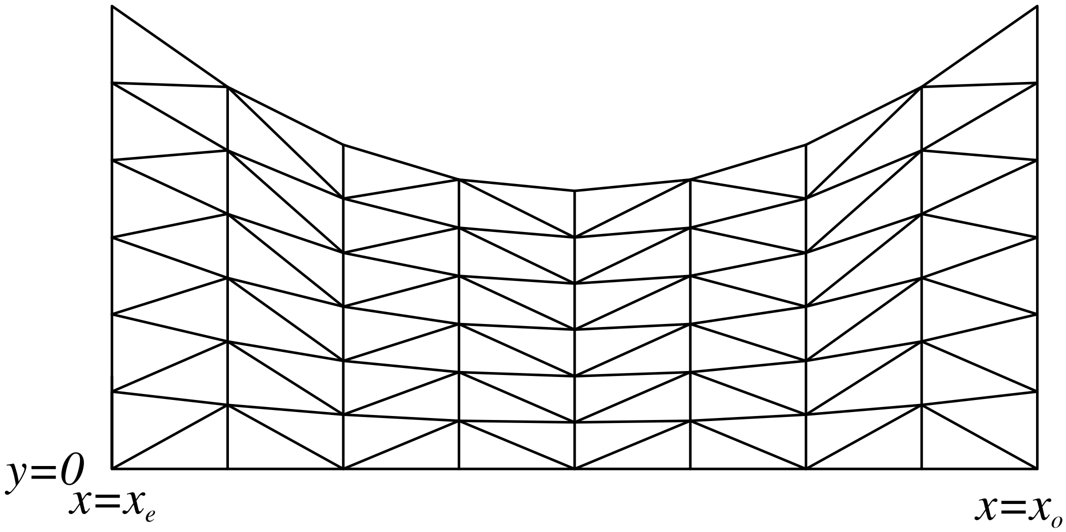

Figure 5.1: Example of triangular grid for $n = 9$ and $m = 7$.

assigned values on the inlet section Σ_e (at $x = x_e$) of the boundary Σ and on the outlet section Σ_o (at $x = x_o$) such that

$$\int_{\Sigma_e} C \, d\Sigma + \int_{\Sigma_o} C \, d\Sigma = 0,$$

for consistency with conservation of mass.

The domain $\tilde{D}$ is discretised into a triangular grid using the $l = mn$ points

$$\begin{aligned} x_{i+(j-1)n} &= \tfrac{i-1}{n-1} (x_o - x_e) + x_e, \\ y_{i+(j-1)n} &= \tfrac{1}{2} \tfrac{j-1}{m-1} B(x_{i+(j-1)n}), \end{aligned} \qquad (5.3)$$

for $i = 1, \ldots, n$ and $j = 1, \ldots, m$. The region D^h covered by the grid is an approximation to $\tilde{D}$. An example of a grid for $n = 9$ and $m = 7$ is given in Figure 5.1.

The approximation method is similar to that in one dimension. The approximate velocity potential and depth functions are expanded in terms of finite element basis functions and substituted into the functionals L_1^c and L_4^c. The finite dimensional versions of the functionals, generated in this way, are to be made stationary with respect to variations in the parameters of the approximations. The

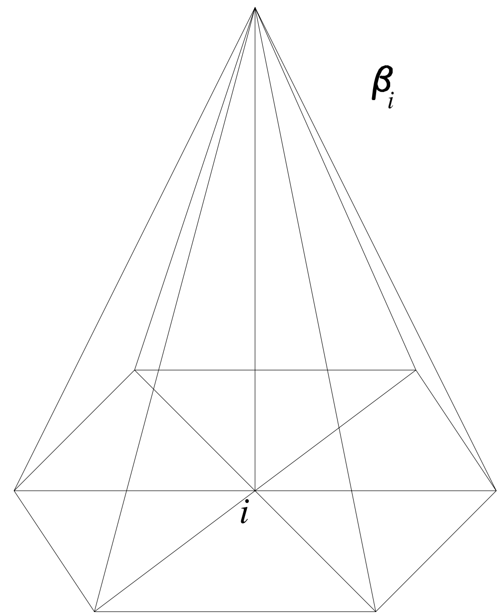

Figure 5.2: A two-dimensional piecewise linear basis function.

basis functions considered here are two-dimensional piecewise linear. The basis function corresponding to a particular node is of magnitude one at the node, zero at all other nodes and linear over each triangular element. An example of a typical basis function for an internal grid node is given in Figure 5.2.

5.1 The Constrained 'p' Principle

The functional L_1^c, defined by (5.1), is used to generate approximations to the velocity potential ϕ. By considering the matrix of second derivatives of L_1^c it can be shown that L_1^c is maximised by subcritical solutions of the shallow water equations and has a saddle point for supercritical solutions.

Let the approximation to the velocity potential be

$$\phi^h(x,y) = \sum_{i=1}^{l} \phi_i \beta_i(x,y), \tag{5.4}$$

where the β_i are the two-dimensional piecewise linear basis functions defined on the grid given by the points (5.3) and the ϕ_i are the values of ϕ^h at the nodes of the grid.

Substituting (5.4) into (5.1) gives the finite dimensional version of the functional, that is,

$$L(\boldsymbol{\phi}) = \iint_{D^h} p(\boldsymbol{\nabla}\phi^h, E)\,dx\,dy + \int_{\Sigma_e} C\phi^h\,d\Sigma + \int_{\Sigma_o} C\phi^h\,d\Sigma,$$

where $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_l)^T$, $E(x,y) = \tilde{E} + gh(x,y)$ and $\tilde{D}$ is approximated by D^h, the region covered by the triangular grid. The finite element solution is given by the $\boldsymbol{\phi}$ which satisfies

$$F_i(\boldsymbol{\phi}) = \frac{\partial L}{\partial \phi_i} = \iint_{D^h} p_{\boldsymbol{\nabla}\phi^h}.\boldsymbol{\nabla}\beta_i\,dx\,dy + \int_{\Sigma_e} C\beta_i\,d\Sigma + \int_{\Sigma_o} C\beta_i\,d\Sigma = 0$$

for $i = 1, \ldots, l$ and is found using Newton's method in the same way as before. The Jacobian is given by

$$J(\boldsymbol{\phi}) = \{J_{ij}\} = \left\{\frac{\partial F_i}{\partial \phi_j}\right\} = \left\{\frac{\partial^2 L}{\partial\phi_j\partial\phi_i}\right\} = \left\{\iint_{D^h} \boldsymbol{\nabla}\beta_i.p_{\boldsymbol{\nabla}\phi^h\,\boldsymbol{\nabla}\phi^h}.\boldsymbol{\nabla}\beta_j\,dx\,dy\right\},$$

where

$$p_{\boldsymbol{\nabla}\phi^h\,\boldsymbol{\nabla}\phi^h} = \begin{pmatrix} -E + \frac{1}{2}\boldsymbol{\nabla}\phi^h.\boldsymbol{\nabla}\phi^h + {\phi_x^h}^2 & \phi_x^h\phi_y^h \\ \phi_y^h\phi_x^h & -E + \frac{1}{2}\boldsymbol{\nabla}\phi^h.\boldsymbol{\nabla}\phi^h + {\phi_y^h}^2 \end{pmatrix},$$

and is negative definite for wholly subcritical flow and indefinite for wholly supercritical flow.

Given an initial approximation $\boldsymbol{\phi}^0$ to the solution $\boldsymbol{\phi}$ a sequence of approximations is generated, using Newton's method, from

$$\boldsymbol{\phi}^{k+1} = \boldsymbol{\phi}^k + \delta\boldsymbol{\phi}^k, \tag{5.5}$$

where

$$J(\boldsymbol{\phi}^k)\,\delta\boldsymbol{\phi}^k = -\mathbf{F}(\boldsymbol{\phi}^k). \tag{5.6}$$

The process is continued until

$$\frac{\max_i \left|\delta\phi_i^k\right|}{\max_i \left|\phi_i^k\right|} < \text{tolerance.} \tag{5.7}$$

Using the piecewise linear basis functions, defined on a triangular grid, the integrands of the Jacobian J and the vector $\mathbf{F} = (F_1, \ldots, F_l)^T$ are constants over each element so J and $\mathbf{F}$ are integrated exactly.

The Jacobian is no longer tridiagonal, as it was in the one-dimensional examples, although it is symmetric and banded. Equation (5.6) may still be solved efficiently for $\delta\boldsymbol{\phi}^k$ using a pre-conditioned conjugate gradient method (Golub and Van Loan (1989)), provided that J is not indefinite. The matrix J is pre-conditioned by its diagonal entries, that is, by the matrix $P = \mathrm{diag}(J_{11}, \ldots, J_{ll})$. The system

$$P^{-1}(\boldsymbol{\phi}^k)\, J(\boldsymbol{\phi}^k)\, P^{-1}(\boldsymbol{\phi}^k)\, \delta\boldsymbol{\psi}^k = -P^{-1}(\boldsymbol{\phi}^k)\, \mathbf{F}(\boldsymbol{\phi}^k)$$

is solved for $\delta\boldsymbol{\psi}^k$ by the conjugate gradient method. Then the solution $\delta\boldsymbol{\phi}^k$ of (5.6) is given by

$$\delta\boldsymbol{\phi}^k = P^{-1}(\boldsymbol{\phi}^k)\, \delta\boldsymbol{\psi}^k.$$

The effect of this pre-conditioning should be to improve the convergence rate of the conjugate gradient iteration. If $p_{\boldsymbol{\nabla}\phi^h\,\boldsymbol{\nabla}\phi^h}$ is a constant in D^h then pre-conditioning the system using the matrix P will improve the convergence rate of the conjugate gradient iteration (Wathen (1987)).

The initial approximation to $\boldsymbol{\phi}$ is given by

$$\phi_i^0 = \left(\frac{x_i - x_1}{x_2 - x_1}\right) v^0 \qquad\qquad i = 1, \ldots, l, \tag{5.8}$$

where v^0 is assigned a value which determines whether the solution being calculated is an approximation to subcritical or to supercritical flow. The energy $\tilde{E}$ is taken to be 50.

The boundary function C is given the value $C(x_e, y) = -K$, where K is a constant, on the inlet boundary Σ_e and $C(x_o, y) = \frac{KB_e}{B_o}$ on the outlet boundary Σ_o, this is consistent with conservation of mass. The use of this boundary function implies that the flow is uniform across the inlet and outlet boundaries. Therefore it should ideally be applied to an infinitely long channel which has straight parallel sides and a horizontal bed for all but a finite section of its length. In order to investigate the effects of applying this boundary function at the ends of a channel of finite length, let the domain D be such that B and h vary only on the interval [0,10] of the x axis and are constant on $[x_e, 0]$ and $[10, x_o]$, where $x_e = -L$ and $x_o = 10 + L$, for some number L. By using several different values of L any inconsistencies caused by using the boundary function C defined above can be studied. The constant K is given the value 10.

The breadth functions used here are

$$B_4(x) = \begin{cases} 6 + 4\left(1 - \frac{x}{5}\right)^2 & x \in [0, 10] \\ 10 & x \in [-L, 0] \cup [10, 10 + L] \end{cases}, \tag{5.9}$$

$$B_5(x) = \begin{cases} 10 & x \in [-L, 0] \\ 6 + 4\left(1 - \frac{x}{5}\right)^2 & x \in [0, 8] \\ 6 + 4\left(\frac{3}{5}\right)^2 & x \in [8, 10 + L] \end{cases}, \tag{5.10}$$

$$B_6(x) = \begin{cases} 6 + 4\left(1 - \frac{x}{3}\right)^2 & x \in [0, 3] \\ 6 + 4\left(\frac{3}{7} - \frac{x}{7}\right)^2 & x \in [3, 10] \\ 10 & x \in [-L, 0] \cup [10, 10 + L] \end{cases}. \tag{5.11}$$

The depth of fluid below the reference level $z = 0$ is taken to be identically zero, corresponding to a horizontal channel bed.

Let the grid of points be defined by (5.3) with $n = 5$ and $m = 3$. Consider the channel with breadth B_4 and let $L = 0$. Newton's method, with a tolerance in (5.7) of 10^{-3}, converges to the subcritical approximation in 4 iterations, requiring 7, 7, 7 and 4 conjugate gradient iterations, with a tolerance of 10^{-4}. On a refined grid with $n = 9$ and $m = 7$, and using the same tolerances, Newton's method converges to the subcritical approximation in 4 iterations requiring 29, 27, 24 and 7 conjugate gradient iterations. In both cases $\boldsymbol{\phi}^0$ is given by (5.8) with $v^0 = 0$.

A piecewise constant velocity approximation $\mathbf{v}^h$ can be generated from the velocity potential approximation ϕ^h using $\mathbf{v}^h = \boldsymbol{\nabla}\phi^h$ on each triangular element. The velocity approximations for the above two cases are shown in Figure 5.3, the length of the arrow in each element being directly proportional to the magnitude of $\mathbf{v}^h$. Both of the approximations exhibit the property of the exact solution that the speed increases as the breadth decreases. They also approximately satisfy the boundary conditions of zero flow across the channel side, $y = \frac{B(x)}{2}$ for $x \in [x_e, x_o]$, and across the axis of symmetry, $y = 0$. The change in the speed, as the breadth decreases then increases, is represented better on the more refined grid, in particular the maximum speed has increased (from 4.00 to 4.42) and the minimum speed has decreased (from 2.48 to 2.33).

Figure 5.4 shows corresponding results for breadth B_5 with $L = 0$. The approximation on the grid with $n = 5$ and $m = 3$ converges in 4 Newton iterations, requiring 14, 14, 12 and 1 conjugate gradient iterations. The approximation for $n = 9$ and $m = 7$ also converges in 4 Newton iterations, requiring 40, 38, 27 and 16 conjugate gradient iterations. In both cases $\boldsymbol{\phi}^0$ is given by (5.8) with $v^0 = 0$ and the tolerances are 10^{-3} for the Newton iterations and 10^{-4} for the conjugate

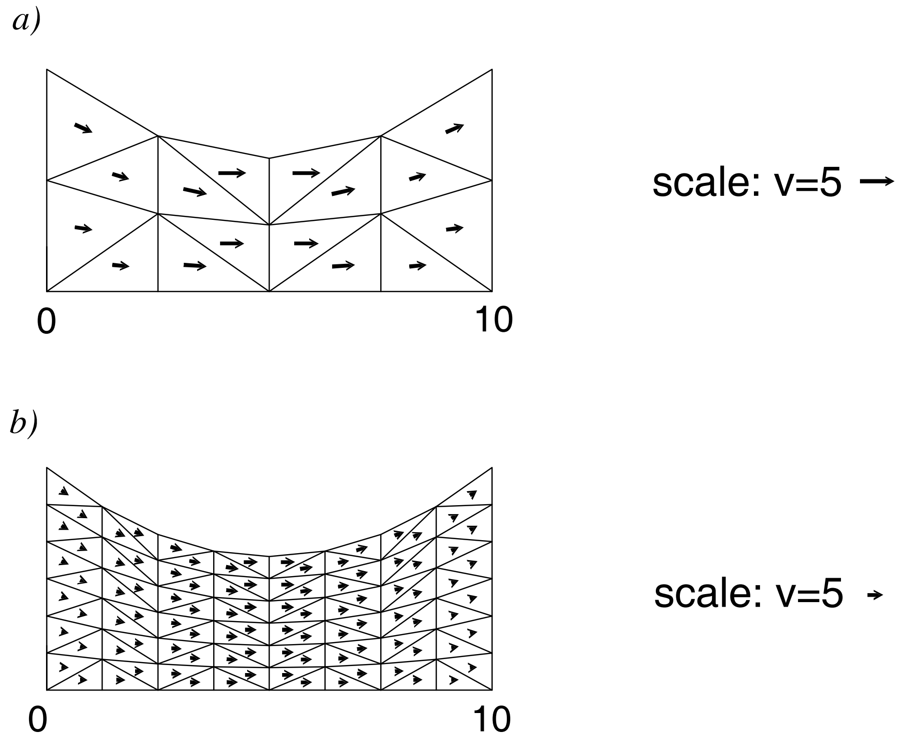

Figure 5.3: Subcritical piecewise constant velocity approximations in a channel with breadth B_4 and $L = 0$ for a) $n = 5$ and $m = 3$ and b) $n = 9$ and $m = 7$.

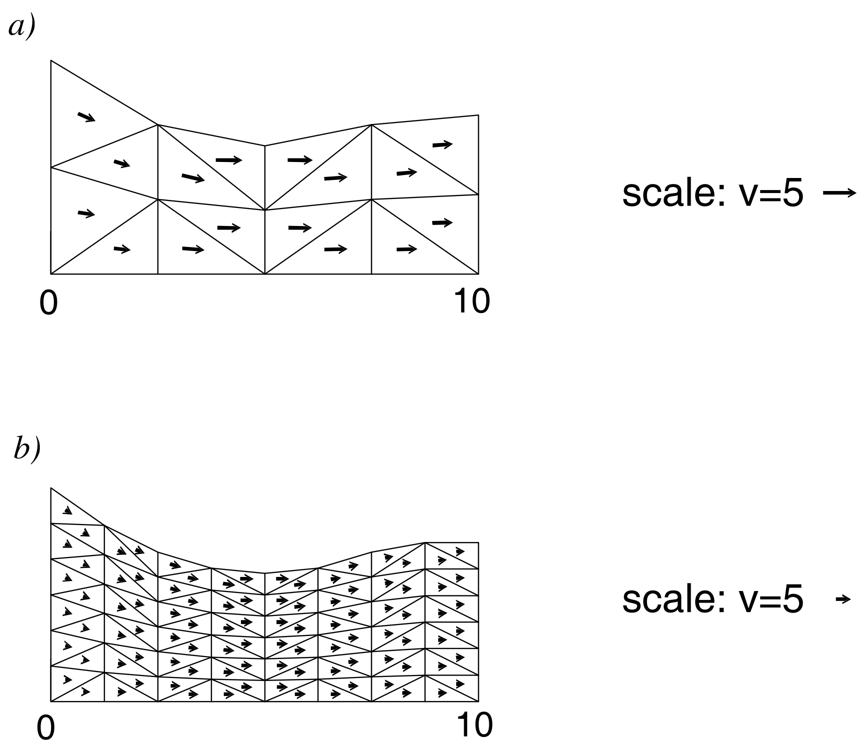

Figure 5.4: Subcritical piecewise constant velocity approximations in a channel with breadth B_5 and $L = 0$ for a) $n = 5$ and $m = 3$ and b) $n = 9$ and $m = 7$.

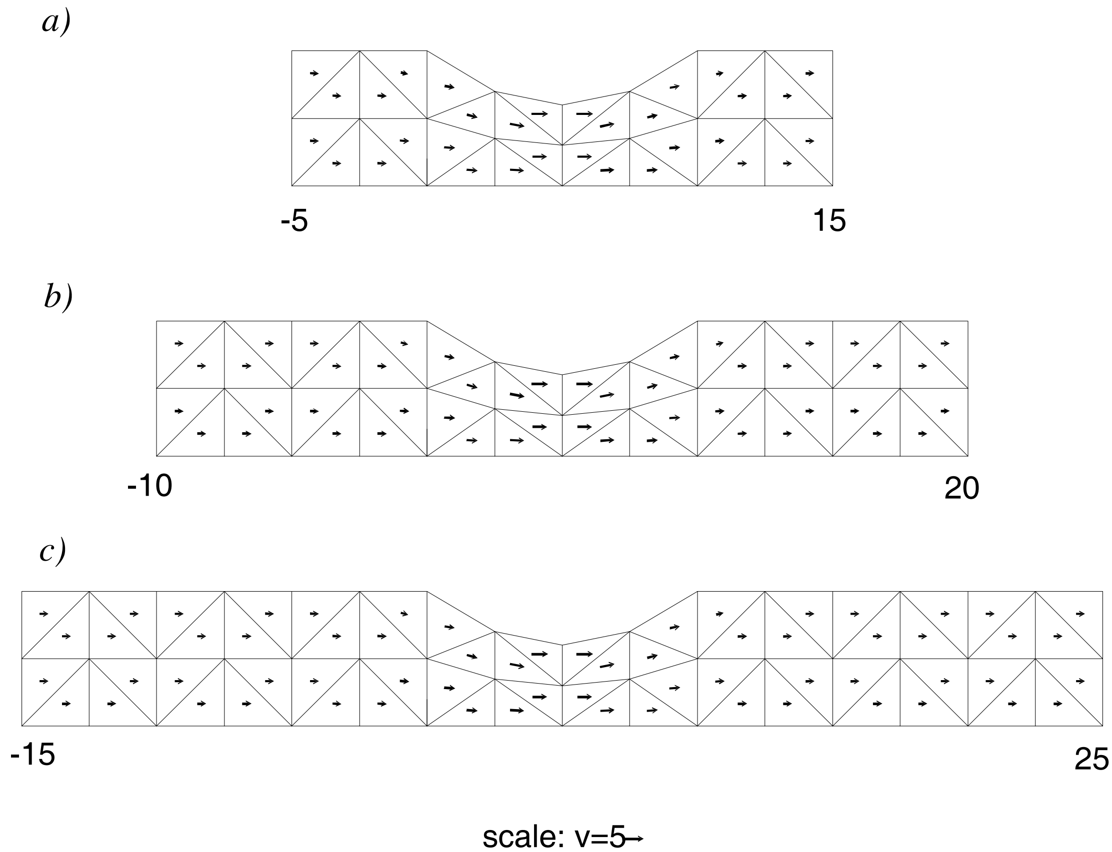

Figure 5.5: Subcritical piecewise constant velocity approximations in a channel with breadth B_4 for $a)$ $L = 5$, $b)$ $L = 10$ and $c)$ $L = 15$.

gradient iterations.

In order to investigate the effects of applying the given boundary function C on a channel of finite length, approximations to flows in channels which have different values of L are generated. Consider the channel with breadth B_4. Approximations for $L = 5$, 10 and 15 are given in Figure 5.5. It can be seen that increasing L from 5 to 15 has very little effect on the approximation in the region between the lines $x = -5$ and $x = 15$. Also notice that the velocity in the regions between $x = -15$ and $x = -5$ and between $x = 15$ and $x = 25$ is virtually uniform and is parallel to the channel sides.

It should be possible to generate approximations to supercritical flows by taking v^0 in the range $\frac{x_o - x_e}{n-1} c_*^{\max} < v^0 < \frac{x_o - x_e}{n-1} \min_{(x,y)\in\tilde{D}} \sqrt{2E}$ in (5.8), where $c_*^{\max}$ is the maximum critical speed in a particular channel for a flow with given values of K and $\tilde{E}$. However, the Jacobian J is indefinite for supercritical approximations and (5.6) must be solved for $\delta\boldsymbol{\phi}^k$ by an alternative to the conjugate gradient method, such as can be found in Golub and Van Loan (1989).

5.2 The Constrained 'R' Principle

The functional of the constrained 'R' principle, given by (5.2), is used to generate approximations to the velocity potential and to the depth of flow. By considering the matrix of second derivatives of L_4^c it can be shown that L_4^c is maximised by subcritical solutions of the shallow water equations and has a saddle point for supercritical solutions.

Let

$$\phi^h(x,y) = \sum_{i=1}^{l} \phi_i \beta_i(x,y) \quad \text{and} \quad d^h(x,y) = \sum_{i=1}^{l} d_i \beta_i(x,y) \qquad (5.12)$$

be approximations to the velocity potential and the depth, where the β_i are the two-dimensional piecewise linear basis functions defined earlier and the ϕ_i and the d_i are parameters of the solutions whose values are to be determined.

Substituting (5.12) into (5.2) gives the finite dimensional version of the functional, that is,

$$L(\boldsymbol{\phi}, \mathbf{d}) = \iint_{D^h} \left(-R(\boldsymbol{\nabla}\phi^h, d^h) + Ed^h\right) dx\, dy + \int_{\Sigma_e} C\phi^h \, d\Sigma + \int_{\Sigma_o} C\phi^h \, d\Sigma,$$

where $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_l)^T$, $\mathbf{d} = (d_1, \ldots, d_l)^T$ and D^h is an approximation to $\tilde{D}$. The parameters of the approximation are those values of $\boldsymbol{\phi}$ and $\mathbf{d}$ for which L is stationary with respect to variations, that is, $\boldsymbol{\phi}$ and $\mathbf{d}$ satisfy

$$\begin{aligned} F_i(\boldsymbol{\phi}, \mathbf{d}) &= \frac{\partial L}{\partial \phi_i} = -\iint_{D^h} R_{\boldsymbol{\nabla}\phi^h} . \boldsymbol{\nabla}\beta_i \, dx\, dy + \int_{\Sigma_e} C\beta_i \, d\Sigma + \int_{\Sigma_o} C\beta_i \, d\Sigma = 0, \\ F_{i+l}(\boldsymbol{\phi}, \mathbf{d}) &= \frac{\partial L}{\partial d_i} = -\iint_{D^h} (R_{d^h} - E)\, \beta_i \, dx\, dy = 0, \end{aligned}$$

for $i = 1, \ldots, l$.

The solution is found using Newton's method. The Jacobian is given by

$$J(\boldsymbol{\phi}, \mathbf{d}) = \{J_{ij}\},$$

where

$$\begin{aligned} J_{ij} &= -\iint_{D^h} R_{\boldsymbol{\nabla}\phi^h \, \boldsymbol{\nabla}\phi^h} \boldsymbol{\nabla}\beta_i . \boldsymbol{\nabla}\beta_j \, dx\, dy, \\ J_{i\, j+l} &= -\iint_{D^h} \boldsymbol{\nabla}\beta_i . R_{\boldsymbol{\nabla}\phi^h \, d^h} \beta_j \, dx\, dy, \\ J_{i+l\, j} &= -\iint_{D^h} \beta_i R_{d^h \, \boldsymbol{\nabla}\phi^h} . \boldsymbol{\nabla}\beta_j \, dx\, dy, \\ J_{i+l\, j+l} &= -\iint_{D^h} \beta_i R_{d^h d^h} \beta_j \, dx\, dy, \end{aligned}$$

for $i = 1, \ldots, l$ and $j = 1, \ldots, l$.

Given initial approximations $\boldsymbol{\phi}^0$ and $\mathbf{d}^0$ to $\boldsymbol{\phi}$ and $\mathbf{d}$ Newton's method yields a sequence of approximations,

$$\begin{pmatrix} \boldsymbol{\phi}^{k+1} \\ \mathbf{d}^{k+1} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\phi}^{k} \\ \mathbf{d}^{k} \end{pmatrix} + \delta \begin{pmatrix} \boldsymbol{\phi}^{k} \\ \mathbf{d}^{k} \end{pmatrix},$$

where

$$J(\boldsymbol{\phi}^k, \mathbf{d}^k)\, \delta \begin{pmatrix} \boldsymbol{\phi}^{k} \\ \mathbf{d}^{k} \end{pmatrix} = -\mathbf{F}(\boldsymbol{\phi}^k, \mathbf{d}^k). \tag{5.13}$$

The sequence ends when

$$\frac{\max_i \left|\delta\phi_i^k\right|}{\max_i \left|\phi_i^k\right|} < \text{tolerance} \quad \text{and} \quad \frac{\max_i \left|\delta d_i^k\right|}{\max_i \left|d_i^k\right|} < \text{tolerance}, \tag{5.14}$$

for some specified tolerance. The Jacobian and the vector $\mathbf{F}$ are evaluated using 7 point Gaussian quadrature for integrating over triangles.

The Jacobian is symmetric and banded and (5.13) is solved, when J is not indefinite, using a pre-conditioned conjugate gradient method, with the pre-conditioning matrix $P = \mathrm{diag}(J_{11}, \ldots, J_{2l\,2l})$.

The initial approximation $\boldsymbol{\phi}^0$ to $\boldsymbol{\phi}$ is given by (5.8) and the initial approximation $\mathbf{d}^0$ to $\mathbf{d}$ is given by $d_i^0 = \hat{d}$, for $i = 1, \ldots, l$, where $\hat{d}$ is a constant. The values of $\hat{d}$ and v^0, in (5.8), must be consistent with one another, that is, if $\hat{d}$ is assigned a value corresponding to a subcritical depth then v^0 must be given a value in the range $0 \leq v^0 < \frac{x_o - x_e}{n-1} c_*^{\min}$, where $c_*^{\min} = \min_{(x,y)\in\tilde{D}} c_*$ and c_* is defined by (2.63). If $\hat{d}$ has a value corresponding to a supercritical depth then v^0 must lie in the range $\frac{x_o - x_e}{n-1} c_*^{\max} < v^o \leq \frac{x_o - x_e}{n-1} \min_{(x,y)\in\tilde{D}} \sqrt{2E}$, where $c_*^{\max} = \max_{(x,y)\in\tilde{D}} c_*$.

The constant $\tilde{E}$ is given the value 50. The boundary function C is defined in the same way as in Section 5.1, that is, $C(x_e, y) = -K$ on Σ_e and $C(x_o, y) = \frac{KB_e}{B_o}$ on Σ_o; K is taken to be 10. The channel breadths considered here are those given by (5.9), (5.10) and (5.11). The depth of fluid below the reference level $z = 0$ is taken to be identically zero.

Consider the channel with breadth B_4 and let $L = 5$. Then, with $n = 9$ and $m = 3$, Newton's method, with a tolerance of 10^{-3} in (5.14), converges in 26 iterations, with a tolerance on the conjugate gradient iterations of 5×10^{-2}. On a refined grid with $n = 17$ and $m = 5$ Newton's method converges in 32 iterations, with the same tolerances as before. In both cases the initial data is $\hat{d} = 4.5$ and $v^0 = 0$.

The results for $n = 9$ and $m = 3$ are given in Figure 5.6. Figure 5.6a shows

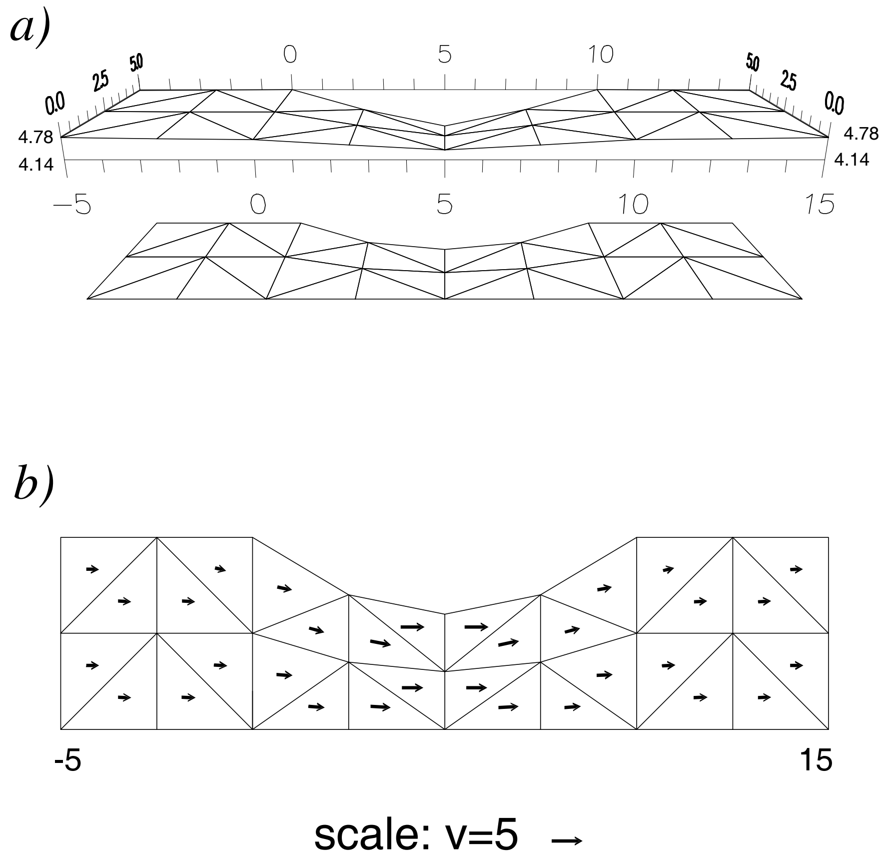

Figure 5.6: Subcritical approximations for $n = 9$ and $m = 3$ a) piecewise linear depth and b) piecewise constant velocity.

the piecewise linear depth approximation and Figure 5.6b the piecewise constant velocity approximation, calculated using $\mathbf{v}^h = \boldsymbol{\nabla}\phi^h$, where the arrows show the approximate flow directions, the lengths being directly proportional to the magnitude of $\mathbf{v}^h$ in each element. The depths lie in the range 4.14 to 4.78 and the speeds lie in the range 1.89 to 3.89.

Figure 5.7 shows the corresponding results for $n = 17$ and $m = 5$. The depths lie between 4.03 and 4.85 and the speeds lie between 1.66 and 4.31.

Supercritical approximations may be generated by using an appropriate method to solve (5.13) for $\delta \begin{pmatrix} \boldsymbol{\phi}^k \\ \mathbf{d}^k \end{pmatrix}$, where $J(\boldsymbol{\phi}^k, \mathbf{d}^k)$ is indefinite, and by choosing consistent values for $\hat{d}$ and v^0, corresponding to supercritical data.

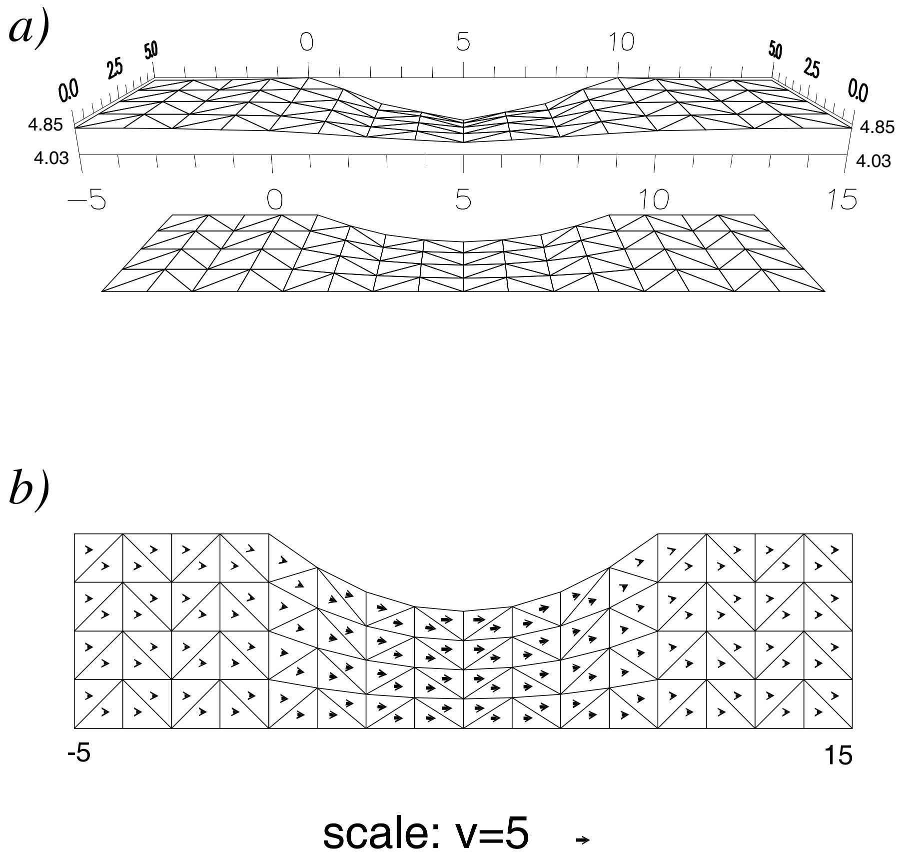

Figure 5.7: Subcritical approximations for $n = 17$ and $m = 5$ a) piecewise linear depth and b) piecewise constant velocity.

In this chapter two algorithms are given which can be used to generate approximations to two-dimensional shallow water flows, where the flows are subcritical. Both methods yield approximations to the velocity of the flow and these approximations can be compared as follows.

Consider the i th triangular element of the discretised domain and let $\triangle_i$ be the area of the element. Let v_i^p be the speed in element i of the velocity approximation generated, as in Section 5.1, by using the 'p' principle based on (5.1). Let v_i^R be the speed in element i of the corresponding approximation generated, as above, by using the 'R' principle based on (5.2). Then

$$e = \frac{\sum_{i=1}^{I} \left| v_i^p - v_i^R \right| \triangle_i}{\sum_{i=1}^{I} \left(v_i^p + v_i^R \right) \triangle_i},$$

where $I = 2(n-1)(m-1)$ is the number of elements in the domain, is a measure of the difference between the two velocity approximations.

Consider the channel with breadth $B_4(x)$, given by (5.9), where $L = 5$, and let the fluid depth below the reference level $z = 0$ be identically zero. The values of e for grids with different values of n and m are given in Table 5.1. The results suggest that, as the number of elements increases, the differences between the approximations derived from the two principles decrease.

n	m	e
9	3	4.9×10^{-3}
13	5	3.6×10^{-3}
17	5	3.2×10^{-3}

Table 5.1: Values of e for various n and m.

Chapter 6

Further Applications

In Chapters 4 and 5 the variational principles for steady state shallow water flows in one and two dimensions, derived in Chapter 3, are used to generate approximations to the corresponding flows. There are, however, other variational principles in Chapter 3 which can be used to generate approximate solutions to other problems. Two such problems are considered here.

The approximations generated so far have all been for solutions of the shallow water equations, in which it is assumed that the component of velocity perpendicular to the xy plane, that is the vertical component, is negligible, see the statement (2.14). The variational principle (3.7), based on Luke's principle (Luke (1967)) is satisfied for solutions of the equations of time-dependent free surface flows of an inviscid, homogeneous fluid in three dimensions. If an approximation to three-dimensional flow can be generated then it can be used to investigate the accuracy of the assumption that, under the conditions of shallow water theory, the magnitude of the vertical component of the velocity is negligible.

In Section 6.1 the functional of (3.7) is reduced to a functional whose corresponding variational principle has as its natural conditions the equations of time-independent free surface flow in two dimensions, that is, the solutions of these equations are functions of the vertical coordinate z and the one horizontal coordinate x. An attempt is made to extend the algorithms of Chapter 4 to generate approximations to free surface flows using the new functional.

In Section 6.2 the 'p' functional for time-dependent quasi one-dimensional shallow water flow (3.98) is used in an attempt to seek approximations to time-dependent flows in a channel of slowly varying breadth. The problems caused by using the functional (3.98) are also mentioned.

6.1 Two-dimensional Free Surface Flows

The functional of the modified version of Luke's principle (3.7) is given by

$$\int_{t_1}^{t_2} \iint_D \int_{-h}^{\eta} \rho \left\{ -\left(\chi_t + gz + \frac{1}{2}\mathbf{u}.\mathbf{u} \right) + \mathbf{u}. \left(\mathbf{u} - \tilde{\boldsymbol{\nabla}} \chi \right) \right\} dz\, dx\, dy\, dt, \qquad (6.1)$$

where $\eta = \eta(x, y, t)$, $h = h(x, y)$, $\chi = \chi(x, y, z, t)$, $\mathbf{u} = \mathbf{u}(x, y, z, t)$ and $\tilde{\boldsymbol{\nabla}}$ is defined by (2.2). The functional (6.1) is used in Section 6.1.1 to derive a functional which has as its natural conditions of the first variation the equations of time-independent motion in the x and z directions.

6.1.1 The Functional

The required functional is generated from (6.1) by first making the assumption that the flow variables are independent of time and evaluating the time integral and then assuming that the domain is a channel of slowly varying breadth so that the variables are functions of the coordinates x and z only and the integral with respect to y can be evaluated. It is also necessary to add in boundary terms so that variations can be allowed which do not necessarily vanish on the inlet and outlet boundaries of the channel.

Time-independent flows

First make the assumption that the free surface flow does not vary with time. Then the flow variable $\mathbf{u}$ and the height of the free surface above the reference level $z = 0$ are independent of time, that is, $\mathbf{u} = \mathbf{u}(x, y, z)$ and $\eta = \eta(x, y)$. The variation of the velocity potential χ with respect to time needs to be deduced. The flow is assumed to be irrotational so, from (2.5),

$$\mathbf{u} = \tilde{\boldsymbol{\nabla}}\chi.$$

By assumption $\mathbf{u}_t \equiv \mathbf{0}$ and so

$$\tilde{\boldsymbol{\nabla}}\chi_t \equiv \mathbf{0},$$

which implies that χ is of the form

$$\chi(x, y, z, t) = \hat{\chi}(x, y, z) + f(t),$$

for arbitrary functions $\hat{\chi}$ and f, where

$$\tilde{\boldsymbol{\nabla}}\chi = \tilde{\boldsymbol{\nabla}}\hat{\chi} \quad \text{and} \quad \chi_t = f'.$$

Let

$$\hat{E} = -\frac{1}{T}\int_{t_1}^{t_2} \chi_t \, dt = -\frac{1}{T}\left(f(t_2) - f(t_1)\right),$$

where $T = t_2 - t_1$. Then, making these substitutions into the functional (6.1) and integrating with respect to time gives

$$\iint_D \int_{-h}^{\eta} \rho T \left(\hat{E} - gz - \frac{1}{2}\mathbf{u}.\mathbf{u} + \mathbf{u}.\left(\mathbf{u} - \tilde{\boldsymbol{\nabla}}\hat{\chi}\right)\right) dz\, dx\, dy. \tag{6.2}$$

Two-dimensional flows

Let D be the domain

$$D = \left\{(x,y) : x \in [x_e, x_o], y \in \left[-\frac{B(x)}{2}, \frac{B(x)}{2}\right]\right\},$$

where $B(x)$, for $x \in [x_e, x_o]$, is the breadth of the channel and is a slowly varying function of x. Let h, the depth of fluid below the reference level $z = 0$, depend on the x coordinate alone.

Now make the assumption that, in this domain, all of the variables are independent of the y coordinate and redefine the variables as follows. The velocity $\mathbf{u}(x,y,z)$ becomes $\mathbf{u} = \mathbf{u}(x,z)$, where $\mathbf{u} = (u, w)$, the velocity potential $\hat{\chi}(x,y,z)$ becomes $\hat{\chi}(x,z)$ and $\eta(x,y)$ becomes $\eta(x)$. The operator $\tilde{\boldsymbol{\nabla}}$ is replaced by its two-dimensional counterpart $\hat{\boldsymbol{\nabla}} \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial z}\right)$.

Making these substitutions in (6.2) and integrating with respect to y gives

$$\int_{x_e}^{x_o} \int_{-h}^{\eta} \rho T \left(\hat{E} - gz - \frac{1}{2}\mathbf{u}.\mathbf{u} + \mathbf{u}.\left(\mathbf{u} - \hat{\boldsymbol{\nabla}}\hat{\chi}\right)\right) B\, dz\, dx.$$

Boundary terms

As the final stage in the construction of the required functional, boundary terms must be added so that variations of the functional do not necessarily have to vanish at the ends of the channel, that is, at $x = x_e$ and $x = x_o$.

The required functional is

$$\begin{aligned} J(\eta, \mathbf{u}, \chi) = &\int_{x_e}^{x_o} \int_{-h}^{\eta} \left(\hat{E} - gz - \frac{1}{2}\mathbf{u}.\mathbf{u} + \mathbf{u}.\left(\mathbf{u} - \hat{\boldsymbol{\nabla}}\chi\right)\right) B\, dz\, dx \\ &+ \int_{-h(x_o)}^{\eta(x_o)} C_o\, \chi|_{x_o}\, dz - \int_{-h(x_e)}^{\eta(x_e)} C_e\, \chi|_{x_e}\, dz, \end{aligned} \tag{6.3}$$

where the constant ρT has been set equal to unity and the ˆ notation on the velocity potential has been dropped for simplicity.

The natural conditions of $\delta J = 0$ are given by

$$\left.\begin{aligned} \mathbf{u} - \hat{\boldsymbol{\nabla}}\chi = \mathbf{0} \\ \hat{\boldsymbol{\nabla}}.(B\mathbf{u}) = 0 \end{aligned}\right\} \quad \text{for } x \in (x_e, x_o);\ z \in (-h(x), \eta(x)),$$

$$\begin{aligned} u\eta_x - w &= 0 \quad \text{on } z = \eta(x) \text{ for } x \in (x_e, x_o), \\ uh_x + w &= 0 \quad \text{on } z = -h(x) \text{ for } x \in (x_e, x_o), \\ C_e - B(x_e)\, u|_{x_e} &= 0 \quad \text{for } z \in (-h(x_e), \eta(x_e)), \\ C_o - B(x_o)\, u|_{x_o} &= 0 \quad \text{for } z \in (-h(x_o), \eta(x_o)), \\ \hat{E} - gz + \frac{1}{2}\mathbf{u}.\mathbf{u} - \mathbf{u}.\hat{\boldsymbol{\nabla}}\chi &= 0 \quad \text{on } z = \eta(x) \text{ for } x \in (x_e, x_o), \end{aligned}$$

which are, respectively, the irrotationality condition and the conservation of mass equation for $x \in (x_e, x_o)$ and $z \in (-h(x), \eta(x))$, the condition of no flow across the free surface, the condition of no flow through the channel bed, boundary conditions on the horizontal component of velocity at the inlet and outlet boundaries and the dynamic free surface condition, as required. Notice that the first six natural conditions are due to variations in the variables χ and $\mathbf{u}$, while the last natural condition is due to the variation in η.

6.1.2 The Algorithm

The basic finite element technique, as used in Chapters 4 and 5, can only be applied to functionals in which the integration is over a fixed region. Various algorithms (for example, Aitchison (1979), Ikegawa and Washizu (1973)) have been developed to approximate flows in domains where the position of the free surface is allowed to vary; the differences are mainly in the treatment of the variation of the free surface. The method used here to position the free surface is based on the method used to approximate the position of a hydraulic jump in Section 4.5.

The functional (6.3) depends on the three functions $\eta(x)$, $\mathbf{u}(x,z)$ and $\chi(x,z)$. A functional depending on only two functions can be derived by making the substitution $\mathbf{u} = \hat{\boldsymbol{\nabla}}\chi$ in (6.3), giving

$$\hat{J}(\eta,\chi) = \int_{x_e}^{x_o}\int_{-h}^{\eta}\left(\hat{E} - gz - \frac{1}{2}\hat{\boldsymbol{\nabla}}\chi.\hat{\boldsymbol{\nabla}}\chi\right) B\, dz\, dx + \int_{-h(x_o)}^{\eta(x_o)} C_o\, \chi|_{x_o}\, dz - \int_{-h(x_e)}^{\eta(x_e)} C_e\, \chi|_{x_e}\, dz. \tag{6.4}$$

The variational principle corresponding to (6.4) is equivalent to the variational principle for (6.3), constrained to satisfy the irrotationality condition. The natural conditions of $\delta\hat{J} = 0$ are

$$\hat{\boldsymbol{\nabla}}.\left(B\hat{\boldsymbol{\nabla}}\chi\right) = 0 \quad \text{for } x \in (x_e, x_o);\ z \in (-h(x), \eta(x)), \tag{6.5}$$

$$\chi_x\eta_x - \chi_z = 0 \quad \text{on } z = \eta(x) \text{ for } x \in (x_e, x_o), \tag{6.6}$$

$$\chi_x h_x + \chi_z = 0 \quad \text{on } z = -h(x) \text{ for } x \in (x_e, x_o), \tag{6.7}$$

$$C_e - B(x_e)\, \chi_x|_{x_e} = 0 \quad \text{for } z \in (-h(x_e), \eta(x_e)), \tag{6.8}$$

$$C_o - B(x_o)\, \chi_x|_{x_o} = 0 \quad \text{for } z \in (-h(x_o), \eta(x_o)), \tag{6.9}$$

$$\hat{E} - gz - \frac{1}{2}\hat{\boldsymbol{\nabla}}\chi.\hat{\boldsymbol{\nabla}}\chi = 0 \quad \text{on } z = \eta(x) \text{ for } x \in (x_e, x_o). \tag{6.10}$$

Let

$$\eta^h(x) = \sum_{i=1}^{n} \eta_i\alpha_i(x) \tag{6.11}$$

be an approximation to $\eta(x)$ and let

$$\chi^h(x,z) = \sum_{i=1}^{l} \chi_i\beta_i(x,z) \tag{6.12}$$

be an approximation to $\chi(x,z)$, where the α_i are the one-dimensional piecewise linear basis functions (4.13), the β_i are two-dimensional piecewise linear basis

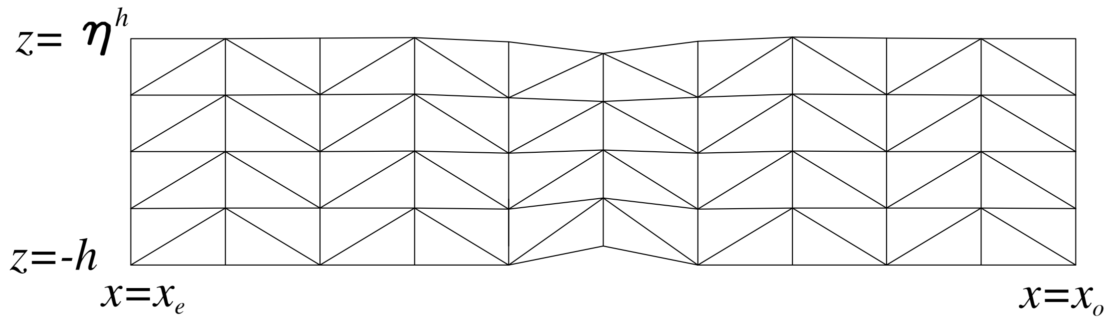

Figure 6.1: Example of a triangular grid for $n = 11$ and $m = 5$.

functions, an example of which is given in Figure 5.2, and the η_i for $i = 1, \ldots, n$ and the χ_i for $i = 1, \ldots, l$ are parameters of the solution and are to be calculated.

The domain of integration in (6.4) is discretised into triangular elements using the following set of $l = mn$ points.

$$
\begin{aligned}
x_{i+(j-1)n} &= \frac{i-1}{n-1}(x_o - x_e) + x_e, \\
z_{i+(j-1)n} &= \frac{j-1}{m-1}\left(\eta^h(x_{i+(j-1)n}) + h(x_{i+(j-1)n})\right) - h(x_{i+(j-1)n}),
\end{aligned}
$$

for $i = 1, \ldots, n$ and $j = 1, \ldots, m$. An example is given in Figure 6.1. Notice that the grid depends on the unknown function η^h. Let $\hat{h}$ be defined by

$$\hat{h}(x) = \sum_{i=1}^{n} h(x_i)\alpha_i(x).$$

Substituting (6.11) and (6.12) into (6.4) gives the finite dimensional version of the functional, that is,

$$
\begin{aligned}
L(\boldsymbol{\eta}, \boldsymbol{\chi}) = \int_{x_1}^{x_n} \int_{-\hat{h}}^{\eta^h} \left(\hat{E} - gz - \frac{1}{2}\hat{\boldsymbol{\nabla}}\chi^h . \hat{\boldsymbol{\nabla}}\chi^h\right) B \, dz \, dx \\
+ \int_{-h(x_n)}^{\eta_n} C_o \, \chi^h\Big|_{x_n} dz - \int_{-h(x_1)}^{\eta_1} C_e \, \chi^h\Big|_{x_1} dz,
\end{aligned}
$$

where $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_n)^T$ and $\boldsymbol{\chi} = (\chi_1, \ldots, \chi_l)^T$. The parameters $\boldsymbol{\eta}$ and $\boldsymbol{\chi}$ are sought such that L is stationary with respect to variations.

Let

$$G_i(\boldsymbol{\eta},\boldsymbol{\chi}) = \frac{\partial L}{\partial \chi_i} = -\int_{x_1}^{x_n}\int_{-\hat{h}}^{\eta^h} \hat{\boldsymbol{\nabla}}\chi^h.\hat{\boldsymbol{\nabla}}\beta_i B\,dz\,dx + \int_{-h(x_n)}^{\eta_n} C_o\,\beta_i|_{x_n}\,dz - \int_{-h(x_1)}^{\eta_1} C_e\,\beta_i|_{x_1}\,dz \quad i=1,\ldots,l, \tag{6.13}$$

then the vector $\mathbf{G} = (G_1,\ldots,G_l)^T$ may be written as

$$\mathbf{G}(\boldsymbol{\eta},\boldsymbol{\chi}) = -A(\boldsymbol{\eta})\boldsymbol{\chi} + \mathbf{b}(\boldsymbol{\eta}),$$

where

$$A(\boldsymbol{\eta}) = \left\{\int_{x_1}^{x_n}\int_{-\hat{h}}^{\eta^h} \hat{\boldsymbol{\nabla}}\beta_j.\hat{\boldsymbol{\nabla}}\beta_i B\,dz\,dx\right\} \tag{6.14}$$

is symmetric, positive definite and banded and $\mathbf{b} = (b_1,\ldots,b_l)^T$,where

$$b_i(\boldsymbol{\eta}) = \int_{-h(x_n)}^{\eta_n} C_o\,\beta_i|_{x_n}\,dz - \int_{-h(x_1)}^{\eta_1} C_e\,\beta_i|_{x_1}\,dz. \tag{6.15}$$

The functional L is stationary with respect to variations in $\boldsymbol{\chi}$ if $\mathbf{G} = \mathbf{0}$, that is, if

$$A(\boldsymbol{\eta})\boldsymbol{\chi} = \mathbf{b}(\boldsymbol{\eta}). \tag{6.16}$$

Therefore, for a fixed $\boldsymbol{\eta}$, $\boldsymbol{\chi}$ can be calculated from (6.16). Let $\eta = \eta^h$ in (6.4). Then the solution $\boldsymbol{\chi}$ of (6.16) gives an approximation to the function χ satisfying (6.5)–(6.9), for the given domain, since these natural conditions are due solely to the variations of χ in $\delta\hat{J} = 0$.

The problem remains to find η^h, that is, to find $\boldsymbol{\eta}$ such that L is stationary with respect to variations in $\boldsymbol{\eta}$. This could be done by adapting the method in Aitchison (1979), which is for a functional written in terms of a stream function. The finite dimensional version of the functional (6.4) can be written as

$$L(\boldsymbol{\eta},\boldsymbol{\chi}) = -\frac{1}{2}\boldsymbol{\chi}^T A(\boldsymbol{\eta})\boldsymbol{\chi} + \mathbf{b}^T(\boldsymbol{\eta})\boldsymbol{\chi} + c(\boldsymbol{\eta}),$$

where $A(\boldsymbol{\eta})$ and $\mathbf{b}(\boldsymbol{\eta})$ are defined by (6.14) and (6.15) and

$$c(\boldsymbol{\eta}) = \int_{x_1}^{x_n} \int_{-\hat{h}}^{\eta^h} \hat{E}B \, dz \, dx.$$

Then, substituting for $\boldsymbol{\chi}$ using (6.16) gives

$$\hat{L}(\boldsymbol{\eta}) = \frac{1}{2}\mathbf{b}^T(\boldsymbol{\eta})A^{-1}(\boldsymbol{\eta})\mathbf{b}(\boldsymbol{\eta}) + c(\boldsymbol{\eta}),$$

and $\boldsymbol{\eta}$ may be found by solving $\frac{\partial \hat{L}}{\partial \eta_i} = 0$ for $i = 1, \ldots, n$.

In this thesis the technique used in Section 4.5 to approximate the position of a hydraulic jump is adapted to the free surface case. In Section 4.5 the natural condition which is generated by the variation in the jump position is used directly to position the approximation to the jump. In the free surface case the natural condition (6.10) is due to the variation in the position of the free surface and an attempt is made to use (6.10) to find the approximation to the free surface.

Let $\boldsymbol{\eta}^k$ be an approximation to $\boldsymbol{\eta}$. Then, using (6.16), an approximation $\boldsymbol{\chi}^k$ to $\boldsymbol{\chi}$ can be calculated from

$$A(\boldsymbol{\eta}^k)\boldsymbol{\chi}^k = \mathbf{b}(\boldsymbol{\eta}^k), \tag{6.17}$$

by the conjugate gradient method.

An updated approximation to $\boldsymbol{\eta}$ is generated using (6.10), as follows. Consider the node of the grid at the position (x_j, z_j), where $z_j = \eta_i^k$ for some i in the range $1, \ldots, n$. Consider the elements of the grid which neighbour node j and let

$$\bar{u}_j = \frac{\sum_{i=1}^{I} \left| \hat{\boldsymbol{\nabla}} \hat{\chi} \right| \triangle_i}{\sum_{i=1}^{I} \triangle_i},$$

where I is the number of elements surrounding node j,

$$\hat{\chi}(x, z) = \sum_{i=1}^{l} \chi_i^k \beta_i(x, z)$$

is an approximation to χ^h and $\triangle_i$ is the area of element i. Then $\bar{u}_j$ is in some way representative of the approximate speed of flow at node j. The position of η_j^k is updated by

$$\eta_j^{k+1} = \left(\frac{1}{g} \left(\hat{E} - \frac{1}{2} \bar{u}_j^2 \right) - \eta_j^k \right) \gamma + \eta_j^k,$$

where $0 < \gamma \leq 1$ is a relaxation parameter. Notice that if $\gamma = 1$ then η_j^{k+1} is the height of the free surface above $z = 0$ where the speed at the point (x_j, z_j) is $\bar{u}_j$, from (6.10). The approximation $\boldsymbol{\chi}^{k+1}$ can now be calculated from (6.17) and the process is repeated until

$$\max_j \left| \eta_j^k - \eta_j^{k+1} \right| < \text{tolerance},$$

for some specified tolerance.

The initial approximation $\boldsymbol{\eta}^0$ to $\boldsymbol{\eta}$ is given by

$$\eta_i^0 = d^h(x_i) - h(x_i) \qquad \text{for } i = 1, \ldots, n,$$

where d^h is the approximation to the shallow water depth d in the given domain, calculated using the constrained 'r' principle (3.117), as in Section 4.1.

The algorithm is implemented in the domain

$$\{(x, z) : x \in [-20, 30],\ z \in [-h(x), \eta(x)]\},$$

where

$$h(x) = \begin{cases} -H & x \in [-20, 0] \cup [10, 30] \\ -H \left(1 - \frac{x}{5}\right)^2 & x \in [0, 10] \end{cases}.$$

The breadth function is defined by $B(x) = 10$ for $x \in [-20, 30]$. The constant $\hat{E}$ is given the value 50, $C_e = \frac{100}{\eta_1^k + h(x_1)}$ and $C_o = \frac{100}{\eta_n^k + h(x_n)}$.

The algorithm appears to work initially with $\max_j \left| \eta_j^k - \eta_j^{k+1} \right|$ decreasing as k increases. On closer inspection however $\max_j \left| \eta_j^k - \eta_j^{k+1} \right|$ does not decrease much

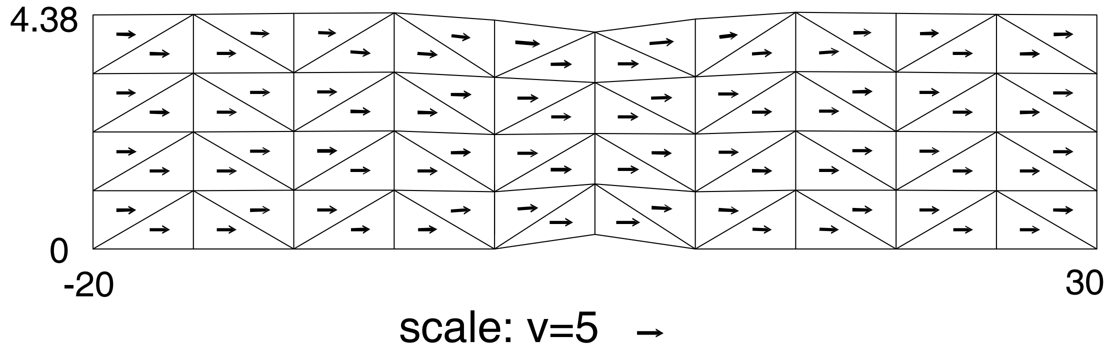

Figure 6.2: Piecewise constant velocity approximation for a subcritical free surface flow.

below approximately 10^{-2}, for the value $\gamma = 0.05$ of the relaxation parameter. The method perhaps should not be expected to generate approximations for any specified accuracy since $\bar{u}_j$ may not be a very good approximation to the speed of flow at the position (x_j, z_j).

Figure 6.2 shows the piecewise constant approximation to the velocity in the given domain, with $H = 0.3$, at the stage in the algorithm where $\max_j \left|\eta_j^k - \eta_j^{k+1}\right|$ reaches its minimum value of 7.0×10^{-3}. It can be seen that the approximation to the velocity appears to satisfy approximately the conditions of no flow through the free surface and through the channel bed. The maximum magnitude of the ratio $\frac{|w|}{|u|}$ in any element is 0.1, that is, in this approximation the vertical component of velocity is less than one tenth of the horizontal component in magnitude.

6.2 Time-dependent Quasi One-dimensional Flows

In this section a constrained version of the 'p' principle based on (3.98) is used to develop an algorithm for generating approximations to time-dependent quasi one-dimensional flows in shallow water.

The functional of the constrained 'p' principle (3.107) is given by

$$K_1^c(\phi) = \int_{t_1}^{t_2} \int_{x_e}^{x_o} \hat{p}(\phi) B \, dx \, dt + \int_{t_1}^{t_2} \left(C_o \left. (B\phi) \right|_{x_o} - C_e \left. (B\phi) \right|_{x_e} \right) dt$$
$$+ \int_{x_e}^{x_o} \left(\left. \phi \right|_{t_2} g_2 - \left. \phi \right|_{t_1} g_1 \right) B \, dx, \qquad (6.18)$$

where

$$\hat{p}(\phi) = \frac{1}{2g} \left(\phi_t - gh + \frac{1}{2} \phi_x^2 \right)^2,$$

$B(x)$ for $x \in [x_e, x_o]$ is the breadth of the channel, $C_e(t)$ and $C_o(t)$ are boundary functions for the magnitude of the mass flow at x_e and x_o, respectively, and $g_1(x)$ and $g_2(x)$ are time boundary functions for the depth of fluid at times t_1 and t_2 respectively.

An approximation to the velocity potential ϕ is generated by adapting the algorithms of Chapter 5. Let the domain of the problem,

$$\{(x,t) : x \in [x_e, x_o]; \, t \in [t_1, t_2]\},$$

be discretised into regular triangular elements using the grid of points defined by

$$x_{i+(j-1)n} = \frac{i-1}{n-1}(x_o - x_e) + x_e,$$
$$T_{i+(j-1)n} = \frac{j-1}{m-1}(t_2 - t_1) + t_1,$$

for $i = 1, \ldots, n$ and $j = 1, \ldots, m$.

Let

$$\phi^h(x,t) = \sum_{i=1}^{l} \phi_i \beta_i(x,t), \tag{6.19}$$

be an approximation to ϕ, where the β_i are two-dimensional piecewise linear basis functions, as in Figure 5.2, the ϕ_i are parameters of the solution, to be determined, and $l = nm$. The finite dimensional version of (6.18) is generated by substituting (6.19) for ϕ in (6.18) to give

$$L(\boldsymbol{\phi}) = \int_{T_1}^{T_l} \int_{x_1}^{x_n} \hat{p}(\phi^h) B \, dx \, dt + \int_{T_1}^{T_l} \left(C_o \left(B\phi^h \right)\Big|_{x_n} - C_e \left(B\phi^h \right)\Big|_{x_1} \right) dt$$
$$+ \int_{x_1}^{x_n} \left(\phi^h\Big|_{T_l} g_2 - \phi^h\Big|_{T_1} g_1 \right) B \, dx,$$

where $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_l)^T$.

The approximation for ϕ is given by (6.19), where $\boldsymbol{\phi}$ satisfies

$$F_i(\boldsymbol{\phi}) = \frac{\partial L}{\partial \phi_i} = \int_{T_1}^{T_l} \int_{x_1}^{x_n} \frac{1}{g} \left(\phi_t^h - gh + \frac{1}{2} {\phi_x^h}^2 \right) \left(\frac{\partial \beta_i}{\partial t} + \phi_x^h \frac{\partial \beta_i}{\partial x} \right) B \, dx \, dt$$
$$+ \int_{T_1}^{T_l} \left(C_o B_o \, \beta_i|_{x_n} - C_e B_e \, \beta_i|_{x_1} \right) dt + \int_{x_1}^{x_n} \left(\beta_i|_{T_l} g_2 - \beta_i|_{T_1} g_1 \right) B \, dx = 0, \tag{6.20}$$

for $i = 1, \ldots, l$.

One way of solving (6.20) is by using Newton's method. Given an approximation $\boldsymbol{\phi}^k$ to $\boldsymbol{\phi}$ an updated approximation is obtained from

$$\boldsymbol{\phi}^{k+1} = \boldsymbol{\phi}^k + \delta \boldsymbol{\phi}^k,$$

where

$$J(\boldsymbol{\phi}^k) \, \delta \boldsymbol{\phi}^k = -\mathbf{F}(\boldsymbol{\phi}^k) \tag{6.21}$$

and

$$J(\boldsymbol{\phi}) = \left\{ \int_{T_1}^{T_l} \int_{x_1}^{x_n} \frac{1}{g} \left[\left(\frac{\partial \beta_j}{\partial t} + \phi_x^h \frac{\partial \beta_j}{\partial x} \right) \left(\frac{\partial \beta_i}{\partial t} + \phi_x^h \frac{\partial \beta_i}{\partial x} \right) \right. \right.$$
$$\left. \left. + \left(\phi_t^h - gh + \frac{1}{2} {\phi_x^h}^2 \right) \frac{\partial \beta_j}{\partial x} \frac{\partial \beta_i}{\partial x} \right] B \, dx \, dt \right\}.$$

The Jacobian J is symmetric and banded and (6.21) is solved, when J is not indefinite, using the pre-conditioned conjugate gradient method, with the pre-conditioning matrix $P = \mathrm{diag}(J_{11}, \ldots, J_{ll})$. Both J and $\mathbf{F} = (F_1, \ldots, F_l)^T$ are evaluated exactly. The process is continued until

$$\frac{\max_i \left|\delta\phi_i^k\right|}{\max_i \left|\phi_i^k\right|} < \text{tolerance}, \tag{6.22}$$

for some specified tolerance.

The initial approximation $\boldsymbol{\phi}^0$ is given by

$$\phi_i^0 = \frac{x_i - x_1}{x_2 - x_1} v^0 + (T_i - T_1)\,\bar{E} \qquad i = 1, \ldots, l, \tag{6.23}$$

for some constants v^0 and $\bar{E}$.

Let $x_e = 0$, $x_o = 10$, $t_1 = 0$ and $t_2 = 10$. The algorithm is implemented in a channel with $B(x) = 10$ and $h(x) = 0$ for $x \in [0, 10]$. The boundary functions $C_e(t)$, $C_o(t)$, $g_1(x)$ and $g_2(x)$ also need to be prescribed. There is an obvious difficulty with defining $g_2(x)$, the depth for $x \in [0, 10]$ at the time t_2. Here $g_1(x)$ and $g_2(x)$ are defined by $g_1(x) = g_2(x) = \hat{d}$, where $\hat{d} > 0$ is either the subcritical or the supercritical root of

$$g\hat{d}^3 - \tilde{E}\hat{d}^2 + \frac{1}{2}C^2 = 0,$$

where $\tilde{E}$ is given the value 50 and $C = 10$, that is, g_1 and g_2 are the depths in the channel for a steady state flow with energy $\tilde{E} = 50$ and mass flow at inlet $C = 10$. The functions $C_e(t)$ and $C_o(t)$ are defined by

$$C_e(t) = \begin{cases} 10 - \hat{C}\,t & t \in [0, 2] \\ 10 - \hat{C}(4 - t) & t \in [2, 4] \\ 10 & t \in [4, 10] \end{cases},$$

$$C_o(t) = \begin{cases} 10 - \hat{C}\,t & t \in [0,2] \\ 10 - \hat{C}(4-t) & t \in [2,4] \\ 10 & t \in [4,10] \end{cases},$$

where $\hat{C}$ is a given constant in the range $0 \leq \hat{C} \leq 10$.

Thus, in this example, any time-dependence of the resulting flow is due to the changes in the mass flow at inlet and outlet with time. The conditions given above could be generated in practice by taking an initial steady flow with energy 50, and values of mass flow at inlet and outlet of 10, where these values are controlled by using a weir or a sluice gate. The values of the mass flow at inlet and outlet could be altered for $t \in [0,4]$ and then returned to their original values. The flow might then be expected eventually to return to a steady state.

Let $n = 9$ and $m = 9$. Then the algorithm converges to a subcritical approximation in 4 Newton iterations for $\hat{C} = 0$, in 5 Newton iterations for $\hat{C} = 1$ and in 5 Newton iterations for $\hat{C} = 3$, for a tolerance in (6.22) of 10^{-3}. The initial approximation $\boldsymbol{\phi}^0$ is, in each case, given by (6.23), with $v^0 = 2.5$ and $\bar{E} = 50$.

The value of ϕ_x^h in each element may be thought of as being an average of the velocity taken over the time period covered by the element. The piecewise constant velocity approximations for $n = 9$ and $m = 9$, in the three cases $\hat{C} = 0$, $\hat{C} = 1$ and $\hat{C} = 3$, are shown in Figure 6.3. The length of the arrow in each element is directly proportional to the approximate velocity, ϕ_x^h, in that element.

Notice that in Figure 6.3*a* the flow is uniform, as is expected for the given channel shape and for the boundary conditions C_e and C_o which are independent of time. In Figure 6.3*b* it can be seen that the effect of decreasing the mass flow at inlet and outlet for $t \in [0,4]$ is to reduce the velocity during this time. The effect is more pronounced in Figure 6.3c, where the reduction in the mass flow at inlet and outlet is larger.

The success of the algorithm is heavily dependent on choosing boundary conditions which are consistent with one another. In particular, for certain choices, there may be no solution at all or the solution may be discontinuous, in which case, the given algorithm will be unsuccessful since in using the functional (6.18) an assumption is made that the variables are continuous.

Supercritical approximations may be generated by using an appropriate method to solve (6.21) for $\delta\boldsymbol{\phi}^k$, where $J(\boldsymbol{\phi})^k$ is indefinite.

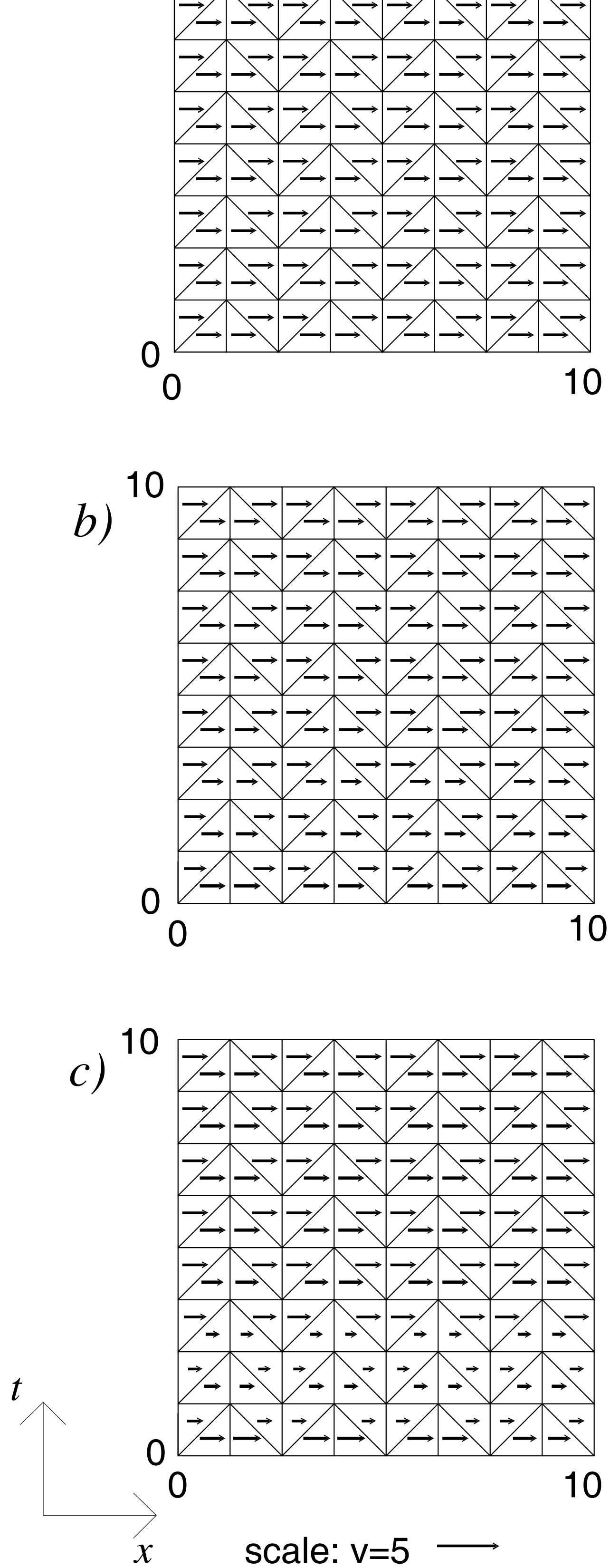

Figure 6.3: Time-dependent subcritical piecewise constant velocity approximations for $a)$ $\hat{C} = 0$, $b)$ $\hat{C} = 1$ and $c)$ $\hat{C} = 3$.

Chapter 7

Concluding Remarks

The central part of the work described in this thesis can be thought of as having two distinct components. The first component deals with the derivation of variational principles for free surface flows while in the second part a selection of these variational principles is used to generate numerical approximations to free surface flows. Variational principles for three-dimensional free surface flows are stated in Chapter 3 and used to derive principles for shallow water flows. Approximations to shallow water flows are generated in Chapters 4, 5 and 6, and Chapter 6 also contains an algorithm for approximating three-dimensional steady flows in a channel of constant breadth.

Two variational principles for general three-dimensional flows are used — one based on Hamilton's principle, (3.12), and the other based on Luke's principle (Luke (1967)). By approximating the variables by their shallow water counterparts, performing the integration with respect to the vertical coordinate z and adding on appropriate boundary terms, these two principles are reduced to give variational principles for shallow water. The process of changing variables in the functionals of the two shallow water principles derived in this way then allows the integrands to be expressed in terms of the p and r functions, defined by (3.27) and (3.29) respectively, and multiples of the conservation laws, (2.20) and (2.24), and the irrotationality condition, (2.15).

The function p has the values of vertically averaged pressure while r may be thought of as a Lagrangian density since its value at a point is the difference between kinetic and potential energy of a particle at that point. By recognising that p and r are related by means of a Legendre transform, two further functions — denoted by P and R — are constructed, so that p, r, P and R constitute a quartet of functions related to one another by a closed set of Legendre transforms, as shown in Figure 3.3. The function P has the values of flow stress and the value of R at a point is the total energy of a particle at that point.

Benjamin and Bowman (1987) consider conservation laws and symmetry properties of Hamiltonian systems, including shallow water, for which they derive four functions, two of which — identified by them as a Hamiltonian density and a flow force — have the values of the functions R and P respectively, apart from constant multipliers. The approach described here is more direct.

A set of four functionals — based on the p, r, P and R functions — is comprised of the two functionals derived from the variational principles for three-dimensional free surface flows and the two functionals generated by substituting P and R for p and r, respectively, in these functionals using the Legendre transforms. By making the assumption that the flow is independent of time, functionals for steady state flows are derived. Then, constraining the variations in the 'p' principle for steady flow to satisfy irrotationality, giving (3.94), and constraining the variations in the 'P' principle for steady flow to satisfy the conservation of mass equation and a boundary condition on the mass flow, giving (3.95), the gas dynamics analogy may be invoked to identify (3.94) and (3.95) as examples of Bateman's functions (Bateman (1929)). Sewell (1963) re-examined the relationships between these principles, in the context of Legendre transforms, for three-dimensional steady flows in perfect fluids. Use has been made of these variational principles, by, for example, Lush and Cherry (1956) and Wixcey (1990), to generate approximate solutions to the equations of motion for compressible gas flows.

For the case of shallow water there exist the extra variational principles — the 'r' and 'R' principles and all of their constrained versions — which may be used to approximate solutions of the shallow water equations. These principles are of particular value since they contain functionals of the depth of flow and can thus be used directly for generating approximations to the depth, unlike the constrained 'p' and 'P' principles.

The implementation of variational principles for finding approximations to time dependent flows reveals several inherent problems, which are discussed below. Therefore, with one exception, the numerical methods are applied to variational principles for steady state flows.

The constrained 'r' principle (3.117) for steady quasi one-dimensional flow depends on only one variable — the depth of flow — which makes it a natural candidate for developing an algorithm to generate approximations to the depth function. The constrained 'p' principle for steady flow (3.94) and the version of the variational principle for steady quasi one-dimensional flow are useful too since they also depend on only one variable each — the velocity potential. Other variational principles are used as well, namely, the unconstrained 'r' principle for steady quasi one-dimensional flow, which depends on the depth, mass flow and velocity potential functions, the 'R' principle for steady flow constrained to satisfy irrotationality, which depends on the depth and velocity potential functions, the constrained 'p' principle for unsteady quasi one-dimensional flow and a version of Luke's free surface principle (Luke (1967)), which depends on the velocity potential and the height of the free surface.

The same basic algorithm is applied to all of the variational principles and is, on the whole, successful. The variables in the variational principles are expressed as expansions in terms of finite element basis functions — piecewise linear and piecewise constant basis functions in one dimension and piecewise linear basis functions in two dimensions. The parameters of the solutions are determined as the values which cause the functionals of the variational principles to be stationary with respect to variations in the finite dimensional space spanned by the finite element basis functions. In each case this leads to one or more sets of equations, at least one of which is non-linear.

The method chosen to solve these non-linear sets of equations is Newton's method, which has quadratic convergence to the approximate solution, given an initial guess sufficiently close to the solution. The Jacobian in each case is symmetric and banded and, in fact, tridiagonal for the equations generated from the functionals for steady quasi one-dimensional flow. For tridiagonal Jacobians the update to the approximation is found using Gaussian elimination and back substitution, while for non-tridiagonal positive or negative definite Jacobians the update is found using a pre-conditioned conjugate gradient method.

In this way it is possible to find approximations to the shallow water variables in cases where the flow is continuous.

A slightly different approach is taken in order to approximate discontinuous flows and in using a version of Luke's principle for free surface flows to approximate flows which do not necessarily satisfy the assumptions of shallow water theory. The constrained 'r' principle for steady quasi one-dimensional flow is used to generate approximations to discontinuous depth functions. In both cases the flow variables — depth in the 'r' principle case and velocity potential in the version of Luke's principle — are expanded in terms of the finite element basis functions and the values of the parameters are found for which the functionals are stationary with respect to variations in the parameters. The positioning of the hydraulic jump in the discontinuous case and the free surface in the case of Luke's principle is carried out by a direct application of the appropriate natural conditions of the corresponding variational principles. The method appears to work well in approximating discontinuous depth functions but is less successful in approximating the height of the free surface for a flow which does not necessarily satisfy the assumptions of shallow water theory. In this last case the method seems initially to be converging to a solution and then the method diverges. This may suggest that the solution algorithm is only capable of converging to the solution from one direction so that, if the algorithm causes an approximation to the solution to overshoot the solution, it will not converge.

The constrained 'r' principle for steady quasi one-dimensional flow is used in Section 4.1.2 to derive an error bound on the piecewise constant approximation to the depth function. The piecewise constant approximation is found to converge linearly, in the L_2 norm, to the depth of flow. Numerical experiments show that the piecewise linear approximation to the depth, generated using the constrained 'r' principle, is quadratically convergent, in the L_2 norm, to the exact solution. The error in the piecewise constant approximation to the velocity, derived from the piecewise linear approximation to the velocity potential generated using the constrained 'p' principle for steady quasi one-dimensional flow, is considered in Section 4.3.2. The piecewise constant approximation can be seen to converge linearly, in the L_2 norm, to the velocity of the flow, using numerical experiments.

The availability of the exact one-dimensional solution for flow in a channel, which can be found by solving (2.55) and (2.56) to give the values of energy and mass flow at each point and then solving (2.34) and (2.35) simultaneously to give the values of the depth and velocity at each point, enables the above conclusions to be drawn about the accuracies of the approximations generated from 'p' and 'r' functionals for steady quasi one-dimensional flow. The exact solutions of the two-dimensional shallow water equations are not known and so the two-dimensional approximations cannot be analysed in this way.

The usual method of finding error bounds for finite element approximations, such as in Strang and Fix (1973) and Hughes (1987), depends on identifying a norm, in which the distance between the exact solution and the finite dimensional space spanned by the finite element basis functions is minimised by the approximation derived as being the function that either minimises or maximises a particular functional. In the shallow water case it has not been possible to identify such a norm using either the functional of the constrained 'p' principle or the functional of the constrained 'R' principle, both of which are used in Chapter 5 to generate approximations to the flow variables in two dimensions. However the two principles each give rise to approximations for the velocity and, by comparing the velocity approximations over several different grids, it can be seen that, as the grid is refined, the difference between the two approximations decreases.

An algorithm is presented in Section 4.4 which uses the constrained 'p' principle for steady quasi one-dimensional flow to generate approximations to the velocity potential on an adaptive grid. The approximation is the finite element expansion such that the 'p' functional is stationary with respect to variations in the parameters of the expansion and with respect to variations in the positions of the internal nodes of the grid on which the expansion is defined. The L_2 error of the velocity approximation, generated from the approximation to the velocity potential, is shown to be reduced, but only slightly, when compared with the corresponding error for an approximation defined on a fixed regular grid with the same number of nodes.

In Chapter 6 an algorithm is described for generating approximations to time-dependent velocity potential functions using the constrained 'p' principle for time-dependent quasi one-dimensional flow. While some success is achieved, that is, for a particular set of prescribed boundary functions approximate solutions are generated, the success is very dependent on defining the boundary functions in a consistent way. In particular the boundary functions must be such that a solution actually exists and, if an approximation is to be found using functional (6.18), this solution must be continuous.

The boundary functions required are the values of mass flow at the inlet and outlet points of the channel, over the whole time interval being considered, and values of the depth at every point in the channel at the initial and final times. In order to prescribe this last condition the solution at the final time must be known in advance of solving the problem. This difficulty is overcome in Chapter 6 by assuming that the flow is initially time-independent and that it returns to its original steady state before the end of the time interval. The boundary functions for the mass flow must then be defined consistently with this, in particular, it is necessary that, over the period of time considered, the same amount of mass leaves the channel as has entered.

A practical problem which might be posed is to find the subsequent flow as a function of time, given the depth of the fluid initially and the variations of the mass flow at inlet and outlet with time. Unless the depth function at the end of the time interval can be deduced from this data the finite element method using the functionals for time-dependent flow derived in Chapter 3 is of little use. One possibility is to consider a very long time interval and to set the boundary function for depth at the final time equal to the asymptotic solution, given as $t \to \infty$. However, this would be computationally expensive because of the size of the domain which would be necessary to accommodate a sufficiently long time.

The problem does not just suddenly appear; it is present in Hamilton's principle (3.12). In its general form, Hamilton's principle, with variations vanishing at the initial and final times, gives rise to differential equations of motion. Once these equations have been integrated any boundary and initial conditions may be applied. In the same way the variational principles for shallow water, deduced from Hamilton's principle and Luke's principle, give rise to the equations of motion for shallow water flows in the domain of integration, assuming that variations vanish at the initial and final times so that the time boundary terms also vanish (the solutions are assumed known on the time boundaries and so these terms are constants).

In order to use the variational principles for time-dependent shallow water to generate time-dependent approximations, boundary terms are added in Chapter 3 so that non-zero variations are allowed on the time boundaries and no assumption need be made about knowing the solution at the final time. However, it can be seen that this only rephrases the basic problem since the solutions at the ends of the time interval are precisely the functions which are required for the boundary terms.

The numerical methods employed in this thesis have been successful in generating approximations to continuous steady flows which are wholly subcritical, for both quasi one-dimensional and two-dimensional flows, or wholly supercritical, for quasi one-dimensional flows. Success is also achieved in approximating discontinuous steady quasi one-dimensional flows. However, as described above, the application of the methods to the time-dependent case presents difficulties.

In order to investigate the accuracy of the shallow water approximation to free surface flows it would be useful to have finite element approximations to free surface flows. The algorithm given in Section 6.1, which attempts to generate such an approximation, fails but by adapting other methods, for example those of Aitchison (1979) and Ikegawa and Washizu (1973), it should be possible to obtain some approximations for comparison.

Other possibilities for future work include generating supercritical approximations to two-dimensional flows and approximations to discontinuous flows in two-dimensions. In the first case, if an attempt is made to solve the non-linear set of equations obtained from the finite dimensional versions of the functionals by Newton's method, the Jacobian is indefinite so that a more sophisticated technique for solving a system with an indefinite matrix must be investigated; there is also a possibility that the Jacobian may go singular. An algorithm to approximate discontinuous flows in two dimensions could be based on the method for approximating discontinuous flows in one dimension, that is, by generating two continuous solutions — one subcritical and one supercritical — and coupling them at a curve, whose position is determined using the two-dimensional jump conditions. Any such algorithm using the variational principles of Section 3.8 would only be able to generate approximations to flows with a discontinuity which does not terminate inside the domain, since, in deriving these variational principles, this assumption is made.

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