Path Properties of Lévy Processes

Adam Barker

Department of Mathematics and Statistics,

University of Reading

A thesis submitted for the degree of

Doctor of Philosophy

May 2019

Arte et labore

Declaration of Original Authorship: I confirm that this is my own work and the use of all material from other sources has been properly and fully acknowledged.

Adam Barker

Abstract

This thesis can be split into two main components, the first of which looks at the fractal dimension, specifically, box-counting dimension, of sets related to subordinators (non-decreasing Lévy processes). It was recently shown in [111] that $\lim_{\delta\to 0} U(\delta)N(t,\delta) = t$ almost surely, where $N(t,\delta)$ is the minimal number of boxes of size at most δ needed to cover a subordinator's range up to time t, and $U(\delta)$ is the subordinator's renewal function. The main result in this section is a central limit theorem (CLT) for $N(t,\delta)$, complementing and refining work in [111].

Box-counting dimension is defined in terms of $N(t,\delta)$, but for subordinators we prove that it can also be defined using a new process obtained by shortening the original subordinator's jumps of size greater than δ. This new process can be manipulated with remarkable ease in comparison to $N(t,\delta)$, and allows better understanding of the box-counting dimension of a subordinator's range in terms of its Lévy measure, improving upon [111, Corollary 1]. We prove corresponding CLT and almost sure convergence results for the new process.

The second main component of this thesis studies Markov processes conditioned so that their local time must grow slower than a prescribed function. Building upon recent work on Brownian motion with constrained local time in [8] and [78], we study whether or not the conditioned process is transient or recurrent, working with a broad class of Markov processes.

In order to understand the local time, it is equivalent to study the inverse local time, which is itself a subordinator. The problem at hand is effectively equivalent to determining the distribution of a subordinator (the inverse local time) conditioned to remain above a given function. In conditioning a subordinator to remain above a curve of the form $g(t), t \geq 0$, the process is restricted to a time-dependent region, in contrast to previous works in which a process is conditioned to remain in a fixed region (e.g. cones in [43] and [60]). This means that we study boundary crossing probabilities for a family of curves, and must obtain uniform asymptotics for such a family.

The main result in this section is a necessary and sufficient condition for transience or recurrence of the conditioned Markov process. We will explicitly determine the distribution of the inverse local time for the conditioned process, and in the transient case, we explicitly determine the law of the conditioned Markov process. In the recurrent case, we characterise the *entropic repulsion envelope* via necessary and sufficient conditions.

Mathematics Subject Classification Primary: 60G51; 28A80; 60J55.
Secondary: 60G75; 60F05; 60F15; 60G17; 60J25.

Keywords and key phrases Lévy processes; subordinators; fractal dimension; box-counting dimension; limit theorems; Markov processes; local time; large deviations; regular variation.

Acknowledgements

Firstly, I would like to thank Mladen Savov for dedicating such a lot of his time to supervise my PhD. Thank you for introducing me to many beautiful mathematical topics, for suggesting interesting research problems, and for encouraging me to pursue my own ideas. It has been a pleasure to work together. Also, a big thank you to Tobias Kuna for all of his guidance and encouragement throughout my PhD. Thank you to my family for their support, and for nurturing my interest in mathematics. I would also like to thank Lance for his endless warmth, loyalty, and compassion. Thank you for helping with morale through difficult times. Thank you to EPSRC for funding the PhD project.

Contents

1 Introduction **1**

1.1 History & Motivating Examples . . . 1

1.2 Applications of Lévy Processes . . . 4

1.3 Preliminary Definitions, Notation, & Results . . . 7

1.3.1 Definition of a Lévy Process . . . 7

1.3.2 The Lévy-Khintchine Formula . . . 7

1.3.3 Jumps & Poisson Point Processes . . . 9

1.3.4 Examples of Lévy Processes . . . 10

1.3.5 Infinitely Divisible Distributions . . . 11

1.3.6 (Strong) Markov Property . . . 12

1.3.7 Definition of a Subordinator . . . 12

1.3.8 Lévy-Khintchine Formula for Subordinators . . . 13

2 Fractal-Dimensional Properties of Subordinators **15**

2.1 Literature Overview . . . 16

2.1.1 Box-Counting Dimension . . . 16

2.1.2 Relevant Results from the Literature . . . 18

2.1.3 An Alternative Box-Counting Scheme . . . 19

2.2 Main Results . . . 21

2.2.1 A Central Limit Theorem for $N(t,\delta)$ . . . 22

2.2.2 An Alternative Box-Counting Scheme, $L(t,\delta)$ . . . 23

2.3 Proof of CLT for $N(t,\delta)$ . . . 27

2.3.1 First Sufficient Condition via Berry-Esseen Theorem . . . 28

2.3.2 Proof of Theorem 2.2.1 . . . 31

2.3.3 Proofs of Lemmas 2.3.9, 2.3.12, 2.3.5 . . . 34

2.4 Proofs of Results on $L(\mathrm{t},\delta)$ . . . 37

2.5 Extensions and Special Cases . . . 42

2.5.1 Extensions: Box-Counting Dimension of the Graph 42
2.5.2 Special Cases: Regular Variation of the Laplace Exponent 43

3 Markov Processes with Constrained Local Time 45
3.1 Introduction & Background 46
3.1.1 Related Problems in Polymer Physics 46
3.1.2 Local Time of a Markov Process 49
3.2 Literature Overview 53
3.2.1 Relevant Results from the Literature 54
3.2.2 Brief Exposition of the Main Result: Necessary and Sufficient Condition 58
3.3 Key Definitions 59
3.4 Statements of Main Results 61
3.4.1 Regularity Conditions 61
3.4.2 Results in the $I(f) < \infty$ Case 66
3.4.3 Results in the $I(f) = \infty$ Case 68
3.5 Proofs of Main Results 71
3.5.1 Proofs in the $I(f) < \infty$ Case 72
3.5.2 Proofs in the $I(f) = \infty$ Case 77
3.5.3 Proof of Proposition 3.4.19 79
3.5.4 Proof of Proposition 3.4.20 84
3.6 Proof of Lemma 3.5.1 94
3.7 Proof of Lemma 3.5.3 102
3.8 Proof of Lemma 3.5.7 126
3.9 Proofs of Auxiliary Lemmas 136
3.10 Extensions 142

Appendix 146
A.1 Supplementary Figures 146
A.2 Key Inequalities 150
A.3 Key Theorems in Measure Theory 150
A.4 Results on Regularly Varying Functions and Related Functions 151

Bibliography 154

Index 166

Glossary of Notation 168

Chapter 1

Introduction

1.1 History & Motivating Examples

The class of Lévy processes is a rather broad generalisation - one which accounts for a wide range of common stochastic processes including Brownian motion, Poisson processes, and *everything in between.* Lévy processes are characterised as the class of stochastic processes with stationary, independent increments, whose sample paths are right continuous with left limits. This seemingly simple definition gives rise to a very rich class of processes, while still allowing us to extend many important results from familiar processes to a much more general setting.

When studying mathematics, there is often a compromise to be met between the *depth* of a theorem and its *generality.* In this regard, the depth of a theorem is a measure of complexity, while its generality describes the scope of the result - perhaps a result which considers only a very specific special case, in contrast to a result which can be drawn upon in a vast number of situations*. With this in mind, the study of Lévy processes can be seen in a very positive light. Indeed, we find many beautiful and deep results in the literature which apply to *all* Lévy processes (perhaps with a caveat excluding cases for which the result does not make sense or is trivial, e.g. fractal properties of a process which does not have any meaningful structure at an arbitrarily

*See [63, Chapters 14-17], for further discussion small scale). Among all common classes of stochastic processes which are continuous in time/space, the Lévy process finds this balance between depth and generality especially well. Before providing statements for some of the most important, fundamental results on Lévy processes, let us first take a look at some history to illustrate the significance of this family of processes. We shall also briefly discuss the applications of Lévy processes before moving on to statements of key results.

Probability Theory The study of probability theory dates back to the 1500s and Cardano's efforts to analyse games of chance and gambling in *Liber de ludo aleae* [39], posthumously published in the 1600s. Cardano's gambling-motivated treatise contained preliminary statements alluding to some important results, albeit without rigorous proofs. For instance, Cardano noted that larger sample sizes tend to yield more accurate results for understanding a population. Eventually, this idea was captured mathematically and formally proven by Jakob Bernoulli as a version of the law of large numbers, appearing in his 1713 book *Ars Conjectandi* [10].

Between the 1600s and 1800s, the foundations for what was to become modern probability theory were gradually laid out by a number of mathematicians, most notably including: Fermat; Pascal; de Moivre; Laplace. However, the main theme of this thesis, stochastic processes, would not appear as a concept until the early 1900s. Thereafter, the interest in the theory of stochastic processes began to expand quite rapidly. Let us illustrate this growth with a few examples which, in turn, motivate the study of the processes which appear in this thesis.

Brownian Motion In 1827, botanist Robert Brown observed the motion of large particles of pollen moving around in water, a phenomenon now understood via the ubiquitous Wiener process, more commonly referred to as Brownian motion. The study of Brownian motion, purely from a probabilistic point of view, has its origins in the early 1900s. This type of process now regularly appears in a vast range of situations across pure and applied mathematics, physics, and economics.

Poisson Processes Also in the early 1900s, interest in the Poisson process developed as a means of modelling a variety of situations: numbers of incoming phone calls in an interval of time [53]; occurences of insurance claims [91]; alpha particle emission [108]. While Poisson himself did not study these processes, the name comes from the inherent relationship between Poisson processes and the Poisson distribution, which *was* studied (discovered, in fact) by Poisson. The connection is that for a Poisson process of rate λ, the distribution of the process at time t has Poisson distribution with parameter λt.

Lévy Processes Lévy processes were first studied as a class in the mid-1900s, capturing the attention of a number of eminent mathematicians including Kiyosi Itô, Andrey Kolmogorov, and Paul Lévy. Many results on the aforementioned Brownian motion and Poisson process can in fact be proven in general for *all* Lévy processes. This is perhaps surprising, given the differences between these processes. For instance, sample paths of Brownian motion are continuous, whereas a Poisson process has discontinuous sample paths. However, both processes have *càdlàg*† (right-continuous with left-limits) sample paths, as do all Lévy processes. Hence it has become apparent in recent decades that studying Lévy processes as a whole class, rather than splitting up into individual cases, presents a highly attractive unified theory which contains a large number of important processes. A variety of books that deal with Lévy processes have been published in recent years: [1]; [14]; [15]; [45]; [82]; [110].

Random Walks One cannot give a satisfactory introduction to Lévy processes without mentioning random walks. Random walks have been the subject of many interesting theoretical studies, and remain an important topic of interest in modern probability theory. The random walk is a highly versatile mathematical tool, and can be used to model a vast range of phenomena such as the total winnings of a gambler playing a number of games, or the fluctuating price of a stock.

While a random walk does not actually satisfy the definition of a Lévy process, the similarity between the two is striking. A random walk is a progressive sum of independent and identically distributed steps. Independence of the steps corresponds to the independent increments property of Lévy processes, while the condition that steps are identically distributed corresponds to stationarity of increments of Lévy proesses. Thus random walks are considered to be the discrete-time analogue of Lévy processes (which exist on a continuum of time), and vice-versa. Moreover, considering only the values taken by a Lévy process at times

†From the French: *continue à droite, limite à gauche*

$1, 2, 3, \ldots$, this gives rise to a random walk (though a lot of information is lost about the original Lévy process).

Various results on random walks can be generalised to Lévy processes, and many papers consider Lévy processes in tandem with random walks. The connection between random walks and Lévy processes is particularly well documented in [45, Chapter 4]. Despite all the similarities between random walks and Lévy processes, the fact that the latter is defined in a continuous setting, rather than a discrete setting, adds some additional features of interest. One of these is the behaviour of the process at arbitrarily small scales, which naturally leads to many interesting questions such as that of the fractal nature of the process, as will be investigated in Chapter 2.

1.2 Applications of Lévy Processes

Mathematical Finance In mathematical finance, the widely studied Black-Scholes model uses a geometric Brownian motion to model the instantaneous log-return of a stock price. While geometric Brownian motion is highly tractable, empirical evidence suggests that it does not adequately capture certain features of market equity returns, such as volatility smile or asymmetry [29, 37, 57, 81, 93]. Moreover, it has been observed that the tails of Brownian motion are too light when compared with market data [61]. An alternative model involves replacing geometric Brownian motion by an exponential of a Lévy process.

A range of Lévy processes other than Brownian motion are used in mathematical finance, each having advantages in different scenarios. Some Lévy processes used to model the price of a stock are: normal inverse Gaussian processes [7]; hyperbolic processes [51]; tempered stable processes (also referred to as CGMY processes) [29, p310]; variance gamma processes [92].

Let us present an example of the kind of problem tackled in mathematical finance. It may be useful to understand whether or not (perhaps in a given time interval) an agent will go bankrupt, or their capital will exceed a given level. If we model the agent's total capital by a random process, then the aforementioned problem corresponds to determining the probability distribution of the time at which a given process crosses a certain boundary (or multiple boundaries).

Such boundary-crossing type problems are particularly useful for understanding *barrier options*. A barrier option is a financial option which may be activated (or extinguished) upon the event of a given stock price crossing a certain level. In Chapter 3, we shall study boundary-crossing behaviour of certain Lévy processes. In particular, we study the probability that a process lies above a given function over a certain time interval, which means that we are looking at the boundary-crossing behaviour for our process with regard to the given function. For further reading on financial applications of Lévy processes, we refer to [1, Section 5.6], [26], [27], [38], [82, Section 1.3], [84], and [112].

Branching Processes In various biological systems, branching processes are used to model the number of individuals in a population and how they are related to each other, with births and deaths occuring at different times. The study of branching processes has its origins in the late 1800s, and the emergence of the Galton-Watson[‡] process [121]. Such a process has a given number of individuals at its first time step, each of which subsequently has a random number of offspring who are added to the population at the next time step, and so on, where the number of offspring for each individual is independent and identically distributed for all individuals and generations.

It is simple to modify the Galton-Watson process to exist on a continuous interval of time, rather than discrete time increments. It is also natural to wish to generalise such branching processes to possess a continuous state space, which motivated Miloslav Jiřina's work in the 1950s [68]. It is through this generalisation to allow for a continuous state space that we see a close, intrinsic connection between branching processes and Lévy processes.

In the 1960s, John Lamperti discovered a relationship between Lévy processes and continuous-state branching processes (CSBPs) [85]. Moreover, in the 1970s, Lamperti derived a further relationship between Lévy processes and positive self-similar processes [86], but we shall focus on the former relationship between Lévy processes and CSBPs. Lamperti's result provides a one-to-one correspondence, via a simple random time change, between CSBPs and Lévy process which have no negative jumps (discontinuities). This has proved highly valuable in the study of CSBPs (see e.g. [20]), and also in the study of superprocesses [87],

[‡]Most commonly known as Galton-Watson processes, they are more recently referred to as Bienaymé-Galton-Watson processes, due to Bienaymé's earlier (but less well-known) works on such processes [65].

as well as for the study of Lévy processes themselves [13]. For further reading on CSBPs one can refer to [82, Chapter 12] and [90, Chapter 3].

Fragmentation Processes Fragmentation processes are mathematical models which describe how an object breaks up into smaller pieces, each of which continues to break up into further, smaller pieces, and so on. The probabilistic study of such processes dates back to the 1940s. In 1940, it was observed by N.K. Razumovskii that the logarithm of particle sizes of minerals, under grinding, approximately followed a normal distribution [103]. This motivated Andrey Kolmogorov to formulate a simple branching random walk model, and subsequently prove a central limit theorem result for such processes [114], thus providing a theoretical explanation for the phenomenon observed by Razumovskii.

The connection between fragmentation processes and Lévy processes, yielding a rigorous framework in which to study fragmentation processes in their full generality, is a particularly recent development, originating in Jean Bertoin's 2001 paper [16]. The fragmentation processes considered in prior works all had the constraints that the time taken for each particle to fracture is positive, and that the number of fragments produced by such a fracture is finite. In order to allow for the instantaneous shattering of a piece into infinitely many fragments, Bertoin used a construction for a fragmentation process based on a Lévy process, the details of which can be found in the monograph [17]. Such pioneering works on fragmentation processes have established a close link between the study of Lévy processes and the study of the fragmentation of blocks of minerals in the mining industry, see [18] and [58].

Polymer Physics & Local Time In Chapter 3, we shall study the large-scale properties of Markov processes under certain constraints, motivated by problems in polymer physics which look at the large-scale behaviour of a long polymer chain. The proofs in this chapter heavily rely upon the intrinsic connection of certain Markov processes to corresponding non-decreasing Lévy processes, via the concept of *local time*. Further discussion of polymer physics and local time shall be postponed until Sections 3.1.1 and 3.1.2.

1.3 Preliminary Definitions, Notation, & Results

In this section we shall introduce and discuss some essential definitions, notation, and key results which form the foundations for this thesis. While it is assumed that the reader is reasonably familiar with measure theoretical probability, some important results from measure theory (integral/limit theorems) and probability theory (key inequalities) are included in the appendix, see Section A.2 and Section A.3. We also include some key results on the theory of *regularly varying functions* in the appendix (Section A.4), although any discussion of regular variation is postponed until Chapters 2 and 3.

There is, of course, a vast literature providing the necessary background on probability theory and measure theory, including for example [19, 50, 55, 69].

1.3.1 Definition of a Lévy Process

Definition 1.3.1. *A Lévy process, denoted by $X = (X_t)_{t\geq 0}$, is a stochastic process in Euclidean space $\mathbb{R}^d$ which starts from* 0 *almost surely, has càdlàg (right-continuous with left-limits) sample paths almost surely, and has stationary, independent increments.*

That is to say, for each $0 < s < t$, the increments $X_t - X_s$ and $X'_{t-s} - X'_0$ have the same distribution, where X' is an independent copy of X (stationarity of increments), and for each $0 < s < t < u < v$, the increments $X_v - X_u$ and $X_t - X_s$ are independent (independence of increments).

In terms of topology, a sample path of a Lévy process exists in the space of all càdlàg functions on $[0, \infty)$ endowed with the Skorohod topology. We refer to [19, Section 16] for further details.

1.3.2 The Lévy-Khintchine Formula

The most important discovery on Lévy processes, laying the foundations for the modern study of such processes, is the celebrated *Lévy-Khintchine formula*, which gives an analytic expression for the characteristic function associated to a Lévy process in full generality, in terms of three key quantities. The characteristic function determines the distribution of a process, giving a simple and elegant way of working with Lévy processes.

However, the characteristic function itself is not usually studied - the convention is to work with the *characteristic exponent*, Ψ, defined by the relation

$$\mathbb{E}[e^{i\lambda X_t}] = e^{-t\Psi(\lambda)}, \text{ for } t \geq 0, \lambda \in \mathbb{R}.$$

The fact that this relation holds for all $t \geq 0$ follows from the fact that Lévy processes have infinitely divisible distributions (see the upcoming Subsection 1.3.5), as well as the stationary, independent increments property.

The Lévy-Khintchine formula allows mathematicians to use analytic tools in order to understand probabilistic or geometric properties of Lévy processes. The vast majority of results on Lévy processes are formulated in terms expressions deriving from this formula. The Lévy-Khintchine formula was first proven in 1934 by Lévy [89], but a much simpler proof was provided in 1937 by Khintchine [72]. We shall only state the 1-dimensional version of the Lévy-Khintchine formula, as the Lévy processes studied in this thesis will typically be 1-dimensional.

Theorem 1.3.2 (Lévy-Khintchine Formula). *For $a \in \mathbb{R}$, $q \in [0, \infty)$, for a measure Π on $\mathbb{R} \setminus \{0\}$ satisfying $\int_{\mathbb{R}\setminus\{0\}} \min(1, x^2)\Pi(dx) < \infty$, and for $\lambda \in \mathbb{R}$, define the function $\Psi(\lambda)$ by*

$$\Psi(\lambda) := ia\lambda + \frac{q\lambda^2}{2} + \int_{\mathbb{R}\setminus\{0\}} \left(1 - e^{i\lambda x} + i\lambda x \mathbb{1}_{\{|x|<1\}}\right) \Pi(dx). \tag{1.1}$$

Then there exists a unique probability measure on the space of all càdlàg functions on $[0, \infty)$, endowed with the Skorohod topology, which defines a Lévy process $X = (X_t)_{t\geq 0}$ with characteristic exponent Ψ, so satisfies $\mathbb{E}[e^{i\lambda X_t}] = e^{-t\Psi(\lambda)}$, for all $t \geq 0, \lambda \in \mathbb{R}$. On the other hand, every Lévy process has a unique characteristic exponent of the above form.

This formula has a very meaningful probabilistic interpretation. Looking at the three terms in the characteristic exponent, one can deduce that the corresponding Lévy process is a sum of a deterministic linear drift, a Gaussian process, and a Poisson point process. The Poisson point process contribution determines the *jumps* (discontinuities) of the process, which we define now:

1.3.3 Jumps & Poisson Point Processes

An important feature of Lévy processes is that they can instantaneously change position, or *jump* from one point to another. We formally define this as follows:

Definition 1.3.3. *The process* $(X_t)_{t\geq 0}$ *has a "jump of size* x *at time* t*" if* $X_t - X_{t-} = x$*, where* X_{t-} *denotes* $\lim_{s\uparrow t} X_s$*.*

The jumps of a Lévy process all come from the Poisson point process contribution, and the distribution of the rate and size of jumps is entirely determined by the *Lévy measure*, $\Pi(dx)$. We now explore how the jumps of a Lévy process are connected to its Lévy measure.

Closely examining the condition $\int_{\mathbb{R}\setminus\{0\}} \min(1, x^2)\Pi(dx) < \infty$, it becomes apparent that it is possible for the Lévy measure to be infinite, in the sense that $\int_{\mathbb{R}\setminus\{0\}} \Pi(dx) = \infty$ (for example, $\Pi(dx) = |x|^{-5/2}dx$). It is, of course, also possible for the Lévy measure to be finite. While Lévy processes can have infinitely many infinitesimally small jumps in a finite amount of time, a process with finite Lévy measure corresponds to the special case where the jumps of the process occur at a finite rate.

Let us first consider the case in which the Lévy measure is finite. The total number of jumps up to any time $t > 0$ has Poisson distribution with parameter $t \times \int_{\mathbb{R}\setminus\{0\}} \Pi(dx)$. One can then deduce that the first jump occurs at a random time T_1, which is exponentially distributed with parameter $\int_{\mathbb{R}\setminus\{0\}} \Pi(dx)$. The time until the next jump also has an exponential distribution with parameter $\int_{\mathbb{R}\setminus\{0\}} \Pi(dx)$, and so on for each successive jump. The sizes of the jumps are independent, and each distributed according to the rescaled measure

$$\mathbb{P}(X_{T_1} - X_{T_1-} \in dy) = \frac{\Pi(dy)}{\int_{\mathbb{R}\setminus\{0\}} \Pi(dx)}. \tag{1.2}$$

If the Lévy measure is infinite, then jumps occur at an infinite rate. This means that the notion of a *first jump* is not well-defined, in the sense that $\inf\{t > 0 : X_t - X_{t-} \neq 0\} = 0$ almost surely. However, we can still understand the behaviour of our process as an infinite sum of independent processes, each of which has a finite Lévy measure. For a measurable subset $A \subset \mathbb{R} \setminus \{0\}$ such that $\int_A \Pi(dx) < \infty$, the first jump whose size is in the set A occurs at an exponentially distributed time T_1^A, with parameter $\int_A \Pi(dx)$. The time until the next jump has the same distribution, and so on for each successive jump. As in (1.2), the size of each jump is determined by the rescaled measure

$$\mathbb{P}(X_{T_1^A} - X_{T_1^A-} \in dy) = \frac{\Pi(dy)}{\int_A \Pi(dx)}. \tag{1.3}$$

This information is especially useful for Chapter 3, wherein we see that the large-scale behaviour of a Lévy process is mostly determined by the distribution of its largest jumps. In this context, our set A is of the form $[a, \infty)$, for $a > 0$.

1.3.4 Examples of Lévy Processes

Now let us take a brief look our two main motivating examples in the light of the Lévy-Khintchine formula, and using the notion of jumps.

Brownian Motion Standard 1-dimensional Brownian motion, denoted by $(B_t)_{t\geq 0}$, is defined as a stochastic process in $\mathbb{R}$ which satisfies the following:

(i) $B_0 = 0$ almost surely;

(ii) $(B_t)_{t\geq 0}$ has independent increments;

(iii) For each $0 \leq s < t$, the increment $B_t - B_s$ has a *normal distribution* with mean 0 and variance $t - s$;

(iv) The function $t \mapsto B_t$ is continuous, almost surely.

From property (iii), one can deduce that our 1-dimensional Brownian motion satisfies $e^{-t\Psi(\lambda)} = \mathbb{E}[e^{i\lambda B_t}] = e^{-t\lambda^2/2}$, so its characteristic exponent is given by

$$\Psi(\lambda) = \frac{\lambda^2}{2}.$$

Comparing this with the general Lévy-Khintchine formula in (1.1), we observe that there is no drift, and the Lévy measure is zero, so the process has no jumps.

Poisson Counting Process A Poisson counting process with rate parameter $\mu > 0$, denoted by $(N_t)_{t\geq 0}$, is a stochastic process taking values in $\mathbb{N}$, for which:

(i) $N_0 = 0$ almost surely;

(ii) $(N_t)_{t\geq 0}$ has independent increments;

(iii) For each $0 \leq s < t$, the increment $N_t - N_s$ has a *Poisson distribution* with parameter $\mu(t-s)$.

From property (iii), one can deduce that our Poisson counting process satisfies $e^{-t\Psi(\lambda)} = \mathbb{E}[e^{i\lambda N_t}] = e^{\mu t(e^{i\lambda}-1)}$, so its characteristic exponent is given by

$$\Psi(\lambda) = \mu(1 - e^{i\lambda}).$$

Comparing this with (1.1), it is clear that there is no drift, nor is there a Gaussian component. The Poisson counting process does have jumps, and they are each of size 1. Indeed, one can verify that the Lévy measure is a point mass at 1, with weight μ. That is to say, $\Pi(dx) = \mu \times \Delta_1(dx)$, where $\Delta_1(dx)$ is the Dirac measure at the point 1 (see [69, p9] for details on the Dirac measure).

So the Poisson counting process is a sum of randomly timed jumps, each of size 1. If we denote the first jump time by T_1, then T_1 has an exponential distribution with parameter μ. The time until the next jump is then independent of T_1, and also has an exponential distribution with parameter μ, and so on for each subsequent jump.

1.3.5 Infinitely Divisible Distributions

Before defining *infinite divisibility*, we remark that normal and Poisson distributions are infinitely divisible. It is well-known that these distributions can be rewritten as sums of normal/Poisson distributions, respectively, from which one can easily verify that they are infinitely divisibile. Here we have two of the simplest examples of infinitely divisible distributions, which correspond to two of the simplest Lévy processes: Brownian motion and Poisson counting processes.

Definition 1.3.4. *A random variable X has an infinitely divisible distribution if for each $n \in \mathbb{N}$, there exist* *independent and identically distributed random variables* $X_1, \ldots, X_n$*, such that* X *has the same distribution as* $X_1 + \cdots + X_n$*.*

The stationary independent increments definition implies that Lévy processes have infinitely divisible distributions. Indeed, for each $n \in \mathbb{N}$,

$$X_t \overset{d}{=} X_{\frac{t}{n}} + \left(X_{\frac{2t}{n}} - X_{\frac{t}{n}}\right) + \left(X_{\frac{3t}{n}} - X_{\frac{2t}{n}}\right) + \cdots + \left(X_{\frac{nt}{n}} - X_{\frac{(n-1)t}{n}}\right),$$

where each of these n increments are independent, with the same distribution. Conversely, every infinitely divisible distribution gives rise to a unique Lévy process, so there is a 1:1 correspondence between Lévy processes and infinitely divisible distributions. See [55, Chapter XVII] for further details.

1.3.6 (Strong) Markov Property

An important feature of Lévy processes is that they satisfy the *Markov property*, so a Lévy process is a *Markov process*. Essentially, this means that the process has no memory of its past behaviour, so its future behaviour only depends on its present state. More formally, a stochastic process satisfies the Markov property if for each $t \geq 0$, the process $(Y_s)_{s \geq 0}$ defined by $Y_s := X_{t+s}$ is conditionally independent of $(X_s)_{0 \leq s \leq t}$, given the value of X_t.

A more restrictive property than the Markov property is the *strong Markov property*, which is almost the same, only with the additional requirement we can replace $t \geq 0$ by a finite *stopping time*. So a process satisfies the strong Markov property if for each finite stopping time T, the process $(Y_s)_{s \geq 0}$ defined by $Y_s := X_{T+s}$ is conditionally independent of $(X_s)_{0 \leq s < T}$, given the value of X_T. We refer to [14, Section I.2] for further details, including rigorous definitions of stopping times.

1.3.7 Definition of a Subordinator

A subordinator is the special case of a *non-decreasing*, real-valued Lévy process. As it is non-decreasing, a subordinator cannot have a Gaussian component, but it can have a non-negative drift. Moreover, it is clear that a subordinator cannot have any negative jumps, as otherwise it would not be non-decreasing. So the

Lévy measure of a subordinator is only supported on $(0, \infty)$. However, imposing monotonicity actually adds a further restriction on the Lévy measure.

Recall that the Lévy measure, $\Pi(dx)$, of a general Lévy process satisfies $\int_{\mathbb{R}\setminus\{0\}} \min\{1, x^2\}\Pi(dx) < \infty$. It turns out that the Lévy measure of a subordinator must satisfy the slightly stronger condition that $\int_0^\infty \min\{1, x\}\Pi(dx) < \infty$. This extra condition is a consequence of the fact that the sample paths of a Lévy process are of *bounded variation* (BV) on each compact time interval almost surely if and only if $q = 0$ and $\int_{\mathbb{R}\setminus\{0\}} \min\{1, x\}\Pi(dx) < \infty$. See, for instance, [1, Section 2.3.3] for more details on bounded variation and Lévy processes. It is straightforward to show that the sample paths of every subordinator are of BV, almost surely, which leads to the stronger constraint on the Lévy measure. We then arrive at the following definition for a subordinator:

Definition 1.3.5. *A subordinator is a non-decreasing Lévy process (almost surely) which takes values in* $\mathbb{R}$*, and whose Lévy measure satisfies* $\int_0^\infty \min\{1, x\}\Pi(dx) < \infty$*.*

Equivalently, one can define a subordinator as a Lévy process which takes values only in $[0, \infty)$ almost surely, because a Lévy process can decrease if and only if it can take negative values (this is straightforward to deduce from Definition 1.3.1).

1.3.8 Lévy-Khintchine Formula for Subordinators

Because subordinators are non-negative, their Laplace transform is well-defined. We can hence study the Laplace exponent ϕ in place of the characteristic exponent Ψ, where the Laplace exponent ϕ is defined by:

$$e^{-t\phi(\lambda)} = \mathbb{E}[e^{-\lambda X_t}], \text{ for all } t \geq 0, \lambda \geq 0.$$

Observe that the characteristic exponent determines the Laplace exponent via the relationship $\phi(\lambda) = \Psi(i\lambda)$. Then we can use the Lévy-Khintchine formula to express the Laplace exponent of a subordinator in a neat, general form. Recalling that a subordinator has no Gaussian component, has non-negative drift, its Lévy measure is not supported on $(-\infty, 0)$, and its Lévy measure satisfies $\int_0^1 x\Pi(dx) < \infty$, we deduce from (1.1) that

$$\begin{aligned}\phi(\lambda) = \Psi(i\lambda) &= -a\lambda \ + \ \int_0^\infty \left(1 - e^{-\lambda x} - \lambda x \mathbb{1}_{\{|x|<1\}}\right)\Pi(dx) \\ &= -a\lambda - \lambda \int_0^1 x\Pi(dx) + \int_0^\infty (1 - e^{-\lambda x})\Pi(dx) \\ &=: \ \mathrm{d}\lambda \ + \ \int_0^\infty (1 - e^{-\lambda x})\Pi(dx), \qquad \mathrm{d} \geq 0. \end{aligned} \tag{1.4}$$

Hereon, the *drift* of a subordinator shall refer to the quantity d, as defined in (1.4), rather than the term "a", which appears in (1.1).

A function which will be key in the following chapters is the *tail* of the Lévy measure of a subordinator. This function, defined by $\overline{\Pi}(y) := \int_{(y,\infty)} \Pi(dx)$, determines the rate of occurence of jumps of size greater than y, as a special case of the relation in (1.3). The tail function offers an alternative formulation of the general form for the Laplace exponent of a subordinator:

$$\phi(\lambda) = \mathrm{d}\lambda + \int_0^\infty (1 - e^{-\lambda x})\Pi(dx) = \mathrm{d}\lambda + \int_0^\infty \lambda e^{-\lambda x}\overline{\Pi}(x)dx. \tag{1.5}$$

This version of the Lévy-Khintchine formula for subordinators will be essential for many of the computations in Chapter 3, as this version allows us to bound the probabilites of various events relating to subordinators in terms of the asymptotic behaviour of the tail function $\overline{\Pi}(x)$.

Chapter 2

Fractal-Dimensional Properties of Subordinators

Abstract

This chapter looks at the box-counting dimension of sets related to subordinators (non-decreasing Lévy processes). It was recently shown in [111] that almost surely $\lim_{\delta\to 0} U(\delta)N(t,\delta) = t$, where $N(t,\delta)$ is the minimal number of boxes of size at most δ needed to cover a subordinator's range up to time t, and $U(\delta)$ is the subordinator's renewal function. Our main result is a central limit theorem (CLT) for $N(t,\delta)$, complementing and refining work in [111].

Box-counting dimension is defined in terms of $N(t,\delta)$, but for subordinators we prove that it can also be defined using a new process obtained by shortening the original subordinator's jumps of size greater than δ. This new process can be manipulated with remarkable ease in comparison to $N(t,\delta)$, and allows better understanding of the box-counting dimension of a subordinator's range in terms of its Lévy measure, improving upon [111, Corollary 1]. Further, we shall prove corresponding CLT and almost sure convergence results for the new process.

2.1 Literature Overview

2.1.1 Box-Counting Dimension

It is clear how to define the dimension of many simple mathematical objects, simply using an integer. For example, a line is 1-dimensional, a square is 2-dimensional, and a cube is 3-dimensional. But if we attempt to generalise this notion of dimension to more complicated sets, such as those with infinitesimally small structure, it is harder to assign an appropriate value to determine their dimension, and the dimension is not always an integer.

To answer these natural questions, we require the notion of *fractal dimension*, which generalises our familiar notion of the dimension of a set. There are many ways to define the fractal dimension of a set, each helpful in different contexts. In this chapter we shall focus on the *box-counting dimension.* Before providing a formal definition of box-counting dimension, we consider some simple illustrative examples.

Consider splitting a line up into pieces of length at most $\delta > 0$. If we proceed to split into smaller pieces of length at most $\delta/2$, then we require twice as many pieces. Reducing the size to δ/n, we require n times more pieces.

For a square, splitting up into squares of side length at most $\delta > 0$, then further reducing the side length to $\delta/2$, we now require *four times* as many pieces. Reducing the size to δ/n, we require n^2 times more pieces.

Similarly, for a cube split into smaller cubes, reducing the side length by a factor of $1/n$ requires n^3 times more pieces.

A clear pattern is emerging here: a power law. For the 1-dimensional object, we have n times more pieces. For the 2-dimensional object, we have n^2 times more pieces, and for the 3-dimensional object, we have n^3 times more pieces. So for a k-dimensional object split up into smaller k-dimensional "boxes" (hypercubes) of side length $1/n$, it is natural to expect the required number of these boxes to increase by a factor of n^k, as n varies. We can make this idea rigorous through the following definition:

Definition 2.1.1. *For a non-empty, bounded subset of $\mathbb{R}^n$, let $N(\delta)$ denote the minimal number of boxes (i.e. lines/squares/cubes/hypercubes) of side length at most $\delta > 0$ required to cover the set. The box-counting* *dimension of the set is defined by*

$$\lim_{\delta\to 0}\frac{\log(N(\delta))}{\log(1/\delta)}.$$

When this limit doesn't exist, we can instead consider the limsup and liminf, which respectively define the upper and lower box-counting dimensions.

Moreover, closely related to these are the upper/lower modified box-counting dimensions, defined as the infimum over all suprema of upper/lower box-counting dimensions of members of countable covers of the set. This modification allows us to consider non-compact sets. See [54, Section 3.3] for more details on upper/lower modified box-counting dimension.

Relation to Other Notions of Dimension Alongside box-counting dimension (and its modifications), two of the most widely used notions of fractal dimension are *Hausdorff dimension* (see [54, Section 2.2]) and *packing dimension* (see [54, Section 3.4]). It turns out that if we know the box-counting dimension of a set, then this gives us a good insight into its Hausdorff dimension and packing dimension. With the obvious shorthand notation, $\overline{\dim}$ denoting upper notion of dimension and $\underline{\dim}$ denoting lower, the following inequalities (proven in [54, Chapter 3]) show how each of these different notions of dimension are related:

$$\dim_H \leq \underline{\dim}_{MB} \leq \underline{\dim}_B; \qquad \underline{\dim}_{MB} \leq \overline{\dim}_{MB} \leq \dim_P \leq \overline{\dim}_B;$$

and moreover, for any subset of $\mathbb{R}^n$, $\dim_P = \overline{\dim}_{MB}$.

Our Sets of Interest The main sets of interest in this chapter are the range $\{X_s : 0 \leq s \leq t\}$, and the graph $\{(s, X_s) : 0 \leq s \leq t\}$, of a subordinator $(X_s)_{s\geq 0}$. The fractal dimensional study of sets such as the range or graph of Lévy processes, and especially subordinators, has a very rich history. There are many works which study the box-counting, Hausdorff, and packing dimensions of sets related to Lévy processes [23, 31, 54, 56, 59, 64, 73–77, 111, 113, 123]. In particular, we refer to [123, Chapters 4-5] for an account of the study of fractal properties of the range and graph of Markov processes.

We shall now give a brief summary of some of the most significant/relevant results in the literature on fractal dimensional properties of subordinators.

2.1.2 Relevant Results from the Literature

Hausforff Dimension We begin by stating some results from the 1960s on the Hausdorff dimension of the range of subordinators. We refer to [54, Section 2.2] for a precise definition of Hausdorff dimension.

Blumenthal and Getoor determined the Hausdorff dimension of the range for the special case of a *stable subordinator* in [23]. For $\alpha \in [0,1]$, a *stable subordinator of index* α is defined as a subordinator with Laplace exponent $\phi(\lambda) = C\lambda^{\alpha}$, for a constant $C > 0$. It makes sense to first consider the special case of stable subordinators. In this case, difficult problems can be tractable due to the fact that the distribution of the subordinator at each fixed time is a stable distribution. Blumenthal and Getoor's result is as follows:

Theorem 2.1.2 (**Blumenthal, Getoor 1960**)**.** For a stable subordinator of index $\alpha \in [0,1]$, the Hausdorff dimension of the range up to any fixed time $t > 0$, $\{X_u : 0 \leq u \leq t\}$, is almost surely equal to α.

This result was generalised by Pruitt in [101], where a formula for the Hausdorff dimension of the range of a general subordinator is determined in terms of the asymptotic behaviour of the Laplace exponent:

Theorem 2.1.3 (**Pruitt, 1969**)**.** For any subordinator, the Hausdorff dimension of its range is almost surely equal to its "lower index", which is defined as:

$$\sup\left\{\alpha > 0 : \lim_{\lambda\to\infty} \lambda^{-\alpha}\phi(\lambda) = \infty\right\}.$$

Box-Counting Dimension The (upper and lower) box-counting dimension of the range of a general subordinator was determined by Bertoin in [15, Theorem 5.1], and the results are again expressed in terms of the asymptotic behaviour of the Laplace exponent:

Theorem 2.1.4 (**Bertoin, 1997**)**.** *For any subordinator, the upper and lower box-counting dimensions of its range are almost surely equal to, respectively, the upper and lower indices of its Laplace exponent:*

$$\overline{\text{ind}}(\phi) = \inf\left\{\alpha > 0 : \lim_{\lambda\to\infty} \lambda^{-\alpha}\phi(\lambda) = 0\right\};$$

$$\underline{\text{ind}}(\phi) = \sup\left\{\alpha > 0 : \lim_{\lambda\to\infty} \lambda^{-\alpha}\phi(\lambda) = \infty\right\}.$$

Prior to the recent results of Savov in [111], most works on box-counting dimension focused only on finding the value of $\lim_{\delta\to 0}\log(N(t,\delta))/\log(1/\delta)$, which defines the box-counting dimension, or the limsup/liminf of this quantity, which determines the upper/lower box-counting dimension. However, working with $N(t,\delta)$ itself allows precise understanding of its fluctuations around its mean, which is inaccessible at the log scale. Savov's main result, [111, Theorem 1.1], precisely determines the asymptotic behaviour of $N(t,\delta)$ as $\delta \to 0$:

Theorem 2.1.5 (Savov, 2014). Let $N(t,\delta)$ denote the minimal number of intervals of size at most δ needed to cover the range $\{X_s : 0 \le s \le t\}$ of any subordinator which is not a compound Poisson process. Then for any $t > 0$, almost surely,

$$\lim_{\delta\to 0} U(\delta)N(t,\delta) = t,$$

where we define the *passage time*, T_δ, of our process above the level $\delta > 0$ as

$$T_\delta := \inf\{t \ge 0 : X_t > \delta\},$$

and the *renewal function* is defined as $U(\delta) := \mathbb{E}[T_\delta]$, the expected passage time.

This is a more refined result than the previous theorem of Bertoin, which only deals with the number of boxes at a logarithmic scale. In this chapter, we shall also work with this number of boxes explicitly, rather than at a logarithmic scale. One of the main results in this chapter, Theorem 2.2.1, is a central limit theorem for $N(t,\delta)$, complementing the almost sure convergence result of Savov.

2.1.3 An Alternative Box-Counting Scheme

So far we have used $N(\delta)$, the minimal number of "boxes" of side length δ in a cover, to define box-counting dimension:

$$\lim_{\delta\to 0}\frac{\log(N(\delta))}{\log(1/\delta)}.$$

However, the log-scale at which box-counting dimension is defined allows other functions to be used in place of $N(\delta)$, while preserving the above limit. For example, we can replace $N(\delta)$ by "$M(\delta)$", defined as the number of boxes in a "mesh" of side length δ to intersect with a fractal. The following three figures illustrate these different box-counting schemes on the *Koch curve*:

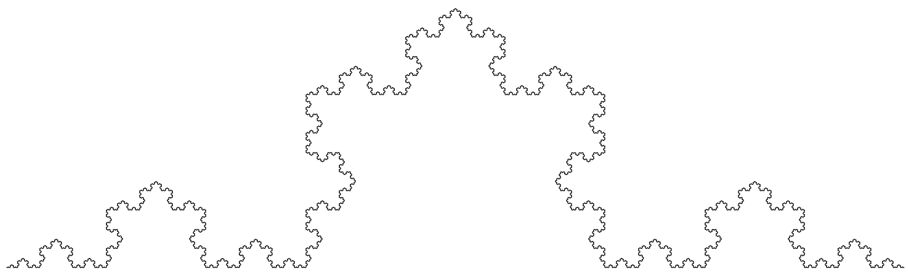

Figure 2.1: Koch curve

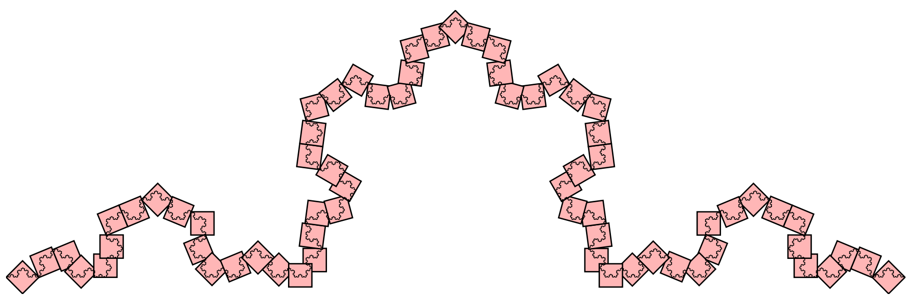

Figure 2.2: Koch curve with optimal covering; $N(\delta) = 67$

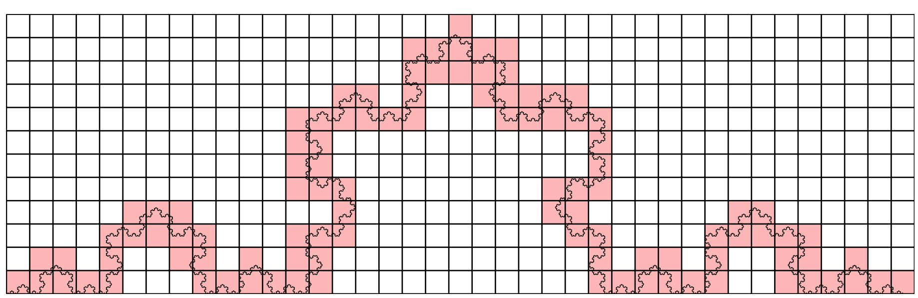

Figure 2.3: Koch curve with mesh covering; $M(\delta) = 103$

The inequalities $N(\delta) \leq M(\delta)$ and $M(\delta) \leq 3^n N(\delta)$ hold for any subset of $\mathbb{R}^n$, which implies that $\lim_{\delta\to 0} \log(N(\delta))/\log(1/\delta) = \lim_{\delta\to 0} \log(M(\delta))/\log(1/\delta)$, and hence one can use $N(\delta)$ and $M(\delta)$ interchangeably when calculating the box-counting dimension of a set. The same result holds with the limsup/liminf, meaning the upper/lower box-counting dimension can also be defined using $M(\delta)$ or $N(\delta)$.

It is easy to prove that $N(\delta) \leq M(\delta) \leq 3^n N(\delta)$ holds for any subset of $\mathbb{R}^n$. Indeed, first observe that $N(\delta) \leq M(\delta)$, since the boxes in the mesh covering, of course, form a cover (but this cover is not necessarily minimal in terms of number of boxes). On the other hand, considering an optimal covering of $N(\delta)$ boxes, one can easily verify that each box in this covering can be contained in at most 3^n boxes in a mesh, and it follows that $M(\delta) \leq 3^n N(\delta)$, as required.

Remark 2.1.6. *In fact, the box-counting dimension of a set can be defined using any function $L(\delta)$, for which $N(\delta) \asymp L(\delta)$ as $\delta \to 0$, where the notation means that there exist constants $\alpha, \beta \in (0, \infty)$ such that for all sufficiently small δ, $\alpha N(\delta) \leq L(\delta) \leq \beta N(\delta)$. A number of different definitions are used in practice, see [54, Section 3.1] for further examples.*

One of the main results in this chapter involves finding an alternative definition of box-counting dimension to $N(t, \delta)$ in Theorem 2.2.4. This allows us to understand the dimension of the range in terms of the Lévy measure, complementing results formulated in terms of the renewal function, $U(\delta)$, such as Theorem 2.2.1 and [111, Theorem 1.1].

Typically, it is preferable to work with quantities formulated in terms of the Laplace exponent of a subordinator (i.e. Lévy measure and drift) than with other quantities such as the renewal function, $U(\delta)$. So for subordinators, working with our alternative box-counting scheme provides a natural, simple alternative to dealing with $N(t, \delta)$.

2.2 Main Results

Let us briefly recall some important results and notation introduced in Chapter 1, before stating the main results for this chapter.

A Lévy process is a stochastic process in $\mathbb{R}^d$ which has stationary, independent increments, and starts at the origin. A subordinator $X := (X_t)_{t\geq 0}$ is a non-decreasing real-valued Lévy process. The Laplace exponent ϕ of a subordinator X is defined by the relation $e^{-t\phi(\lambda)} = \mathbb{E}[e^{-\lambda X_t}]$ for $t, \lambda \geq 0$. By the Lévy Khintchine formula as formulated in (1.4), ϕ can always be expressed as

$$\phi(\lambda) = \mathrm{d}\lambda + \int_0^\infty (1 - e^{-\lambda x})\Pi(dx), \tag{2.1}$$

where $\mathrm{d} \geq 0$ is the linear drift, and Π is the Lévy measure, which determines the size and intensity of the jumps (discontinuities) of X. The Lévy measure must also satisfy the condition $\int_0^\infty (1 \wedge x)\Pi(dx) < \infty$, with the standard notation $a \wedge b := \min\{a, b\}$.

If the Lévy measure is infinite, in the sense that $\int_0^\infty \Pi(dx) = \infty$, then the process will have infinitely many (infinitesimally small) jumps in each finite time interval, almost surely. In this chapter, we shall not study processes with finite Lévy measure, for which $\int_0^\infty \Pi(dx) < \infty$, because such processes have only finitely many jumps, and hence there is no interesting fractal structure.

2.2.1 A Central Limit Theorem for $N(t, \delta)$

The first result of this chapter is our central limit theorem for the quantity $N(t, \delta)$ as $\delta \to \infty$. For a subordinator with no drift, we require a mild regularity condition on the Lévy measure:

$$\liminf_{\delta\to 0} \frac{I(2\delta)}{I(\delta)} > 1, \tag{2.2}$$

where $I(u) := \int_0^u \overline{\Pi}(x)dx$, and $\overline{\Pi}(x) := \Pi((x, \infty))$. The condition (2.2) has many equivalent formulations, and can also be expressed in terms of the tail function $\overline{\Pi}$, the Laplace exponent ϕ, or the first derivative ϕ', see [14, Ex. III.7]. and [21, Section 2.1]. We emphasise that this condition is far less restrictive than regular variation, or even $\mathcal{O}$-regular variation of the Laplace exponent, because our condition is a one-sided inequality rather than upper and lower bounds. The condition (2.2) also appears naturally in the context of the law of the iterated logarithm, see e.g. [14, p87].

Recall the definitions $T_\delta := \inf\{t \geq 0 : X_t > \delta\}$ and $U(\delta) := \mathbb{E}[T_\delta]$ as introduced in Theorem 2.1.5. These are required to formulate the central limit theorem result, which we are now ready to state.

Theorem 2.2.1. *For each driftless subordinator with Lévy measure satisfying the regularity condition* (2.2)*, for any $t > 0$, $N(t, \delta)$ satisfies the central limit theorem*

$$\frac{N(t,\delta) - ta(\delta)}{t^{\frac{1}{2}} b(\delta)} \xrightarrow{d} \mathcal{N}(0,1) \tag{2.3}$$

as $\delta \to 0$, where $a(\delta) := U(\delta)^{-1}$, $b(\delta) := U(\delta)^{-\frac{3}{2}} \mathrm{Var}(T_\delta)^{\frac{1}{2}}$, and $\mathcal{N}(0,1)$ is the standard normal distribution.

We will prove Theorem 2.2.1 in Section 2.3. Next we shall state the other main results of this chapter.

2.2.2 An Alternative Box-Counting Scheme, $L(t, \delta)$

The following definition is important, as it gives rise to a new quantity, denoted $L(t, \delta)$, which is related to $N(t, \delta)$. This new quantity, $L(t, \delta)$, will be used to provide a simpler alternative definition for the box-counting dimension of the range of a subordinator.

Definition 2.2.2. *The process of δ-shortened jumps, $\tilde{X}^\delta := (\tilde{X}^\delta_t)_{t\geq 0}$, is obtained by shortening all jumps of X of size larger than δ to instead have size δ. That is, $\tilde{X}^\delta$ is the subordinator with Laplace exponent $\tilde{\phi}^\delta(u) = \mathrm{d}u + \int_0^\delta (1 - e^{-ux}) \tilde{\Pi}^\delta(dx)$ and Lévy measure $\tilde{\Pi}^\delta(dx) = \Pi(dx) \mathbb{1}_{\{x<\delta\}} + \overline{\Pi}(\delta) \Delta_\delta(dx)$, where Δ_δ denotes a unit point mass at δ, and Π is the Lévy measure of the original subordinator X.*

The idea of shortening the jumps is simple and intuitive, as illustrated in the following figure:

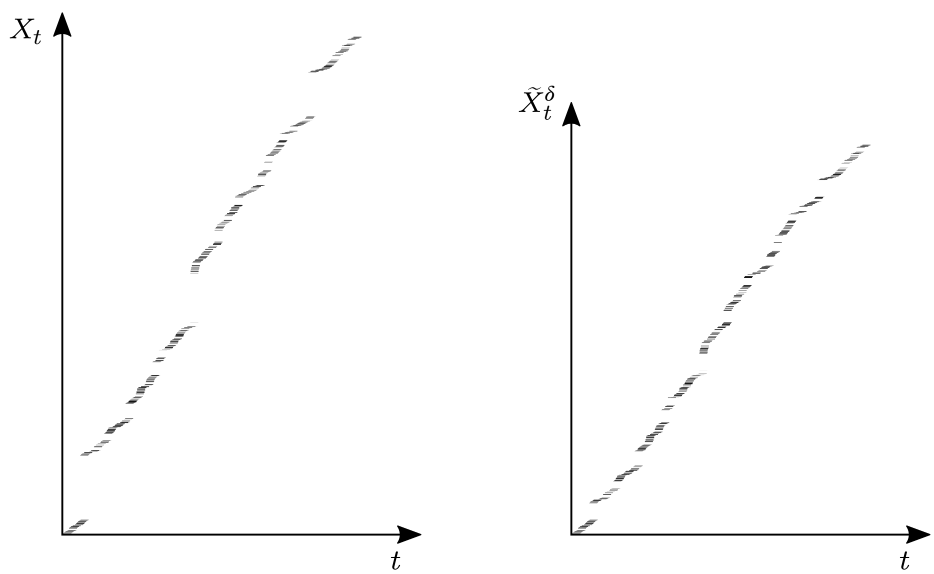

Figure 2.4: Process of δ-shortened jumps, X^δ, alongside original process X

Definition 2.2.3. *For $\delta, t > 0$, the key quantity $L(t, \delta)$ is defined by $L(t, \delta) := \frac{1}{\delta}\tilde{X}_t^\delta$.*

We will see in Theorem 2.2.4 that $L(t, \delta)$ can replace $N(t, \delta)$ in the definition $\lim_{\delta\to 0} \log(N(t, \delta))/\log(1/\delta)$ of the box-counting dimension of the range of X. Then we will prove almost sure convergence and CLT results for $L(t, \delta)$. Recall the notation $f(x) \asymp g(x)$, as defined in Remark 2.1.6.

Theorem 2.2.4. *For all $\delta, t > 0$, for every subordinator, $N(t, \delta) \asymp L(t, \delta)$. In particular, by Remark 2.1.6, $L(t, \delta)$ can be used to define the box-counting dimension of the range, i.e.* $\lim_{\delta\to 0} \log(N(t, \delta))/\log(1/\delta) = \lim_{\delta\to 0} \log(L(t, \delta))/\log(1/\delta)$.

The following three figures illustrate our three different box-counting schemes of interest on the range of the subordinator in Figure 2.4 above:

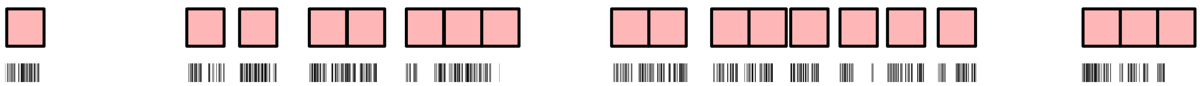

Figure 2.5: $N(t,\delta) = 19$

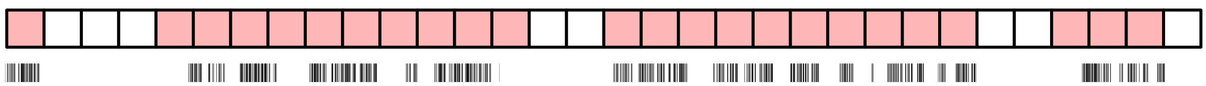

Figure 2.6: $M(t,\delta) = 24$

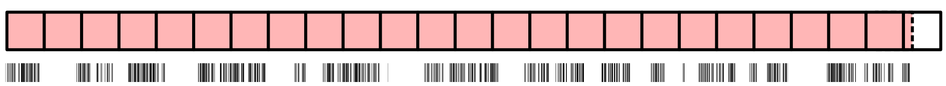

Figure 2.7: $L(t,\delta) \approx 24.2$

Theorem 2.2.5. *For every subordinator with infinite Lévy measure, i.e.* $\int_0^\infty \Pi(dx) = \infty$, *for all* $t > 0$,

$$\lim_{\delta \to 0} \frac{L(t,\delta)}{\mu(\delta)} = t, \tag{2.4}$$

almost surely, where $\mu(\delta) := \frac{1}{\delta}(\mathrm{d} + I(\delta))$, *and* $I(\delta) := \int_0^\delta \overline{\Pi}(y)dy$.

Theorem 2.2.6. *For every subordinator with infinite Lévy measure, for all* $t > 0$,

$$\frac{L(t,\delta) - t\mu(\delta)}{t^{\frac{1}{2}} v(\delta)} \xrightarrow{d} \mathcal{N}(0,1) \tag{2.5}$$

as $\delta \to 0$, *where* $\mu(\delta) := \frac{1}{\delta}(\mathrm{d} + I(\delta))$, *and* $v(\delta) := \frac{1}{\delta}\left[\int_0^\infty (x \wedge \delta)^2 \Pi(dx)\right]^{\frac{1}{2}}$. *Recall that* $a \wedge b := \min\{a, b\}$.

Let us conclude this results section by discussing benefits of working with $L(t,\delta)$ rather than $N(t,\delta)$.

Remark 2.2.7. Observe that for each integrable function f, with the notation $a \wedge b := \min\{a,b\}$, we have

$$\int_0^\infty f(x)\ \tilde{\Pi}^\delta(dx) = \int_0^\delta f(x)\ \tilde{\Pi}^\delta(dx) = \int_0^\infty f(x\wedge\delta)\ \Pi(dx).$$

Then applying Definition 2.2.2 and the Lévy Khintchine formula (2.1), it follows that for all $\delta, t > 0$, the mean and variance of $L(t,\delta)$ are given by

$$\begin{aligned}\mathbb{E}[L(t,\delta)] &= t\mu(\delta) = \frac{t}{\delta}(\mathrm{d} + I(\delta)),\\ \mathrm{Var}(L(t,\delta)) &= tv(\delta) = \frac{t}{\delta}\left[\int_0^\infty (x\wedge\delta)^2\Pi(dx)\right]^{\frac{1}{2}}.\end{aligned}$$

Computing the moments of $L(t,\delta)$ is remarkably simple in comparison to computing the moments of $N(t,\delta)$, which are not well known. We emphasise that for $L(t,\delta)$, the moments are all formulated in terms of the *characteristics* of the subordinator (i.e. the drift and Lévy measure), which are typically known. Most results on Lévy processes are typically formulated in terms of their characteristics, so our results on $L(t,\delta)$ fit in naturally with a lot of work in the literature on Lévy processes. On the other hand, results on $N(t,\delta)$ are harder to reconcile with the literature, as they are formulated in terms of the renewal function, $U(\delta)$.

Remark 2.2.8. Theorem 2.2.5 is formulated in terms of the characteristics of the subordinator. For $N(t,\delta)$, the almost sure behaviour in Theorem 2.1.5 is formulated in terms of the renewal function ($U(\delta)$, and in order to write this in terms of the characteristics, the expression is more complicated than for $L(t,\delta)$. For details, see [111, Corollary 1] and [49, Prop 1], the latter of which is very powerful for understanding the asymptotics of $U(\delta)$ for subordinators with a positive drift, significantly improving upon results in [30].

Remark 2.2.9. It should also be emphasised that the results on $L(t,\delta)$, Theorems 2.2.5 and 2.2.6, hold in full generality (excluding the trivial case of a subordinator with finite Lévy measure, which has no fractal structure). In contrast, the CLT result for $N(t,\delta)$ in Theorem 2.2.1 requires a slight regularity condition to exclude particularly poorly-behaved processes, and also for the drift to be zero. Even with this regularity condition, it is clear that the more general proofs for $L(t,\delta)$ are much shorter and rely upon less complicated mathematical tools. This is a strong indication that $L(t,\delta)$ is indeed the best quantity to consider for the purposes of studying box-counting dimension of the range of a subordinator in detail.

Recall the notation $f(x) \asymp g(x)$, as defined in Remark 2.1.6.

Remark 2.2.10. From Theorem 2.1.5, it is known that the asymptotic behaviour of $N(t,\delta)$ is like that of $U(\delta)^{-1}$. It is also known, see [15, Prop 1.4], that $U(\delta)^{-1} \asymp \mu(\delta) = \frac{1}{\delta}(\mathrm{d} + I(\delta))$ for every subordinator. It is hence natural to ask if there is a definition of box-counting dimension using a quantity which behaves like $\mu(\delta)$ asymptotically, as we determine in Theorem 2.2.5. Moreover, Theorem 2.2.4 allows us to understand the relationship $U(\delta)^{-1} \asymp \mu(\delta)$ in terms of geometric properties of subordinators.

Remark 2.2.11. Another consequence of [15, Prop 1.4] is that $U(\delta)^{-1} \asymp \phi(1/\delta) \asymp \mu(\delta)$. It is also possible to understand this relationship geometrically by considering the quantity $\phi(1/\delta)$. Applying (2.1), one can deduce that

$$t\phi(1/\delta) = t\frac{1}{\delta}\left[\mathrm{d} + \int_0^\infty x\left(\frac{\delta(1-e^{-\frac{x}{\delta}})}{x}\right)\Pi(dx)\right],$$

so that $t\phi(1/\delta) = \mathbb{E}[\overline{X}_t^\delta/\delta]$, where $(\overline{X}_t^\delta)_{t\geq 0}$ is the subordinator with Lévy measure $\left(\delta(1-e^{-x/\delta})/x\right)\Pi(dx)$. The new subordinator $(\overline{X}_t^\delta)_{t\geq 0}$ corresponds to an alternative truncation of jumps, but the geometric meaning behind this truncation is not as transparent as for the truncation used in Definition 2.2.2.

2.3 Proof of Theorem 2.2.1

In this section, we shall prove Theorem 2.2.1, the CLT result for $N(t,\delta)$. The proof begins with the Berry-Esseen theorem, see Lemma 2.3.2, through which we find a sufficient condition for our CLT to hold. We then proceed by finding a chain of further sufficient conditions, each of which implies the former. At the end of this chain, we are able to show that our sufficient conditions follow from the imposed regularity condition (2.2), and hence the CLT result is proven under this condition. This chain of sufficient conditions can be seen through Figure A.1 (it should be noted that Figure A.1 uses notation not defined until later in this chapter).

2.3.1 First Sufficient Condition via Berry-Esseen Theorem

We will first work towards proving the following sufficient condition:

Lemma 2.3.1. *For every subordinator with infinite Lévy measure, a sufficient condition for the convergence in distribution* (2.3)*, with* $\sigma_\delta^2 := \mathrm{Var}(T_\delta)$*, is*

$$\lim_{\delta \to 0} \frac{U(\delta)^{\frac{7}{3}}}{\sigma_\delta^2} = 0. \tag{2.6}$$

The proof of Lemma 2.3.1 relies upon the Berry-Esseen theorem, which is stated here in Lemma 2.3.2. The Berry-Esseen theorem is a very powerful result for studying central limit theorem type limiting behaviour, as it provides the speed of convergence. See e.g. [55, p542] for more details.

Lemma 2.3.2. *(Berry-Esseen Theorem) Let* $Z \sim \mathcal{N}(0,1)$*. There exists a finite constant* $c > 0$ *such that for every collection of iid random variables* $(Y_k)_{k \in \mathbb{N}}$ *with the same distribution as* Y*, where* Y *has finite mean, finite absolute third moment, and finite non-zero variance, for all* $n \in \mathbb{N}$ *and* $x \in \mathbb{R}$*,*

$$\left| \mathbb{P}\left(\frac{Y_1 - \mathbb{E}[Y] + \cdots + Y_n - \mathbb{E}[Y]}{\mathrm{Var}(Y)^{\frac{1}{2}} \sqrt{n}} \geq x \right) - \mathbb{P}(Z \geq x) \right| \leq \frac{c\mathbb{E}[|Y - \mathbb{E}[Y]|^3]}{\mathrm{Var}(Y)^{\frac{3}{2}} \sqrt{n}}. \tag{2.7}$$

Now we are ready to prove Lemma 2.3.1, then to proceed with our chain of sufficient conditions to prove Theorem 2.2.1. For brevity, we will only provide calculations for $t = 1$. The proofs for different values of t are essentially the same.

Recall the definitions of T_δ and $U(\delta)$, provided in Theorem 2.1.5. Recall the notation $a(\delta) := U(\delta)^{-1}$, $\sigma_\delta^2 := \mathrm{Var}(T_\delta)$, and $b(\delta) := U(\delta)^{-\frac{3}{2}} \sigma_\delta$. To prove Theorem 2.2.1, we shall aim to prove that for all $x \in \mathbb{R}$,

$$\lim_{\delta \to 0} \left| \mathbb{P}\left(\frac{N(1,\delta) - a(\delta)}{b(\delta)} \leq x \right) - \mathbb{P}\left(Z \leq x\right) \right| = 0. \tag{2.8}$$

For each $\delta > 0$, the inequality (2.7) in the Berry-Esseen theorem provides an upper bound on the difference of probabilities in (2.8). Then we shall find further upper bounds through our chain of sufficient conditions, and we will conclude by showing that these converge to zero as $\delta \to 0$. First, we prove Lemma 2.3.1, which requires the auxiliary Lemma 2.3.3 (this is stated and proven after the proof of Lemma 2.3.1).

Proof of Lemma 2.3.1. Let $T_\delta^{(k)}$ denote the kth time at which $N(t,\delta)$ increases, i.e. $T_\delta^{(1)} := 0$ and $T_\delta^{(n)} := \inf\{t \geq T_\delta^{(n-1)} : X_t > X_{T_\delta^{(n-1)}} + \delta\}$, and let $T_{\delta,k}$, $k \in \mathbb{N}$, denote iid copies of $T_\delta^{(1)}$. By the strong Markov property, $T_\delta^{(k)}$ and $\sum_{i=1}^k T_{\delta,i}$ have the same distribution, since the nth "box" in an optimal cover is the set $[X_{T_\delta^{(n)}}, X_{T_\delta^{(n)}} + \delta]$, and the time taken to exit this set from $X_{T_\delta^{(n)}}$ has the same law as the time to exit $[X_{T_\delta^{(1)}}, X_{T_\delta^{(1)}} + \delta]$ from $X_{T_\delta^{(1)}}$. Then, with $n := \lceil a(\delta) + xb(\delta) \rceil$, where $\lceil \cdot \rceil$ denotes the ceiling function,

$$\mathbb{P}\left(\frac{N(1,\delta) - a(\delta)}{b(\delta)} \leq x\right) = \mathbb{P}\left(N(1,\delta) \leq a(\delta) + xb(\delta)\right), \tag{2.9}$$

and since $N(1,\delta)$ only takes integer values and increases only at times $X_{T_\delta^{(k)}}$, $k \in \mathbb{N}$, using the fact that $T_\delta^{(n)}$ has the same distribution as the sum of n iid copies of $T_\delta^{(1)}$, it follows that

$$\begin{aligned}(2.9) = \mathbb{P}\big(N(1,\delta) \leq n\big) = \mathbb{P}\left(T_\delta^{(n)} \geq 1\right) \;&=\; \mathbb{P}\left(\sum_{i=1}^n T_{\delta,i} \geq 1\right) \\ &= \mathbb{P}\left(\sum_{i=1}^n \big(T_{\delta,i} - U(\delta)\big) \geq 1 - nU(\delta)\right) \\ &= \mathbb{P}\left(\frac{\sum_{i=1}^n \big(T_{\delta,i} - U(\delta)\big)}{\sqrt{n\sigma_\delta^2}} \geq \frac{1 - nU(\delta)}{\sqrt{n\sigma_\delta^2}}\right).\end{aligned} \tag{2.10}$$

It follows from Lemma 2.3.3 that $\sigma_\delta^2 \leq \mathbb{E}[T_\delta^2] \leq CU(\delta)^2$ for a constant $C > 0$, which then implies that $b(\delta) = o(a(\delta))$ as $\delta \to 0$. Then, as $\delta \to 0$, the asymptotic behaviour of n is

$$n = \lceil a(\delta) + xb(\delta) \rceil \sim a(\delta) + xb(\delta) = a(\delta) + o(a(\delta)) \sim a(\delta) = U(\delta)^{-1}.$$

It follows, with x'_δ depending on x and δ, that as $\delta \to 0$,

$$-x'_\delta := \frac{1 - nU(\delta)}{\sqrt{n\sigma_\delta^2}} = \frac{1 - \lceil a(\delta) + xb(\delta) \rceil U(\delta)}{\big(\lceil a(\delta) + xb(\delta) \rceil\big)^{\frac{1}{2}} \sigma_\delta} \sim \frac{1 - (a(\delta) + xb(\delta))U(\delta)}{(a(\delta) + xb(\delta))^{\frac{1}{2}} \sigma_\delta} \tag{2.11}$$

$$= \frac{1 - 1 - xb(\delta)U(\delta)}{(a(\delta) + xb(\delta))^{\frac{1}{2}} \sigma_\delta} \sim \frac{-xb(\delta)U(\delta)}{U(\delta)^{-\frac{1}{2}} \sigma_\delta} = \frac{-xb(\delta)U(\delta)^{\frac{3}{2}}}{\sigma_\delta} = -x. \tag{2.12}$$

Now, by the triangle inequality and symmetry of the normal distribution, combining (2.10) and (2.12), and using the fact that $\lim_{\delta\to 0} x'_\delta = x$, it follows that as $\delta \to 0$, for any $x \in \mathbb{R}$,

$$\begin{aligned}\left|\mathbb{P}\left(\frac{N(1,\delta)-a(\delta)}{b(\delta)} \le x\right) - \mathbb{P}(Z \le x)\right| &\le \left|\mathbb{P}(Z \ge -x'_\delta) - \mathbb{P}(Z \ge -x)\right| \\ &+ \left|\ \mathbb{P}\left(\frac{1}{\sqrt{n\sigma_\delta^2}}\sum_{i=1}^{n}(T_{\delta,i}-U(\delta)) \ge -x'_\delta\right) - \ \mathbb{P}(\ Z \ge -x'_\delta)\ \right| \\ &= \left|\ \mathbb{P}\left(\frac{1}{\sqrt{n\sigma_\delta^2}}\sum_{i=1}^{n}(T_{\delta,i}-U(\delta)) \ge -x'_\delta\right) - \ \mathbb{P}(\ Z \ge -x'_\delta)\ \right| + o(1).\end{aligned} \tag{2.13}$$

Recall that we wish to show that (2.13) converges to zero. By the Berry-Esseen Theorem and the fact that $n \sim U(\delta)^{-1}$ as $\delta \to 0$, it follows that as $\delta \to 0$,

$$(2.13) \le c\frac{\mathbb{E}[|T_\delta - U(\delta)|^3]}{\sigma_\delta^3 n^{\frac{1}{2}}} + o(1) \sim c\frac{U(\delta)^{\frac{1}{2}}\mathbb{E}[|T_\delta - U(\delta)|^3]}{\sigma_\delta^3}.$$

Applying the triangle inequality, then Lemma 2.3.3 with $m = 2$ and $m = 3$ to $\mathbb{E}[|T_\delta - U(\delta)|^3]$, as $\delta \to 0$,

$$(2.13) \le c\frac{\mathbb{E}[T_\delta^3] + 3U(\delta)\mathbb{E}[T_\delta^2] + 3U(\delta)^2\mathbb{E}[T_\delta] + U(\delta)^3}{\sigma_\delta^3} \le 8c'\frac{U(\delta)^{\frac{1}{2}}U(\delta)^3}{\sigma_\delta^3} = 8c'\left(\frac{U(\delta)^{\frac{7}{3}}}{\sigma_\delta^2}\right)^{\frac{2}{3}},$$

for some constant $c' < \infty$. Therefore if the condition (2.6) as in the statement of Lemma 2.3.1 holds, then the desired convergence in distribution (2.3) follows, as required.

□

Lemma 2.3.3. *For every subordinator with infinite Lévy measure, for all $m \ge 1$,*

$$\limsup_{\delta\to 0}\frac{\mathbb{E}[T_\delta^m]}{U(\delta)^m} < \infty.$$

Proof of Lemma 2.3.3. First, by the moments and tails lemma (see Lemma A.3.3),

$$\frac{\mathbb{E}[T_\delta^m]}{U(\delta)^m} = \mathbb{E}\left[\left(\frac{T_\delta}{U(\delta)}\right)^m\right] = \int_0^\infty my^{m-1}\mathbb{P}\left(\frac{T_\delta}{U(\delta)} > y\right)dy.$$

By the definition of T_δ, it follows that $X_u \le \delta$ if and only if $T_\delta > u$, and then

$$\frac{\mathbb{E}[T_\delta^m]}{U(\delta)^m} = \int_0^\infty m y^{m-1} \mathbb{P}(X_{yU(\delta)} \le \delta) dy = \int_0^\infty m y^{m-1} \mathbb{P}(e^{-\frac{1}{\delta} X_{yU(\delta)}} \ge e^{-1}) dy.$$

Now, applying Markov's inequality (Theorem A.2.1), the definition $\mathbb{E}[e^{-\lambda X_t}] = e^{-t\phi(\lambda)}$, and the fact that $U(\delta)\phi(1/\delta) \ge K$ for some constant $K > 0$ (see [15, Prop 1.4]),

$$\frac{\mathbb{E}[T_\delta^m]}{U(\delta)^m} \le \int_0^\infty m y^{m-1} e^{1-yU(\delta)\phi(1/\delta)} dy \le \int_0^\infty m y^{m-1} e^{1-Ky} dy,$$

which is finite and independent of δ. Therefore the lim sup is finite, as required.

□

2.3.2 Proof of Theorem 2.2.1

We shall now present the proof of Theorem 2.2.1, which requires two lemmas. Theorem 2.2.1 is proven by a contradiction, using the upcoming Lemma 2.3.6 to show that the sufficient condition in Lemma 2.3.5 holds. The fact that Lemma 2.3.5 is a sufficient condition for the CLT result in Theorem 2.2.1 is proven through a chain of lemmas (see Figure A.1), but we shall postpone this series of proofs until after the proof of Theorem 2.2.1, for the sake of clarity. First, we shall introduce some important notation in Definition 2.3.4.

Definition 2.3.4. *Recalling from Definition 2.2.2 that the process $\tilde{X}^\delta$ has Laplace exponent $\tilde{\phi}^\delta(u) = \mathrm{d}u + \int_0^\delta (1 - e^{-ux})\Pi(dx) + (1 - e^{-u\delta})\overline{\Pi}$, we define:*

(i) $g(u) := \frac{d}{du}\tilde{\phi}^\delta(u) = \mathrm{d} + \int_0^\delta x e^{-ux} \tilde{\Pi}^\delta(dx)$,

(ii) $R(u) := \tilde{\phi}^\delta(u) - ug(u) = \int_0^\delta \left(1 - e^{-ux}(1 + ux)\right) \tilde{\Pi}^\delta(dx)$,

(iii) λ_δ denotes the unique solution to $g(\lambda_\delta) = x_\delta$, for $\mathrm{d} < x_\delta < \mathrm{d} + \int_0^\delta x\tilde{\Pi}^\delta(dx)$.

We refer to [67, p93] for further details on these important quantities. In fact, one can ignore the drift d in Definition 2.3.4, since $\mathrm{d} = 0$ throughout Section 2.3. Now we are ready to state Lemma 2.3.5, which is our final sufficient condition in the chain of implications in Figure A.1.

Lemma 2.3.5. *For $\alpha > 0$, $t = (1+\alpha)U(\delta)$, and $g(\lambda_\delta) = x_\delta = \delta/t$, if*

$$\limsup_{\delta\to 0} \ \delta\lambda_\delta < \infty, \tag{2.14}$$

then the desired convergence in distribution (2.3)*, as in Theorem 2.2.1, holds.*

So in order to prove that Theorem 2.2.1 holds, we just need to verify that (2.14) holds under our imposed regularity condition (2.2). This is made possible by the following lemma.

Lemma 2.3.6. *Recall the definition $I(\delta) := \int_0^\delta \overline{\Pi}(x)dx$. The condition* (2.2) *implies that for each $\eta \in (0,1)$, there exists a sufficiently large integer n such that*

$$\liminf_{\delta\to 0} \frac{I(\delta)}{I(2^{-n}\delta)} > \frac{1}{\eta}. \tag{2.15}$$

Proof of Lemma 2.3.6. The integral condition (2.2) imposes that for some $B > 1$,

$$\liminf_{\delta\to 0} \frac{I(\delta)}{I(\delta/2)} = \liminf_{\delta\to 0} \frac{\int_0^\delta \overline{\Pi}(y)(dy)}{\int_0^{\delta/2} \overline{\Pi}(y)dy} = B. \tag{2.16}$$

Then, by effectively replacing $1/2$ with 2^{-n} (so $1/2$ is replaced by a smaller constant), we can replace B with B^n, which can be made arbitrarily large by choice of n. This follows by splitting up the fraction,

$$\begin{aligned}
\liminf_{\delta\to 0} \frac{I(\delta)}{I(2^{-n}\delta)} &= \liminf_{\delta\to 0} \left(\frac{I(\delta)}{I(2^{-1}\delta)} \frac{I(2^{-1}\delta)}{I(2^{-2}\delta)} \cdots \frac{I(2^{-(n-1)}\delta)}{I(2^{-n}\delta)} \right) \\
&\geq \liminf_{\delta\to 0} \left(\frac{I(\delta)}{I(2^{-1}\delta)} \right) \liminf_{\delta\to 0} \left(\frac{I(2^{-1}\delta)}{I(2^{-2}\delta)} \right) \cdots \liminf_{\delta\to 0} \left(\frac{I(2^{-(n-1)}\delta)}{I(2^{-n}\delta)} \right) \\
&= B^n > \frac{1}{\eta},
\end{aligned}$$

where we simply take n sufficiently large that $B^n > 1/\eta$.

□

Using Lemma 2.3.6 for a contradiction is the step in the proof of Theorem 2.2.1 which requires the condition (2.2). We are now ready to prove Theorem 2.2.1.

Proof of Theorem 2.2.1. Assume for a contradiction that there exists a sequence $(\delta_m)_{m\geq 1}$ converging to zero, such that $\lim_{m\to\infty} \lambda_{\delta_m}\delta_m = \infty$. That is to say, assume that the sufficient condition in Lemma 2.3.5 does not hold. For brevity, we shall omit the dependence of δ_m on m. The assumption (for contradiction) implies that for each fixed $\eta, n > 0$, $\eta \geq e^{-\lambda_\delta 2^{-n}\delta}$ for all small enough $\delta > 0$. By Fubini's theorem (Theorem A.3.2), $I(\delta) = \int_0^\delta \overline{\Pi}(y)dy = \int_0^\delta \int_y^\infty \Pi(dx)dy = \int_0^\infty \int_0^{x\wedge\delta} dy\Pi(dx) = \int_0^\delta x\tilde{\Pi}^\delta(dx)$, so

$$\begin{aligned}\eta I(\delta) + I(2^{-n}\delta) &\geq e^{-\lambda_\delta 2^{-n}\delta} I(\delta) + I(2^{-n}\delta) \geq e^{-\lambda_\delta 2^{-n}\delta}\int_0^\delta x\tilde{\Pi}^\delta(dx) + \int_0^{2^{-n}\delta} x\Pi(dx) \\ &= e^{-\lambda_\delta 2^{-n}\delta}\delta\overline{\Pi}(\delta) + e^{-\lambda_\delta 2^{-n}\delta}\int_0^\delta x\Pi(dx) + \int_0^{2^{-n}\delta} x\Pi(dx). \end{aligned} \tag{2.17}$$

Removing part of the first integral and noting $1 \geq e^{-\lambda_\delta x}$ for all $x > 0$,

$$(2.17) \geq e^{-\lambda_\delta 2^{-n}\delta}\delta\overline{\Pi}(\delta) + \int_{2^{-n}\delta}^\delta e^{-\lambda_\delta 2^{-n}\delta} x\Pi(dx) + \int_0^{2^{-n}\delta} e^{-\lambda_\delta x} x\Pi(dx).$$

Now, $e^{-\lambda_\delta 2^{-n}\delta} \geq e^{-\lambda_\delta x}$ for $x \geq 2^{-n}\delta$. So for $g(\lambda_\delta) = x_\delta = \frac{\delta}{(1+\alpha)U(\delta)}$, where $\alpha > 0$ is fixed and chosen sufficiently large that $x_\delta < \int_0^\delta x\tilde{\Pi}^\delta(dx)$ for all δ (this is possible by the relation $U(\delta)^{-1} \asymp I(\delta)/\delta$, see [15, p74] for the result and Remark 2.1.6 for the notation),

$$\begin{aligned}(2.17) &\geq e^{-\lambda_\delta 2^{-n}\delta}\delta\overline{\Pi}(\delta) + \int_{2^{-n}\delta}^\delta e^{-\lambda_\delta x} x\Pi(dx) + \int_0^{2^{-n}\delta} e^{-\lambda_\delta x} x\Pi(dx) \\ &= e^{-\lambda_\delta 2^{-n}\delta}\delta\overline{\Pi}(\delta) + \int_0^\delta e^{-\lambda_\delta x} x\Pi(dx) \geq g(\lambda_\delta) = \frac{\delta}{(1+\alpha)U(\delta)} \geq \frac{I(\delta)}{(1+\alpha)K},\end{aligned}$$

where the last two inequalities respectively follow by Definition 2.2.2, Definition 2.3.4 (i) with $\mathrm{d} = 0$, and the relation $U(\delta)^{-1} \asymp I(\delta)/\delta$. So for a constant $K > 0$, for all sufficiently small $\delta > 0$, we have shown $\eta I(\delta) + I(2^{-n}\delta) \geq \frac{I(\delta)}{(1+\alpha)K}$.

Taking $\eta > 0$ small enough that $\frac{1}{(1+\alpha)K} \geq 2\eta$, it follows that $I(2^{-n}\delta) \geq \eta I(\delta)$, and hence $I(\delta)/I(2^{-n}\delta) \leq 1/\eta$. But in Lemma 2.3.6 we showed that for each fixed $\eta > 0$, there is sufficiently large n such that $\liminf_{\delta\to 0} I(\delta)/I(2^{-n}\delta) > 1/\eta$, which is a contradiction, so the sufficient condition as in Lemma 2.3.5 must hold. □

Remark 2.3.7. For a driftless subordinator, Theorem 2.2.1 holds under the same condition (2.2) applied to the function $H(y) := \int_0^y x\Pi(dx)$ rather than the integrated tail function I. The integrated tail $I(y) = H(y) + y\overline{\Pi}(y)$ depends on the large jumps of X since $\overline{\Pi}(x) = \Pi((x,\infty))$, but H does not depend on the large jumps, so these conditions are substantially different.

With only minor changes, the argument as in the proof of Theorem 2.2.1 works with H in place of I. Under condition (2.2) for H in place of I, one can prove that Lemma 2.3.6 holds with H in place of I. Then we assume for a contradiction that there exists a sequence $(\delta_m)_{m\geq 1}$ converging to zero, such that $\lim_{m\to\infty} \lambda_{\delta_m}\delta_m = \infty$. But then as in the proof of Theorem 2.2.1, one can deduce that $\eta H(\delta) + H(2^{-n}\delta) \geq \frac{1}{(1+\alpha)K'} H(\delta)$, which contradicts the analogous Lemma 2.3.6 result with H in place of I.

Remark 2.3.8. Theorem 2.2.1 can also be proven for subordinators with a drift $\mathrm{d} > 0$, under a stronger regularity condition. For $Y_t := X_t - \mathrm{d}t$, define ϕ_Y as the Laplace exponent of Y. The convergence in distribution (2.3) holds whenever $\limsup_{x\to 0} x^{-5/6}\phi_Y(x) < \infty$. This is proven using Remark 2.3.10, the inequality $\mathbb{P}(Y_t < a) \geq 1 - Cth(a)$ for all Lévy processes (see [102, p954] for details, including the definition of the function $h(a)$), and the asymptotic expansion of $U(\delta)$ as in [49, Theorem 4].

2.3.3 Proofs of Lemmas 2.3.9, 2.3.12, 2.3.5

Lemmas 2.3.9, 2.3.12, and 2.3.5 give successive sufficient conditions for Theorem 2.2.1 to hold (see Figure A.1). The proofs for these lemmas are facilitated by Lemma 2.3.11, which was proven in 1987 by Jain and Pruitt [67, p94]. Recall that $\tilde{X}^\delta$ denotes the process with δ-shortened jumps, as defined in Definition 2.2.2.

Lemma 2.3.9. *The convergence in distribution* (2.3) *as in Theorem 2.2.1 holds if for some* $\alpha \in (0,1]$, $\liminf_{\delta\to 0}\left[\mathbb{P}\left(\tilde{X}^\delta_{(1+\alpha)U(\delta)} \leq \delta\right) + \mathbb{P}\left(\tilde{X}^\delta_{(1-\alpha)U(\delta)} \geq \delta\right)\right] > 0.$

Proof of Lemma 2.3.9. This result builds upon Lemma 2.3.1. For all $\alpha > 0$, recalling that $\mathbb{E}[T_\delta] = U(\delta)$,

$$\sigma_\delta^2 = \mathrm{Var}(T_\delta) \geq \mathrm{Var}(T_\delta; |T_\delta - U(\delta)| \geq \alpha U(\delta))$$

$$\geq \alpha^2 U(\delta)^2[\mathbb{P}(T_\delta \geq (1+\alpha)U(\delta)) + \mathbb{P}(T_\delta \leq (1-\alpha)U(\delta))].$$

For the desired convergence in distribution (2.3) to hold, it is sufficient by Lemma 2.3.1 to show that $\lim_{\delta\to 0} U(\delta)^{\frac{7}{3}}/\sigma_\delta^2 = 0$. Now,

$$\frac{U(\delta)^{\frac{7}{3}}}{\sigma_\delta^2} \le \frac{U(\delta)^{\frac{1}{3}}}{\alpha^2[\mathbb{P}(T_\delta \ge (1+\alpha)U(\delta)) + \mathbb{P}(T_\delta \le (1-\alpha)U(\delta))]}.$$

Note that $T_\delta \ge t$ if and only if $\tilde{X}_t^\delta \le \delta$ since jumps of size larger than δ do not occur in either case, and so $X_t = \tilde{X}_t^\delta$ when $T_\delta \ge t$, so that $\mathbb{P}(T_\delta \ge (1+\alpha)U(\delta)) = \mathbb{P}(\tilde{X}^\delta_{(1+\alpha)U(\delta)} \le \delta)$. On the other hand, one can verify that $\mathbb{P}(T_\delta \le (1-\alpha)U(\delta)) = \mathbb{P}(\tilde{X}^\delta_{(1-\alpha)U(\delta)} \ge \delta)$, by splitting up according to when the first jump of size $\ge \delta$ occurs. If this jump occurs before time $1-\alpha$, then the process passes above δ, regardless of the size of this jump, so the subordinator is unaffected by the truncation. If the large jump does not occur before time $1-\alpha$, then $X_{1-\alpha} = \tilde{X}^\delta_{1-\alpha}$, so again the truncation has no effect on the probability. Hence we conclude that the desired convergence in distribution (2.3) holds if

$$\liminf_{\delta\to 0} \left[\mathbb{P}\left(\tilde{X}^\delta_{(1+\alpha)U(\delta)} \le \delta\right) + \mathbb{P}\left(\tilde{X}^\delta_{(1-\alpha)U(\delta)} \ge \delta\right)\right] > 0.$$

□

Remark 2.3.10. *The condition in Lemma 2.3.9 is not strictly optimal, in the sense that a weaker sufficient condition for* (2.3) *exists. One can verify that if there exists* $\varepsilon \in (0, 1/6)$ *for which*

$$\lim_{\delta\to 0} U(\delta)^{2\varepsilon-\frac{1}{3}} \left[\mathbb{P}\left(\tilde{X}^\delta_{U(\delta)+U(\delta)^{1+\varepsilon}} \le \delta\right) + \mathbb{P}\left(\tilde{X}^\delta_{U(\delta)-U(\delta)^{1+\varepsilon}} \ge \delta\right)\right] = \infty,$$

then the convergence in distribution (2.3) *follows too. However, using our method of proof, this weaker condition does not lead to any more generality than the condition* (2.2) *for driftless subordinators.*

Lemma 2.3.11 (Jain, Pruitt [67, Lemma 5.2])**.** *Recall the notation introduced in Definition 2.3.4. There exists* $c > 0$ *such that for every* $\varepsilon > 0$*,* $t \ge 0$ *and* $x_\delta > 0$ *satisfying* $\mathrm{d} = g(\infty) < x_\delta < g(0) = \mathrm{d} + \int_0^\delta x\tilde{\Pi}^\delta(dx)$*,*

$$\mathbb{P}\left(\tilde{X}_t^\delta \le tx_\delta\right) \ge \left(1 - \frac{(1+\varepsilon)c}{\varepsilon^2 tR(\lambda_\delta)}\right) e^{-(1+2\varepsilon)tR(\lambda_\delta)}. \tag{2.18}$$

Lemma 2.3.12. *For* $\alpha > 0$, $t = (1+\alpha)U(\delta)$, *and* $g(\lambda_\delta) = x_\delta = \delta/t$, *if*

$$\limsup_{\delta\to 0}\ tR(\lambda_\delta) < \infty,$$

then the desired convergence in distribution (2.3), *as in Theorem 2.2.1, holds.*

Proof of Lemma 2.3.12. This result build upon Lemma 2.3.9. Applying inequality (2.18) from Lemma 2.3.11,

$$\mathbb{P}\left(\tilde{X}^{\delta}_{(1+\alpha)U(\delta)} \le \delta\right) \ge \left(1 - \frac{(1+\varepsilon)c}{\varepsilon^2 tR(\lambda_\delta)}\right) e^{-(1+2\varepsilon)tR(\lambda_\delta)}. \tag{2.19}$$

Now, letting $\limsup_{\delta\to 0} tR(\lambda_\delta) < \infty$, we will consider two separate cases:

(i) If $\liminf_{\delta\to 0} tR(\lambda_\delta) = \beta > 0$, then by choice of $\varepsilon > 0$ such that $\frac{1+\varepsilon}{\varepsilon^2} = \frac{\beta}{2c}$, the lower bound in (2.19) is larger than a positive constant as $\delta \to 0$.

(ii) If $\liminf_{\delta\to 0} tR(\lambda_\delta) = 0$, then imposing $\varepsilon = 2c/(tR(\lambda_\delta))$, the lower bound in (2.19) is again larger than a positive constant as $\delta \to 0$.

The desired convergence in distribution (2.3) then follows immediately in each case by Lemma 2.3.9.

□

Proof of Lemma 2.3.5. This result builds upon Lemma 2.3.12. Noting that $1 - e^{-y}(1+y) \le y$ for all $y > 0$,

$$tR(\lambda_\delta) = (1+\alpha)U(\delta)\int_0^\delta (1 - e^{-\lambda_\delta x}(1+\lambda_\delta x))\tilde{\Pi}^\delta(dx)$$

$$\le (1+\alpha)U(\delta)\int_0^\delta \lambda_\delta x\tilde{\Pi}^\delta(dx) = (1+\alpha)U(\delta)\left(\int_0^\delta x\Pi(dx) + \delta\overline{\Pi}(\delta)\right)\lambda_\delta. \tag{2.20}$$

Then by the relation $U(\delta)I(\delta) \le C\delta$ for a constant C (see [15, Prop 1.4]), it follows that

$$(2.20) = (1+\alpha)U(\delta)I(\delta)\lambda_\delta \le (1+\alpha)C\delta\lambda_\delta.$$

Hence if $\limsup_{\delta\to 0}\delta\lambda_\delta < \infty$, then the desired convergence in distribution (2.3) follows by Lemma 2.3.12. □

2.4 Proofs of Results on $L(t,\delta)$

Now we shall prove the main results on the quantity $L(t,\delta)$. Firstly, we prove Theorem 2.2.4, which confirms that $L(t,\delta)$ can replace $N(t,\delta)$ in the definition of the box-counting dimension of the range. This is done by showing that $L(t,\delta) \asymp N(t,\delta)$, which is known to be sufficient by Remark 2.1.6, where we recall that the notation notation $f(x) \asymp g(x)$, is defined in Remark 2.1.6.

Proof of Theorem 2.2.4. The jumps of the original subordinator X and the process with shortened jumps $\tilde{X}^\delta$ are all the same size, other than jumps bigger than size δ. The optimal number of intervals to cover the range, $N(X,t,\delta)$, always increases by 1 at the time of each jump bigger than size δ, regardless of its size, so it follows that $N(X,t,\delta) = N(\tilde{X}^\delta,t,\delta)$, with the obvious notation.

Instead of counting the number $N(X,t,\delta)$ of boxes needed to cover the range of X, consider counting those needed for the range of the subordinator $X^{(0,\delta)}$ with Lévy measure $\Pi(dx)\mathbb{1}_{\{x<\delta\}}$ (so all jumps of size larger than δ are removed), and then adding Y_t^δ, which counts the number of jumps of size at least δ up to time t. This new counting method corresponds to a cover which is not necessarily optimal, so that $N(X,t,\delta) \le N(X^{(0,\delta)},t,\delta) + Y_t^\delta$.

For a bound in the other direction, consider the fact that X can have at most $N(X,t,\delta)-1$ large jumps of size at least δ by time t, so that $Y_t^\delta \le N(X,t,\delta)-1$. Moreover, it is clear that $N(X^{(0,\delta)},t,\delta) \le N(X,t,\delta)$, and hence $N(X^{(0,\delta)},t,\delta) + Y_t^\delta \le 2N(X,t,\delta)$, so that $N(X,t,\delta) \asymp N(X^{(0,\delta)},t,\delta) + Y_t^\delta$.

Consider $M(X^{(0,\delta)},t,\delta)$, the number of intervals in a lattice of side length δ to intersect with the range of $X^{(0,\delta)}$. It is easy to show that $N(X^{(0,\delta)},t,\delta) \asymp M(X^{(0,\delta)},t,\delta)$, see [54, p42] for a detailed proof. Also, $M(X^{(0,\delta)},t,\delta) = \lceil \frac{1}{\delta} X_t^{(0,\delta)} \rceil$, since $X^{(0,\delta)}$ has no jumps of size larger than δ. Now, observe that $\frac{1}{\delta}X_t^{(0,\delta)} \asymp \lceil \frac{1}{\delta} X_t^{(0,\delta)} \rceil$ for all small enough δ, and hence

$$\begin{aligned} L(X,t,\delta) = \frac{1}{\delta}\tilde{X}_t^\delta = \frac{1}{\delta}X_t^{(0,\delta)} + Y_t^\delta &\asymp M(X^{(0,\delta)},t,\delta) + Y_t^\delta \\ &\asymp \; N(X^{(0,\delta)},t,\delta) + Y_t^\delta \asymp N(X,t,\delta). \end{aligned}$$

It then follows immediately (see Remark 2.1.6) that

$$\lim_{\delta\to 0}\frac{\log(L(t,\delta))}{\log(1/\delta)}=\lim_{\delta\to 0}\frac{\log(N(t,\delta))}{\log(1/\delta)},$$

and hence $L(t,\delta)$ can be used to define the box-counting dimension of the range of a subordinator.

□

Next we will prove the CLT result for $L(t,\delta)$, working with $t=1$ for brevity (the proof is essentially the same for other values of $t>0$). We will show convergence of the Laplace transform of $\frac{1}{v(\delta)}(L(1,\delta)-\mu(\delta))$ to that of the standard normal distribution, which is sufficient for convergence in distribution [69, Theorem 4.3]. Recall that $Z\sim\mathcal{N}(0,1)$ has Laplace transform $\mathbb{E}[e^{-\lambda Z}]=e^{\lambda^2/2}$.

Proof of Theorem 2.2.6. Recall that by Definition 2.2.2 and (2.1), $\delta L(t,\delta)=\tilde{X}^\delta_t$ is a subordinator with Laplace exponent $\tilde{\phi}^\delta(u)=\mathrm{d}u+\int_0^\delta(1-e^{-ux})\tilde{\Pi}^\delta(dx)$ and Lévy measure $\tilde{\Pi}^\delta(dx)=\Pi(dx)\mathbb{1}_{\{x<\delta\}}+\overline{\Pi}(\delta)\Delta_\delta(dx)$, where Δ_δ denotes a unit point mass at δ, and Π is the Lévy measure of the original subordinator X. Therefore, for all $\lambda\geq 0$, the following two statements are equivalent:

$$\lim_{\delta\to 0}\mathbb{E}\left[\exp\left(-\lambda\frac{L(1,\delta)-\mu(\delta)}{v(\delta)}\right)\right]=e^{\frac{\lambda^2}{2}}\iff\lim_{\delta\to 0}\left(\frac{\lambda\mu(\delta)}{v(\delta)}-\tilde{\phi}^\delta\left(\frac{\lambda}{\delta v(\delta)}\right)\right)=\frac{\lambda^2}{2}.$$

Recalling the definition $\mu(\delta)=\frac{1}{\delta}(\mathrm{d}+I(\delta))$, where $I(\delta):=\int_0^\delta x\tilde{\Pi}^\delta(dx)$, and writing $\tilde{\phi}^\delta$ in the Lévy Khintchine representation as in (2.1), it follows that

$$\begin{aligned}\frac{\lambda\mu(\delta)}{v(\delta)}-\tilde{\phi}^\delta\left(\frac{\lambda}{\delta v(\delta)}\right)&=\frac{\lambda(\mathrm{d}+I(\delta))}{\delta v(\delta)}-\frac{\mathrm{d}\lambda}{\delta v(\delta)}-\int_0^\delta(1-e^{-\frac{\lambda x}{\delta v(\delta)}})\tilde{\Pi}^\delta(dx)\\&=\frac{\lambda I(\delta)}{\delta v(\delta)}-\int_0^\delta(1-e^{-\frac{\lambda x}{\delta v(\delta)}})\tilde{\Pi}^\delta(dx)\\&=\int_0^\delta\frac{\lambda x}{\delta v(\delta)}\tilde{\Pi}^\delta(dx)-\int_0^\delta(1-e^{-\frac{\lambda x}{\delta v(\delta)}})\tilde{\Pi}^\delta(dx).\end{aligned}\tag{2.21}$$

Then applying the fact that $\frac{y^2}{2} - \frac{y^3}{6} \leq y - 1 + e^{-y} \leq \frac{y^2}{2}$ for all $y > 0$, we obtain the bounds

$$\int_0^\delta \left(\frac{\lambda^2 x^2}{2\delta^2 v(\delta)^2} - \frac{\lambda^3 x^3}{6\delta^3 v(\delta)^3} \right) \tilde{\Pi}^\delta(dx) \leq (2.21) \leq \int_0^\delta \frac{\lambda^2 x^2}{2\delta^2 v(\delta)^2} \tilde{\Pi}^\delta(dx).$$

Now, by the definition of $v(\delta)$ as in the statement of Theorem 2.2.6, $v(\delta)^2 = \frac{1}{\delta^2}\int_0^\delta x^2 \tilde{\Pi}^\delta(dx)$, and hence

$$\int_0^\delta \frac{\lambda^2 x^2}{2\delta^2 v(\delta)^2} \tilde{\Pi}^\delta(dx) = \frac{\lambda^2}{2}.$$

It is then sufficient, in order to show that (2.21) converges to $\frac{\lambda^2}{2}$, to prove that

$$\lim_{\delta \to 0} \int_0^\infty \frac{x^3}{\delta^3 v(\delta)^3} \; \tilde{\Pi}^\delta(dx) = 0. \tag{2.22}$$

Again by the definition of $v(\delta)$, for (2.22) to hold we require both

$$\lim_{\delta \to 0} \frac{\int_0^\delta x^3 \Pi(dx)}{(\int_0^\delta x^2 \Pi(dx) + \delta^2 \overline{\Pi}(\delta))^{\frac{3}{2}}} = 0, \tag{2.23}$$

$$\lim_{\delta \to 0} \frac{\delta^3 \overline{\Pi}(\delta)}{(\int_0^\delta x^2 \Pi(dx) + \delta^2 \overline{\Pi}(\delta))^{\frac{3}{2}}} = 0. \tag{2.24}$$

Squaring the expression in (2.23), since $x \leq \delta$ within each integral, it follows that

$$\frac{\left(\int_0^\delta x^3 \Pi(dx)\right)^2}{\left(\int_0^\delta x^2 \Pi(dx) + \delta^2 \overline{\Pi}(\delta)\right)^3} \leq \frac{\delta^2 \left(\int_0^\delta x^2 \Pi(dx)\right)^2}{\left(\int_0^\delta x^2 \Pi(dx) + \delta^2 \overline{\Pi}(\delta)\right)^3}.$$

Now, by the binomial series expansion, we have $(a+b)^3 \geq 3a^2 b$ for $a, b > 0$, and therefore as $\delta \to 0$,

$$(2.23) \leq \frac{\delta^2 \left(\int_0^\delta x^2 \Pi(dx)\right)^2}{3\left(\int_0^\delta x^2 \Pi(dx)\right)^2 \left(\delta^2 \overline{\Pi}(\delta)\right)} = \frac{1}{3\overline{\Pi}(\delta)} \to 0,$$

where we use the assumption that $\int_0^\infty \Pi(dx) = \infty$, which implies that $\lim_{\delta \to 0} \overline{\Pi}(\delta) = \infty$. For (2.24), we again use that $\lim_{\delta\to 0}\overline{\Pi}(\delta)=\infty$, and simply observe that as $\delta\to 0$,

$$\frac{\delta^3\overline{\Pi}(\delta)}{(\int_0^\delta x^2\Pi(dx)+\delta^2\overline{\Pi}(\delta))^{\frac{3}{2}}}\le\frac{\delta^3\overline{\Pi}(\delta)}{(\delta^2\overline{\Pi}(\delta))^{\frac{3}{2}}}=\frac{1}{\overline{\Pi}(\delta)^{\frac{1}{2}}}\to 0.$$

□

Next we will prove Theorem 2.2.5, the almost sure convergence result for $L(t,\delta)$. Using a Borel-Cantelli argument (see Lemma A.3.4 for details), we shall prove that $\liminf_{\delta\to 0}L(t,\delta)/\mu(\delta)=\limsup_{\delta\to 0}L(t,\delta)/\mu(\delta)=t$ almost surely. First, we will prove the almost sure convergence to t along a subsequence δ_n converging to zero. Then, using the fact that $\mu(\delta)$ and $L(t,\delta)$ are monotone in δ, we will deduce that for all δ between δ_n and δ_{n+1}, $L(t,\delta)/\mu(\delta)$ also tends to t as $\delta_n\to 0$, which completes the proof.

Proof of Theorem 2.2.5. If there is a drift and the Lévy measure is finite, then as $\delta\to 0$, $L(t,\delta)\sim \mathrm{d}t/\delta$ almost surely, and moreover $\mu(\delta)\sim \mathrm{d}/\delta$, so the desired result, $\lim_{\delta\to 0}L(t,\delta)/\mu(\delta)=t$, follows immediately. Hence we need only consider cases with infinite Lévy measure.

Consider a monotone decreasing subsequence $(\delta_n)_{n\ge 0}$ for which $\lim_{n\to\infty}\delta_n=0$. We shall choose a specific subsequence later in the proof. Then for all $\varepsilon>0$, by Chebyshev's inequality (Theorem A.2.2) and Remark 2.2.7,

$$\begin{aligned}
\sum_n \mathbb{P}\left(\left|\frac{L(t,\delta_n)}{t\mu(\delta_n)}-1\right|>\varepsilon\right) &\le \frac{1}{\varepsilon^2}\sum_n\frac{\mathrm{Var}\big(L(t,\delta_n)\big)}{t^2\mu(\delta_n)^2}\\
&= \frac{1}{\varepsilon^2}\sum_n\frac{\frac{t}{\delta_n^2}\big(\int_0^{\delta_n}x^2\Pi(dx)+\delta_n^2\overline{\Pi}(\delta_n)\big)}{\frac{t^2}{\delta_n^2}\big(\mathrm{d}+\int_0^{\delta_n}x\Pi(dx)+\delta_n\overline{\Pi}(\delta_n)\big)^2}\\
&\le \frac{1}{\varepsilon^2}\sum_n\frac{\frac{t}{\delta_n^2}\big(\int_0^{\delta_n}x^2\Pi(dx)+\delta_n^2\overline{\Pi}(\delta_n)\big)}{\frac{t^2}{\delta_n^2}\big(\int_0^{\delta_n}x\Pi(dx)+\delta_n\overline{\Pi}(\delta_n)\big)^2}\\
&= \frac{1}{t\varepsilon^2}\sum_n\frac{\big(\int_0^{\delta_n}x^2\Pi(dx)+\delta_n^2\overline{\Pi}(\delta_n)\big)}{\big(\int_0^{\delta_n}x\Pi(dx)+\delta_n\overline{\Pi}(\delta_n)\big)^2}\\
&\le \frac{1}{t\varepsilon^2}\sum_n\frac{\delta_n\big(\int_0^{\delta_n}x\Pi(dx)+\delta_n\overline{\Pi}(\delta_n)\big)}{\big(\int_0^{\delta_n}x\Pi(dx)+\delta_n\overline{\Pi}(\delta_n)\big)^2}=:\frac{1}{t\varepsilon^2}\sum_n\frac{1}{\hat{\mu}(\delta_n)},
\end{aligned}\tag{2.25}$$

where $\hat{\mu}(\delta) := \int_0^\infty ((x \wedge \delta)/\delta)\ \Pi(dx)$ denotes the expectation of $L(t, \delta)$ without the drift term, and we use the notation $a \wedge b := \min\{a, b\}$. Then since $(x \wedge \delta)/\delta$ is non-decreasing as δ decreases, it follows that $\hat{\mu}(\delta)$ is non-decreasing as δ decreases. Now, $\lim_{\delta \to 0} \hat{\mu}(\delta) = \infty$, and $\hat{\mu}$ is continuous, so for each fixed $r \in (0, 1)$ there is a decreasing sequence δ_n such that $\hat{\mu}(\delta_n) = r^{-n}$ for each n. Then (2.25) is finite, so by the Borel-Cantelli lemma (see Lemma A.3.4), we conclude that $\lim_{n\to\infty} L(t, \delta_n)/\mu(\delta_n) = t$ almost surely. Next we shall prove that this almost sure convergence holds beyond our specific subsequence, as $\delta \to \infty$.

Whether or not there is a drift, $L(t, \delta)$ is obtained by changing the original subordinator's jump sizes from y to $(y \wedge \delta)/\delta$. By monotonicity of this map, it follows that for a fixed sample path of the original subordinator, each individual jump of the process $L(t, \delta_{n+1})$ is at least as big as the corresponding jump of the process $L(t, \delta_n)$. So $L(t, \delta)$ is non-decreasing as δ decreases, and similarly for $\mu(\delta)$. Hence for all $\delta_{n+1} \le \delta \le \delta_n$,

$$\frac{L(t, \delta_n)}{t\mu(\delta_n)} \frac{\mu(\delta_n)}{\mu(\delta_{n+1})} \le \frac{L(t, \delta)}{t\mu(\delta)} \le \frac{L(t, \delta_{n+1})}{t\mu(\delta_n)} = \frac{L(t, \delta_{n+1})}{t\mu(\delta_{n+1})} \frac{\mu(\delta_{n+1})}{\mu(\delta_n)}.$$

Then by our choice of the subsequence δ_n, it follows that for all $\delta_{n+1} \le \delta \le \delta_n$,

$$r\frac{L(t, \delta_n)}{t\mu(\delta_n)} \le \frac{L(t, \delta)}{t\mu(\delta)} \le \frac{1}{r}\frac{L(t, \delta_{n+1})}{t\mu(\delta_{n+1})}, \tag{2.26}$$

and since $\lim_{n\to\infty} L(t, \delta_n)/\mu(\delta_n) = t$, it follows that for all $r \in (0, 1)$,

$$rt \le \liminf_{\delta \to 0} \frac{L(t, \delta)}{\mu(\delta)} \le \limsup_{\delta \to 0} \frac{L(t, \delta)}{\mu(\delta)} \le \frac{t}{r}.$$

Taking limits as $r \to 1$, it follows that $\lim_{\delta \to 0} L(t, \delta)/\mu(\delta) = t$ almost surely.

□

2.5 Extensions and Special Cases

2.5.1 Extensions: Box-Counting Dimension of the Graph

Having extensively studied the range of a subordinator, we now direct our attention to the graph of a subordinator X up to time t is the set $\{(s, X_s) : 0 \leq s \leq t\}$. The box-counting dimensions of the range and graph are closely related. Indeed, the graph of any Lévy process $(X_t)_{t\geq 0}$ can be interpreted as the range of the bivariate Lévy process $(t, X_t)_{t\geq 0}$. This is particularly evident when we consider the mesh box counting schemes $M_G(t, \delta)$, $M_R(t, \delta)$, denoting graph and range respectively. The mesh box-counting scheme counts the number of boxes in a lattice of side length δ to intersect with a set.

Remark 2.5.1. For every subordinator with infinite Lévy measure or a positive drift, $M_G(t, \delta) = \lfloor t/\delta \rfloor + M_R(t, \delta)$, where $\lfloor \cdot \rfloor$ denotes the floor function. Indeed, $M_R(t, \delta)$ increases by 1 if and only if $M_G(t, \delta)$ increases by 1 in such a way that the new box for the graph lies directly above the previous box. For each integer n, $M_G(t, \delta)$ also increases at time $n\delta$, the new box directly to the right of the previous box.

Remark 2.5.2. It follows that the graph of every subordinator X has the same box-counting dimension as the range of $X'_t := t + X_t$, the original process plus a unit drift.

Proposition 2.5.3. *For every subordinator with drift* $\mathrm{d} > 0$*, the box-counting dimensions of the range and graph agree almost surely.*

Proof of Proposition 2.5.3. Letting $T_{(\delta,\infty)}$ denote the first passage time of the subordinator above δ, consider an optimal covering of the graph with squares of side length δ as follows:

Starting with $[0, \delta] \times [0, \delta]$, at time $T_1 := \min(T_{(\delta,\infty)}, \delta)$, add a new box $[T_1, T_1 + \delta] \times [X_{T_1}, X_{T_1} + \delta]$, and so on. Denote the number of these boxes by $N_G(t, \delta)$, and write $N_R(t, \delta)$ as the optimal number of boxes needed to cover the range.

If $\mathrm{d} \geq 1$, then we have $T_1 = T_{(\delta,\infty)}$ because $X_\delta \geq \mathrm{d}\delta$. It follows that each time $N_G(t, \delta)$ increases by 1, so does $N_R(t, \delta)$, and vice versa, so $N_G(t, \delta) = N_R(t, \delta)$, and the box-counting dimension of the range and graph are equal when $\mathrm{d} \geq 1$.

For $\mathrm{d} \in (0,1)$, a similar argument applies with a covering of $\frac{\delta}{\mathrm{d}} \times \delta$ rectangles rather than $\delta \times \delta$ squares. Starting with $[0, \frac{\delta}{\mathrm{d}}] \times [0, \delta]$, at time T_1, add a new box $[T_1, T_1 + \frac{\delta}{\mathrm{d}}] \times [X_{T_1}, X_{T_1} + \delta]$, and so on. The number of these boxes is again $N_R(\delta, t)$, since $X_{\frac{\delta}{\mathrm{d}}} \geq \delta$. By Remark 2.1.6 , this covering of rectangles can still be used to define the box-counting dimension of the range, since for $k := \lceil \frac{1}{\mathrm{d}} \rceil$, writing $N_G(t,\delta)$ and $N'_G(t,\delta)$ for the number of squares and of rectangles respectively, we have

$$N'_G(t,\delta) \leq N_G(t,\delta) \leq k \; N'_G(t, \delta/k).$$

□

Remark 2.5.4. The box-counting dimension of the graph of every subordinator is 1 almost surely, since the sample paths of subordinators have bounded variation (BV) almost surely. See [1, Section 2.3.3] for more details on bounded variation. The same is true for the graph of all BV functions/processes, including in particular every Lévy process without a Gaussian component, whose Lévy measure satisfies the condition $\int (1 \wedge |x|)\Pi(dx) < \infty$. By Proposition 2.5.3, we see that the box-counting dimension of the range of every subordinator with a non-zero drift is 1 almost surely.

2.5.2 Special Cases: Regular Variation of the Laplace Exponent

Corollary 2.5.5 is analogous to [111, Corollary 2], with $L(t,\delta)$ in place of $N(t,\delta)$. This allows very fine comparisons, not visible at the log-scale, to be made between subordinators whose Laplace exponents are regularly varying with the same index.

Corollary 2.5.5. *Consider a subordinator whose Laplace exponent is regularly varying at infinity, such that $\phi(\lambda) \sim \lambda^\alpha F(\lambda)$ as $\lambda \to \infty$, for $\alpha \in (0,1)$, where $F(\cdot)$ is a slowly varying function. Then for all $t > 0$, almost surely as $\delta \to 0$,*

$$L(t,\delta) \sim \frac{t\delta^{-\alpha} F\left(\frac{1}{\delta}\right)}{\Gamma(2-\alpha)}.$$

Proof of Corollary 2.5.5. Note that $\mathrm{d} = 0$, i.e. there is no drift, when the Laplace exponent is regularly varying of index $\alpha \in (0,1)$. By Theorem 2.2.5, as $\delta \to 0$,

$$L(t,\delta) \sim t\mu(\delta) = \frac{tI(\delta)}{\delta} = \frac{t}{\delta}\int_0^\delta \overline{\Pi}(x)dx.$$

Since ϕ is regularly varying at 0, as $x \to 0$, $\overline{\Pi}(x) \sim \phi(\frac{1}{x})/\Gamma(1-\alpha)$ (see [14, p75]). Then by Karamata's Theorem (see Theorem A.4.3), almost surely as $\delta \to 0$,

$$L(t,\delta) \sim \frac{t\delta^{-\alpha}F\left(\frac{1}{\delta}\right)}{\Gamma(2-\alpha)}.$$

□

Corollary 2.5.6 strengthens the result of Theorem 2.2.4 when the Laplace exponent ϕ is regularly varying. The result can not be strengthened in general, as the relationship between $\mu(\delta)$ and $U(\delta)^{-1}$ is "$\asymp$" rather than "$\sim$" (see [15, Prop. 1.4]), where we recall that the notation "$\asymp$" is defined in Remark 2.1.6.

Corollary 2.5.6. *For a subordinator with Laplace exponent ϕ regularly varying at infinity, of index $\alpha \in (0,1)$, for all $t > 0$, almost surely as $\delta \to 0$,*

$$N(t,\delta) \sim \Gamma(2-\alpha)\Gamma(1+\alpha)L(t,\delta).$$

Corollary 2.5.6 follows immediately from Corollary 2.5.5 and [111, Corollary 2], which says that when the Laplace exponent ϕ is regularly varying at infinity, such that $\phi(\lambda) \sim \lambda^\alpha F(\lambda)$ for $\alpha \in (0,1)$, where $F(\cdot)$ is a slowly varying function, for all $t > 0$, almost surely as $\delta \to 0$,

$$N(t,\delta) \sim \Gamma(1+\alpha)t\delta^{-\alpha}F\left(\frac{1}{\delta}\right).$$

Remark 2.5.7. *For $\alpha \in (0,1)$, $\Gamma(2-\alpha)\Gamma(1+\alpha)$ takes values between $\pi/4$ and 1. So $L(t,\delta)$ and $N(t,\delta)$ are closely related when the Laplace exponent is regularly varying, but as $\delta \to 0$, $L(t,\delta)$ grows to infinity slightly faster than $N(t,\delta)$.*

Chapter 3

Markov Processes with Constrained Local Time

Abstract

In this chapter, we study Markov processes conditioned so that their local time must grow slower than a prescribed function. Building upon recent work on Brownian motion with constrained local time in [8, 78], we study transience and recurrence for a broad class of Markov processes.

Through the notion of inverse local time, this problem is equivalent to studying a non-decreasing Lévy process (the inverse local time process), conditioned to remain above a given level which varies in time. We study a time-dependent region, in contrast to previous works in which a process is conditioned to remain in a fixed region (e.g. in [43, 60]), so we must study boundary crossing probabilities for a family of curves, and thus obtain uniform asymptotics for such a family.

Main results include necessary and sufficient conditions for transience or recurrence of the conditioned Markov process. We will explicitly determine the distribution of the inverse local time for the conditioned process, and in the transient case, we explicitly determine the law of the conditioned Markov process. In the recurrent case, we characterise the *entropic repulsion envelope* via necessary and sufficient conditions.

3.1 Introduction & Background

The study of stochastic processes under various constraints is important in a range of theoretical and applied settings, and there have been many such studies in recent years. We shall discuss the literature on constrained processes in more detail in Section 3.2.

In this chapter, the specific constraint of interest is that we shall restrict the rate at which a certain type of stochastic process returns to the origin. The original process is chosen to be *recurrent*, in the sense that prior to applying a constraint, it (almost surely) continues to return to the origin at arbitrarily large times. We will determine, depending on the strength of the constraint, whether or not the constrained process will continue to return to the origin at arbitrarily large times. If the constraint is too stringent, we will see that it is possible for the process to never return to the origin again after a certain time, in which case we say the process is *transient*. The notions of transience and recurrence are formally defined in Section 3.3.

Let us now proceed by motivating the upcoming research through a brief discussion of some relevant problems in polymer physics, specific to the modelling of a long polymer chain. Discussion of these problems is included here as a means of helping the reader to establish an intuitive picture for the upcoming research questions. Long polymer chain models are particularly relevant here because there is a remarkable similarity between the questions we ask in this chapter and common research questions on polymers (see e.g. [8, p5]). In particular, we will see that determining whether a process is transient or recurrent is analogous to determining if a polymer chain is *delocalised* or *localised*, respectively. It should, however, be noted that this is just one of many applications which justify theoretical studies such as those in this chapter and further theoretical studies in the literature, as discussed in Section 3.2.

3.1.1 Related Problems in Polymer Physics

Motivation: Study of Transience and Recurrence We shall consider a specific modelling scenario in detail, to give a clear idea of the relevant problems of interest in polymer physics. This serves as an overall motivation for similar problems in polymer physics. For a much more detailed account of a variety of related polymer models, one can refer to [28, 118], for instance.

The setting we shall consider is that of a a polymer made up of two different types of monomer, near a boundary between two fluids. Further details on this particular example can be found in [28, Section 5].

The monomers may be attracted to or repelled by the fluids, and the different kinds of monomer will feel a different strength of attractive or repulsive force to each fluid (e.g. hydrophobic/hydrophilic monomers, if one fluid is water). In a number of scenarios, the polymer tends to place as many monomers in their preferred fluid as possible, which requires it to remain near the fluid boundary. However, this reduces entropy, and depending on the strengths of the attractive/repulsive forces involved, a *phase transition* may occur, wherein entropic forces cause the polymer to move away from the boundary. The transition is from a *localised phase* to a *delocalised phase*. The following figure depicts a polymer in the localised phase, i.e. one which remains close to the boundary between two fluids:

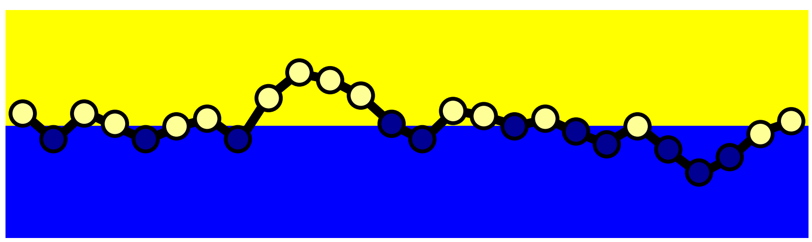

Figure 3.1: Polymer at boundary between oil and water. Blue monomers are attracted to water, yellow monomers are attracted to oil.

Polymer physicists are especially interested in the occurrence of this *phase transition* between the localised phase, where the polymer remains close to the interface, and the delocalised phase, where it moves away from the interface. The goal is often to understand when the transition occurs as underlying model parameters vary, as in a vast number of works, e.g. [9, 25, 28, 35, 62, 66, 118, 119]. The problem of determining exactly when a phase transition occurs motivates much of the work in this chapter.

Indeed, we shall study random processes constrained to avoid the origin, where the origin is analogous to the fluid boundary. We will determine exactly how strong a constraint needs to be in order to change the behaviour of a given process from localised to delocalised, which corresponds to finding when this phase transition occurs in a polymer model, as model parameters vary. Moving from the context of polymers to a formal setting of stochastic processes, the notions of delocalisation and localisation correspond to *transience* and *recurrence*, respectively (see Definition 3.3.1).

To briefly justify the use of random models for polymers, we note that many polymers, such as fats, contain chains of hydrophobic/hydrophilic monomers which are arranged in an erratic manner. The disorderly nature in which such a polymer chain is formed means that a random model is particularly appropriate for modelling purposes. Further suitable examples include surfactants, emulsifiers, and foaming agents. See [28, Section 5] for further discussion.

Motivation: Further Questions In the example of a polymer at a boundary between fluids, determining if a process is transient or recurrent is analagous to determining whether or not the polymer remains close to the boundary. However, there are still many more natural questions which can be asked in each case.

For instance, in the case where a polymer moves away from the boundary, one can ask how and when the polymer moves away. In Section 3.4.2, for the transient case, we address the analogous theoretical question for Markov processes by determining the distribution of the time at which the process returns to the origin for the last time, see Theorem 3.4.12 and Remark 3.4.13. We also discuss the distribution of the process after this last return time in Remark 3.4.17.

For the case where a polymer remains close to the boundary, if a lot of the monomers are placed at the boundary, then the entropy of the system is reduced, and an *entropic force* is felt by the polymer. This is a repulsive force which encourages the polymer to place more of its monomers away from the boundary. We call this phenomenon *entropic repulsion*. It is then natural to ask how many monomers are situated at (or near) the boundary. We address an analogous question for recurrent Markov processes by determining the rate of return to the origin. In Theorem 3.4.22, we find the almost sure asymptotic behaviour of the rate of growth of the local time, which determines the rate of return to the origin of the Markov process. See Section 3.1.2 for more details on the notion of local time.

This can be all seen as an extended analogy of work in this chapter, but we will not tackle specific applied problems on polymers. Rather, the results here add to a body of theoretical works which set out to improve our understanding of various stochastic processes, which can then be used in a variety of modelling scenarios.

3.1.2 Local Time of a Markov Process

As discussed in Section 3.1.1, we are going to study Markov processes under a certain type of constraint. Specifically, we shall impose a restriction on the rate at which our Markov process returns to the origin, and then we determine exactly how strong the constraint needs to be in order to change the behaviour of a given process from recurrent (localised) to transient (delocalised). This constraint is made rigorous through the notion of *local time*, which we shall now define, after providing a formal definition of a Markov process:

Definition 3.1.1. *A Markov process* $(M_t)_{t\geq 0}$*, is a* $\mathbb{R}^d$*-valued stochastic process such that for each (almost surely) finite stopping time* T*, under the conditional law* $\mathbb{P}\left(\cdot|M_T = x\right) = \mathbb{P}_x\left(\cdot\right)$*, the shifted process* $(M_{s+T})_{s\geq 0}$ *is independent of* $\mathcal{F}_T$ *and has the same as the law,* $\mathbb{P}_x$*, as the process* M *started from* x*. Moreover, we impose that* M *has right-continuous sample paths,* $M_0 = 0$*, and that the origin is regular and instantaneous:*

Regular means that for each (almost surely) finite stopping time T*, if* $M_T = 0$*, then* $\inf\left\{t > T : M_t = 0\right\} = T$ *almost surely. Instantaneous means that for each (almost surely) finite stopping time* T*, if* $M_T = 0$*, then* $\inf\left\{t > T : M_t \neq 0\right\} = T$ *almost surely.*

This definition ensures that, conditionally given the present state of the process, the future behaviour is independent of the past behaviour (the future only depends on the present). There are many different definitions for Markov processes in the literature. The above definition is particularly helpful because it ensures that our process is homogeneous in time (although it does not necessarily have to be homogeneous in space). We have further imposed that the origin is regular and instantaneous so that the behaviour of the process is suitably interesting, in the sense that the structure of the local time has a fractal nature (there is non-trivial structure at an arbitrarily small scale). Hereon, $(M_t)_{t\geq 0}$ shall always denote a Markov process in the above sense.

Local Time: Discrete State Space While we are going to constrain the local time of a process on a continuous state space, it is first helpful to introduce the notion of local time in the simpler discrete setting.

The local time (at 0) of a Markov process $(M_t)_{t\geq 0}$ on a discrete state space is the process $(L_t)_{t\geq 0}$, defined by

$$L_t := \int_0^t \mathbb{1}_{\{M_s=0\}} ds.$$

The local time (here equivalent to *occupation time*) simply records how long M spends at 0, in the sense that L_t is the total time spent at 0, up to time t. The following figure demonstrates how the local time grows relative to the position of its associated Markov process, with matching time axes:

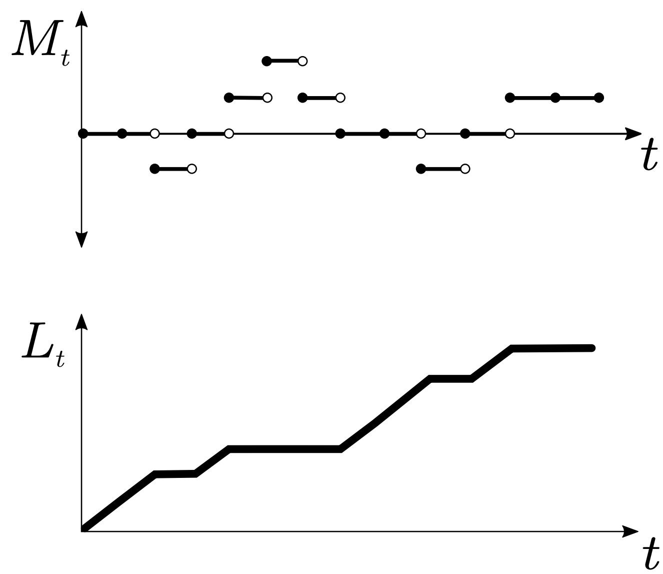

Figure 3.2: Local time $(L_t)_{t\geq 0}$ of a Markov process $(M_t)_{t\geq 0}$ on a discrete state space

Local Time: Continuous State Space This chapter concerns Markov processes on a continuous state space, and many such processes (e.g. Brownian motion) only spend an infinitesimally small period of time at any single point. That is to say, for many Markov processes of interest (on a continuous state space), the set of times at which it visits the origin will have zero Lebesgue measure (almost surely). Hence a subtly different definition is required to capture something non-trivial, which is still analogous to the notion of local time of a process on a discrete state space, see the upcoming Definition 3.3.2. Informally, the local time can be thought of as a rescaled measure of how much time the Markov process $(M_t)_{t\geq 0}$ spends near the origin.

Connection to Lévy Processes The local time process $(L_t)_{t\geq 0}$ is not usually studied directly. Instead, it is generally best to work with its inverse. The *inverse local time* process, $(X_s)_{s\geq 0}$, is defined for $s \geq 0$ by $X_s := \inf\{u \geq 0 : L_u > s\}$. Remarkably, for all Markov processes in the sense of Definition 3.1.1, the (right-continuous) inverse local time process is in fact a subordinator. The jumps of the inverse local time subordinator correspond to *excursions* of $(M_t)_{t\geq 0}$ away from 0, which we shall now define:

An excursion interval of a Markov process (away from the origin) is defined as an interval of time, $(g, d) \subset [0, \infty)$, such that $M_s \neq 0$ for all $s \in (g, d)$, and (g, d) is maximal (locally).

The value of the local time remains constant when M is in an excursion interval, and increases otherwise. In our cases of interest, the fact that we have imposed that the origin is regular and instantaneous (see Definition 3.1.1) ensures that there are inifinitely many excursions of infinitesimally small size (almost surely). This ensures that the behaviour of the Markov process is suitably *interesting*, in the sense that there is non-trivial behaviour at an arbitrarily small scale (in both time and space). At the level of the inverse local time subordinator, this condition ensures that the subordinator has infinitely many infinitesimally small jumps in each finite window of time (almost surely).

For a Markov process (on a continuous state space) in the sense of Definition 3.1.1, the following figure illustrates the relationship between the Markov process and its local time, with matching time axes. Some large excursion intervals are highlighted in red:

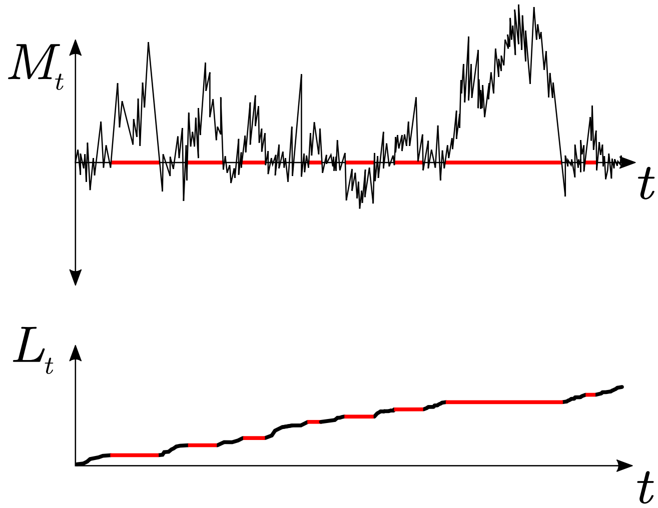

Figure 3.3: Local time of Markov process on a continuous state space, large excursions coloured in red

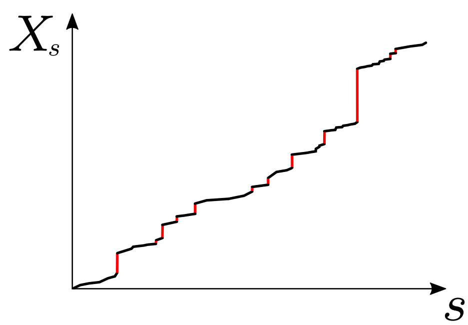

Figure 3.4: Inverse of local time, jumps correspond to excursions in Figure 3.3

Remark 3.1.2. *Throughout this chapter, we will assume that the inverse local time subordinator has zero drift. Consider the result [14, Corollary IV.6], which states that for each Markov process in the sense of*

Definition 3.1.1, there exists a constant $\mathrm{d} \geq 0$ *such that its local time can be characterised for all* $t \geq 0$ *by*

$$\mathrm{d}L_t = \int_0^t \mathbb{1}_{\{M_s=0\}} ds.$$

It follows that the inverse local time $(X_t)_{t\geq 0}$ *has linear drift* $\mathrm{d} \geq 0$*, see [14, Theorem IV.8]. So for our Markov processes of interest, set of times at which the process takes the value 0 will have zero Lebesgue measure.*

We direct the reader to [14, Chapter IV] for a more detailed introduction to local time, including proof of the fact that the inverse local time process is always a (possibly killed) subordinator [14, Theorem IV.8], where a *killed* subordinator takes the value infinity from an independent exponentially distributed random time onwards (this corresponds to transience of the Markov process). One can refer to e.g. [69, Chapter 19] or [104, Chapter VI] for discussion of the importance of the study of local time, with particular emphasis on connections to semimartingales and stochastic calculus.

3.2 Literature Overview

We are going to study the asymptotic behaviour of a Markov process whose local time is constrained to grow slower than f, an increasing function. Underlying this work is the problem of determining how the local time behaves. As the (right-continuous) inverse of the local time process is a subordinator, our study of the behaviour of the local time process is in fact equivalent to studying a subordinator conditioned to grow faster than the inverse function f^{-1}, i.e. conditioned to remain in a specific region in space and time.

This chapter is hence related to a number of works on stochastic processes conditioned to remain in a certain fixed region, such as processes conditioned to stay in a cone [43,60,83], or in a Weyl chamber [41,80]. We highlight the fact that our subordinator is conditioned to remain in a region which varies in time, whereas the aforementioned works consider fixed regions, as appears to be the case for all works prior to [78] and [6].

We thus say that our constraint is a *weak* constraint, in constrast to *strong* constraints such as conditioning a process to avoid a point, where the constraint does not change (see e.g. [11, 24, 33]). When our conditioned Markov process is transient, the constraint varies for only a finite window of time, whereas when our conditioned Markov process is recurrent, the constraint varies over all time. So our results and methodology, especially in the recurrent case, offer a significant contrast to many prior works which deal with a strong constraint.

Many works, e.g. [2–4, 22, 40, 42, 46, 70, 71, 88, 94, 100, 120], do consider a time-dependent region, in the sense that they study the time at which a process will first cross a specific boundary. However, in this chapter we study the boundary crossing probability for a family of curves, and study the time at which this crossing occurs (the function f^{-1} forms our boundary of interest). Moreover, we study these asymptotics, uniformly among such a family of curves in Lemma 3.5.3, and we consider the deeper problem of determining the law of a subordinator conditioned to remain in this time-dependent region.

Remark 3.2.1. *Our weak constraint is a weak repellence constraint, in the sense that the process is repelled away from the origin (without the constraint, the process would typically spend more time at the origin). This links back to our previous discussion of polymer models in Section 3.1.1. Many processes with imposed weak repellence are studied in works related to polymer physics, see e.g. [9, 28, 62, 115–118, 122].*

The work in this chapter is also similar, in spirit, to various other works on Brownian motion [79, 98, 105, 107], Lévy processes [11, 12, 24, 32, 33, 48, 97, 125, 126], and more general diffusions [106, 109] with restricted path behaviour.

3.2.1 Relevant Results from the Literature

Let us briefly summarise some important results from the literature on constrained Markov processes. We begin by considering the example of a process conditioned to avoid the origin. It is important to note here that the event $\{M_t \neq 0, \text{ for all } t \geq 0\}$ typically has zero probability. When we talk about conditioning upon this event, the terminology is somewhat informal, but is made rigorous by taking limits, see (3.17). In this chapter, we use the limiting procedure to make sense of a Markov process with constrained local time, since the imposed constraint typically has zero probability.

Processes Conditioned to Avoid Zero For a 1-dimensional process with continuous sample paths, such as Brownian motion, this condition ensures that the process stays on one side of a boundary (i.e. the process is conditioned to stay positive/negative). The analogous polymer model for this type of process is a polymer constrained to stay on one side of an interface, which acts like a hard wall, see [28, Section 3.4]. This contrasts the previous setting of a polymer at a boundary between fluids (see Section 3.1.1), where the polymer is allowed to cross the boundary, although it may be discouraged from crossing too often.

In 1975, Pitman studied the distribution of a 1-dimensional Brownian motion conditioned to avoid zero (i.e. to stay positive or negative), and found a remarkable connection with Bessel processes [98]. For $d \in \mathbb{N}$, we define a *Bessel process of order* d as the real-valued process $(B_t^{(d)})_{t\geq 0}$, given by $B_t^{(d)} := |W_t|$, where W_t is a d-dimensional Brownian motion, and $|\cdot|$ is the standard Euclidean norm. Pitman's result is as follows:

Pitman (1975) A 1-dimensional Brownian motion, $(B_t)_{t\geq 0}$, conditioned to stay positive is equal in distribution to a Bessel process of order 3. In particular, the process $(B_t - 2\underline{B}_t)_{t\geq 0}$ has the same distribution as a Bessel process of order 3, where $\underline{B}$ denotes the running infimum, $\underline{B}_t := \inf\{B_s : 0 \leq s \leq t\}$.

A natural generalisation of this result, to allow for a 1-dimensional Brownian motion with a linear drift, was then found by Pitman and Rogers in [99].

Pitman, Rogers (1981) Let $(B_t^\mu)_{t\geq 0}$ be a 1-dimensional Brownian motion with a linear drift μ. Then B^μ conditioned to stay positive is equal in distribution to $|W_t|$, where $(W_t)_{t\geq 0}$ is a 3-dimensional Brownian motion with a linear drift of absolute value $|\mu|$. The conditioned process again has the same distribution as $(B_t^\mu - 2\underline{B}_t^\mu)_{t\geq 0}$, with the notation as above, but applied to the Brownian motion with a drift.

For a more general Lévy process with jumps, avoiding the origin is no longer equivalent to staying positive or negative, as the process can jump over the origin without visiting it. This means that the problem of conditioning to avoid the origin becomes much harder. However, for a 1-dimensional Lévy process with no jumps in the negative direction (a *spectrally positive* Lévy process), conditioning to stay negative is still a tractable problem, thanks to the simplifying assumption that there are no negative jumps. The distribution of a spectrally positive Lévy process conditioned to stay negative was determined by Bertoin in [11].

Bertoin (1992) Let $(X_t)_{t\geq 0}$ be a spectrally positive Lévy process. Then X conditioned to stay negative has the same distribution as the process

$$(X_t - 2\overline{X}_t^c - J_t)_{t\geq 0},$$

where $\overline{X}_t := \sup\{X_s : 0 \leq s \leq t\}$ denotes the running maximum of the process X, while $\overline{X}^c_t$ denotes the continuous part of $\overline{X}$ (i.e. $\overline{X}$ with its jumps removed), and finally J is the process of all jumps of the original process across its previous maximum, defined by

$$J_t := \sum_{s \leq t} \mathbb{1}_{\{X_s > \overline{X}_{s-}\}}(X_s - X_{s-}),$$

with the standard notation $X_{s-} := \lim_{u \uparrow s} X_u$.

There is a striking similarity between this result and the previous results (Pitman 1975), (Pitman, Rogers, 1981). The fact that the process $(J_t)_{t \geq 0}$ is needed reflects the fact that this is a much more difficult problem for a process with jumps. A spectrally positive Lévy process conditioned to stay negative is equivalent to a *spectrally negative Lévy process* (with no positive jumps) conditioned to stay positive, as we can simply consider the *dual* process, $(-X_t)_{t \geq 0}$, which is spectrally positive.

The more difficult problem of determining the distribution of a Lévy process conditioned to avoid the origin, where the process has positive and negative jumps, has recently been addressed by Pantí, see [97, Theorem 2.6] for further details. We refer to [34] for more on Lévy processes conditioned to stay positive.

Brownian Motion with Constrained Local Time The aforementioned results are closely related to the work in this chapter. However, this chapter specifically builds upon two works on Brownian motion with constrained local time, [8, 78]. In these works, the inverse local time subordinator is stable with index 1/2.

Let us first discuss the work of Benjamini and Berestycki [8], in which a 1-dimensional Brownian motion is conditioned so that its local time at the origin, $(L_t)_{t \geq 0}$, satisfies $L_t \leq f(t)$ for all $t \geq 0$ for a given function f, and then a sufficient condition for the constraint to yield a transient process is found.

Benjamini, Berestycki (2011) Consider a 1-dimensional Brownian motion conditioned so that its local time $(L_t)_{t \geq 0}$ satisfies $L_t \leq f(t)$ for all $t \geq 0$. If the following integral converges,

$$\int_1^\infty \frac{f(s)}{s^{\frac{3}{2}}} ds < \infty, \tag{3.1}$$

then (under some regularity conditions on the function f) any resultant process which arises as a weak subsequential limit of the conditioning is a transient process.

Remark 3.2.2. *The local time (at 0) of an unconditioned 1-dimensional Brownian motion (almost surely) grows proportionally to* $\sqrt{t}$ *as* $t \to \infty$ *(this can be verified using e.g. Brownian scaling, see [8, p2]). The unconditioned process is known to be recurrent, but inspecting the criterion in* (3.1) *it follows that restricting the local time so that* $L_t \leq f(t) = \sqrt{t}\log(t)^{-1-\varepsilon}$ *for some* $\varepsilon > 0$*, the conditioned process then becomes transient. Remarkably, a very minor restriction on the local time results in a significant change to the conditioned process, compared with the original Brownian motion.*

It was then proven by Kolb and Savov that the criterion in (3.1) is in fact a necessary and sufficient for transience of the conditioned Brownian motion, see [78, Theorem 3].

Kolb, Savov (2016) Consider a 1-dimensional Brownian motion conditioned so that its local time $(L_t)_{t\geq 0}$ satisfies $L_t \leq f(t)$ for all $t \geq 0$ (under some regularity conditions). The conditioned process is transient if and only if the integral in (3.1) converges. The conditioned process is recurrent otherwise.

In the transient case, an explicit formulation for the distribution of the conditioned Brownian motion is found, see [78, Section 3]. In the recurrent case, understanding the distribution of the conditioned Brownian motion is much harder. However, in the recurrent case, Kolb and Savov determine a very deep result about the rate of growth of the inverse local time L_t, as $t \to \infty$. Note that in the transient case, the conditioned Brownian motion never visits the origin again after a certain finite time. The local time never grows beyond this time, which corresponds to *explosion* of the inverse local time, i.e. reaching an infinite value in a finite time. Hence one can only make sense of the long-term rate of growth of the local time in the recurrent case.

Entropic Repulsion For conditioned Brownian motion in the recurrent case where $\int_1^\infty f(s)/s^{3/2}ds = \infty$, the conditioned process typically doesn't use all of its allowance of local time, in the sense that while $L_t \leq f(t)$ is required, it does not usually come close to the boundary at which $L_t = f(t)$. This phenomenon is related to entropic effects, which cause the process to stay far away from breaking the constraint to allow for more fluctuations. The term *entropic repulsion* is used to describe this kind of situation, borrowing the term from physics (see e.g. [36]). In fact, the most likely way for the conditioned Brownian motion process to satisfy an imposed condition is for it to satisfy an even stronger condition, in the sense that $L_t = o(f(t))$ as $t \to \infty$.

This is made rigorous by Kolb and Savov in [78, Theorem 4] through their characterisation of the *entropic repulsion envelope*. Consider the set of non-decreasing functions w such that $\lim_{t\to\infty} w(t) = \infty$, for which $\lim_{t\to\infty} \mathbb{Q}(X_t \geq w(t)f^{-1}(t)) = 1$, where $\mathbb{Q}(\cdot)$ denotes the probability measure obtained by conditioning on $L_t \leq f(t)$ for all $t \geq 0$. The set of such functions is called the entropic repulsion envelope for the function f, and is denoted by

$$R_{f^{-1}} := \left\{ w : \lim_{t\to\infty} w(t) = \infty, w \text{ is non-decreasing, } \lim_{t\to\infty} \mathbb{Q}(X_t \geq w(t)f^{-1}(t)) = 1 \right\}.$$

Kolb and Savov's result is as follows:

Kolb, Savov (2016) If the conditioned process as above is recurrent, then (under some regularity conditions) the entropic repulsion envelope for the function f is given by:

$$w \in R_{f^{-1}} \iff \lim_{h\to\infty} \int_h^{f(f^{-1}(h)w(h))} \frac{1}{\sqrt{f^{-1}(s)}} ds = 0.$$

This chapter specifically builds upon Kolb and Savov's work in [78], which itself builds upon Benjamini and Berestycki's work in [8]. In particular, we shall provide results analogous to those in [78], but for a much broader class of processes than Brownian motion. It was conjectured in [78, Remark 9] that such analogous results hold when the Lévy measure of the inverse local time process (see the upcoming Section 3.1.2) has a regularly varying tail function, which we confirm in this chapter. We shall extend further beyond this conjecture by including a much more general setting, see Definition 3.4.8.

3.2.2 Brief Exposition of the Main Result: Necessary and Sufficient Condition

Now we provide a brief exposition of the main result, before introducing some key definitions. Starting with a recurrent Markov process, we constrain its local time $(L_t)_{t\geq 0}$ so that $L_t \leq f(t)$ for all t, using a limiting procedure. The following necessary and sufficient condition tells us if the constraint is strong enough to change the behaviour of the process from recurrent to transient: it is transient if

$$\int_1^\infty f(x)\Pi(dx) < \infty, \tag{3.2}$$

and the process remains recurrent otherwise. Here $\Pi(dx)$ denotes the Lévy measure of the inverse local time subordinator (see the upcoming Section 3.1.2). For a large class of subordinators and functions f, our criterion (3.2) can be understood in terms of the rate of growth of the inverse local time $(X_s)_{s\geq 0}$ as $s \to \infty$, as it is known [14, Theorem III.13] that if $\mathbb{E}[X_1] = \infty$, f^{-1} is increasing, and $t \mapsto f^{-1}(t)/t$ is increasing,

$$\int_1^\infty f(x)\Pi(dx) < \infty \iff \lim_{s\to\infty} \frac{X_s}{f^{-1}(s)} = 0, \text{ almost surely.}$$

So the boundary choice of f, at which the conditioned process changes from recurrent to transient, coincides with the boundary at which X_s grows to infinity faster or slower than $f^{-1}(s)$.

Remark 3.2.3. *In constraining the local time of a Markov process, the extent to which our constraint affects the process varies over time, depending on the past behaviour of the process. For instance, if at a certain time, the Markov process has used little of its allowance of local time, then the local time can subsequently grow very rapidly for a short period. This is not possible if the full allowance of local time has been used, and we remark that the conditioned process M is hence no longer Markovian. Still, as for a Markov process, exactly one of transience and recurrence holds here (see Proposition 3.4.14, Proposition 3.4.19).*

Remark 3.2.4. *The conditioned process is no longer homogeneous in time unless f is linear, in which case the inverse local time subordinator conditioned to grow faster than $f^{-1}(t) = at+b$ is equivalent to a spectrally positive Lévy process with drift $-a$ conditioned to stay above 0, as dealt with in [11].*

The remainder of the chapter is structured as follows: Section 3.3 provides key definitions; Section 3.4 outlines the statements of the main results and the conditions under which they hold, including the necessary and sufficient conditions for transience/recurrence, the distribution of the conditioned process, and the characterisation of the entropic repulsion envelope; Section 3.5 contains the proofs of the main results; Sections 3.6, 3.7, and 3.8 contain the proofs of 3 key lemmas required for the main results; Section 3.9 contains the proofs of the remaining auxiliary lemmas.

3.3 Key Definitions

We shall provide some definitions, following conventions of [14, Chapter IV].

Definition 3.3.1. *A process Y is transient under the law $\mathbb{P}(\cdot)$ if $\mathbb{P}$-almost surely,* $\sup\{t \geq 0 : Y_t = 0\}$ *is finite, or recurrent if this supremum is infinite. By Kolmogorov's 0-1 law [69, Theorem 2.13], exactly one of transience or recurrence holds for Markov processes in the sense of Definition 3.1.1.*

Definition 3.3.2. *For a Markov process, and for an arbitrary choice of $c \in (0, \infty)$, let $l_1(x)$ denote the length of the first excursion interval (away from zero) of length $l > x > 0$, and define*

$$P(a) := \begin{cases} 1/\mathbb{P}(l_1(a) > c), & 0 < a \leq c, \\ \mathbb{P}(l_1(c) > a), & a > c. \end{cases}$$

Let $g_n(a)$ be the start time of the nth excursion of length $l > a > 0$, write $N_a(t) := \sup\{n \in \mathbb{N} : g_n(a) < t\}$. Then the local time process, $(L_t)_{t\geq 0}$, of M at zero is defined by $L_t := \lim_{a\to 0} N_a(t)/P(a), t \geq 0$.

See [14, Chapter IV] for more details on the definition of local time. A subordinator is defined to be a non-decreasing real-valued stochastic process with stationary independent increments, started from 0. The right-continuous inverse local time process, defined by $X_t := \inf\{s > 0 : L_s > t\}$, is a subordinator. The jumps of $(X_t)_{t\geq 0}$ correspond to excursions of $(M_t)_{t\geq 0}$ away from zero.

The Laplace exponent ϕ of a subordinator X is defined by $e^{-\phi(\lambda)} = \mathbb{E}[e^{-\lambda X_1}]$, $\lambda \geq 0$. By the Lévy-Khintchine formula (1.4), ϕ can always be written

$$\phi(\lambda) = \mathrm{d}\lambda + \int_0^\infty (1 - e^{-\lambda x})\Pi(dx),$$

where $\mathrm{d} \geq 0$ is the linear drift, and Π is the Lévy measure, which determines the size and rate of the jumps of X, and satisfies $\int_0^\infty (1 \wedge x)\Pi(dx) < \infty$. For our purposes in this chapter, $\mathrm{d} = 0$, so we aim to formulate our results in terms of the measure $\Pi(dx)$. We refer to [14, 15, 45] for background on subordinators.

Next, we define some important classes of functions with which we shall work.

Definition 3.3.3 (Regular Variation and Related Properties)**.**

1. *A function $h : \mathbb{R} \to \mathbb{R}$ is regularly varying at ∞ with index $\alpha \in \mathbb{R}$ if for all $\lambda > 0$,* $\lim_{t\to\infty} h(\lambda t)/h(t) =$

λ^α. We refer to [21] for background on regular variation.

2. *A function $L : \mathbb{R} \to \mathbb{R}$ is slowly varying at ∞ if $\lim_{t\to\infty} L(\lambda t)/L(t) = 1$ for each $\lambda > 0$. A function h, regularly varying at ∞ of index α, can always be written as $h(x) = x^\alpha L(x)$, where L is slowly varying at ∞, see [21].*

3. *The lower index, $\beta(h)$, of a function $h : \mathbb{R} \to \mathbb{R}$ is the supremum of $\beta \in \mathbb{R}$ for which there exists $C > 0$ so that for all $\Lambda > 1$, $h(\lambda x)/h(x) \geq (1 + o(1))C\lambda^\beta$, uniformly in $\lambda \in [1, \Lambda]$, as $x \to \infty$, see [21, p68].*

4. *A function h is CRV at ∞ if $\lim_{\lambda\to 1} \lim_{t\to\infty} h(\lambda t)/h(t) = 1$. The class of CRV functions lies between extended regularly varying functions and $\mathcal{O}$-regularly varying functions. See [44] for details.*

5. *A function $h : \mathbb{R} \to \mathbb{R}$ is $\mathcal{O}$-regularly varying at ∞ if for each $\lambda > 0$, both $\limsup_{t\to\infty} h(\lambda t)/h(t) < \infty$ and $\liminf_{t\to\infty} \; h(\lambda t)/h(t) > 0$. See [44] for further details.*

3.4 Statements of Main Results

We aim to constrain the local time so that $L_t \leq f(t)$ for all $t \geq 0$, where $f : [0, \infty) \to (0, \infty)$ is increasing, $f(0) \in (0, 1)$, and $\lim_{t\to\infty} f(t) = \infty$. This work concerns the behaviour of our process as $t \to \infty$, which is unaffected by the condition on $f(0)$. Before stating our main results, we define the regularity conditions under which these results hold.

3.4.1 Regularity Conditions

We shall impose regularity conditions on the function f, its inverse function $g := f^{-1}$ (extended so that for $x \in [0, f(0))$, $g(x) = 0$), and the tail $\overline{\Pi}(x) := \Pi(x, \infty)$ in two main cases of interest. Our conditions are imposed on the inverse local time subordinator rather than directly on the Markov process. Now let us define the main cases of interest for our results:

Definition 3.4.1 (Case (i)). *We impose on our subordinator that the drift is zero, and the tail $\overline{\Pi}(x) = \Pi(x,\infty)$ is regularly varying at ∞ with index $-\alpha \in (-1,0)$, so $\overline{\Pi}(x) = x^{-\alpha}L(x)$ for L slowly varying at ∞. We further impose there exist $B, N > 0$ such that the function $x \mapsto x^N L(x)$ is non-decreasing on (B,∞).*

We impose that $f(0) \in (0,1)$, $\lim_{t\to\infty} f(t) = \infty$, f is differentiable, $tf'(t)\overline{\Pi}(t)$ decreases to 0 as $t \to \infty$ (so f is increasing), the inverse $g := f^{-1}$ satisfies $\lim_{t\to\infty} g(t+\varepsilon)/g(t) = 1$ for all $\varepsilon > 0$, and there exists some value $\beta > (1+2\alpha)/(2\alpha+\alpha^2) > 1$ such that

$$\lim_{t\to\infty} t\overline{\Pi}\left(\frac{g(t)}{\log(t)^\beta}\right) = 0. \tag{3.3}$$

Remark 3.4.2. *Case (i) includes stable subordinators, and subordinators whose Lévy measures have similarly well-behaved tail asymptotics. Thus the set of Markov processes corresponding to case (i) includes Bessel processes, stable Lévy processes of index $\alpha \in (1,2)$, and other Markov processes with similarly well-behaved asymptotics. Considering $\overline{\Pi}(x) = x^{-\alpha}\log(x)^\delta$, with $f(t) = 1/2 + t^\alpha \log(e+t)^\gamma$, so $g(t) \sim t^{1/\alpha}\log(t)^{-\gamma/\alpha}$ as $t \to \infty$, for $\gamma < -\delta - \alpha\beta$ and $\alpha \in (0,1)$, one can verify that this class of examples is included in case (i).*

Remark 3.4.3. *The results for case (i) in the upcoming Section 3.4.2 do not make use of the assumption that f is differentiable or that $tf'(t)\overline{\Pi}(t)$ decreases to 0 as $t \to \infty$. We can also weaken the assumption* (3.3) *by replacing β by* 1*, and all the proofs in Section 3.4.2 are still valid, as is that of Lemma 3.5.1.*

Remark 3.4.4. *For all results in this chapter, we can replace the condition "$tf'(t)\overline{\Pi}(t)$ decreases to 0" by a weaker combination of two conditions: $\lim_{t\to\infty} tf'(t)\overline{\Pi}(t) = 0$; there exists $\kappa > 0$ such that $t^\kappa f'(t)\overline{\Pi}(t)$ decreases to zero as $t \to \infty$. We use the condition "$tf'(t)\overline{\Pi}(t)$ decreases to 0" for brevity.*

Definition 3.4.5 (Case (ia)). *Under the assumptions of case (i), define "case (ia)" by imposing f, f' are $\mathcal{O}$-regularly varying at ∞, the densities $f_t(x)dx := \mathbb{P}(X_t \in dx)$ and $u(x)dx := \Pi(dx)$ exist, u has bounded increase and bounded decrease (see Definition A.4.5), and there exist constants $a, x_0 \in (0,\infty)$, such that for all $t \in (0,\infty)$ and $x \geq g(t) + x_0$, where $g = f^{-1}$,*

$$f_t(x) \leq atu(x). \tag{3.4}$$

Remark 3.4.6. *If $\overline{\Pi}$ is regularly varying at ∞ and the density f_t exists, then* (3.4) *holds for each fixed t and*

$x > x(t)$, where $x(t)$ may depend on t (see e.g. [124, Theorem 1]). Here we further impose a bound on $x(t)$, so that (3.4) *holds uniformly among sufficiently many x and t for us to prove Theorem 3.4.22. For a stable subordinator of index $\alpha \in (0,1)$, the density f_t exists (see [14, p227]) and* (3.4) *holds for all functions $g = f^{-1}$ satisfying the conditions for case (ia) (see Corollary 3.4.23), so case (ia) includes stable subordinators.*

Remark 3.4.7. *Imposing that f and f' are $\mathcal{O}$-regularly varying at ∞ is by no means a restrictive condition. It effectively removes particularly fast-growing functions, e.g. $f(t) \sim e^t$ as $t \to \infty$, for which conditioning upon $\{X_t \geq f^{-1}(t), t \geq 0\}$ does not have much of an interesting effect as $t \to \infty$, since the rate of growth of X_t is typically extremely fast compared with that of $f^{-1}(t)$.*

Definition 3.4.8 (Case (ii))**.** *We impose on our subordinator that the drift is zero, and the tail function $\overline{\Pi}(x) = \Pi(x, \infty)$ is CRV at ∞, with lower index $\beta(\overline{\Pi}) > -1$.*

We impose that $f(0) \in (0,1)$, f is increasing, and that there exists $\varepsilon > 0$ such that for $g := f^{-1}$,

$$\lim_{t\to\infty} t^{1+\varepsilon}\overline{\Pi}\left(g(t)\right) = 0. \tag{3.5}$$

Remark 3.4.9. *In Definition 3.4.8, we impose $\beta(\overline{\Pi}) > -1$, which is equivalent to imposing that the function $\int_0^x \overline{\Pi}(y)dy$ has positive increase as $x \to \infty$ (see [21, Section 2.1] for a definition). This has many equivalent formulations [14, Ex. III.7], [21, Section 2.1], and appears naturally in a range of contexts [5, p2], [14, p87].*

Remark 3.4.10. *Considering $\overline{\Pi}(x) = x^{-\alpha}\log(x)^{\delta}$, $\delta > 0$, with $f(t) = 1/2 + t^{1/(\alpha+\tau)}$, $\tau > 0$, one can verify that this class of examples is included in case (ii).*

Now let us introduce some notation required to formulate our results. Recall that $f : [0,\infty) \to (0,\infty)$ is increasing, $f(0) \in (0,1)$, and $g := f^{-1}$ is the inverse of f, where we take $g(x) = 0$ for $x \in [0, f(0))$. The event $\mathcal{O}_u$ corresponds to bounding the inverse local time until time u (or equivalently, bounding the local time until time $g(u)$). We will study the asymptotics of $\mathbb{P}(\mathcal{O}_u)$ as $u \to \infty$, and those of the integral $\Phi(s)$ of this probability.

$$\mathcal{O}_u := \{X_s \geq g(s), \forall\ 0 \leq s \leq u\}, \tag{3.6}$$

$$\Phi(s) := \int_0^s \mathbb{P}(\mathcal{O}_u)du. \tag{3.7}$$

We shall also study the event $\mathcal{O}_u$ for the process $X^{(0,a)}$ with truncated Lévy measure $\Pi(dx)\mathbb{1}_{\{x\in(0,a)\}}$,

$$\mathcal{O}_{u,X^{(0,a)}} := \left\{X_s^{(0,a)} \geq g(s), \forall\ 0 \leq s \leq u\right\}. \tag{3.8}$$

The time of our subordinator's first jump of size larger than $x > 0$, or in the interval (a,b) for $b > a > 0$, are respectively denoted by

$$\Delta_1^x := \inf\{t \geq 0 : X_t - X_{t-} > x\}, \tag{3.9}$$

$$\Delta_1^{(a,b)} := \inf\{t \geq 0 : X_t - X_{t-} \in (a,b)\}. \tag{3.10}$$

In Theorem 3.4.16 and Proposition 3.4.19, we determine that $I(f) < \infty$ is a necessary and sufficient condition for transience of the conditioned process (when such a process exists), where

$$I(f) := \int_1^\infty f(x)\Pi(dx). \tag{3.11}$$

Remark 3.4.11. *The necessary and sufficient condition $I(f) < \infty$ arises naturally in a number of contexts, including rate of growth of subordinators [14, Theorem III.13] and spectrally negative Lévy processes [95, Theorem 3].*

For $h < t$, by the stationary independent increments property, the conditional event $\{X_t > g(t) | X_h = y\}$ is the same as $\{X_{t-h} > g(t) - y\} = \{X_{t-h} > g((t-h)+h) - y\}$, in the sense that they have the same probability. This new boundary for X to stay above is given by $g_y^h(\cdot)$, with $\mathcal{O}_u^{g_y^h}$, $\Phi_y^h(s)$ corresponding to $\mathcal{O}_u$, $\Phi(s)$.

$$g_y^h(t) := g(t+h) - y, \tag{3.12}$$

$$\mathcal{O}_u^{g_y^h} := \{X_s \geq g_y^h(s), \forall\, 0 \leq s \leq u\}, \tag{3.13}$$

$$\Phi_y^h(s) := \int_0^s \mathbb{P}(\mathcal{O}_u^{g_y^h}) du. \tag{3.14}$$

The functions $\rho(\cdot)$, $\rho_y^h(\cdot)$ are error terms in the upcoming equations (3.18) and (3.19).

$$\rho(t) := \frac{\mathbb{P}\left(\mathcal{O}_t\right)}{\Phi(t)} - \overline{\Pi}(g(t)), \tag{3.15}$$

$$\rho_y^h(t) := \frac{\mathbb{P}(\mathcal{O}_t^{g_y^h})}{\Phi_y^h(t)} - \overline{\Pi}(g_y^h(t)). \tag{3.16}$$

The law of our conditioned process will be found by taking limits. Recall the notation (3.6) and (3.13). For the measure $\mathbb{Q}(\cdot) := \lim_{t\to\infty} \mathbb{P}(\cdot|\mathcal{O}_t)$, for all $\mathcal{B}_h \subseteq \mathcal{O}_h, \mathcal{B}_h \in \mathcal{F}_h$, where $(\mathcal{F}_u)_{u\geq 0}$ is the natural filtration of X,

$$\begin{aligned}
\mathbb{Q}(X_h \in dy; \mathcal{B}_h) := \lim_{t\to\infty} \mathbb{P}\left(X_h \in dy; \mathcal{B}_h | \mathcal{O}_t\right) &= \lim_{t\to\infty} \mathbb{P}\left(X_h \in dy; \mathcal{B}_h; \mathcal{O}_h | \mathcal{O}_t\right) \\
&= \lim_{t\to\infty} \frac{\mathbb{P}\left(\mathcal{O}_t | X_h \in dy; \mathcal{B}_h; \mathcal{O}_h\right) \mathbb{P}\left(X_h \in dy; \mathcal{B}_h; \mathcal{O}_h\right)}{\mathbb{P}(\mathcal{O}_t)} \\
&= \mathbb{P}\left(X_h \in dy; \mathcal{B}_h\right) \lim_{t\to\infty} \frac{\mathbb{P}(\mathcal{O}_{t-h}^{g_y^h})}{\mathbb{P}(\mathcal{O}_t)}.
\end{aligned} \tag{3.17}$$

To understand the behaviour of X under $\mathbb{Q}$, we study the probabilities $\mathbb{P}(\mathcal{O}_t^{g_y^h})$ and $\mathbb{P}(\mathcal{O}_t)$ as $t \to \infty$. Lemma 3.5.1, proven in Section 3.6, relates the asymptotics of $\mathbb{P}(\mathcal{O}_t)$ to $\Phi(t)$, and $\mathbb{P}(\mathcal{O}_t^{g_y^h})$ to $\Phi_y^h(t)$, and as a result, by (3.3), (3.5) respectively in cases (i), (ii), $\lim_{t\to\infty} \mathbb{P}(\mathcal{O}_t) = \lim_{t\to\infty} \mathbb{P}(\mathcal{O}_t^{g_y^h}) = 0$. We obtain the equations

$$\mathbb{P}(\mathcal{O}_t) = \frac{d}{dt}\Phi(t) = \left(\overline{\Pi}(g(t)) + \rho(t)\right)\Phi(t), \tag{3.18}$$

$$\mathbb{P}(\mathcal{O}_t^{g_y^h}) = \frac{d}{dt}\Phi_y^h(t) = \left(\overline{\Pi}(g_y^h(t)) + \rho_y^h(t)\right)\Phi_y^h(t). \tag{3.19}$$

We can determine $\Phi(t), \Phi_y^h(t)$ from these equations, yielding (for $B > 0$ as in Definition 3.4.1) the expressions

$$\Phi(t) = \Phi(1)\exp\left(\int_1^t \overline{\Pi}(g(s))ds + \int_1^t \rho(s)ds\right), \tag{3.20}$$

$$\Phi_y^h(t) = \Phi_y^h(t_0(y))\exp\left(\int_{t_0(y)}^t \overline{\Pi}(g_y^h(s))ds + \int_{t_0(y)}^t \rho_y^h(s)ds\right), \tag{3.21}$$

$$t_0(y) := f(Ay) \vee f(1+2/A), \qquad A > 3 \vee (B-1). \tag{3.22}$$

The error terms $\rho(\cdot)$ and $\rho_y^h(\cdot)$ are later shown to be integrable (the latter uniformly in y and h), which is key for determining the distribution of our conditioned process. The required bound for $\rho(\cdot)$ is given in Remark 3.5.4, and we provide a uniform bound for $\rho_y^h(\cdot)$ in Lemma 3.5.3, proven in Section 3.7.

3.4.2 Results in the $I(f) < \infty$ Case

We shall see in Theorem 3.4.16 that when $I(f) < \infty$, any possible weak limit of our conditioned process M is transient. Theorem 3.4.12 explicitly finds the distribution of the process X in this case.

Theorem 3.4.12. *With assumptions on f and $\overline{\Pi}$ in case (i) or case (ii), if $I(f) < \infty$, then the measure $\mathbb{Q}(\cdot) := \lim_{t\to\infty} \mathbb{P}(\cdot|\mathcal{O}_t)$ exists for the space $\mathcal{D}[0,\infty)$, to which X belongs, of càdlàg paths on $[0,\infty)$, in the sense that for all $h > 0$, $y > g(h)$, for all $\mathcal{B}_h \subseteq \mathcal{O}_h, \mathcal{B}_h \in \mathcal{F}_h$, where $(\mathcal{F}_u)_{u\geq 0}$ is the natural filtration of X,*

$$\mathbb{Q}(X_h \in dy; \mathcal{B}_h) = \frac{\Phi_y^h(\infty)}{\Phi(\infty)}\mathbb{P}\left(X_h \in dy; \mathcal{B}_h\right),$$

where $\Phi(\infty) < \infty$, $\Phi_y^h(\infty) < \infty$. Define, independently of M or X, the random variable $\mathfrak{C}$ by

$$\mathbb{P}\left(\mathfrak{C} \in ds\right) := \frac{\mathbb{P}(\mathcal{O}_s)}{\Phi(\infty)}ds, \quad s \geq 0, \tag{3.23}$$

which exists since $\Phi(\infty) < \infty$. Then for all $h \geq 0$, $\mathbb{Q}(X_h < \infty) = \mathbb{P}(\mathfrak{C} > h)$.

Remark 3.4.13. *Since $\mathbb{Q}(X_h < \infty) = \mathbb{P}(\mathfrak{C} > h)$ for all $h > 0$, the process X under $\mathbb{Q}(\cdot)$ is finite until a random time, which we denote by T_∞, and which has the same distribution under $\mathbb{Q}(\cdot)$ as $\mathfrak{C}$ under $\mathbb{P}(\cdot)$. In particular, $\mathbb{Q}(T_\infty \in ds) = \mathbb{P}(\mathfrak{C} \in ds)$ for all $s \geq 0$. In Theorem 3.4.16, we will show that under any possible* *weak limit measure, the process M never returns to 0 after time $X_{T_\infty-} = \lim_{s\uparrow T_\infty} X_s$.*

Proposition 3.4.14. *In cases (i) and (ii), $((X_u)_{\Delta_1^{g(t)}>u\geq 0}, \Delta_1^{g(t)}), t \geq 0$ under $\mathbb{P}(\cdot|\mathcal{O}_t)$ converges as $t \to \infty$, in the sense that there exists a unique limit measure $\mathbb{Q}'(\cdot)$, on the space $\mathcal{D}[0,\infty) \times (0,\infty)$, where $\mathcal{D}[0,\infty)$ is the space of càdlàg paths on $[0,\infty)$, such that for all $y > g(x)$, $t > b > a > x > 0$, with a, b, x, y fixed, and for all events $\mathcal{B}_X \in \mathcal{F}_x$, where $(\mathcal{F}_u)_{u\geq 0}$ is the natural filtration of X, such that $\mathcal{B}_X \subseteq \mathcal{O}_x$, for $\mathfrak{C}$ as in (3.23),*

$$\lim_{t\to\infty} \mathbb{P}(X_x \in dy; \mathcal{B}_X; \Delta_1^{g(t)} \in (a,b)|\mathcal{O}_t) = \int_a^b \mathbb{P}(X_x \in dy; \mathcal{B}_X|\mathcal{O}_s)\mathbb{P}(\mathfrak{C} \in ds) =: \mathbb{Q}'\left(X_x \in dy; \mathcal{B}_X; T'_\infty \in (a,b)\right), \tag{3.24}$$

for the explosion time T'_∞ defined under $\mathbb{Q}'(\cdot)$ as T_∞ is defined under $\mathbb{Q}(\cdot)$ in Remark 3.4.13. The projection of $\mathbb{Q}'(\cdot)$ onto $(0,\infty)$ agrees with $\mathbb{Q}(\cdot)$ in the sense that $\mathbb{Q}'(T'_\infty \in ds) = \mathbb{Q}(T_\infty \in ds) = \mathbb{P}(\mathfrak{C} \in ds)$.

We shall now determine the behaviour of the conditioned process M until a time corresponding to the point at which X becomes infinite. Theorem 3.4.16 and Remark 3.4.17 consider the behaviour after this time. Proposition 3.4.15 requires some understanding of excursion theory of Markov processes. For background on excursion theory, we direct the reader to [14, Chapter IV].

Proposition 3.4.15. *In cases (i) and (ii), if $I(f) < \infty$ then there exists a measure $\mathbb{Q}''(\cdot)$ on the product space of the space containing the excursion process with $\mathcal{D}[0,\infty)\times(0,\infty)$, where $\mathcal{D}[0,\infty)$ is the space of càdlàg paths on $[0,\infty)$, such that for all fixed $b > a > h > 0$, and for $\mathcal{B} \subseteq \mathcal{O}_h$, $\mathcal{B} \in \mathcal{F}_h$, where $\mathcal{F}$ denotes the natural filtration of X, with F_1 a bounded continuous functional on the excursion process $(\varepsilon_s)_{s\geq 0}$ of M, defining the operator $\pi_h((Z_u)_{u\geq 0}) := (Z_u)_{h\geq u\geq 0}$, let F_1 satisfy $F_1((\varepsilon_s)_{s\geq 0}) = F_1(\pi_h((\varepsilon_s)_{s\geq 0})))$ (F_1 depends only on the excursion process of M up to time h, or equivalently, the excursions in the first h units of time),*

$$\begin{aligned} &\lim_{t\to\infty} \mathbb{E}\left[\ F_1(\pi_h((\varepsilon_s)_{s\geq 0}))\ \mathbb{1}_{\{\pi_h(X)\in\mathcal{B}\}}\mathbb{1}_{\{\Delta_1^{g(t)}\in(a,b)\}}\ \middle|\ \mathcal{O}_t\ \right] \\ &= \int_{\nu\in\mathcal{B}}\int_{u\in(a,b)} \mathbb{E}\left[F_1(\pi_h((\varepsilon_s)_{s\geq 0}))\middle|\pi_h(X) = \nu\right]\mathbb{Q}'\left(\pi_h(X) \in d\nu; T'_\infty \in du\right) \\ &=: \mathbb{E}_{\mathbb{Q}''}\left[F_1(\pi_h((\varepsilon_s)_{s\geq 0}))\ \mathbb{1}_{\{\pi_h(X)\in\mathcal{B}\}}\ \mathbb{1}_{\{T''_\infty\in(a,b)\}}\right], \end{aligned} \tag{3.25}$$

where T''_∞ is the explosion time under the measure $\mathbb{Q}''(\cdot)$, and the projection of $\mathbb{Q}''(\cdot)$ onto $\mathcal{D}[0,\infty) \times (0,\infty)$ agrees with $\mathbb{Q}'(\cdot)$. In particular, with $\Delta := \Delta_1^{g(t)}$, $((M_t)_{X_{\Delta-}>t\geq 0}, (X_s)_{\Delta>s\geq 0}, \Delta)$ under $\mathbb{P}(\cdot|\mathcal{O}_t)$ converges *weakly as $t \to \infty$ to $((M_t)_{X_{T''_\infty -} > t \geq 0}, (X_s)_{\infty > s \geq 0}, T''_\infty)$ under $\mathbb{Q}''(\cdot)$. The behaviour of M under $\mathbb{Q}''(\cdot)$ before time $X_{T''_\infty -}$ has the same distribution as the following construction, expressed in terms of the original measure $\mathbb{P}$ as follows: sampling the random time $\mathfrak{C} = s$ under $\mathbb{P}(\cdot)$, we run X conditioned on $\mathcal{O}_s$ until time s, take $X_u = \infty$ for all $u \geq s$, then construct M via its excursions using $(X_u)_{\infty > u \geq 0}$ to determine the timing and length of each excursion, where we sample each excursion of M until time X_{s-} using the excursion measure conditional on the given excursion length.*

Theorem 3.4.16. *In cases (i) and (ii), if $I(f) < \infty$ then M is transient under any possible weak limit of the measure $\mathbb{P}(\cdot|\mathcal{O}_t)$ as $t \to \infty$.*

Remark 3.4.17. *While the last excursion of the Markov process M is not dealt with explicitly here, the behaviour of M from time $X_{\mathfrak{C}-}$ onwards should be the same as that of M conditioned to avoid zero. Proving this requires existence of the limit as $g(t) \to \infty$ of the excursion measure conditioned on the length (lifetime) of an excursion being longer than $g(t)$, which is beyond the scope of this work. This is verified in the simple, single case where M is a 1-dimensional Brownian motion in [78, p8]. When M is a Lévy process, the behaviour of the process conditioned to avoid zero is well understood, see [97, Theorem 8]. There is some technical difficulty in applying results from [97] to our final excursion. The measures $\mathbb{Q}, \mathbb{Q}', \mathbb{Q}''$ are constructed by conditioning until a deterministic time $t \to \infty$, but in [97], the measure is constructed by conditioning until an independent exponential random time with parameter $q \to 0$. Equivalence of such deterministic and random limits is a separate matter, beyond the scope of the work in this chapter.*

3.4.3 Results in the $I(f) = \infty$ Case

We now restrict our attention to case (i). We will see that when $I(f) = \infty$, our conditioned Markov process is recurrent. Theorem 3.4.18 determines the distribution of the conditioned inverse local time subordinator in this case.

Theorem 3.4.18. *In case (i), if $I(f) = \infty$, then the law $\mathbb{Q}(\cdot) = \lim_{t\to\infty} \mathbb{P}(\cdot|\mathcal{O}_t)$ exists for the process X in the sense that for all $h > 0$ and $y \geq g(h)$, for all $h > 0$, $y > g(h)$, for all $\mathcal{B}_h \subseteq \mathcal{O}_h, \mathcal{B}_h \in \mathcal{F}_h$, where $(\mathcal{F}_u)_{u\geq 0}$*

is the natural filtration of X*,*

$$\mathbb{Q}\left(X_h \in dy; \mathcal{B}_h\right) = \lim_{t\to\infty} \frac{\mathbb{P}(\mathcal{O}^{g_y^h}_{t-h})}{\mathbb{P}(\mathcal{O}_t)} \mathbb{P}\left(X_h \in dy; \mathcal{B}_h\right)$$
$$= q_h(y)\mathbb{P}\left(X_h \in dy; \mathcal{B}_h\right), \tag{3.26}$$

where $q_h(y)$ *is finite, non-decreasing in* y*, and satisfies*

$$q_h(y) = \frac{\Phi^h_y(t_0(y))}{\Phi(1)} \lim_{t\to\infty} \exp\left(\int_{t_0(y)}^t \left(\overline{\Pi}(g^h_y(s)) + \rho^h_y(s)\right) ds - \int_1^t \left(\overline{\Pi}(g(s)) + \rho(s)\right) ds\right),$$

$$t_0(y) := f(Ay) \vee f(1+2/A), \qquad A > 3 \vee (B-1),$$

for $B > 0$ *as in Definition 3.4.1.*

We now verify that when $I(f) = \infty$, M is recurrent under the new measure $\mathbb{Q}''(\cdot)$, as X never hits infinity at a finite time, $\mathbb{Q}$-almost surely, and then M under $\mathbb{Q}''(\cdot)$ is constructed from its excursion process and X.

Proposition 3.4.19. *In case (i), if* $I(f) = \infty$*, then for each* $h > 0$*,* $\mathbb{Q}\left(X_h \in (g(h), \infty)\right) = 1$*.*

Proposition 3.4.20. *In case (i), if* $I(f) = \infty$*, then there exists a measure* $\mathbb{Q}''(\cdot)$ *on the product space of the space containing the excursion process with the space* $\mathcal{D}[0,\infty)$ *of càdlàg paths on* $[0,\infty)$*, such that for all fixed* $h > 0$*, and for* $\mathcal{B} \subseteq \mathcal{O}_h$*,* $\mathcal{B} \in \mathcal{F}_h$*, where* $\mathcal{F}$ *denotes the natural filtration of* X*, let* F_1 *be a bounded continuous functional on the excursion process* $(\varepsilon_s)_{s\geq 0}$ *of* M*, defining the operator* $\pi_h((Z_u)_{u\geq 0}) := (Z_u)_{h\geq u\geq 0}$*, with* F_1 *such that* $F_1((\varepsilon_s)_{s\geq 0}) = F_1(\pi_h((\varepsilon_s)_{s\geq 0}))$*,*

$$\lim_{t\to\infty} \mathbb{E}\left[\ F_1(\pi_h((\varepsilon_s)_{s\geq 0}))\ \mathbb{1}_{\{\pi_h(X)\in\mathcal{B}\}}\ \middle|\ \mathcal{O}_t\ \right] \tag{3.27}$$
$$= \int_{\nu\in\mathcal{B}} \mathbb{E}\left[F_1(\pi_h((\varepsilon_s)_{s\geq 0}))\middle|\pi_h(X) = \nu\right] \mathbb{Q}\left(\pi_h(X) \in d\nu\right)$$
$$=: \mathbb{E}_{\mathbb{Q}''}\left[F_1(\pi_h((\varepsilon_s)_{s\geq 0}))\ \mathbb{1}_{\{\pi_h(X)\in\mathcal{B}\}}\right],$$

where the projection of $\mathbb{Q}''(\cdot)$ *onto* $\mathcal{D}[0,\infty)$ *agrees with* $\mathbb{Q}(\cdot) = \lim_{t\to\infty}\mathbb{P}(\cdot|\mathcal{O}_t)$*. In particular, as* $t \to \infty$*,* $((M_t)_{\infty>t\geq 0}, (X_s)_{\infty>s\geq 0})$ *under* $\mathbb{P}(\cdot|\mathcal{O}_t)$ *converges weakly to* $((M_t)_{\infty>t\geq 0}, (X_s)_{\infty>s\geq 0})$ *under* $\mathbb{Q}''(\cdot)$*. We con-*

struct M via its excursions using $(X_u)_{\infty>u\geq 0}$ to determine the timing and length of each excursion, where we sample the excursions of M using the excursion measure conditional on each excursion length. Moreover, M visits 0 at arbitrarily large times, so M is recurrent under $\mathbb{Q}''(\cdot)$.

Now we shall determine the entropic repulsion envelope through Theorem 3.4.22.

Definition 3.4.21. *A non-decreasing function w, with $\lim_{h\to\infty} w(h) = \infty$, is in the entropic repulsion envelope R_g (for the function $g = f^{-1}$) if*

$$\lim_{h\to\infty} \mathbb{Q}'' \left(X_h \geq w(h)g(h)\right) = 1. \tag{3.28}$$

Theorem 3.4.22. *In case (ia), a necessary and sufficient condition for a non-decreasing function w, for which $\lim_{h\to\infty} w(h) = \infty$, to be in R_g, for $g = f^{-1}$ is*

$$w \in R_g \iff \lim_{h\to\infty} \int_h^{f(w(h)g(h))} \overline{\Pi}(g(s))ds = 0.$$

We illustrate the generality of Theorem 3.4.22 via the following corollary, expanding upon [78, Theorem 4].

Corollary 3.4.23. *In case (i), with f, f' $\mathcal{O}$-regularly varying at ∞, for a stable subordinator of index $\alpha \in (0,1)$, a necessary and sufficient condition for non-decreasing w, with $\lim_{h\to\infty} w(h) = \infty$, to be in R_g, for $g = f^{-1}$ is*

$$w \in R_g \iff \lim_{h\to\infty} \int_h^{f(w(h)g(h))} g(s)^{-\alpha} ds = 0.$$

Consider a stable subordinator of index α, with $f(t) = t^\alpha \log(t)^{-\gamma}$, and so $g(t) \sim t^{1/\alpha}\log(t)^{\gamma/\alpha}$ as $t \to \infty$, where $\gamma \in (\alpha\beta, 1)$, for β as in (3.3). One can verify that this scenario is included in case (ia), and the entropic repulsion envelope contains the function $w(t) := e^{\ln(t)^\gamma}$, see [78, Remark 5] for further details.

3.5 Proofs of Main Results

This section contains the proofs of the results stated in Section 3.4. First we state Lemmas 3.5.1 and 3.5.3, which are proven in Sections 3.6 and 3.7, respectively.

Lemma 3.5.1. *In cases (i) and (ii), as $t \to \infty$,*

$$\mathbb{P}\left(\mathcal{O}_t\right) = \left(\overline{\Pi}(g(t)) + \rho(t)\right)\Phi(t) = (1 + o(1))\ \overline{\Pi}(g(t))\Phi(t). \tag{3.29}$$

Definition 3.5.2. *In this chapter we use the following asymptotic notation:*

$f(x) \sim g(x)$ as $x \to \infty$ if $\lim_{x\to\infty} f(x)/g(x) = 1$.

$f(x) \lesssim g(x)$ if there exists $C \in (0, \infty)$ such that for all large enough x, $f(x) \leq Cg(x)$.

Moreover, we write $f(x) \gtrsim g(x)$ if $g(x) \lesssim f(x)$, and

$f(x) \asymp g(x)$ if both $f(x) \gtrsim g(x)$ and $f(x) \lesssim g(x)$.

Lemma 3.5.3. *In cases (i) and (ii), there exists a function $u(t)$ with $\lim_{t\to\infty} u(t) = 0$, and there exists $\varepsilon > 0$ such that uniformly for all $A > 3$, $h > 0$, $y > g(h)$, and $t > t_0(y)$, and $t > t_0(y)$ as defined in* (3.22)*, we have*

$$\rho_y^h(t) \lesssim \frac{1}{t\log(t)^{1+\varepsilon}}\left(1 + \frac{1}{f(y) - h}\right), \tag{3.30}$$

$$\rho_y^h(t) \leq u(t)\overline{\Pi}(g(t))\left(1 + \frac{1}{f(y) - h}\right). \tag{3.31}$$

Remark 3.5.4. *The inequality* (3.30) *also holds when $y = h = 0$ (recall that we have imposed $f(0) > 0$). The proof of this fact is omitted, as computations are much simpler without dependence on y, h. This implies $\int_{t_0}^{\infty} \rho(s)ds < \infty$, so by* (3.20)*, uniformly as $t \to \infty$,*

$$\Phi(t) = \Phi(1)\exp\left(\int_1^t (\overline{\Pi}(g(s)) + \rho(s))ds\right) \asymp \exp\left(\int_1^t \overline{\Pi}(g(s))ds\right). \tag{3.32}$$

Lemma 3.5.5. *In case (i), for the function ρ as defined in* (3.15)*, $\liminf_{t\to\infty} \rho(t) \geq 0$.*

3.5.1 Proofs in the $I(f) < \infty$ Case

Proof of Theorem 3.4.12

Proof of Theorem 3.4.12. First, let us verify that $\Phi(\infty) < \infty$. Recalling that $g := f^{-1}$, we have

$$I(f) = \int_1^\infty f(x)\Pi(dx) = \int_1^\infty \int_0^{f(x)} dy\Pi(dx) = \int_0^\infty \int_{1\vee g(y)}^\infty \Pi(dx)dy = \int_0^\infty \overline{\Pi}(1 \vee g(y))dy. \tag{3.33}$$

Now, recall from (3.20) that

$$\Phi(\infty) = \Phi(1)\exp\left(\int_1^\infty \overline{\Pi}(g(s))ds + \int_1^\infty \rho(s)ds\right).$$

By Lemma 3.5.1, as $s \to \infty$, $\rho(s) = o(\overline{\Pi}(g(s)))$. Then by (3.33), since $I(f) < \infty$,

$$\int_1^\infty \overline{\Pi}(g(s))ds + \int_1^\infty \rho(s)ds \overset{3.5.1}{\lesssim} \int_1^\infty \overline{\Pi}(g(s))ds \overset{(3.33)}{<} \infty,$$

so $\Phi(\infty) < \infty$. Now, it follows by Lemma 3.5.5 that $\int_1^\infty \rho(s)ds > -\infty$, and hence

$$I(f) < \infty \iff \Phi(\infty) < \infty. \tag{3.34}$$

To show $\Phi_y^h(\infty) < \infty$, with $t_0(y)$ as defined in (3.22), recall that by (3.21),

$$\Phi_y^h(\infty) = \Phi_y^h(t_0(y))\exp\left(\int_{t_0(y)}^\infty \overline{\Pi}(g_y^h(s))ds + \int_{t_0(y)}^\infty \rho_y^h(s)ds\right).$$

Now, observe that for each fixed $y, h > 0$, $g(s) \sim g_y^h(s)$ as $s \to \infty$, by (3.12) and the properties of g introduced in Definition 3.4.1, so $\overline{\Pi}(g(s)) \sim \overline{\Pi}(g_y^h(s))$ as $s \to \infty$, since $\overline{\Pi}$ is CRV at ∞. Now, applying (3.31) and (3.33), since y and h are fixed and $y > g(h)$ implies $f(y) - h > 0$, noting that $s > t_0(y)$ ensures $g_y^h(s) > 0$, we get

$$\int_{t_0(y)}^\infty \overline{\Pi}(g_y^h(s))ds + \int_{t_0(y)}^\infty \rho_y^h(s)ds \overset{(3.31)}{\lesssim} \left(1 + \frac{1}{f(y) - h}\right)\int_{t_0(y)}^\infty \overline{\Pi}(g(s))ds \overset{(3.33)}{<} \infty,$$

and hence $\Phi_y^h(\infty) < \infty$. By (3.17) and Lemma 3.5.1, since $\overline{\Pi}(g(t)) \sim \overline{\Pi}(g_y^h(t-h))$ as $t \to \infty$,

$$\begin{aligned}\mathbb{Q}(X_h \in dy; \mathcal{B}_h) &\overset{(3.17)}{=} \mathbb{P}(X_h \in dy; \mathcal{B}_h) \lim_{t\to\infty} \frac{\mathbb{P}(\mathcal{O}_{t-h}^{g_y^h})}{\mathbb{P}(\mathcal{O}_t)} \\ &\overset{3.5.1}{=} \mathbb{P}(X_h \in dy; \mathcal{B}_h) \lim_{t\to\infty} \frac{\overline{\Pi}(g_y^h(t-h))\Phi_y^h(t-h)}{\overline{\Pi}(g(t))\Phi(t)} = \mathbb{P}(X_h \in dy; \mathcal{B}_h) \frac{\Phi_y^h(\infty)}{\Phi(\infty)}. \end{aligned} \tag{3.35}$$

Now we show that $\mathbb{Q}(X_h < \infty) = \mathbb{P}(\mathfrak{C} > h)$. By (3.35),

$$\begin{aligned}\mathbb{Q}(X_h < \infty) &= \int_{g(h)}^{\infty} \mathbb{Q}(X_h \in dy) = \int_{g(h)}^{\infty} \frac{\Phi_y^h(\infty)}{\Phi(\infty)} \mathbb{P}(X_h \in dy; \mathcal{O}_h) \\ &= \frac{1}{\Phi(\infty)} \int_{g(h)}^{\infty} \int_0^{\infty} \mathbb{P}\left(\mathcal{O}_v^{g_y^h}\right) dv \mathbb{P}(X_h \in dy; \mathcal{O}_h) \\ &= \frac{1}{\Phi(\infty)} \int_0^{\infty} \int_{g(h)}^{\infty} \mathbb{P}\left(\mathcal{O}_v^{g_y^h}\right) \mathbb{P}(X_h \in dy; \mathcal{O}_h)\, dv.\end{aligned}$$

Now, $\mathbb{P}(\mathcal{O}_v^{g_y^h})\mathbb{P}(X_h \in dy; \mathcal{O}_h) = \mathbb{P}(\mathcal{O}_{v+h}; X_h \in dy)$ by (3.13). Then by the definition (3.6) of $\mathcal{O}_{v+h}$,

$$\begin{aligned}\mathbb{Q}(X_h < \infty) &= \frac{1}{\Phi(\infty)} \int_0^{\infty} \int_{g(h)}^{\infty} \mathbb{P}(\mathcal{O}_{v+h}; X_h \in dy)\, dv = \frac{1}{\Phi(\infty)} \int_0^{\infty} \mathbb{P}(\mathcal{O}_{v+h}; X_h > g(h))\, dv \\ &\overset{(3.6)}{=} \frac{1}{\Phi(\infty)} \int_0^{\infty} \mathbb{P}(\mathcal{O}_{v+h})\, dv = \frac{1}{\Phi(\infty)} \int_h^{\infty} \mathbb{P}(\mathcal{O}_u)\, du =: \mathbb{P}(\mathfrak{C} > h).\end{aligned}$$

□

Proof of Proposition 3.4.14

Proof of Proposition 3.4.14. For $y > g(x)$, $t > b > a > x > 0$, with a, b, x, y fixed, and an event $\mathcal{B}_X \in \mathcal{F}_x$, where $(\mathcal{F}_u)_{u\geq 0}$ is the natural filtration of X, such that $\mathcal{B}_X \subseteq \mathcal{O}_x$, consider

$$\lim_{t\to\infty} \mathbb{P}\left(X_x \in dy; \mathcal{B}_X; \Delta_1^{g(t)} \in (a,b) | \mathcal{O}_t\right). \tag{3.36}$$

If $\Delta_1^{g(t)} \in ds$, then $X_s > g(t)$, so $\mathcal{O}_t$ is fully attained by time s, and thus $\mathcal{O}_t$ can be replaced by $\mathcal{O}_s$, yielding

$$\begin{aligned}(3.36) &= \lim_{t\to\infty} \frac{1}{\mathbb{P}(\mathcal{O}_t)} \int_a^b \mathbb{P}\left(X_x \in dy; \mathcal{B}_X; \Delta_1^{g(t)} \in ds; \mathcal{O}_t\right) \\ &= \lim_{t\to\infty} \frac{1}{\mathbb{P}(\mathcal{O}_t)} \int_a^b \mathbb{P}\left(X_x \in dy; \mathcal{B}_X; \Delta_1^{g(t)} \in ds; \mathcal{O}_s\right).\end{aligned}$$

Recall the definition (3.8). Given $\Delta_1^{g(t)} > a > x$, we can replace $\{X_x \in dy\}, \mathcal{B}_X$, $\mathcal{O}_s$ by corresponding events $\{X_x^{(0,g(t))} \in dy\}, \mathcal{B}_{X^{(0,g(t))}}$, $\mathcal{O}_{s,X^{(0,g(t))}}$ for the process $X^{(0,g(t))}$ with Lévy measure restricted to $(0, g(t))$, i.e. all jumps larger than $g(t)$ are removed. These events are each independent of $\Delta_1^{g(t)}$, and since $\Delta_1^{g(t)}$ is exponentially distributed with parameter $\overline{\Pi}(g(t))$, we get

$$\begin{aligned}(3.36) &= \lim_{t\to\infty} \frac{1}{\mathbb{P}(\mathcal{O}_t)} \int_a^b \mathbb{P}\left(X_x^{(0,g(t))} \in dy; \mathcal{B}_{X^{(0,g(t))}}; \Delta_1^{g(t)} \in ds; \mathcal{O}_{s,X^{(0,g(t))}}\right) \\ &= \lim_{t\to\infty} \frac{1}{\mathbb{P}(\mathcal{O}_t)} \int_a^b \mathbb{P}\left(X_x^{(0,g(t))} \in dy; \mathcal{B}_{X^{(0,g(t))}}; \mathcal{O}_{s,X^{(0,g(t))}}\right) \mathbb{P}\left(\Delta_1^{g(t)} \in ds\right) \\ &= \lim_{t\to\infty} \frac{\overline{\Pi}(g(t))}{\mathbb{P}(\mathcal{O}_t)} \int_a^b \mathbb{P}\left(X_x^{(0,g(t))} \in dy; \mathcal{B}_{X^{(0,g(t))}}; \mathcal{O}_{s,X^{(0,g(t))}}\right) e^{-\overline{\Pi}(g(t))s} ds. \qquad (3.37)\end{aligned}$$

Now, since $\lim_{t\to\infty} e^{-\overline{\Pi}(g(t))s} = 1$, uniformly among $s \in (a, b)$,

$$(3.36) = \lim_{t\to\infty} \frac{\overline{\Pi}(g(t))}{\mathbb{P}(\mathcal{O}_t)} \int_a^b \mathbb{P}\left(X_x^{(0,g(t))} \in dy; \mathcal{B}_{X^{(0,g(t))}}; \mathcal{O}_{s,X^{(0,g(t))}}\right) ds.$$

Applying Lemma 3.5.1, and recalling from (3.34) that $\Phi(\infty) < \infty$ when $I(f) < \infty$,

$$\begin{aligned}(3.36) &= \frac{1}{\Phi(\infty)} \lim_{t\to\infty} \int_a^b \mathbb{P}\left(X_x^{(0,g(t))} \in dy; \mathcal{B}_{X^{(0,g(t))}}; \mathcal{O}_{s,X^{(0,g(t))}}\right) ds \qquad (3.38) \\ &= \frac{1}{\Phi(\infty)} \lim_{t\to\infty} \int_a^b \mathbb{P}\left(X_x^{(0,g(t))} \in dy; \mathcal{B}_{X^{(0,g(t))}} \middle| \mathcal{O}_{s,X^{(0,g(t))}}\right) \mathbb{P}\left(\mathcal{O}_{s,X^{(0,g(t))}}\right) ds.\end{aligned}$$

Now, $\lim_{t\to\infty} \mathbb{P}(\mathcal{O}_{s,X^{(0,g(t))}})/\mathbb{P}(\mathcal{O}_s) = 1$, uniformly among $s \in (a, b)$, so

$$(3.36) = \lim_{t\to\infty} \int_a^b \mathbb{P}\left(X_x^{(0,g(t))} \in dy; \mathcal{B}_{X^{(0,g(t))}} \middle| \mathcal{O}_{s,X^{(0,g(t))}}\right) \frac{\mathbb{P}(\mathcal{O}_s)}{\Phi(\infty)} ds,$$

and similarly $\mathbb{P}(X_x^{(0,g(t))} \in dy; \mathcal{B}_{X^{(0,g(t))}} \big| \mathcal{O}_{s,X^{(0,g(t))}}) \sim \mathbb{P}\left(X_x \in dy; \mathcal{B}_X \big| \mathcal{O}_s\right)$ as $t \to \infty$, uniformly among $s \in (a,b)$. Then by the definition of $\mathfrak{C}$ in (3.23),

$$\begin{aligned}(3.36) = \int_a^b \mathbb{P}\left(X_x \in dy; \mathcal{B}_X \;\middle|\; \mathcal{O}_s\right) \frac{\mathbb{P}\left(\mathcal{O}_s\right)}{\Phi(\infty)} ds = \int_a^b \mathbb{P}\left(X_x \in dy; \mathcal{B}_X \;\middle|\; \mathcal{O}_s\right) \mathbb{P}\left(\mathfrak{C} \in ds\right) \\ =: \mathbb{Q}'\left(X_x \in dy; \mathcal{B}_X; T'_\infty \in (a,b)\right).\end{aligned} \tag{3.39}$$

It is clear that (3.39) uniquely determines the limit measure $\mathbb{Q}'(\cdot)$ on $\mathcal{D}[0,\infty) \times (0,\infty)$. To verify that T'_∞ under $\mathbb{Q}'(\cdot)$ has the desired properties, by (3.39) with $\mathcal{B}_X = \mathcal{O}_x$, since $x < a < s$,

$$\mathbb{Q}'(T'_\infty \in (a,b)) = \int_{g(x)}^\infty \mathbb{Q}'\left(X_x \in dy; \mathcal{O}_x; T'_\infty \in (a,b)\right) = \int_{g(x)}^\infty \int_a^b \mathbb{P}\left(X_x \in dy; \mathcal{O}_x \;\middle|\; \mathcal{O}_s\right) \mathbb{P}\left(\mathfrak{C} \in ds\right)$$

$$= \int_a^b \mathbb{P}\left(X_x > g(x); \mathcal{O}_x \;\middle|\; \mathcal{O}_s\right) \mathbb{P}\left(\mathfrak{C} \in ds\right) = \int_a^b \mathbb{P}\left(\mathfrak{C} \in ds\right) = \mathbb{P}(\mathfrak{C} \in (a,b)) = \mathbb{Q}(T_\infty \in (a,b)) = \mathbb{P}(\mathfrak{C} \in (a,b)).$$

Similarly, by (3.39) with $\mathcal{B}_X = \mathcal{O}_x$, taking limits as $a \to x$ and $b \to \infty$, since $x < s$, we also have

$$\mathbb{Q}'(X_x < \infty) = \int_{g(x)}^\infty \int_x^\infty \mathbb{P}(X_x \in dy; \mathcal{O}_x | \mathcal{O}_s) \mathbb{P}(\mathfrak{C} \in ds)$$

$$= \int_x^\infty \mathbb{P}\left(X_x > g(x); \mathcal{O}_x \;\middle|\; \mathcal{O}_s\right) \mathbb{P}\left(\mathfrak{C} \in ds\right) = \int_x^\infty \mathbb{P}\left(\mathfrak{C} \in ds\right) = \mathbb{P}(\mathfrak{C} > x) = \mathbb{Q}'(T'_\infty > x),$$

so that T'_∞ is indeed the explosion time for the process X under $\mathbb{Q}'(\cdot)$. □

Proof of Proposition 3.4.15

Proof of Proposition 3.4.15. Recall $\Delta_1^{g(t)}$ is the time of X's first jump bigger than $g(t)$, $\pi_h(X)$ is the sample path of X up to time h, F_1 is a functional on the excursion process, and $\mathcal{B} \subseteq \mathcal{O}_h$, $\mathcal{B} \in \mathcal{F}_h$, where $(\mathcal{F}_u)_{u \geq 0}$ is X's natural filtration. For fixed $b > a > h > 0$, disintegrating on the values of $\Delta_1^{g(t)}$ and $\pi_h(X)$,

$$\mathbb{E}\left[\mathbb{1}_{\{\pi_h(X) \in \mathcal{B}\}}\ \mathbb{1}_{\{\Delta_1^{g(t)} \in (a,b)\}} F_1(\pi_h((\varepsilon_s)_{s \geq 0})) \;\middle|\; \mathcal{O}_t\right] \tag{3.40}$$

$$= \int_{\nu\in\mathcal{B}} \int_{u\in(a,b)} \mathbb{E}\big[F_1(\pi_h((\varepsilon_s)_{s\geq 0}))\big|\mathcal{O}_t; \pi_h(X)=\nu; \Delta_1^{g(t)}=u\big]\mathbb{P}\big(\pi_h(X)\in d\nu; \Delta_1^{g(t)}\in du|\mathcal{O}_t\big).$$

Given a fixed path $\pi_h(X)=\nu$, $\pi_h((\varepsilon_s)_{s\geq 0})$ depends only on ν, so $\pi_h((\varepsilon_s)_{s\geq 0})$ is conditionally independent of $\Delta_1^{g(t)}$ and $\mathcal{O}_t$. Here, $h<a<u$, so the excursion process $(\varepsilon_s)_{s\geq 0}$ contains only excursions of length at most $g(t)$, so we may replace $\pi_h((\varepsilon_s)_{s\geq 0})$ by $\pi_h((\varepsilon_s^{g(t)})_{s\geq 0})$, where $(\varepsilon_s^{g(t)})_{s\geq 0}$ is the excursion process sampled using the conditional excursion measure on the space of excursions of length at most $g(t)$, so

$$(3.40) = \int_{\mathcal{B}}\int_a^b \mathbb{E}\left[F_1(\pi_h((\varepsilon_s^{g(t)})_{s\geq 0}))\big|\pi_h(X)=\nu\right]\mathbb{P}\left(\pi_h(X)\in d\nu; \Delta_1^{g(t)}\in du|\mathcal{O}_t\right),$$

Now, $\lim_{t\to\infty} g(t)=\infty$, so $\lim_{t\to\infty}\mathbb{E}[F_1(\pi_h((\varepsilon_s^{g(t)})_{s\geq 0}))\mid \pi_h(X)=\nu\ ] = \mathbb{E}[F_1(\pi_h((\varepsilon_s)_{s\geq 0}))\big|\pi_h(X)=\nu]$, and by Proposition 3.4.14, $\lim_{t\to\infty}\mathbb{P}(\pi_h(X)\in d\nu; \Delta_1^{g(t)}\in du|\mathcal{O}_t) = \mathbb{Q}'\left(\pi_h(X)\in d\nu; T'_\infty\in du\right)$, so we conclude

$$\begin{aligned}\lim_{t\to\infty}(3.40) &= \int_{\mathcal{B}}\int_a^b \mathbb{E}\left[F_1(\pi_h((\varepsilon_s)_{s\geq 0}))\big|\pi_h(X)=\nu\right]\mathbb{Q}'\left(\pi_h(X)\in d\nu; T'_\infty\in du\right)\\ &=: \mathbb{E}_{\mathbb{Q}''}\left[F_1(\pi_h((\varepsilon_s)_{s\geq 0}))\ \mathbb{1}_{\{\pi_h(X)\in\mathcal{B}\}}\ \mathbb{1}_{\{T''_\infty\in(a,b)\}}\right],\end{aligned} \tag{3.41}$$

where we are able to exchange the order of limits and integration since F_1 is bounded. Taking $F_1\equiv 1$, it follows immediately that $\mathbb{Q}''(\cdot)$ and $\mathbb{Q}'(\cdot)$ agree on $\mathcal{D}[0,\infty)\times(0,\infty)$. The weak convergence as $t\to\infty$ of $((M_t)_{X_{\Delta-}>t\geq 0}, (X_s)_{\Delta>s\geq 0}, \Delta)$ under $\mathbb{P}(\cdot|\mathcal{O}_t)$ to $((M_t)_{X_{T''_\infty-}>t\geq 0}, (X_s)_{\infty>s\geq 0}, T''_\infty)$ under $\mathbb{Q}''(\cdot)$ then follows immediately from the fact (see e.g. [14, Ex. IV.6.3] or [78, p4113]) that for all $x>0$, $(M_t)_{X_{x-}>t\geq 0}$ is uniquely determined by $(\varepsilon_s)_{x>s\geq 0}$ and $(X_s)_{x>s\geq 0}$, and both of $(\varepsilon_s)_{x>s\geq 0}$ and $(X_s)_{x>s\geq 0}$ have weak limits as determined in (3.41). That is, we construct M pathwise via its excursions using $(X_u)_{\infty>u\geq 0}$ to determine the timing and length of each excursion, where we sample the excursions of M until time X_{s-} using the excursion measure conditional on each excursion length. Similarly, the explicit description of the behaviour of M until time $X_{T''_\infty-}$ under $\mathbb{Q}''(\cdot)$ follows immediately from the definition of $\mathbb{Q}'(\cdot)$ in (3.24), using the fact that $\mathbb{Q}''(\cdot)$ and $\mathbb{Q}'(\cdot)$ agree on $\mathcal{D}[0,\infty)\times(0,\infty)$.

□

Proof of Theorem 3.4.16

Proof of Theorem 3.4.16. As X determines the lengths and timings of excursions of M (see [14, Ex. IV.6.3]), it follows that for all $K > 0$ and $b > a > 0$, for all t large enough that $g(t) > K$,

$$\{\Delta_1^{g(t)} \in (a,b)\} = \{\Delta_1^{g(t)} \in (a,b)\} \cap \{M_v \neq 0, \text{ for all } v \in (X_{\Delta_1^{g(t)}-}, X_{\Delta_1^{g(t)}-} + K)\}. \tag{3.42}$$

Let us assume that a weak limit measure $\hat{\mathbb{Q}}(\cdot) = \lim_{t\to\infty} \mathbb{P}(\cdot|\mathcal{O}_t)$ exists on the space containing $(M_t)_{t\geq 0}$. Such a measure must agree with $\mathbb{Q}'(\cdot)$ on $\mathcal{D}[0,\infty) \times (0,\infty)$, as we proved in Proposition 3.4.14 that any such limit measure is uniquely determined on $\mathcal{D}[0,\infty) \times (0,\infty)$ by (3.24). It follows that for all $K > 0$ and $b > a > 0$,

$$\lim_{t\to\infty} \mathbb{P}(M_v \neq 0, \text{ for all } v \in (X_{\Delta_1^{g(t)}-}, X_{\Delta_1^{g(t)}-} + K); \Delta_1^{g(t)} \in (a,b)|\mathcal{O}_t) \tag{3.43}$$

$$= \hat{\mathbb{Q}}(M_v \neq 0, \text{ for all } v \in (X_{\hat{T}_\infty-}, X_{\hat{T}_\infty-} + K); \hat{T}_\infty \in (a,b)),$$

where $\hat{T}_\infty$ is the explosion time for X under $\hat{\mathbb{Q}}(\cdot)$. But also by (3.42) and uniqueness of the limit measure on $\mathcal{D}[0,\infty) \times (0,\infty)$, we have for all $K > 0$ and $b > a > 0$,

$$\hat{\mathbb{Q}}(M_v \neq 0, \text{ for all } v \in (X_{\hat{T}_\infty-}, X_{\hat{T}_\infty-}+K); \hat{T}_\infty \in (a,b)) = (3.43) = \lim_{t\to\infty} \mathbb{P}(\Delta_1^{g(t)} \in (a,b)|\mathcal{O}_t) = \hat{\mathbb{Q}}(\hat{T}_\infty \in (a,b)),$$

from which it follows immediately that M is transient under $\hat{\mathbb{Q}}(\cdot)$, as required. □

3.5.2 Proofs in the $I(f) = \infty$ Case

The next three proofs require Lemma 3.5.6, proven in Section 3.9.

Lemma 3.5.6. *In case (i), for $t_0(y)$ as defined in* (3.22)*, uniformly in $h > 0, y > g(h)$, and $t \in (t_0(y), \infty]$,*

$$\int_{t_0(y)}^{t} \left(\overline{\Pi}(g(s+h) - y) - \overline{\Pi}(g(s))\right) ds \lesssim y f'(y) \overline{\Pi}(y). \tag{3.44}$$

Proof of Theorem 3.4.18

Proof of Theorem 3.4.18. For each fixed $h > 0$, $y > g(h)$, we will prove $q_h(y) := \lim_{t\to\infty} \mathbb{P}(\mathcal{O}_{t-h}^{g_y^h})/\mathbb{P}(\mathcal{O}_t) < \infty$. For each $h > 0$, $y > g(h)$, note $g(t) \sim g_y^h(t-h)$ by the properties of $g(t)$ given in Definition 3.4.1. Hence $\overline{\Pi}(g(t)) \sim \overline{\Pi}(g_y^h(t-h))$ as $t \to \infty$, since $\overline{\Pi}$ is CRV at ∞. Thus, applying Lemma 3.5.1,

$$\lim_{t\to\infty} \frac{\mathbb{P}\big(\mathcal{O}_{t-h}^{g_y^h}\big)}{\mathbb{P}(\mathcal{O}_t)} = \lim_{t\to\infty} \frac{\overline{\Pi}(g_y^h(t-h))\Phi_y^h(t)}{\overline{\Pi}(g(t))\Phi(t)} = \lim_{t\to\infty} \frac{\Phi_y^h(t)}{\Phi(t)}. \tag{3.45}$$

Then by (3.20) and (3.21), for $t_0(y)$ as defined in (3.22),

$$(3.45) = \frac{\Phi_y^h(t_0(y))}{\Phi(1)} \lim_{t\to\infty} \exp\left(\int_{t_0(y)}^t \left(\overline{\Pi}(g_y^h(s)) + \rho_y^h(s)\right) ds - \int_1^t \left(\overline{\Pi}(g(s)) + \rho(s)\right) ds\right).$$

By (3.30) in Lemma 3.5.3, the integral $\int_{t_0(y)}^\infty \rho_y^h(s)ds$ is uniformly bounded for all $h > 0$, $y > g(h)$. By Lemma 3.5.5, it follows that $-\int_1^\infty \rho(s)ds < \infty$, so

$$\begin{aligned}(3.45) &\lesssim \frac{\Phi_y^h(t_0(y))}{\Phi(1)} \lim_{t\to\infty} \exp\left(\int_{t_0(y)}^t \overline{\Pi}(g_y^h(s))ds - \int_1^t \overline{\Pi}(g(s))ds\right) \\ &\lesssim \frac{\Phi_y^h(t_0(y))}{\Phi(1)} \lim_{t\to\infty} \exp\left(\int_{t_0(y)}^t \left(\overline{\Pi}(g_y^h(s)) - \overline{\Pi}(g(s))\right) ds\right).\end{aligned}$$

Applying Lemma 3.5.6, and recalling that $yf'(y)\overline{\Pi}(y)$ decreases to zero as $y \to \infty$, we get

$$(3.45) \lesssim \frac{\Phi_y^h(t_0(y))}{\Phi(1)} \exp\left(yf'(y)\overline{\Pi}(y)\right) < \infty.$$

Now, $q_h(y) := \lim_{t\to\infty} \mathbb{P}(\mathcal{O}_{t-h}^{g_y^h})/\mathbb{P}(\mathcal{O}_t)$ is non-decreasing in y since for all $y < y'$, $g_y^h(t) = g(t+h) - y > g(t+h) - y' = g_{y'}^h(t)$, and so $\mathbb{P}(\mathcal{O}_{t-h}^{g_y^h}) \leq \mathbb{P}(\mathcal{O}_{t-h}^{g_{y'}^h})$. Finally, we conclude by (3.17) that $\mathbb{Q}\left(X_h \in dy; \mathcal{B}_h\right) = \mathbb{P}\left(X_h \in dy; \mathcal{B}_h\right) q_h(y)$, as required.

□

3.5.3 Proof of Proposition 3.4.19

Proof of Proposition 3.4.19. For $h > 0$, $\lim_{t\to\infty} \mathbb{P}(X_h \in (g(h),\infty)|\mathcal{O}_t) = 1$. We will prove by dominated convergence (Theorem A.3.1) that limits and integration can be exchanged from (3.46) to (3.47), so by (3.17), for each $h > 0$,

$$\begin{aligned} 1 = \lim_{t\to\infty} \mathbb{P}(X_h \in (g(h),\infty)|\mathcal{O}_t) &= \lim_{t\to\infty} \int_{g(h)}^{\infty} \mathbb{P}(X_h \in dy|\mathcal{O}_t) \\ &\overset{(3.17)}{=} \lim_{t\to\infty} \int_{g(h)}^{\infty} \frac{\mathbb{P}(\mathcal{O}_{t-h}^{g_y^h})}{\mathbb{P}(\mathcal{O}_t)} \mathbb{P}(X_h \in dy; \mathcal{O}_h) &(3.46) \\ &= \int_{g(h)}^{\infty} \lim_{t\to\infty} \frac{\mathbb{P}(\mathcal{O}_{t-h}^{g_y^h})}{\mathbb{P}(\mathcal{O}_t)} \mathbb{P}(X_h \in dy; \mathcal{O}_h) &(3.47) \\ &= \mathbb{Q}(X_h \in (g(h),\infty)), &(3.48) \end{aligned}$$

as required. For $A > 3 \vee (B-1)$, we will bound the integral over $(g(h),\infty)$ via:

$$\left[\frac{g(t-h)}{A}, \infty\right) \cup (g(h), g(h+1)] \cup \left(g(h+1), \frac{g(t-h)}{A}\right) =: I_1 \cup I_2 \cup I_3. \tag{3.49}$$

Proof for I_1 Recall that $g = f^{-1}$. Since $y \in I_1$ if and only if $t \le f(Ay) + h$, by Lemma 3.5.1, we get

$$\int_{g(h)}^{\infty} \mathbb{1}_{\{y\in I_1\}} \frac{\mathbb{P}(\mathcal{O}_{t-h}^{g_y^h})}{\mathbb{P}(\mathcal{O}_t)} \mathbb{P}(X_h \in dy; \mathcal{O}_h) \lesssim \int_{g(h)}^{\infty} \frac{\mathbb{1}_{\{y\in I_1\}}}{\overline{\overline{\Pi}}(g(t))\Phi(t)} \mathbb{P}(X_h \in dy) = \frac{\mathbb{P}\left(X_h \ge \frac{g(t-h)}{A}\right)}{\overline{\overline{\Pi}}(g(t))\Phi(t)}.$$

Now, $I(f) = \infty$, so $\lim_{t\to\infty} \Phi(t) = \infty$ by (3.34), and it suffices to show that

$$\limsup_{t\to\infty} \frac{\mathbb{P}\left(X_h \ge \frac{g(t-h)}{A}\right)}{\overline{\overline{\Pi}}(g(t))} \tag{3.50}$$

is finite for each fixed $h > 0$, as the integal in (3.46) over the region I_1 tends to 0 as $t \to \infty$, so the dominated convergence theorem applies, trivially, on I_1, since the limit of the integral is simply zero, and the integral over I_1 does not contribute to the value of (3.46).

Recall the notation in (3.9). Observe that $\Delta_1^{g(t)}$ has exponential distribution of rate $\overline{\Pi}(g(t))$, so

$$\begin{aligned}\mathbb{P}\left(X_h \geq \frac{g(t-h)}{A}\right) &= \mathbb{P}\left(X_h \geq \frac{g(t-h)}{A}; \Delta_1^{g(t)} > h\right) + \mathbb{P}\left(X_h \geq \frac{g(t-h)}{A}; \Delta_1^{g(t)} \leq h\right) \\ &\leq \mathbb{P}\left(X_h^{(0,g(t))} \geq \frac{g(t-h)}{A}\right) + \mathbb{P}\left(\Delta_1^{g(t)} \leq h\right) \\ &= \mathbb{P}\left(X_h^{(0,g(t))} \geq \frac{g(t-h)}{A}\right) + 1 - e^{-h\overline{\Pi}(g(t))} \\ &\leq \mathbb{P}\left(X_h^{(0,g(t))} \geq \frac{g(t-h)}{A}\right) + h\overline{\Pi}(g(t)),\end{aligned} \tag{3.51}$$

where $X^{(0,g(t))}$ has the same Lévy measure as X, but restricted to $(0, g(t))$, so $X^{(0,g(t))}$ has no jumps larger than $g(t)$. By (3.51) and Markov's inequality (Theorem A.2.1),

$$(3.50) - h \leq \limsup_{t\to\infty} \frac{\mathbb{P}\left(X_h^{(0,g(t))} \geq \frac{g(t-h)}{A}\right)}{\overline{\Pi}(g(t))} \lesssim \limsup_{t\to\infty} \frac{A\mathbb{E}[X_h^{(0,g(t))}]}{\overline{\Pi}(g(t))g(t-h)} = \limsup_{t\to\infty} \frac{Ah\int_0^{g(t)} x\Pi(dx)}{\overline{\Pi}(g(t))g(t-h)}.$$

Now, observe that

$$\int_0^{g(t)} x\Pi(dx) = \int_{x=0}^{g(t)}\int_{y=0}^{x} dy\Pi(dx) = \int_{y=0}^{g(t)}\int_{x=y}^{g(t)} \Pi(dx)dy \leq \int_0^{g(t)} \overline{\Pi}(y)dy. \tag{3.52}$$

Then as $\overline{\Pi}$ is regularly varying at ∞, $\lim_{t\to\infty} g(t-h)\overline{\Pi}(g(t)) = \infty$, and $g(t) \sim g(t-h)$ as $t \to \infty$ (see Definition 3.4.1), by Karamata's theorem (Theorem A.4.3), we deduce, as required for dominated convergence on I_1, that for each $h > 0$,

$$\begin{aligned}(3.50) &\overset{(3.52)}{\lesssim} h + \limsup_{t\to\infty} \frac{Ah\int_0^1 \overline{\Pi}(y)dy + Ah\int_1^{g(t)} \overline{\Pi}(y)dy}{\overline{\Pi}(g(t))g(t-h)} \\ &\overset{A.4.3}{\lesssim} h + \limsup_{t\to\infty} \frac{Ahg(t)\overline{\Pi}(g(t))}{\overline{\Pi}(g(t))g(t-h)} \overset{3.4.1}{<} \infty.\end{aligned}$$

Proof for I_2 By Theorem 3.4.18, $q_h(y)$ is non-decreasing in y, so by (3.26), $\mathbb{P}(\mathcal{O}_{t-h}^{g_y^h})$ is non-decreasing in y. Now, $\lim_{t\to\infty} \mathbb{P}(\mathcal{O}_{t-h}^{g_{g(h+1)}^h})/\mathbb{P}(\mathcal{O}_t) = q_h(g(h+1)) < \infty$ for each fixed h by Theorem 3.4.18, and we conclude that

$$\lim_{t\to\infty}\int_{g(h)}^{\infty}\mathbb{1}_{\{y\in I_2\}}\frac{\mathbb{P}(\mathcal{O}_{t-h}^{g_y^h})}{\mathbb{P}(\mathcal{O}_t)}\mathbb{P}\left(X_h\in dy;\mathcal{O}_h\right)\le\lim_{t\to\infty}\frac{\mathbb{P}(\mathcal{O}_{t-h}^{g_{g(h+1)}^h})}{\mathbb{P}(\mathcal{O}_t)}\int_{g(h)}^{\infty}\mathbb{1}_{\{y\in I_2\}}\mathbb{P}\left(X_h\in dy;\mathcal{O}_h\right)$$
$$=q_h(g(h+1))\mathbb{P}\left(X_h\in I_2;\mathcal{O}_h\right),$$

which is finite for each $h>0$, so dominated convergence applies on I_2.

Proof for I_3 By (3.19) and Lemma 3.5.1, for all large enough t,

$$\frac{\mathbb{P}(\mathcal{O}_{t-h}^{g_y^h})}{\mathbb{P}(\mathcal{O}_t)}\le 2\frac{\left[\overline{\Pi}(g_y^h(t-h))+\rho_y^h(t-h)\right]\Phi_y^h(t-h)}{\overline{\Pi}(g(t))\Phi(t)}.$$

For $y\in I_3$, $y>g(h+1)$, so $f(y)-h>f(g(h+1))-h=1$, and $1+1/(f(y)-h)<2$. By (3.31), as $\lim_{t\to\infty}u(t)=0$, for all large enough t and for all $y\in I_3$,

$$\frac{\mathbb{P}(\mathcal{O}_{t-h}^{g_y^h})}{\mathbb{P}(\mathcal{O}_t)}\le 2\frac{\Big(1+u(t)\big(1+\frac{1}{f(y)-h}\big)\Big)\overline{\Pi}(g_y^h(t-h))\Phi_y^h(t-h)}{\overline{\Pi}(g(t))\Phi(t)}$$
$$\frac{\mathbb{P}(\mathcal{O}_{t-h}^{g_y^h})}{\mathbb{P}(\mathcal{O}_t)}\le 6\frac{\overline{\Pi}(g_y^h(t-h))\Phi_y^h(t-h)}{\overline{\Pi}(g(t))\Phi(t)}.$$

Now, recall $A>3$, so $y<g(t-h)/A<g(t)/3$ for $y\in I_3$, and by (3.12), since $\overline{\Pi}$ is regularly varying at ∞, $\overline{\Pi}(g_y^h(t-h))=\overline{\Pi}(g(t)-y)\lesssim\overline{\Pi}(g(t))$, uniformly among $y\in I_3$. So for each fixed $h>0$, for all large enough t, uniformly in $y\in I_3$,

$$\frac{\mathbb{P}(\mathcal{O}_{t-h}^{g_y^h})}{\mathbb{P}(\mathcal{O}_t)}\lesssim\frac{\Phi_y^h(t-h)}{\Phi(t)}. \tag{3.53}$$

Now, by (3.21), for $t_0(y)$ as defined in (3.22), we have

$$\Phi_y^h(t-h)=\Phi_y^h(t_0(y))\exp\left(\int_{t_0(y)}^{t-h}\overline{\Pi}(g_y^h(s))ds+\int_{t_0(y)}^{t-h}\rho_y^h(s)ds\right).$$

Applying (3.30) and recalling that $1 + 1/(f(y) - h) < 2$ for $y \in I_3$, the integral $\int_{t_0(y)}^\infty \rho_y^h(s)ds$ is uniformly bounded among y, so uniformly among $y \in I_3$,

$$\Phi_y^h(t-h) \lesssim \Phi_y^h(t_0(y)) \exp\left(\int_{t_0(y)}^{t-h} \overline{\Pi}(g_y^h(s))ds\right).$$

By Lemma 3.5.5, $\liminf_{t\to\infty} \rho(t) \geq 0$, so $\liminf_{t\to\infty} \int_1^t \rho(s)ds > -\infty$, then by (3.18),

$$\Phi(t) = \Phi(1) \exp\left(\int_1^t \overline{\Pi}(g(s))ds + \int_1^t \rho(s)ds\right) \gtrsim \exp\left(\int_1^t \overline{\Pi}(g(s))ds\right).$$

Since $y > g(h+1)$ in I_3, recalling (3.22), $t_0(y) \geq f(Ay) > f(y) \geq h+1 > 1$, so

$$\begin{aligned}\frac{\Phi_y^h(t-h)}{\Phi(t)} &\lesssim \Phi_y^h(t_0(y)) \exp\left(\int_{t_0(y)}^{t-h} \overline{\Pi}(g_y^h(s))ds - \int_1^t \overline{\Pi}(g(s))ds\right) \\ &\leq \Phi_y^h(t_0(y)) \exp\left(\int_{t_0(y)}^{t} \left(\overline{\Pi}(g_y^h(s)) - \overline{\Pi}(g(s))\right)ds - \int_1^{t_0(y)} \overline{\Pi}(g(s))ds\right).\end{aligned}$$

Now, by Lemma 3.5.6, since $yf'(y)\overline{\Pi}(y)$ decreases to zero as $y \to \infty$, we have uniformly among $y > g(h)$,

$$\int_{t_0(y)}^{t} \left(\overline{\Pi}(g_y^h(s)) - \overline{\Pi}(g(s))\right)ds \leq \int_{t_0(y)}^{\infty} \left(\overline{\Pi}(g_y^h(s)) - \overline{\Pi}(g(s))\right)ds \lesssim \sup_{y>g(h)} yf'(y)\overline{\Pi}(y) < \infty.$$

So for each fixed $h > 0$, uniformly among $y > g(h)$, by (3.14),

$$\lim_{t\to\infty} \frac{\Phi_y^h(t-h)}{\Phi(t)} \lesssim \Phi_y^h(t_0(y))e^{-\int_1^{t_0(y)} \overline{\Pi}(g(s))ds} \leq t_0(y)e^{-\int_1^{t_0(y)} \overline{\Pi}(g(s))ds}. \tag{3.54}$$

Now, it follows from (3.53) and (3.54) that for each fixed $h > 0$, uniformly for all t large enough,

$$\int_{g(h)}^\infty \mathbb{1}_{\{y\in I_3\}} \frac{\mathbb{P}(\mathcal{O}_{t-h}^{g_y^h})}{\mathbb{P}(\mathcal{O}_t)} \mathbb{P}\left(X_h \in dy; \mathcal{O}_h\right) \lesssim \int_{g(h)}^\infty \mathbb{1}_{\{y\in I_3\}} t_0(y)e^{-\int_1^{t_0(y)} \overline{\Pi}(g(s))ds} \mathbb{P}\left(X_h \in dy; \mathcal{O}_h\right). \tag{3.55}$$

Now (by choice of $A = A(f)$ sufficiently large if necessary) we have $t_0(y) = f(Ay) \vee (1 + 2/A) = f(Ay)$ for all $y > g(h+1) > g(1)$. Then writing $\zeta(x) := xe^{-\int_1^x \overline{\Pi}(g(u))du}$, noting $\zeta(\cdot)$ is differentiable with $\zeta'(x) = (1 - x\overline{\Pi}(g(x)))e^{-\int_1^x \overline{\Pi}(g(u))du} > 0$ for all large enough x since $\lim_{x\to\infty} x\overline{\Pi}(g(x)) = 0$, and noting $\zeta(1) = 1$,

$$\begin{aligned}
(3.55) &= \int_{g(h)}^{\infty} \mathbb{1}_{\{y \in I_3\}} f(Ay) e^{-\int_1^{f(Ay)} \overline{\Pi}(g(s))ds} \mathbb{P}\left(X_h \in dy; \mathcal{O}_h\right) \\
&= \int_{g(h+1)}^{\infty} f(Ay) e^{-\int_1^{f(Ay)} \overline{\Pi}(g(s))ds} \mathbb{P}\left(X_h \in dy; \mathcal{O}_h\right) \\
&\leq \int_{g(h+1)}^{\infty} f(Ay) e^{-\int_1^{f(Ay)} \overline{\Pi}(g(s))ds} \mathbb{P}\left(X_h \in dy\right) = \int_{g(h+1)}^{\infty} \zeta(f(Ay)) \mathbb{P}\left(X_h \in dy\right) \\
&= \int_{g(h+1)}^{\infty} \left(\int_1^{f(Ay)} \zeta'(x)dx + 1 \right) \mathbb{P}\left(X_h \in dy\right) \\
&= \mathbb{P}\left(X_h \geq g(h+1)\right) + \int_{g(h+1)}^{\infty} \int_1^{f(Ay)} \zeta'(x)dx \mathbb{P}\left(X_h \in dy\right).
\end{aligned}$$

Changing the order of integration, and applying the result that for each fixed $c, h > 0$, $\mathbb{P}(X_h \geq z) \asymp \overline{\Pi}(z)$, uniformly in $z > c$, since $\overline{\Pi}$ is regularly varying, see [52, Theorem 1 (iii)], we get

$$\begin{aligned}
\int_1^{\infty} \zeta'(x) \int_{g(h+1) \vee \frac{g(x)}{A}}^{\infty} \mathbb{P}\left(X_h \in dy\right) dx &\leq \int_1^{\infty} \zeta'(x) \int_{\frac{g(x)}{A}}^{\infty} \mathbb{P}\left(X_h \in dy\right) dx = \int_1^{\infty} \zeta'(x) \mathbb{P}\left(X_h \geq \frac{g(x)}{A} \right) dx \\
&\asymp \quad \int_1^{\infty} \zeta'(x) \overline{\Pi}\left(\frac{g(x)}{A} \right) dx \quad \lesssim \int_1^{\infty} \zeta'(x) \overline{\Pi}\left(g(x)\right) dx \\
&= \int_1^{\infty} \left(e^{-\int_1^x \overline{\Pi}(g(u))du} - x\overline{\Pi}(g(x)) e^{-\int_1^x \overline{\Pi}(g(u))du} \right) \overline{\Pi}\left(g(x)\right) dx \\
&\leq \int_1^{\infty} e^{-\int_1^x \overline{\Pi}(g(u))du} \overline{\Pi}\left(g(x)\right) dx = \int_1^{\infty} \frac{d}{dx} \left(-e^{-\int_1^x \overline{\Pi}(g(u))du} \right) dx < \infty,
\end{aligned}$$

and therefore $(3.55) < \infty$, so the dominated convergence theorem applies on I_3, and the order of limits and integration can be swapped between (3.46) and (3.47), as required.

□

3.5.4 Proof of Proposition 3.4.20

Proof of Proposition 3.4.20. Recall $\pi_h(X)$ is the sample path of X up to time h, F_1 is a functional on the excursion process, and $\mathcal{B} \subseteq \mathcal{O}_h$, $\mathcal{B} \in \mathcal{F}_h$, where $(\mathcal{F}_u)_{u\geq 0}$ is X's filtration. Disintegrating on the value of $\pi_h(X) \in \mathcal{B}$,

$$\mathbb{E}\left[F_1(\pi_h((\varepsilon_s)_{s\geq 0}))\ \mathbb{1}_{\{\pi_h(X)\in\mathcal{B}\}}\ \middle|\ \mathcal{O}_t\ \right] \tag{3.56}$$

$$= \int_{\nu\in\mathcal{B}} \mathbb{E}\big[F_1(\pi_h((\varepsilon_s)_{s\geq 0}))\big|\mathcal{O}_t; \pi_h(X) = \nu\big]\mathbb{P}\big(\pi_h(X) \in d\nu|\mathcal{O}_t\big).$$

Given a fixed path $\pi_h(X) = \nu$, $\pi_h((\varepsilon_s)_{s\geq 0})$ depends only on ν, so $\pi_h((\varepsilon_s)_{s\geq 0})$ is conditionally independent of $\mathcal{O}_t$, and then since $\lim_{t\to\infty}\mathbb{P}(\pi_h(X) \in d\nu|\mathcal{O}_t) = \mathbb{Q}(\pi_h(X) \in d\nu)$ by Theorem 3.4.18,

$$\begin{aligned}
\lim_{t\to\infty}(3.56) \quad &= \quad \lim_{t\to\infty}\int_{\mathcal{B}} \mathbb{E}\left[F_1(\pi_h((\varepsilon_s)_{s\geq 0}))\big|\pi_h(X) = \nu\right]\mathbb{P}\left(\pi_h(X) \in d\nu|\mathcal{O}_t\right) \\
&\overset{3.4.18}{=} \int_{\mathcal{B}} \mathbb{E}\left[F_1(\pi_h((\varepsilon_s)_{s\geq 0}))\big|\pi_h(X) = \nu\right]\mathbb{Q}(\pi_h(X) \in d\nu) \qquad (3.57)\\
&=: \quad \mathbb{E}_{\mathbb{Q}''}\left[F_1(\pi_h((\varepsilon_s)_{s\geq 0}))\ \mathbb{1}_{\{\pi_h(X)\in\mathcal{B}\}}\right],
\end{aligned}$$

where we can swap the order of limits and integration since F_1 is bounded. Taking $F_1 \equiv 1$, it follows immediately that $\mathbb{Q}''(\cdot)$ and $\mathbb{Q}(\cdot)$ agree on $\mathcal{D}[0,\infty)$. The weak convergence as $t \to \infty$ of $((M_t)_{t\geq 0}, (X_s)_{s\geq 0})$ under $\mathbb{P}(\cdot|\mathcal{O}_t)$ to $((M_t)_{t\geq 0}, (X_s)_{s\geq 0})$ under $\mathbb{Q}''(\cdot)$ then follows immediately from the fact (see e.g. [14, Ex. IV.6.3] or [78, p4113]) that for all $x > 0$, $(M_t)_{t\geq 0}$ is uniquely determined by $(\varepsilon_s)_{s\geq 0}$ and $(X_s)_{s\geq 0}$, and both of $(\varepsilon_s)_{s\geq 0}$ and $(X_s)_{s\geq 0}$ have weak limits as determined in (3.57). That is, we construct M pathwise via its excursions using $(X_u)_{u\geq 0}$ to determine the timing and length of each excursion, where we sample the excursions of M using the excursion measure conditional on each excursion length. The fact that M is recurrent under $\mathbb{Q}''(\cdot)$ follows immediately from this construction, since X does not explode to infinity by Proposition 3.4.19.

□

Proof of Theorem 3.4.22

Lemmas 3.5.7, 3.5.8, 3.5.9, and 3.5.10, proven in Sections 3.8, 3.9, are required for the proof of Theorem 3.4.22. We shall use the notation $\mu(ds) \asymp \nu(ds)$ for measures, meaning that there exist $\alpha, \beta > 0$ such that for each measurable set A, $\alpha\mu(A) \le \nu(A) \le \beta\mu(A)$.

Lemma 3.5.7. *For each subordinator and $g = f^{-1}$ in case (ia), there exist $K, h_0 > 0$ such that for all $h > h_0$, with $\overline{\Pi}(y) = y^{-\alpha}L(y)$, uniformly in $y > K$,*

$$\mathbb{P}\left(X_h \in g(h)dy; \mathcal{O}_h\right) \asymp y^{-1-\alpha}\frac{L(g(h)y)}{L(g(h))}\mathbb{P}(\mathcal{O}_h)dy. \tag{3.58}$$

Lemma 3.5.8. *In case (i), for $\delta > 0$ small enough that $0 < f(0) < f(\delta) < 1$, uniformly for all $h > 0$ and $y > g(h + f(\delta))$,*

$$q_h(y) \asymp \Phi_y^h(f(Ay)) \exp\left(-\int_1^{f(Ay)} \overline{\Pi}(g(s))ds\right).$$

Lemma 3.5.9. *For a subordinator and a function $g = f^{-1}$ as in case (ia), let $S_{\Delta_1^{g(h)}}$ denote the size of its first jump of size greater than $g(h)$. Then there exists $h_0 > 0$ such that uniformly for all $h > h_0$ and $v > 1$,*

$$\mathbb{P}\left(S_{\Delta_1^{g(h)}} \in g(h)dv\right) = \frac{\Pi(g(h)dv)}{\overline{\Pi}(g(h))} \asymp \frac{L(g(h)v)}{L(g(h))}v^{-1-\alpha}dv.$$

In particular there is $x_0 \in (0, \infty)$ so that for all $x > x_0$, with $\Pi(dx) = u(x)dx$,

$$u(x) \asymp x^{-1}\overline{\Pi}(x) = L(x)x^{-1-\alpha}. \tag{3.59}$$

Lemma 3.5.10. *Recall the notation* (3.14) , (3.22)*. If $h > 0$, $y > g(h)$, and $t \ge f(Ay)$, for $A > 3 \vee (B-1)$, then $\Phi_y^h(t) \ge f(y) - h$.*

Proof of Theorem 3.4.22. By Proposition 3.4.19, $\mathbb{Q}''(X_h \in (g(h), \infty)) = \mathbb{Q}(X_h \in (g(h), \infty)) = 1$, $h \ge 0$, when $I(f) = \infty$, so since $\mathbb{Q}''(\cdot)$ and $\mathbb{Q}(\cdot)$ agree on the space $\mathcal{D}[0, \infty)$ containing X, by Definition 3.4.21,

$$w \in R_g \iff \lim_{h\to\infty} \mathbb{Q}\left(X_h \in (g(h), w(h)g(h))\right) = 0. \tag{3.60}$$

Now, by Theorem 3.4.18,

$$\lim_{h\to\infty} \mathbb{Q}\left(X_h \in (g(h), w(h)g(h))\right) = \lim_{h\to\infty} \int_{g(h)}^{w(h)g(h)} q_h(y)\mathbb{P}\left(X_h \in dy; \mathcal{O}_h\right). \tag{3.61}$$

We begin by showing that if $\lim_{h\to\infty} \int_h^{f(w(h)g(h))} \overline{\Pi}(g(s))ds = 0$, then $w \in R_g$.

Proof of Sufficient Condition Let $\lim_{h\to\infty} \int_h^{f(w(h)g(h))} \overline{\Pi}(g(s))ds = 0$. To show $w \in R_g$, we will show that the limit of the integral in (3.61) is zero on each of

$$[g(h), g(h+1)] \cup [g(h+1), Kg(h)] \cup [g(h+1) \vee Kg(h), w(h)g(h)] =: R_1 \cup R_2 \cup R_3$$

separately, where K is the constant as in Lemma 3.5.7. Note that if $g(h+1) > Kg(h)$, then we need only consider $R_1 \cup R_3$. Since g is non-decreasing, we only need to consider the value K if $K > 1$.

Proof for R_1 By Theorem 3.4.18, $q_h(y)$ is non-decreasing in y, so

$$\begin{aligned} &\lim_{h\to\infty} \int_{g(h)}^{g(h+1)} q_h(y)\mathbb{P}\left(X_h \in dy; \mathcal{O}_h\right) \\ &\leq \lim_{h\to\infty} q_h(g(h+1)) \int_{g(h)}^{g(h+1)} \mathbb{P}\left(X_h \in dy; \mathcal{O}_h\right) \leq \lim_{h\to\infty} q_h(g(h+1))\mathbb{P}\left(\mathcal{O}_h\right). \end{aligned} \tag{3.62}$$

Applying Lemma 3.5.8, then applying Lemma 3.5.1,

$$\begin{aligned} (3.62) &\lesssim \lim_{h\to\infty} \Phi^h_{g(h+1)}(f(Ag(h+1)))e^{-\int_1^{f(Ag(h+1))} \overline{\Pi}(g(s))ds}\mathbb{P}\left(\mathcal{O}_h\right) \\ &= \lim_{h\to\infty} \Phi^h_{g(h+1)}(f(Ag(h+1)))e^{-\int_1^{f(Ag(h+1))} \overline{\Pi}(g(s))ds}\Phi(h)\overline{\Pi}(g(h)). \end{aligned}$$

Now, $\Phi^h_{g(h+1)}(f(Ag(h+1))) \leq f(Ag(h+1)) \lesssim h$, and $f(Ag(h+1)) \geq h$, as $f = g^{-1}$ is $\mathcal{O}$-regularly varying, increasing, and $A > 1$. By (3.32) and (3.3),

$$(3.62) \lesssim \lim_{h\to\infty} he^{-\int_1^{f(Ag(h+1))}\overline{\Pi}(g(s))ds}\Phi(h)\overline{\Pi}(g(h))$$
$$\leq \lim_{h\to\infty} he^{-\int_1^{h}\overline{\Pi}(g(s))ds}\Phi(h)\overline{\Pi}(g(h)) \overset{(3.32)}{\lesssim} \lim_{h\to\infty} h\overline{\Pi}(g(h)) \overset{(3.3)}{=} 0.$$

Proof for R_2 Recall that we only need to consider R_2 when $g(h+1) < Kg(h)$, in which case K must satisfy $K > 1$. By Theorem 3.4.18, $q_h(y)$ is non-decreasing in y, so

$$\lim_{h\to\infty}\int_{g(h+1)}^{Kg(h)} q_h(y)\mathbb{P}\left(X_h \in dy; \mathcal{O}_h\right) \tag{3.63}$$
$$\leq \lim_{h\to\infty} q_h(Kg(h))\int_{g(h+1)}^{Kg(h)}\mathbb{P}\left(X_h \in dy; \mathcal{O}_h\right) \leq \lim_{h\to\infty} q_h(Kg(h))\mathbb{P}\left(\mathcal{O}_h\right).$$

Applying Lemma 3.5.1, then Lemma 3.5.8. (note $g(h+f(\delta)) < g(h+1) < Kg(h)$ for $\delta > 0$ in Lemma 3.5.8),

$$(3.63) \overset{3.5.1}{\leq} \lim_{h\to\infty} q_h(Kg(h))\Phi(h)\overline{\Pi}(g(h)).$$
$$\overset{3.5.8}{\lesssim} \lim_{h\to\infty} \Phi^h_{Kg(h)}(f(AKg(h)))e^{-\int_1^{f(AKg(h))}\overline{\Pi}(g(s))ds}\Phi(h)\overline{\Pi}(g(h)).$$

Observe that $\Phi^h_{Kg(h)}(t) \leq t$ for all $t > 0$, by (3.14), and $f(AKg(h)) \geq f(g(h)) = h$ since f increasing and $A, K > 1$. Moreover, as f is $\mathcal{O}$-regularly varying at ∞, $f(AKg(h)) \lesssim f(g(h)) = h$, as $h \to \infty$, so

$$(3.63) \leq \lim_{h\to\infty} f(AKg(h))\exp\left(-\int_1^{f(AKg(h))}\overline{\Pi}(g(s))ds\right)\Phi(h)\overline{\Pi}(g(h)).$$
$$\leq \lim_{h\to\infty} f(AKg(h))\exp\left(-\int_1^{h}\overline{\Pi}(g(s))ds\right)\Phi(h)\overline{\Pi}(g(h))$$
$$\lesssim \lim_{h\to\infty} h\exp\left(-\int_1^{h}\overline{\Pi}(g(s))ds\right)\Phi(h)\overline{\Pi}(g(h)).$$

By (3.32), $\Phi(h) \asymp \exp(\int_1^h \overline{\Pi}(g(s))ds)$ as $h \to \infty$, so as $\lim_{h\to\infty} h\overline{\Pi}(g(h)) = 0$ by (3.3), we conclude that (3.63) $= 0$, so the integral over R_2 is zero, and thus

$$\lim_{h\to\infty} \int_{R_1 \cup R_2} q_h(y)\mathbb{P}\left(X_h \in dy; \mathcal{O}_h\right) = 0. \tag{3.64}$$

Proof for R_3 Now we wish to show convergence to zero of

$$\int_{g(h+1)\vee Kg(h)}^{w(h)g(h)} q_h(y)\mathbb{P}\left(X_h \in dy; \mathcal{O}_h\right) = \int_{\frac{g(h+1)}{g(h)}\vee K}^{w(h)} q_h(g(h)v)\mathbb{P}\left(X_h \in g(h)dv; \mathcal{O}_h\right). \tag{3.65}$$

Applying Lemma 3.5.7, then changing variables back to $y = g(h)v$, recalling that $\overline{\Pi}(g(h)) = g(h)^{-\alpha}L(g(h))$, then applying Lemma 3.5.1, as $h \to \infty$,

$$\begin{aligned}
(3.65) &\overset{3.5.7}{\asymp} \int_{\frac{g(h+1)}{g(h)}\vee K}^{w(h)} q_h(g(h)v)v^{-1-\alpha}\frac{L(g(h)v)}{L(g(h))}\mathbb{P}(\mathcal{O}_h)dv \\
&= \frac{\mathbb{P}\left(\mathcal{O}_h\right)g(h)^{\alpha}}{L(g(h))}\int_{\frac{g(h+1)}{g(h)}\vee K}^{w(h)} q_h(g(h)v)g(h)^{-\alpha}v^{-1-\alpha}L(g(h)v)dv \\
&= \frac{\mathbb{P}\left(\mathcal{O}_h\right)}{\overline{\Pi}(g(h))}\int_{g(h+1)\vee Kg(h)}^{w(h)g(h)} q_h(y)y^{-1-\alpha}L(y)dy \overset{3.5.1}{\sim} \Phi(h)\int_{g(h+1)\vee Kg(h)}^{w(h)g(h)} q_h(y)y^{-1-\alpha}L(y)dy.
\end{aligned}$$

Changing variables ($u = Ay$), applying Lemma 3.9 and then the uniform convergence theorem (Theorem A.4.1), as L is slowly varying at ∞ and $\Phi_y^h(f(u)) \leq f(u)$, it follows that as $h \to \infty$,

$$\begin{aligned}
(3.65) &\overset{3.9}{\asymp} \Phi(h)\int_{g(h+1)\vee Kg(h)}^{w(h)g(h)} \Phi_y^h(f(Ay))\exp\left(-\int_1^{f(Ay)}\overline{\Pi}(g(s))ds\right)y^{-1-\alpha}L(y)dy \\
&\leq \Phi(h)\int_{Kg(h)}^{w(h)g(h)} \Phi_y^h(f(Ay))\exp\left(-\int_1^{f(Ay)}\overline{\Pi}(g(s))ds\right)y^{-1-\alpha}L(y)dy \\
&\lesssim \Phi(h)\int_{AKg(h)}^{Aw(h)g(h)} \Phi_y^h(f(u))\exp\left(-\int_1^{f(u)}\overline{\Pi}(g(s))ds\right)u^{-1-\alpha}L(u)du \\
&\leq \Phi(h)\int_{AKg(h)}^{Aw(h)g(h)} f(u)\exp\left(-\int_1^{f(u)}\overline{\Pi}(g(s))ds\right)u^{-1-\alpha}L(u)du.
\end{aligned}$$

Since $A, K > 1$, we can split up the integral as follows, and we will deal with each term separately:

$$
\begin{aligned}
(3.65) &\lesssim \Phi(h) \int_{g(h)}^{w(h)g(h)} f(u) \exp\left(-\int_1^{f(u)} \overline{\Pi}(g(s))ds\right) u^{-1-\alpha} L(u)du \\
&+ \Phi(h) \int_{w(h)g(h)}^{Aw(h)g(h)} f(u) \exp\left(-\int_1^{f(u)} \overline{\Pi}(g(s))ds\right) u^{-1-\alpha} L(u)du \\
&=: J_1(h) + J_2(h). \qquad (3.66)
\end{aligned}
$$

Proof for $J_2(h)$ As $f(u) \geq f(w(h)g(h)) \geq f(g(h)) = h$ for $u \geq w(h)g(h)$, by (3.32),

$$
\begin{aligned}
J_2(h) &= \Phi(h) \int_{w(h)g(h)}^{Aw(h)g(h)} f(u) e^{-\int_1^{f(u)} \overline{\Pi}(g(s))ds} u^{-1-\alpha} L(u)du \\
&= \Phi(h) e^{-\int_1^h \overline{\Pi}(g(s))ds} \int_{w(h)g(h)}^{Aw(h)g(h)} f(u) e^{-\int_h^{f(u)} \overline{\Pi}(g(s))ds} u^{-1-\alpha} L(u)du \\
&\overset{(3.32)}{\lesssim} \int_{w(h)g(h)}^{Aw(h)g(h)} f(u) e^{-\int_h^{f(u)} \overline{\Pi}(g(s))ds} u^{-1-\alpha} L(u)du \leq \int_{w(h)g(h)}^{Aw(h)g(h)} f(u) u^{-1-\alpha} L(u)du.
\end{aligned}
$$

Since f and f' are $\mathcal{O}$-regularly varying at ∞, one can verify that for all sufficiently large u, $f(u)/u \asymp f'(u)$, see Theorem A.4.9. Now, in case (i), $uf'(u)\overline{\Pi}(u)$ decreases to 0 as $u \to \infty$, so as $h \to \infty$,

$$
\begin{aligned}
J_2(h) &\lesssim \int_{w(h)g(h)}^{Aw(h)g(h)} f'(u) u^{-\alpha} L(u)du = \int_{w(h)g(h)}^{Aw(h)g(h)} uf'(u)\overline{\Pi}(u) u^{-1} du. \\
&\leq w(h)g(h)\ f'\big(w(h)g(h)\big)\ \overline{\Pi}\big(w(h)g(h)\big) \int_{w(h)g(h)}^{Aw(h)g(h)} u^{-1} du \\
&= o(1) \times \int_{w(h)g(h)}^{Aw(h)g(h)} u^{-1} du = o(1) \times \log(A) = o(1), \qquad (3.67)
\end{aligned}
$$

so $\lim_{h\to\infty} J_2(h) = 0$, and $J_2(h)$ never contributes. Now we consider $J_1(h)$.

Proof for $J_1(h)$ First, changing variables from s to $v := g(s)$, so that $s = f(v)$, we have

$$J_1(h) = \Phi(h) \int_{g(h)}^{w(h)g(h)} f(u) \exp\left(- \int_1^{f(u)} \overline{\Pi}(g(s))ds \right) u^{-1-\alpha} L(u) du$$
$$= \Phi(h) \int_{g(h)}^{w(h)g(h)} f(u) \exp\left(- \int_{g(1)}^{u} \overline{\Pi}(v) f'(v) dv \right) u^{-1-\alpha} L(u) du.$$

Recall $u^{-\alpha}L(u) = \overline{\Pi}(u)$, and $f(u) \asymp u f'(u)$ as $u \to \infty$, so as $h \to \infty$,

$$J_1(h) \asymp \Phi(h) \int_{g(h)}^{w(h)g(h)} f'(u) \exp\left(- \int_{g(1)}^{u} \overline{\Pi}(v) f'(v) dv \right) u^{-\alpha} L(u) du$$
$$= \Phi(h) \int_{g(h)}^{w(h)g(h)} f'(u) \overline{\Pi}(u) \exp\left(- \int_{g(1)}^{u} \overline{\Pi}(v) f'(v) dv \right) du.$$

Changing variables from u to $z := e^{-\int_{g(1)}^{u} \overline{\Pi}(v) f'(v) dv}$ and applying (3.32), it follows that as $h \to \infty$,

$$\begin{aligned} J_1(h) &\asymp \Phi(h) \left[e^{-\int_{g(1)}^{g(h)} \overline{\Pi}(v) f'(v) dv} - e^{-\int_{g(1)}^{w(h)g(h)} \overline{\Pi}(v) f'(v) dv} \right] \\ &= \Phi(h) \left[e^{-\int_1^h \overline{\Pi}(g(s)) ds} - e^{-\int_1^{f(w(h)g(h))} \overline{\Pi}(g(s)) ds} \right] \overset{(3.32)}{\asymp} 1 - e^{-\int_h^{f(w(h)g(h))} \overline{\Pi}(g(s)) ds}. \end{aligned} \tag{3.68}$$

Thus by (3.64), (3.67) and (3.68), whenever $\lim_{h\to\infty} \int_h^{f(w(h)g(h))} \overline{\Pi}(g(s))ds = 0$, w is in the entropic repulsion envelope R_g, as required for the sufficient condition. In this case, $\lim_{h\to\infty} J_1(h) = 0$.

Now we will prove the converse, that is, if $w \in R_g$, then $\lim_{h\to\infty} \int_h^{f(w(h)g(h))} \overline{\Pi}(g(s))ds = 0$.

Proof of Necessary Condition Let $w \in R_g$. Then by (3.26),

$$\begin{aligned} 0 &= \lim_{h\to\infty} \mathbb{Q}\left(X_h \in (g(h), w(h)g(h)) \right) \\ &= \lim_{h\to\infty} \int_{g(h)}^{w(h)g(h)} q_h(y) \mathbb{P}\left(X_h \in dy; \mathcal{O}_h \right) = \lim_{h\to\infty} \int_{Kg(h)}^{w(h)g(h)} q_h(y) \mathbb{P}\left(X_h \in dy; \mathcal{O}_h \right), \end{aligned}$$

since the limit of the integral over $R_1 \cup R_2 = (g(h), Kg(h))$ is always zero by (3.64), regardless of whether or not $\lim_{h\to\infty} \int_h^{f(w(h)g(h))} \overline{\Pi}(g(s))ds = 0$. Changing variables to $v = y/g(h)$, then applying Lemma 3.5.7,

$$0 = \lim_{h\to\infty} \int_K^{w(h)} q_h(g(h)v)\mathbb{P}\left(X_h \in g(h)dv; \mathcal{O}_h\right) \overset{3.5.7}{=} \lim_{h\to\infty} \mathbb{P}(\mathcal{O}_h) \int_K^{w(h)} q_h(g(h)v)v^{-1-\alpha} \frac{L(g(h)v)}{L(g(h))} dv.$$

Changing variables to $y = g(h)v$ and recallling that $\overline{\Pi}(g(h)) = g(h)^{-\alpha} L(g(h))$, then applying Lemmas 3.5.1, 3.5.8, and 3.5.10,

$$\begin{aligned}
0 &= \lim_{h\to\infty} \frac{\mathbb{P}(\mathcal{O}_h)}{\overline{\Pi}(g(h))} \int_{Kg(h)}^{w(h)g(h)} q_h(y)y^{-1-\alpha}L(y)dy \overset{3.5.1}{=} \lim_{h\to\infty} \Phi(h) \int_{Kg(h)}^{w(h)g(h)} q_h(y)y^{-1-\alpha}L(y)dy \\
&\overset{3.5.8}{=} \lim_{h\to\infty} \Phi(h) \int_{Kg(h)}^{w(h)g(h)} \Phi_y^h(f(Ay))e^{-\int_1^{f(Ay)} \overline{\Pi}(g(s))ds} y^{-1-\alpha}L(y)dy
\end{aligned}$$

$$\begin{aligned}
&\overset{3.5.10}{\geq} \lim_{h\to\infty} \left[\Phi(h) \int_{Kg(h)}^{w(h)g(h)} f(y)e^{-\int_1^{f(Ay)} \overline{\Pi}(g(s))ds} y^{-1-\alpha}L(y)dy \right. \\
&\qquad \left. - h\Phi(h) \int_{Kg(h)}^{w(h)g(h)} e^{-\int_1^{f(Ay)} \overline{\Pi}(g(s))ds} y^{-1-\alpha}L(y)dy \right] \\
&=: \lim_{h\to\infty} \left[I_1 - I_2\right].
\end{aligned}$$

First we consider I_2. Note that $AK > 1$, so $f(Ay) \geq f(AKg(h)) \geq h$ for all $y \geq Kg(h)$. Then since $\Phi(h) \asymp e^{-\int_1^h \overline{\Pi}(g(s))ds}$ by (3.32),

$$\lim_{h\to\infty} |I_2| \leq \lim_{h\to\infty} h\Phi(h) \int_{Kg(h)}^{w(h)g(h)} e^{-\int_1^h \overline{\Pi}(g(s))ds} y^{-1-\alpha}L(y)dy \lesssim \lim_{h\to\infty} h \int_{Kg(h)}^{w(h)g(h)} y^{-1-\alpha}L(y)dy.$$

By (3.59), $y^{-1-\alpha}L(y)dy \asymp \Pi(dy)$, so as $\overline{\Pi}$ is regularly varying at ∞, by (3.3),

$$\lim_{h\to\infty} |I_2| \overset{(3.59)}{\lesssim} \lim_{h\to\infty} h \int_{Kg(h)}^{w(h)g(h)} \Pi(dy) \leq \lim_{h\to\infty} h\overline{\Pi}(Kg(h)) \lesssim \lim_{h\to\infty} h\overline{\Pi}(g(h)) \overset{(3.3)}{=} 0,$$

so $I_2 = 0$, and thus $\lim_{h\to\infty} I_1 \leq 0$. As I_1 is non-negative, $\lim_{h\to\infty} I_1 = 0$. Now, changing variables to $v := Ay$, as f is $\mathcal{O}$-regularly varying at ∞ and L is slowly varying at ∞, by the uniform convergence theorem

(Theorem A.4.1),

$$\begin{aligned} 0=\lim_{h\to\infty} I_1 &= \lim_{h\to\infty}\Phi(h)\int_{AKg(h)}^{Aw(h)g(h)} f\left(\frac{v}{A}\right)e^{-\int_1^{f(v)}\overline{\Pi}(g(s))ds}v^{-1-\alpha}A^{\alpha}L\left(\frac{v}{A}\right)dv \\ &\overset{A.4.1}{=} \lim_{h\to\infty}\Phi(h)\int_{AKg(h)}^{Aw(h)g(h)} f(v)e^{-\int_1^{f(v)}\overline{\Pi}(g(s))ds}v^{-1-\alpha}L(v)dv. \end{aligned}$$

Recall $v^{-\alpha}L(v)=\overline{\Pi}(v)$, and $f(v)\asymp vf'(v)$ for all large enough v, because f, f' are $\mathcal{O}$-regularly varying at ∞, see Theorem A.4.9. Then

$$\begin{aligned} 0 &= \lim_{h\to\infty}\Phi(h)\int_{AKg(h)}^{Aw(h)g(h)} f(v)e^{-\int_1^{f(v)}\overline{\Pi}(g(s))ds}v^{-1}\overline{\Pi}(v)dv \\ &\overset{A.4.9}{=} \lim_{h\to\infty}\Phi(h)\int_{AKg(h)}^{Aw(h)g(h)} f'(v)\overline{\Pi}(v)e^{-\int_1^{f(v)}\overline{\Pi}(g(s))ds}dv. \end{aligned}$$

Now, one can verify that $P(v):=\int_{g(1)}^{v}\overline{\Pi}(u)f'(u)du=\int_1^{f(v)}\overline{\Pi}(g(s))ds$ by changing variables from u to $s=f(u)$. Then as $A>3$ and $P'(v)=\overline{\Pi}(v)f'(v)\geq 0$,

$$\begin{aligned} 0 &= \lim_{h\to\infty}\Phi(h)\int_{AKg(h)}^{Aw(h)g(h)} P'(v)e^{-P(v)}dv \\ &\geq \lim_{h\to\infty}\left[\Phi(h)\int_{g(h)}^{w(h)g(h)} P'(v)e^{-P(v)}dv-\Phi(h)\int_{g(h)}^{AKg(h)} P'(v)e^{-P(v)}dv\right] \\ &=: \lim_{h\to\infty}\left[K_1-K_2\right]. \end{aligned}$$

Now, recall that by (3.68), for all large enough h,

$$K_1\asymp J_1\asymp\left(1-e^{-\int_h^{f(w(h)g(h))}\overline{\Pi}(g(s))ds}\right). \tag{3.69}$$

So if we prove $\lim_{h\to\infty}K_1=0$, then $\lim_{h\to\infty}\int_h^{f(w(h)g(h))}\overline{\Pi}(g(s))ds=0$, and the proof is complete. As K_1 is always non-negative, it suffices to prove that $\lim_{h\to\infty}K_1\leq 0$. To prove this, we will show that

$\lim_{h\to\infty}|K_2| = 0$. Since $g = f^{-1}$, note $f(v) > h$ for $v > g(h)$, then as $\Phi(h) \asymp e^{\int_1^h \overline{\Pi}(g(s))ds}$ by (3.32),

$$\begin{aligned}\lim_{h\to\infty}|K_2| &= \lim_{h\to\infty}\Phi(h)\int_{g(h)}^{AKg(h)}\overline{\Pi}(v)f'(v)e^{-\int_1^{f(v)}\overline{\Pi}(g(s))ds}dv \\ &\overset{(3.32)}{\asymp} \lim_{h\to\infty}\int_{g(h)}^{AKg(h)}\overline{\Pi}(v)f'(v)e^{-\int_h^{f(v)}\overline{\Pi}(g(s))ds}dv \leq \lim_{h\to\infty}\int_{g(h)}^{AKg(h)} vf'(v)\overline{\Pi}(v)v^{-1}dv.\end{aligned}$$

Recall that by assumption, $vf'(v)\overline{\Pi}(v)$ decreases to 0 as $v\to\infty$, and hence

$$\begin{aligned}\lim_{h\to\infty}|K_2| &\lesssim \lim_{h\to\infty} g(h)\ f'(g(h))\ \overline{\Pi}(g(h))\int_{g(h)}^{AKg(h)} v^{-1}dv \\ &= \lim_{h\to\infty} g(h)\ f'(g(h))\ \overline{\Pi}(g(h))\times\log(AK) = 0.\end{aligned}$$

□

Proof of Corollary 3.4.23

Proof of Corollary 3.4.23. We need to verify that a stable subordinator of index $\alpha\in(0,1)$ satisfies (3.4), so Theorem 3.4.22 applies. For $t>0$ and $x>g(t)+x_0$, by the scaling property of stable subordinators (see [14, p227]),

$$f_t(x) = t^{-\frac{1}{\alpha}}f_1\left(\frac{x}{t^{\frac{1}{\alpha}}}\right). \tag{3.70}$$

Now consider the result (see [96, Theorem 1.12]) that for a stable subordinator of index $\alpha\in(0,1)$, $f_1(v)\sim c_\alpha v^{-1-\alpha}$ as $v\to\infty$, for $c_\alpha>0$ constant. In particular, for all large enough v, $f_1(v)$ is arbitrarily close to $c_\alpha v^{-1-\alpha}$. Taking e.g. $a' = 2c_\alpha$, it follows that there exist $a', C\in(0,\infty)$ such that for all $v>C$, $f_1(v)\leq a'v^{-1-\alpha}$.

As $\Pi(dv) = u(v)dv = cv^{-1-\alpha}dv$ for a constant $c>0$, if we can show that $x/t^{1/\alpha}\geq C$ for all $t>0$, $x>g(t)+x_0$, with a suitable choice of $x_0>0$, then by (3.70),

$$f_t(x) = t^{-\frac{1}{\alpha}}f_1\left(\frac{x}{t^{\frac{1}{\alpha}}}\right)\leq a'ctx^{-1-\alpha} = atu(x),$$

for $a = a'c$, so condition (3.4) is satisfied, and the proof will be complete. Indeed, by (3.3), $\lim_{t\to\infty} t\overline{\Pi}(g(t)) = \lim_{t\to\infty} tg(t)^{-\alpha} = 0$, so there exists $D \in (0,\infty)$ such that for all $t > D$, $tg(t)^{-\alpha} \le C^{-\alpha}$, so $t^{1/\alpha} \le C^{-1}g(t)$, and hence for all $t > D$,

$$\frac{x}{t^{\frac{1}{\alpha}}} \ge \frac{g(t)+x_0}{t^{\frac{1}{\alpha}}} \ge \frac{g(t)}{t^{\frac{1}{\alpha}}} \ge C.$$

On the other hand, if $t \le D$, then $x/t^{1/\alpha} \ge (g(t)+x_0)/D^{1/\alpha} \ge x_0/D^{1/\alpha}$, and (choosing x_0 large enough that $x_0/D^{1/\alpha} > C$ if necessary), we conclude that $x/t^{1/\alpha} > C$ for all $t > 0$, $x > g(t) + x_0$, so [96, Theorem 1.12] applies to (3.70). It follows that condition (3.4) is satisfied, and so Theorem 3.4.22 applies, as required.

□

3.6 Proof of Lemma 3.5.1

Before proving Lemma 3.5.1, let us begin by stating Lemma 3.6.1, which is key to proving Lemmas 3.5.1 and 3.5.3, and is itself proven in Section 3.9.

Lemma 3.6.1. *Let $(X_t)_{t\ge0}$ be a subordinator satisfying the assumptions in case (i) or (ii). Then there exists a constant $C > 0$, which depends only on the law of X, such that for all $t > 0$, $A(t) \in (1,\infty)$, $B(t) > 0$, and $H(t) \in (0,1)$,*

$$\mathbb{P}\left(X_t^{(0,A(t))} > B(t)\right) \le \exp\left(Ct\log\left(\frac{1}{H(t)}\right)H(t)^{-\frac{A(t)}{B(t)}}\overline{\Pi}(A(t))\frac{A(t)}{B(t)}\right)H(t). \tag{3.71}$$

Proof of Lemma 3.5.1. The strategy for proving Lemma 3.5.1 involves splitting up the probability $\mathbb{P}(\mathcal{O}_t)$ into several smaller pieces, which we bound separately. We refer the reader to Figure A.2, which provides a guide for following the structure of the proof. Since $g = f^{-1}$ is continuous, for each $x > 0$, $t > 0$,

$$\mathbb{P}\big(\mathcal{O}_{t,X^{(0,x)}}\big) = \mathbb{P}\big(\mathcal{O}_{t-,X^{(0,x)}}\big), \tag{3.72}$$

where $\mathcal{O}_{t-,X^{(0,x)}} := \bigcap_{u<t} \mathcal{O}_{u,X^{(0,x)}}$, and moreover by the definition (3.8), for all $s < t$,

$$\mathbb{P}(\mathcal{O}_{t,X^{(0,x)}}) \leq \mathbb{P}\big(\mathcal{O}_{s,X^{(0,x)}}; X_t^{(0,x)} > g(t)\big) \leq \mathbb{P}\big(X_t^{(0,x)} > g(t)\big). \tag{3.73}$$

Now, we partition and disintegrate on the value of $\Delta_1^{g(t)}$, which is exponentially distributed with rate $\overline{\Pi}(g(t))$. Then by (3.6), (3.72) and (3.8), using that $X \overset{d}{=} \tilde{X}^{(0,g(t))} + \tilde{X}^{[g(t),\infty)}$, where the process $\tilde{X}^{(0,g(t))}$ is independent of $\tilde{X}^{[g(t),\infty)}$ and hence independent of any jumps of size in $[g(t), \infty)$, we get

$$\begin{aligned} \mathbb{P}(\mathcal{O}_t|\Delta_1^{g(t)}=s) &\overset{(3.6)}{=} \mathbb{P}(\mathcal{O}_s|\Delta_1^{g(t)}=s) \overset{(3.72)}{=} \mathbb{P}(\mathcal{O}_{s-}|\Delta_1^{g(t)}=s) \overset{(3.8)}{=} \mathbb{P}(\mathcal{O}_{s-,X^{(0,g(t))}}) \overset{(3.72)}{=} \mathbb{P}(\mathcal{O}_{s,X^{(0,g(t))}}); \\ \mathbb{P}(\mathcal{O}_t; \Delta_1^{g(t)} = s) &\overset{(3.6)}{=} \mathbb{P}(\mathcal{O}_s; \Delta_1^{g(t)} = s) \end{aligned} \tag{3.74}$$

so it follows that

$$\begin{aligned} \mathbb{P}(\mathcal{O}_t) &= \mathbb{P}(\mathcal{O}_t; \Delta_1^{g(t)} \leq t) + \mathbb{P}(\mathcal{O}_t; \Delta_1^{g(t)} > t) \\ &= \overline{\Pi}(g(t)) \int_0^t \mathbb{P}(\mathcal{O}_t|\Delta_1^{g(t)} = s) e^{-\overline{\Pi}(g(t))s} ds + \mathbb{P}(\mathcal{O}_t; \Delta_1^{g(t)} > t) \\ &= \overline{\Pi}(g(t)) \int_0^t \mathbb{P}(\ \mathcal{O}_{s,X^{(0,g(t))}}\ )\ e^{-\overline{\Pi}(g(t))s} ds + \mathbb{P}(\mathcal{O}_t; \Delta_1^{g(t)} > t). \end{aligned} \tag{3.75}$$

Now, observe that by the definition (3.8), $\mathbb{P}(\mathcal{O}_s|\Delta_1^{g(t)} > s) = \mathbb{P}(\mathcal{O}_{s,X^{(0,g(t))}})$, so

$$\begin{aligned} \mathbb{P}(\mathcal{O}_s) &= \mathbb{P}(\mathcal{O}_s; \Delta_1^{g(t)} \leq s) + \mathbb{P}(\mathcal{O}_s; \Delta_1^{g(t)} > s) \\ &= \mathbb{P}(\mathcal{O}_s; \Delta_1^{g(t)} \leq s) + \mathbb{P}(\mathcal{O}_s|\Delta_1^{g(t)} > s)\mathbb{P}(\Delta_1^{g(t)} > s) \\ &= \mathbb{P}(\mathcal{O}_s; \Delta_1^{g(t)} \leq s) + \mathbb{P}\left(\mathcal{O}_{s,X^{(0,g(t))}}\right) e^{-\overline{\Pi}(g(t))s}. \end{aligned} \tag{3.76}$$

Disintegrating on $\Delta_1^{g(t)}$, recalling the notation (3.7), (3.9), by (3.75), (3.76), and (3.74),

$$\begin{aligned}
\mathbb{P}(\mathcal{O}_t) &= \overline{\Pi}(g(t)) \int_0^t \left[\mathbb{P}(\mathcal{O}_s) - \mathbb{P}(\mathcal{O}_s; \Delta_1^{g(t)} \le s)\right] ds + \mathbb{P}(\mathcal{O}_t; \Delta_1^{g(t)} > t) \\
&= \overline{\Pi}(g(t))\Phi(t) - \overline{\Pi}(g(t))^2 \int_0^t \int_0^s \mathbb{P}(\mathcal{O}_s | \Delta_1^{g(t)} = v) e^{-\overline{\Pi}(g(t))v} dv ds + \mathbb{P}(\mathcal{O}_t; \Delta_1^{g(t)} > t) \\
&\overset{(3.74)}{=} \overline{\Pi}(g(t))\Phi(t) - \overline{\Pi}(g(t))^2 \int_0^t \int_0^s \mathbb{P}(\mathcal{O}_{v, X^{(0,g(t))}}) e^{-\overline{\Pi}(g(t))v} dv ds + \mathbb{P}(\mathcal{O}_t; \Delta_1^{g(t)} > t) \\
&=: (a1) - (a2) + (a3). \qquad (3.77)
\end{aligned}$$

Firstly, let us show that $(a2) = o(\overline{\Pi}(g(t))\Phi(t))$ as $t \to \infty$. Indeed,

$$|(a2)| \le \overline{\Pi}(g(t))^2 \int_0^t \int_0^s \mathbb{P}(\mathcal{O}_{v, X^{(0,g(t))}}) dv ds \le \overline{\Pi}(g(t))^2 \int_0^t \int_0^t \mathbb{P}(\mathcal{O}_{v, X^{(0,g(t))}}) dv ds$$

$$= t\overline{\Pi}(g(t))^2 \int_0^t \mathbb{P}(\mathcal{O}_{v, X^{(0,g(t))}}) dv \le t\overline{\Pi}(g(t))^2 \int_0^t \mathbb{P}(\mathcal{O}_{v,X}) dv = t\overline{\Pi}(g(t))^2 \Phi(t) = o(\overline{\Pi}(g(t))\Phi(t)),$$

where the last equality holds by (3.3) in case (i), or by (3.5) in case (ii), since both imply $\lim_{t\to\infty} t\overline{\Pi}(g(t)) = 0$. Now, for $t \ge 1$,

$$\Phi(t) \ge \Phi(1) = \text{constant} > 0. \qquad (3.78)$$

We shall use (3.78) to prove $(a3) = o(\Phi(t)\overline{\Pi}(g(t)))$ as $t \to \infty$. For suitably large t, we partition $(a3)$ as

$$\begin{aligned}
(a3) &= \mathbb{P}(\mathcal{O}_t; \Delta_1^{\theta g(t)} < t; \Delta_1^{g(t)} > t) + \mathbb{P}(\mathcal{O}_t; \Delta_1^{g_1(t)} < t; \Delta_1^{\theta g(t)} > t) + \mathbb{P}(\mathcal{O}_t; \Delta_1^{g_1(t)} > t) \\
&=: (3A) + (3B) + (3C), \qquad (3.79)
\end{aligned}$$

with $g_1(t) := g(t)/\log(t)$, and $\theta \in (0,1)$. Later we split $(3B)$ into more pieces.

Proof for $(3A)$ We shall disintegrate on the value of $\Delta_1^{(\theta g(t),g(t))}$, the time of the first jump of size between $\theta g(t)$ and $g(t)$. As this time is exponentially distributed with rate parameter $\overline{\Pi}(\theta g(t)) - \overline{\Pi}(g(t))$,

$$\begin{aligned}(3A) &= e^{-t\overline{\Pi}(g(t))}\mathbb{P}(\mathcal{O}_t; \Delta_1^{(\theta g(t),g(t))} < t | \Delta_1^{g(t)} > t) = e^{-t\overline{\Pi}(g(t))}\mathbb{P}(\mathcal{O}_{t,X^{(0,g(t))}}; \Delta_1^{(\theta g(t),g(t))} < t)\\ &= e^{-t\overline{\Pi}(g(t))}\int_0^t \mathbb{P}(\mathcal{O}_{t,X^{(0,g(t))}} | \Delta_1^{(\theta g(t),g(t))} = s)\Pi((\theta g(t), g(t)))e^{-(\overline{\Pi}(\theta g(t)) - \overline{\Pi}(g(t)))s}ds\\ &\leq (\overline{\Pi}(\theta g(t)) - \overline{\Pi}(g(t)))\int_0^t \mathbb{P}(\mathcal{O}_{t,X^{(0,g(t))}} | \Delta_1^{\theta g(t)} = s)ds\\ &\leq (\overline{\Pi}(\theta g(t)) - \overline{\Pi}(g(t)))\int_0^t \mathbb{P}(\mathcal{O}_{s,X^{(0,\theta g(t))}})ds \leq \left(\frac{\overline{\Pi}(\theta g(t))}{\overline{\Pi}(g(t))} - 1\right)\overline{\Pi}(g(t))\int_0^t \mathbb{P}(\mathcal{O}_s)ds. \end{aligned} \tag{3.80}$$

Now, $\lim_{\theta\to 1}\lim_{t\to\infty}\left(\overline{\Pi}(\theta g(t))/\overline{\Pi}(g(t)) - 1\right) = 0$, as $\overline{\Pi}$ is CRV at ∞ in both cases, and therefore by (3.80), $(3A) = o(\overline{\Pi}(g(t))\Phi(t))$ as $t \to \infty$ then $\theta \to 1$.

Partitioning $(3B)$ Recall the notation (3.6). Disintegrating on $\Delta_1^{g_1(t)}$, using the stationary independent increments property to write $X_t \stackrel{d}{=} X_s + \hat{X}_{t-s}$, for $\hat{X}$ an independent copy of X, as there are no jumps bigger than $\theta g(t)$ we have $X_t \stackrel{d}{=} X_s + \hat{X}_{t-s}^{(0,\theta g(t))}$, and so by (3.73) and (3.74), we can bound

$$\begin{aligned}(3B) &= \mathbb{P}(\mathcal{O}_t; \Delta_1^{g_1(t)} < t; \Delta_1^{\theta g(t)} > t)\\ &= \overline{\Pi}(g_1(t))\int_0^t \mathbb{P}(\mathcal{O}_t; \Delta_1^{\theta g(t)} > t | \Delta_1^{g_1(t)} = s)e^{-\overline{\Pi}(g_1(t))s}ds\\ &\leq \overline{\Pi}(g_1(t))\int_0^t \mathbb{P}(\mathcal{O}_s; X_t > g(t); \Delta_1^{\theta g(t)} > t | \Delta_1^{g_1(t)} = s)ds\\ &= \overline{\Pi}(g_1(t))\int_0^t \mathbb{P}(\mathcal{O}_s; \hat{X}_{t-s}^{(0,\theta g(t))} + X_{s-} + (X_s - X_{s-}) > g(t); \Delta_1^{\theta g(t)} > t | \Delta_1^{g_1(t)} = s)ds,\end{aligned}$$

where $\hat{X}^{(0,\theta g(t))}$ is an independent copy of X with no jumps bigger than $\theta g(t)$ as in the definition (3.8). Then as the jump at time $\Delta_1^{g_1(t)} = s$ has size $X_s - X_{s-} \leq \theta g(t)$,

$$\begin{aligned}(3B) &\leq \overline{\Pi}(g_1(t))\int_0^t \mathbb{P}(\mathcal{O}_{s,X^{(0,g_1(t))}}; \hat{X}_{t-s}^{(0,\theta g(t))} + X_{s-}^{(0,g_1(t))} > (1-\theta)g(t); \Delta_1^{\theta g(t)} > t)ds\\ &\leq \overline{\Pi}(g_1(t))\int_0^t \mathbb{P}(\mathcal{O}_{s,X^{(0,g_1(t))}}; \hat{X}_{t-s}^{(0,\theta g(t))} + X_{s-}^{(0,g_1(t))} > (1-\theta)g(t))ds.\end{aligned}$$

Partitioning according to $\mathcal{A} := \{X_{s-}^{(0,g_1(t))} > (1-\theta)g(t)/2\}$ and $\mathcal{A}^c$, using that X and $\hat{X}$ are independent,

$$\begin{aligned}
(3B) &\leq \overline{\Pi}(g_1(t)) \int_0^t \mathbb{P}(\mathcal{O}_{s,X^{(0,g_1(t))}}; \hat{X}_{t-s}^{(0,\theta g(t))} + X_{s-}^{(0,g_1(t))} > (1-\theta)g(t); \mathcal{A}) ds \\
&+ \overline{\Pi}(g_1(t)) \int_0^t \mathbb{P}(\mathcal{O}_{s,X^{(0,g_1(t))}}; \hat{X}_{t-s}^{(0,\theta g(t))} + X_{s-}^{(0,g_1(t))} > (1-\theta)g(t); \mathcal{A}^c) ds \\
&\leq \overline{\Pi}(g_1(t)) \int_0^t \mathbb{P}\left(X_{s-}^{(0,g_1(t))} > \frac{(1-\theta)g(t)}{2}\right) ds \\
&+ \overline{\Pi}(g_1(t)) \int_0^t \mathbb{P}\left(\mathcal{O}_{s,X^{(0,g_1(t))}}; \hat{X}_{t-s}^{(0,\theta g(t))} > \frac{(1-\theta)g(t)}{2}\right) ds \\
&\leq t\overline{\Pi}(g_1(t))\mathbb{P}\left(X_t^{(0,g_1(t))} > \frac{(1-\theta)g(t)}{2}\right) \\
&+ \overline{\Pi}(g_1(t))\mathbb{P}\left(X_t^{(0,\theta g(t))} > \frac{(1-\theta)g(t)}{2}\right) \int_0^t \mathbb{P}\left(\mathcal{O}_s\right) ds. \\
&=: (J1) + (J2). \qquad (3.81)
\end{aligned}$$

Proof for $(J1)$, Case (i) Recall that $\overline{\Pi}(x) = x^{-\alpha}L(x)$ for L slowly varying. Then by Potter's theorem (Theorem A.4.2), $\overline{\Pi}(g_1(t)) \lesssim \log(t)^{2\alpha}\overline{\Pi}(g(t))$ as $t \to \infty$. Applying Lemma 3.6.1 with $H(t) = t^{-n}, n > 1$, as $t \to \infty$,

$$\begin{aligned}
(J1) &\lesssim t\log(t)^{2\alpha}\overline{\Pi}(g(t))\mathbb{P}\left(X_t^{(0,g_1(t))} > \frac{(1-\theta)g(t)}{2}\right) \overset{3.6.1}{\lesssim} t\log(t)^{2\alpha}\overline{\Pi}(g(t)) \exp\left((*)\right) t^{-n}, \\
(*) &\lesssim nt\log(t)t^{\frac{2n}{(1-\theta)\log(t)}}\overline{\Pi}(g_1(t))\frac{2g_1(t)}{(1-\theta)g(t)} \lesssim te^{\frac{2n}{1-\theta}}\overline{\Pi}(g_1(t)) \lesssim t\overline{\Pi}(g_1(t)).
\end{aligned}$$

By (3.3), $\lim_{t\to\infty}(*) = 0$ for each $n > 1$ and $\theta \in (0,1)$, so in case (i), using (3.78), we can conclude that $(J1) \lesssim \log(t)^{2\alpha}\overline{\Pi}(g(t))t^{-(n-1)} = o(\overline{\Pi}(g(t)\Phi(t))$ as $t \to \infty$ then $\theta \to 1$.

Proof for $(J1)$, Case (ii) In case (ii), for each fixed θ, let us replace $g_1(t)$ by $cg(t)$, for small $c \in (0,\theta)$, and replace $(J1)$ by $(J1')$ with $cg(t)$ in place of $g_1(t)$. As $\overline{\Pi}$ is $\mathcal{O}$-regularly varying at ∞, as $t \to \infty$,

$$\begin{aligned}
(J1') &= t\overline{\Pi}(cg(t))\mathbb{P}\left(X_t^{(0,cg(t))} > \frac{(1-\theta)g(t)}{2}\right) \\
&\lesssim t\overline{\Pi}(g(t))\mathbb{P}\left(X_t^{(0,cg(t))} > \frac{(1-\theta)g(t)}{2}\right).
\end{aligned}$$

By Lemma 3.6.1 with $H(t) = t^{-n}$, for $\varepsilon > 0$ as in (3.5) and $n > 1 + \varepsilon > 1$, taking c small enough that $2cn/(1-\theta) \le 1$, it follows that as $t \to \infty$,

$$\begin{aligned}(J1') &\lesssim \overline{\Pi}(g(t)) \exp\left((*)\right) t^{-(n-1)},\\ (*) &\lesssim t \log(t^n) t^{\frac{2cn}{(1-\theta)}} \overline{\Pi}(cg(t)) \frac{2cg(t)}{(1-\theta)g(t)}\\ &= tn \log(t) e^{\frac{2cn}{1-\theta}} \overline{\Pi}(cg(t))\\ &\lesssim t \log(t) \overline{\Pi}(g(t))\\ &\lesssim t^{1+\varepsilon} \overline{\Pi}(g(t)).\end{aligned}$$

By (3.5), $\lim_{t\to\infty}(*) = 0$ by so for each fixed $\theta \in (0,1)$, $(J1') = o(\overline{\Pi}(g(t)))$ as $t \to \infty$ in case (ii), and by (3.78), $(J1') = o(\overline{\Pi}(g(t))\Phi(t))$ as $t \to \infty$ then $\theta \to 1$.

Proof for ($J2$), Case (i) We begin with case (i). Recall that $\Phi(t) := \int_0^t \mathbb{P}(\mathcal{O}_{s,X})ds$. Applying Markov's inequality (Theorem A.2.1),

$$\begin{aligned}(J2) = {} & \overline{\Pi}(g_1(t))\Phi(t)\mathbb{P}\left(X_t^{(0,\theta g(t))} > \frac{(1-\theta)g(t)}{2}\right) \le \frac{2\overline{\Pi}\left(g_1(t)\right)\Phi(t)}{(1-\theta)g(t)} \mathbb{E}\left[X_t^{(0,\theta g(t))}\right]\\ = {} & \frac{2t\overline{\Pi}\left(g_1(t)\right)\Phi(t)}{(1-\theta)g(t)} \int_0^{\theta g(t)} x\Pi(dx) = \frac{2t\overline{\Pi}\left(g_1(t)\right)\Phi(t)}{(1-\theta)g(t)} \int_0^{\theta g(t)} \left(\overline{\Pi}(x) - \overline{\Pi}(\theta g(t))\right) dx\\ \le {} & \frac{2t\overline{\Pi}(g_1(t))\Phi(t)}{(1-\theta)g(t)} \int_0^{\theta g(t)} \overline{\Pi}(x)dx.\end{aligned}$$

As $\overline{\Pi}$ is regularly varying at ∞, by Karamata's theorem (Theorem A.4.3), as $t \to \infty$,

$$(J2) \lesssim \frac{2t\overline{\Pi}(g_1(t))\Phi(t)}{(1-\theta)g(t)} \theta g(t) \overline{\Pi}(\theta g(t)) \lesssim t\overline{\Pi}(g_1(t))\Phi(t)\overline{\Pi}(g(t)).$$

By (3.3), $\lim_{t\to\infty} t\overline{\Pi}(g_1(t)) = 0$, so $(J2) = o(\overline{\Pi}(g(t))\Phi(t))$ as $t \to \infty$ for each fixed $\theta \in (0,1)$.

Proof for $(J2)$, Case (ii) For case (ii), recall we have $(J2')$ with $cg(t)$ in place of $g_1(t)$, for $c \in (0, \theta)$. By Lemma 3.6.1 with $H(t) = 1/\log(\log(t))$, since $\overline{\Pi}(c(g(t)) \lesssim \overline{\Pi}(g(t))$ as $t \to \infty$,

$$
\begin{aligned}
(J2') &= \overline{\Pi}(cg(t))\Phi(t)\mathbb{P}\left(X_t^{(0,\theta g(t))} > \frac{(1-\theta)g(t)}{2}\right) \\
&\lesssim \frac{\overline{\Pi}(g(t))\Phi(t)}{\log(\log(t))} \exp\left((*)\right), \\
(*) &= t\log(\log(\log(t)))\log(\log(t))^{\frac{2\theta}{1-\theta}}\overline{\Pi}(\theta g(t))\frac{2\theta}{1-\theta} \lesssim t\log(t)^2\overline{\Pi}(g(t)).
\end{aligned}
$$

For $\varepsilon > 0$ as in (3.5), observe that $\log(t)^2 = o(t^\varepsilon)$ as $t \to \infty$, and it follows that $(*) = o(t^{1+\varepsilon}\overline{\Pi}(g(t)))$ as $t \to \infty$, for each fixed $\theta \in (0,1)$. Then by (3.5), $(J2') = o(\overline{\Pi}(g(t))\Phi(t))$ as $t \to \infty$ for each fixed $\theta \in (0,1)$, in case (ii).

Proof for $(3C)$, Case (i) In case (i), by (3.73) and Lemma 3.6.1 with $H(t) = \overline{\Pi}(g(t))^2$, as $t \to \infty$,

$$
(3C) = \mathbb{P}(\mathcal{O}_t; \Delta_1^{g_1(t)} > t) \le \mathbb{P}\left(X_t^{(0,g_1(t))} > g(t)\right) \le \exp\left((*)\right)\overline{\Pi}(g(t))^2, \tag{3.82}
$$

$$
(*) \lesssim t\log\left(\overline{\Pi}(g(t))^{-2}\right)\overline{\Pi}(g(t))^{-\frac{2}{\log(t)}}\frac{\overline{\Pi}(g_1(t))}{\log(t)} \lesssim \frac{t\log\left(\frac{1}{\overline{\Pi}(g(t))}\right)}{\log(t)}\left(\frac{1}{\overline{\Pi}(g(t))}\right)^{\frac{2}{\log(t)}}\overline{\Pi}(g_1(t)).
$$

Now we split into subsets of $t > 0$ (if one subset is bounded, we need only consider the other). Fix $M > 6/\alpha$. First we consider all $t > 0$ for which $g(t) \le t^M$. Recall $\overline{\Pi}(x) = x^{-\alpha}L(x)$ with L slowly varying at ∞, and moreover $\lim_{t\to\infty} t\overline{\Pi}(g_1(t)) = 0$ by (3.3). Then as $g(t)^{1/M} \le t \to \infty$, we have

$$
(*) \lesssim \frac{\log\left(\frac{1}{\overline{\Pi}(g(t))}\right)}{\log(t)}\left(\frac{1}{\overline{\Pi}(g(t))}\right)^{\frac{2}{\log(t)}} = \frac{\log\left(g(t)^\alpha L(g(t))^{-1}\right)}{\log(t)}g(t)^{\frac{2\alpha}{\log(t)}}L(g(t))^{-\frac{2}{\log(t)}}.
$$

As L is slowly varying at ∞, $g(t)^\alpha L(g(t))^{-1} \le g(t)^{2\alpha}$, so as $g(t)^{1/M} \le t \to \infty$,

$$(*) \lesssim \frac{\log\left(g(t)^{2\alpha}\right)}{\log(t)} g(t)^{\frac{4\alpha}{\log(t)}} \lesssim \frac{\log\left(t^{2\alpha M}\right)}{\log(t)} t^{\frac{4\alpha M}{\log(t)}} = 2\alpha M e^{4\alpha M} < \infty.$$

Therefore for all $t > 0$ for which $g(t) \le t^M$, we can conclude by (3.82) and (3.78) that $(3C) \lesssim \overline{\Pi}(g(t))^2 = o(\overline{\Pi}(g(t))\Phi(t))$ as $t \to \infty$.

For all $t > 0$ such that $g(t) \ge t^M$, as $t \to \infty$, $\log(1/\overline{\Pi}(g(t))) \le \overline{\Pi}(g(t))^{-1/3}$, $\overline{\Pi}(g(t))^{-2/\log(t)} \le \overline{\Pi}(g(t))^{-1/3}$, and $\overline{\Pi}(g(t)) = g(t)^{-\alpha} L(g(t)) \lesssim g(t)^{-\alpha} g(t)^{\alpha/2} = g(t)^{-\alpha/2} \le t^{-\alpha M/2}$. Applying Potter's theorem (Theorem A.4.2) to $\overline{\Pi}(g_1(t))/\overline{\Pi}(g(t)) = \log(t)^{-\alpha} L(g_1(t))/L(g(t))$, for each $\delta > 0$, as $g(t)^{1/M} \ge t \to \infty$,

$$\begin{aligned}
(*) &\lesssim \frac{t \log\left(\frac{1}{\overline{\Pi}(g(t))}\right)}{\log(t)} \left(\frac{1}{\overline{\Pi}(g(t))}\right)^{\frac{2}{\log(t)}} \overline{\Pi}(g_1(t)) \lesssim \frac{t\overline{\Pi}(g(t))^{-\frac{1}{3}}}{\log(t)} \overline{\Pi}(g(t))^{-\frac{1}{3}} \overline{\Pi}(g_1(t)) \\
&\lesssim t\overline{\Pi}(g(t))^{\frac{1}{3}} \log(t)^{-1-\alpha+\delta} = t g(t)^{-\frac{\alpha}{3}} L(g(t))^{\frac{1}{3}} \log(t)^{-1-\alpha+\delta} \\
&\lesssim t g(t)^{-\frac{\alpha}{6}} \log(t)^{-1-\alpha+\delta} \lesssim t^{1-\frac{\alpha M}{6}} \log(t)^{-\alpha+\delta}.
\end{aligned}$$

Recall $M > 6/\alpha$, so $1 - \alpha M/6 < 0$, and $\lim_{t\to\infty}(*) = 0$. Then by (3.78) and (3.82), $(3C) = o(\overline{\Pi}(g(t))\Phi(t))$ as $t \to \infty$ in case (i), as required.

Proof for ($3C$), Case (ii) In case (ii), recall we have ($3C'$) with $cg(t)$ in place of $g_1(t)$. By (3.73), applying Lemma 3.6.1 with $H(t) = \overline{\Pi}(g(t))^2$, for arbitrarily small $\eta > 0$, as $t \to \infty$,

$$\begin{aligned}(3C') &= P(\mathcal{O}_t; \Delta_1^{cg(t)} > t) \overset{(3.73)}{\leq} \mathbb{P}(X_t^{(0,cg(t))} > g(t)) \\ &\lesssim \exp\left((*)\right)\overline{\Pi}(g(t))^2, \\ (*) &= t\log\left(\overline{\Pi}(g(t))^{-2}\right)\overline{\Pi}(g(t))^{-2c}c\overline{\Pi}(cg(t)) \\ &\lesssim t\log(1/\overline{\Pi}(g(t)))\overline{\Pi}(g(t))^{-2c}\overline{\Pi}(g(t)) \\ &\lesssim t\overline{\Pi}(g(t))^{1-\eta-2c}.\end{aligned}$$

By (3.5), taking c, η sufficiently small, $\lim_{t\to\infty}(*) = 0$, then by (3.78), $(3C') \lesssim \overline{\Pi}(g(t))^2 = o(\overline{\Pi}(g(t))\Phi(t))$ as $t \to \infty$ then $\theta \to 1$ in case (ii), as required.

□

3.7 Proof of Lemma 3.5.3

In order to prove Lemma 3.5.3, we require Lemma 3.7.1, which is proven in Section 3.9.

Lemma 3.7.1. *Recalling* (3.12) *and* (3.22)*, if* $t > t_0(y)$*, then for all* $y, h > 0$*, we have* $g_y^h(t) \geq (1 - 1/A)\, g(t)$.

Lemma 3.5.3 shall be proven by splitting up $\rho_y^h(t)$ into smaller pieces, and then showing that the inequalities (3.30) and (3.31) hold for each piece separately.

Proof of Lemma 3.5.3. The strategy for proving Lemma 3.5.3 involves splitting up the probability $\mathbb{P}(\mathcal{O}_t^{g_y^h})$ into several smaller pieces, which we bound separately. We thus refer the reader to Figure A.3, which provides a guide for following the structure of the proof in case (i). Similarly, Figure A.4 shows the structure of the proof in case (ii). As with Lemma 3.5.1, Lemma 3.5.3 is simpler to prove in case (ii) than case (i) thanks to the condition (3.5). Now, recall the notation introduced in (3.8) and (3.14). First, repeating the argument as in (3.77), but now with $g_y^h(t)$ in place of $g(t)$, we get

$$\begin{aligned}\mathbb{P}(\mathcal{O}_t^{g_y^h}) &= \overline{\Pi}(g_y^h(t))\Phi_y^h(t) + \mathbb{P}\left(\mathcal{O}_t^{g_y^h}; \Delta_1^{g_y^h(t)} > t\right) - \overline{\Pi}(g_y^h(t))^2 \int_0^t \int_0^s \mathbb{P}(\mathcal{O}_v^{g_y^h}) e^{-\overline{\Pi}(g_y^h(t))v} dvds \\ &\leq \overline{\Pi}(g_y^h(t))\Phi_y^h(t) + \mathbb{P}\left(\mathcal{O}^{g_y^h}_{t,X^{(0,g_y^h(t))}}\right) - \overline{\Pi}(g_y^h(t))^2 \int_0^t \int_0^s \mathbb{P}(\mathcal{O}_v^{g_y^h}) e^{-\overline{\Pi}(g_y^h(t))v} dvds \\ &\leq \overline{\Pi}(g_y^h(t))\Phi_y^h(t) + \mathbb{P}\left(\mathcal{O}^{g_y^h}_{t,X^{(0,g_y^h(t))}}\right). \end{aligned} \tag{3.83}$$

Recall the notation (3.9), (3.10), and (3.16). By (3.83), partitioning on the value of $\Delta_1^{\left(\frac{g_y^h(t)}{\log(t)}, g_y^h(t)\right)}$, the time of the first jump of size between $g_y^h(t)/\log(t)$ and $g_y^h(t)$, as there are no jumps bigger than $g_y^h(t)$, we have

$$\begin{aligned}\rho_y^h(t) :&= \frac{\mathbb{P}(\mathcal{O}_t^{g_y^h})}{\Phi_y^h(t)} - \overline{\Pi}(g_y^h(t)) \overset{(3.83)}{\leq} \frac{1}{\Phi_y^h(t)}\mathbb{P}\left(\mathcal{O}^{g_y^h}_{t,X^{(0,g_y^h(t))}}\right) \\ &= \frac{1}{\Phi_y^h(t)}\left[\mathbb{P}\Big(\mathcal{O}^{g_y^h}_{t,X^{(0,g_y^h(t))}}; \Delta_1^{\frac{g_y^h(t)}{\log(t)}} > t\Big) + \mathbb{P}\Big(\mathcal{O}^{g_y^h}_{t,X^{(0,g_y^h(t))}}; \Delta_1^{\left(\frac{g_y^h(t)}{\log(t)}, g_y^h(t)\right)} \leq t\Big)\right] \\ &=: \frac{1}{\Phi_y^h(t)}\left[(a) + (b)\right]. \end{aligned} \tag{3.84}$$

So to prove (3.30), we need to prove, uniformly in $h > 0, y > g(h)$, as $t \to \infty$,

$$(a) + (b) \lesssim \frac{\Phi_y^h(t)}{t\log(t)^{1+\varepsilon}}\left(1 + \frac{1}{f(y) - h}\right). \tag{3.85}$$

For (3.31), we need suitable u so that uniformly in $h > 0, y > g(h)$, as $t \to \infty$,

$$(a) + (b) \leq \Phi_y^h(t)u(t)\overline{\Pi}(g(t))\left(1 + \frac{1}{f(y) - h}\right). \tag{3.86}$$

Recall that in case (ii), the inequality (3.31) implies the inequality (3.30), so we need only prove that (3.86) holds in case (ii).

Proof for (a), Case (i) Recall the notation (3.13), (3.8). By Lemma 3.7.1, for all $h > 0$, $y > g(h)$, $g_y^h(t)/\log(t) \geq (1 - A^{-1})g(t)/\log(t) > 1$ as $t \to \infty$, so by (3.73), Lemma 3.5.10, and Lemma 3.6.1 with

$H(t) = t^{-n}$, for $n > 1$, uniformly in $h > 0$, $y > g(h)$, as $t \to \infty$,

$$\begin{aligned}
(a) &= \mathbb{P}\Big(\mathcal{O}^{g_y^h}_{t,X^{(0,g_y^h(t))}}; \Delta_1^{\frac{g_y^h(t)}{\log(t)}} > t\Big) \overset{(3.73)}{\leq} \mathbb{P}\Big(X_t^{(0,\frac{g_y^h(t)}{\log(t)})} > g_y^h(t)\Big) \\
&\overset{3.5.10}{\leq} \frac{\Phi_y^h(t)}{f(y)-h}\mathbb{P}\Big(X_t^{(0,\frac{g_y^h(t)}{\log(t)})} > g_y^h(t)\Big) \overset{3.6.1}{\leq} \frac{\Phi_y^h(t)}{f(y)-h}\exp\left((*)\right)t^{-n}, \qquad (3.87) \\
(*) &\lesssim t\log(t^n)\, t^{n\frac{g_y^h(t)}{g_y^h(t)\log(t)}}\ \overline{\Pi}\left(\frac{g_y^h(t)}{\log(t)}\right)\frac{g_y^h(t)}{g_y^h(t)\log(t)} \\
&= ne^n t\overline{\Pi}\left(\frac{g_y^h(t)}{\log(t)}\right). \qquad (3.88)
\end{aligned}$$

Now, by Lemma 3.7.1 and (3.3), $\lim_{t\to\infty} t\overline{\Pi}\left(g_y^h(t)/\log(t)\right) \leq \lim_{t\to\infty} t\overline{\Pi}\left((1-1/A)g(t)/\log(t)\right) = 0$, uniformly in $h > 0$ and $y > g(h)$, and thus $\lim_{t\to\infty}(*) = 0$. Then it follows that as $t \to \infty$, uniformly in y and h, $(a) \lesssim t^{-n}\Phi_y^h(t)/(f(y)-h) \leq t^{-1}\log(t)^{-1-\varepsilon}\Phi_y^h(t)/(f(y)-h)$, as required for (3.85).

To show $(a) \leq \Phi_y^h(t)u(t)\overline{\Pi}(g(t))/(f(y)-h)$, we split into two cases according to the size of $g(t)$. Fix large $M > 0$. By (3.87), for all $t > 0$ such that $g(t) \leq t^M$, we have $(a) \lesssim t^{-n}\Phi_y^h(t)/(f(y)-h)$, and then by choice of $n > 1$ large enough that $t^{-n} \leq \overline{\Pi}(g(t))^2$ (which is possible since $\overline{\Pi}(g(t)) \geq \overline{\Pi}(t^M) = t^{-\alpha M}L(t^M) \gtrsim t^{-\alpha M/2}$), it follows that $(a) \lesssim \Phi_y^h(t)u(t)\overline{\Pi}(g(t))/(f(y)-h)$ for suitable u (i.e. $\overline{\Pi}(g(t)) \leq u(t) = o(1)$ as $t \to \infty$), as $t \to \infty$, uniformly in $h > 0, y > g(h)$, as required for (3.86).

For all $t > 0$ with $g(t) \geq t^M$, by Lemma 3.6.1 with $H(t) = \overline{\Pi}(g(t))^2$, applying Lemma 3.7.1, as $\overline{\Pi}$ is regularly varying at ∞, uniformly in y, h, as $t \to \infty$,

$$\begin{aligned}
(a) &\overset{3.6.1}{\leq} \exp\left((*)\right)\overline{\Pi}(g(t))^2, \qquad (3.89) \\
(*) &\lesssim t\log\left(\overline{\Pi}(g(t))^{-2}\right)\ \overline{\Pi}(g(t))^{-\frac{2}{\log(t)}}\overline{\Pi}\left(\frac{g_y^h(t)}{\log(t)}\right)\frac{g_y^h(t)}{g_y^h(t)\log(t)} \\
&\overset{3.7.1}{\lesssim} t\overline{\Pi}(g(t))^{-\frac{2}{\log(t)}}\frac{\log\left(\overline{\Pi}(g(t))^{-2}\right)}{\log(t)}\overline{\Pi}\left(\frac{g(t)}{\log(t)}\right).
\end{aligned}$$

Now, note that as $t \to \infty$, for arbitrarily small $\eta > 0$, $\overline{\Pi}(g(t))^{-2/\log(t)} \le \overline{\Pi}(g(t))^{-\eta}$ and $\log\left(\overline{\Pi}(g(t))^{-2}\right) \le \overline{\Pi}(g(t))^{-\eta}$. Since $\overline{\Pi}(x) = x^{-\alpha}L(x)$, for L slowly varying at ∞, we have as $t \to \infty$,

$$(*) \lesssim t\frac{1}{\log(t)}\overline{\Pi}\left(\frac{g(t)}{\log(t)}\right)\overline{\Pi}\left(g(t)\right)^{-2\eta} = t\frac{1}{\log(t)}g(t)^{-(1-2\eta)\alpha}\log(t)^{\alpha}L\left(\frac{g(t)}{\log(t)}\right)L(g(t))^{-2\eta}$$
$$\lesssim tg(t)^{-(1-2\eta)\alpha}L\left(\frac{g(t)}{\log(t)}\right)L(g(t))^{-2\eta}.$$

As L is slowly varying, observe that $g(t) \ge t^M$ implies $L(g(t)/\log(t)) \lesssim (g(t)/\log(t))^{(1-2\eta)\alpha/8} \lesssim g(t)^{(1-2\eta)\alpha/4}$, and moreover $L(g(t))^{-2\eta} \lesssim g(t)^{(1-2\eta)\alpha/4}$. Then it follows that

$$(*) \lesssim tg(t)^{-\frac{(1-2\eta)\alpha}{2}}. \tag{3.90}$$

Choosing $M > 4/((1-2\eta)\alpha)$, we get $g(t)^{-(1-2\eta)\alpha/2} \le t^{-(1-2\eta)\alpha M/2} \le t^{-2}$, so $\lim_{t\to\infty}(*) = 0$. Applying (3.89) then Lemma 3.5.10, $(a) \lesssim \overline{\Pi}(g(t))^2 \le \Phi_y^h(t)u(t)\overline{\Pi}(g(t))/(f(y)-h)$ as $t \to \infty$, uniformly in $h > 0$, $y > g(h)$, for suitable u (i.e. $\overline{\Pi}(g(t)) \le u(t) = o(1)$ as $t \to \infty$), as required for (3.86).

In case (ii), we partition $\mathbb{P}\Big(\mathcal{O}^{g_y^h}_{t,X^{(0,g_y^h(t))}}\Big)$ differently to in case (i), which means we have (a') and (b') in place of (a) and (b). For small $c \in (0,1)$,

$$\mathbb{P}\left(\mathcal{O}^{g_y^h}_{t,X^{(0,g_y^h(t))}}\right) = \mathbb{P}\left(\mathcal{O}^{g_y^h}_{t,X^{(0,g_y^h(t))}};\Delta_1^{cg_y^h(t)} > t\right) + \mathbb{P}\left(\mathcal{O}^{g_y^h}_{t,X^{(0,g_y^h(t))}};\Delta_1^{(cg_y^h(t),g_y^h(t))} \le t\right)$$
$$=: (a') + (b'). \tag{3.91}$$

Proof for (a'), Case (ii) By (3.73) and Lemma 3.6.1 with $H(t) = \overline{\Pi}(g(t))^2$, as $t \to \infty$,

$$(a') \overset{(3.73)}{\le} \mathbb{P}\left(X_t^{(0,cg_y^h(t))} > g_y^h(t)\right) \overset{3.6.1}{\le} \exp\left((*)\right)\overline{\Pi}(g(t))^2, \tag{3.92}$$
$$(*) \lesssim t\log\left(\overline{\Pi}(g_y^h(t))^{-2}\right)\overline{\Pi}(g(t))^{-2c}\frac{cg_y^h(t)}{g_y^h(t)}\overline{\Pi}(g_y^h(t)).$$

By Lemma 3.7.1, $\overline{\Pi}(g_y^h(t)) \lesssim \overline{\Pi}(g(t))$ as $t \to \infty$, uniformly among y, h. So for arbitrarily small $\eta > 0$, as $t \to \infty$, uniformly among $h > 0, y > g(h)$,

$$(*) \quad \lesssim \quad t\overline{\Pi}(g(t))^{1-\eta-2c}.$$

Choosing c, η small enough, it follows from (3.5) that $\limsup_{t\to\infty}(*) = 0$, so by Lemma 3.5.10 and (3.92), $(a') \lesssim \overline{\Pi}(g(t))^2 \leq \Phi_y^h(t)u(t)\overline{\Pi}(g(t))/(f(y) - h)$ uniformly in $h > 0$, $y > g(h)$, for a suitable choice of u (i.e. $\overline{\Pi}(g(t)) \leq u(t) = o(1)$ as $t \to \infty$), as required for (3.86) in case (ii).

Partitioning (b), Case (i) Now we partition (b) in case (i). Let $\Delta_m^{(a,b)}$ denote the time of our subordinator's mth jump of size larger in (a, b), as in (3.10). With $\beta > 1$ as in (3.3), for $m > 0$ such that $m > \beta/(\beta - 1)$ and $m > 1/(\alpha(\beta - 1))$, for $c \in (0, 1)$ such that $1 - (m - 1)c > 0$, and for all t large enough that $1/\log(t) \leq c$,

$$\begin{aligned}
(b) &= \mathbb{P}\left(\mathcal{O}^{g_y^h}_{t,X^{(0,g_y^h(t))}}; \Delta_1^{\left(\frac{g_y^h(t)}{\log(t)}, g_y^h(t)\right)} \leq t\right) \\
&= \mathbb{P}\left(\mathcal{O}^{g_y^h}_{t,X^{(0,g_y^h(t))}}; \Delta_1^{\left(\frac{g_y^h(t)}{\log(t)}, cg_y^h(t)\right)} \leq t; \Delta_m^{\left(\frac{g_y^h(t)}{\log(t)}, cg_y^h(t)\right)} > t; \Delta_1^{cg_y^h(t)} > t\right) \\
&+ \mathbb{P}\left(\mathcal{O}^{g_y^h}_{t,X^{(0,g_y^h(t))}}; \Delta_m^{\left(\frac{g_y^h(t)}{\log(t)}, cg_y^h(t)\right)} \leq t; \Delta_1^{cg_y^h(t)} > t\right) \\
&+ \mathbb{P}\left(\mathcal{O}^{g_y^h}_{t,X^{(0,g_y^h(t))}}; \Delta_1^{(cg_y^h(t), g_y^h(t))} \leq t\right) \\
&=: (2A) + (2B) + (2C). \qquad (3.93)
\end{aligned}$$

Note we do not need to consider $(2A)$ or $(2B)$ in case (ii), since $(b') = (2C)$.

Proof for $(2A)$, Case (i) Disintegrating on the value of $\overline{\Delta}_1 := \Delta_1^{\left(\frac{g_y^h(t)}{\log(t)}, cg_y^h(t)\right)}$, as defined in (3.10), which is exponentially distributed with rate $\overline{\Pi}(g_y^h(t)/\log(t)) - \overline{\Pi}(cg_y^h(t))$,

$$\begin{aligned}(2A) &= \mathbb{P}\left(\mathcal{O}^{cg_y^h}_{t,X^{(0,g_y^h(t))}}; \Delta_1^{\left(\frac{g_y^h(t)}{\log(t)}, cg_y^h(t)\right)} \leq t; \Delta_m^{\left(\frac{g_y^h(t)}{\log(t)}, g_y^h(t)\right)} > t; \Delta_1^{cg_y^h(t)} > t\right) \\ &= \int_0^t \mathbb{P}\Big(\mathcal{O}^{g_y^h}_{t,X^{(0,g_y^h(t))}}; \Delta_m^{\left(\frac{g_y^h(t)}{\log(t)}, cg_y^h(t)\right)} > t; \Delta_1^{cg_y^h(t)} > t \;\Big|\; \overline{\Delta}_1 = s\Big)\mathbb{P}\Big(\overline{\Delta}_1 \in ds\Big) \\ &\leq \overline{\Pi}\left(\frac{g_y^h(t)}{\log(t)}\right)\int_0^t \mathbb{P}\Big(\mathcal{O}^{g_y^h}_{t,X^{(0,cg_y^h(t))}}; \Delta_m^{\left(\frac{g_y^h(t)}{\log(t)}, cg_y^h(t)\right)} > t \;\Big|\; \overline{\Delta}_1 = s\Big)ds.\end{aligned}$$

Now, with $\overline{\Delta}_k$ denoting the time of the kth jump of size between $g_y^h(t)/\log(t)$ and $cg_y^h(t)$, we have

$$\begin{aligned}&\mathbb{P}\Big(\mathcal{O}^{g_y^h}_{t,X^{(0,cg_y^h(t))}}; \Delta_m^{\left(\frac{g_y^h(t)}{\log(t)}, cg_y^h(t)\right)} > t \;\Big|\; \overline{\Delta}_1 = s\Big) \\ &= \sum_{k=1}^{m-1} \mathbb{P}\Big(\mathcal{O}^{g_y^h}_{t,X^{(0,cg_y^h(t))}} \;\Big|\; \overline{\Delta}_1 = s; \overline{\Delta}_{k+1} > t; \overline{\Delta}_k \leq t\Big)\mathbb{P}\Big(\overline{\Delta}_{k+1} > t; \overline{\Delta}_k \leq t \;\Big|\; \overline{\Delta}_1 = s\Big) \\ &\leq \sum_{k=1}^{m-1} \mathbb{P}\Big(\mathcal{O}^{g_y^h}_{t,X^{(0,cg_y^h(t))}} \;\Big|\; \overline{\Delta}_1 = s; \overline{\Delta}_{k+1} > t; \overline{\Delta}_k \leq t\Big),\end{aligned}$$

and then it follows that

$$(2A) \quad \leq \quad \sum_{k=1}^{m-1} \overline{\Pi}\left(\frac{g_y^h(t)}{\log(t)}\right)\int_0^t \mathbb{P}\Big(\mathcal{O}^{g_y^h}_{t,X^{(0,cg_y^h(t))}} \;\Big|\; \overline{\Delta}_1 = s; \overline{\Delta}_{k+1} > t; \overline{\Delta}_k \leq t\Big)ds.$$

Now, by (3.73), given that by time t there are k jumps of size $J_i \in [g_y^h(t)/\log(t), cg_y^h(t)]$, $1 \le i \le k$, we have $X_t^{(0,cg_y^h(t))} \overset{d}{=} X_t^{\left(0,\frac{g_y^h(t)}{\log(t)}\right)} + J_1 + \cdots + J_k \le X_t^{\left(0,\frac{g_y^h(t)}{\log(t)}\right)} + kcg_y^h(t)$, and so

$$\begin{aligned}
(2A) &\overset{(3.73)}{\le} \sum_{k=1}^{m-1} \overline{\Pi}\left(\frac{g_y^h(t)}{\log(t)}\right) \int_0^t \mathbb{P}\Big(X_t^{(0,cg_y^h(t))} > g_y^h(t) \Big| \overline{\Delta}_1 = s; \overline{\Delta}_{k+1} > t; \overline{\Delta}_k \le t\Big) ds \\
&\le \sum_{k=1}^{m-1} \overline{\Pi}\left(\frac{g_y^h(t)}{\log(t)}\right) \int_0^t \mathbb{P}\Big(X_t^{\left(0,\frac{g_y^h(t)}{\log(t)}\right)} > (1-kc)g_y^h(t)\Big) ds \\
&= \sum_{k=1}^{m-1} \overline{\Pi}\left(\frac{g_y^h(t)}{\log(t)}\right) t\, \mathbb{P}\Big(X_t^{\left(0,\frac{g_y^h(t)}{\log(t)}\right)} > (1-kc)g_y^h(t)\Big) \\
&\le (m-1)\overline{\Pi}\left(\frac{g_y^h(t)}{\log(t)}\right) t\, \mathbb{P}\Big(X_t^{\left(0,\frac{g_y^h(t)}{\log(t)}\right)} > (1-(m-1)c)g_y^h(t)\Big).
\end{aligned}$$

Now, $\lim_{t\to\infty} t\overline{\overline{\Pi}}(g_y^h(t)/\log(t)) \le \lim_{t\to\infty} t\overline{\overline{\Pi}}((1-1/A)g(t)/\log(t)) = 0$ by (3.3), uniformly in $h > 0, y > g(h)$ by Lemma 3.7.1. Applying Lemma 3.6.1 with $H(t) = t^{-2}$, as $g_y^h(t) \ge (1-1/A)g(t)$, uniformly in $h > 0, y > g(h)$, as $t \to \infty$,

$$\begin{aligned}
(2A) &\lesssim \mathbb{P}\Big(X^{\left(0,\frac{g_y^h(t)}{\log(t)}\right)} > (1-(m-1)c)g_y^h(t)\Big) \le \exp\left((*)\right) t^{-2}, \qquad (3.94) \\
(*) &\lesssim t\log(t^2) t^{\frac{2}{(1-(m-1)c)\log(t)}} \overline{\Pi}\left(\frac{g_y^h(t)}{\log(t)}\right) \frac{g_y^h(t)}{\log(t)(1-(m-1)c)g_y^h(t)} \\
&= \frac{2}{1-(m-1)c} e^{\frac{2}{1-(m-1)c}} t\overline{\Pi}\left(\frac{g_y^h(t)}{\log(t)}\right).
\end{aligned}$$

Again using that $\lim_{t\to\infty} t\overline{\overline{\Pi}}(g_y^h(t)/\log(t)) \le \lim_{t\to\infty} t\overline{\overline{\Pi}}((1-1/A)g(t)/\log(t)) = 0$ by (3.3) and Lemma 3.7.1, it follows that $\lim_{t\to\infty}(*) = 0$, uniformly among $h > 0$ and $y > g(h)$, so by Lemma 3.5.10,

$$(2A) \lesssim t^{-2} \lesssim t^{-1}\log(t)^{-1-\varepsilon} \overset{3.5.10}{\le} t^{-1}\log(t)^{-1-\varepsilon}\Phi_y^h(t)/(f(y)-h)$$

as $t \to \infty$, uniformly in $h > 0$, $y > g(h)$, as required for (3.85).

Now we prove $(2A) \le \Phi_y^h(t)u(t)\overline{\overline{\Pi}}(g(t))$ in case (i). Fixing $M > 0$, we split into two subsets of $t > 0$. For all $t > 0$ such that $g(t) \le t^M$, by (3.94) and Lemma 3.6.1 with $H(t) = t^{-n}$, uniformly in $h > 0, y > g(h)$, as

$g(t)^{1/M} \leq t \to \infty$,

$$\begin{aligned}(2A) &\overset{(3.94)}{\lesssim} \mathbb{P}\left(X^{(0,\frac{g_y^h(t)}{\log(t)})} > (1-(m-1)c)g_y^h(t)\right) \leq \exp\left((*)\right) t^{-n},\\ (*) &\lesssim t\log(t^n)e^{\frac{n}{1-(m-1)c}}\overline{\Pi}\left(\frac{g_y^h(t)}{\log(t)}\right)\frac{g_y^h(t)}{\log(t)(1-(m-1)c)g_y^h(t)} \lesssim t\overline{\Pi}\left(\frac{g_y^h(t)}{\log(t)}\right),\end{aligned}$$

then by (3.3) and Lemma 3.7.1, it follows that $\lim_{t\to\infty}(*) = 0$, uniformly among $h > 0$, $y > g(h)$. Then choosing n large enough that $t^{-n} \leq \overline{\Pi}(g(t))^2$ (this is possible as $\overline{\Pi}$ is regularly varying at ∞ and $g(t) \leq t^M$), by Lemma 3.5.10, $(2A) \lesssim \overline{\Pi}(g(t))^2 \leq \Phi_y^h(t)u(t)\overline{\Pi}(g(t))/(f(y)-h)$ as $g(t)^{1/M} \leq t \to \infty$ for suitable u (i.e. $\overline{\Pi}(g(t)) \leq u(t) = o(1)$ as $t \to \infty$), uniformly in $h > 0, y > g(h)$, as required for (3.86).

For all $t > 0$ for which $g(t) \geq t^M$, by (3.94) and Lemma 3.6.1 with $H(t) = \overline{\Pi}(g(t))^2$, uniformly in $h > 0, y > g(h)$, as $g(t)^{1/M} \geq t \to \infty$,

$$\begin{aligned}(2A) &\overset{(3.94)}{\lesssim} \mathbb{P}\left(X^{(0,\frac{g_y^h(t)}{\log(t)})} > (1-(m-1)c)g_y^h(t)\right) \leq \exp\left((*)\right)\overline{\Pi}(g(t))^2, \qquad (3.95)\\ (*) &\lesssim t\log(\overline{\Pi}(g(t))^{-2})\overline{\Pi}(g(t))^{-\frac{2}{(1-(m-1)c)\log(t)}}\frac{\overline{\Pi}\big(\frac{g_y^h(t)}{\log(t)}\big)g_y^h(t)}{\log(t)(1-(m-1)c)g_y^h(t)}\\ &\lesssim t\log(\overline{\Pi}(g(t))^{-2})\overline{\Pi}(g(t))^{-\frac{2}{(1-(m-1)c)\log(t)}}\frac{\overline{\Pi}\big(\frac{g_y^h(t)}{\log(t)}\big)}{\log(t)}.\end{aligned}$$

As $\overline{\Pi}$ is regularly varying at ∞, $\overline{\Pi}(x) = x^{-\alpha}L(x)$, for L slowly varying at ∞. Observe that for each $\eta > 0$, as $t \to \infty$, $\overline{\Pi}(g(t))^{-2/((1-(m-1)c)\log(t))} \leq \overline{\Pi}(g(t))^{-\eta} = g(t)^{\alpha\eta}L(g(t))^{-\eta}$, and moreover for all large enough t, $\log(\overline{\Pi}(g(t))^{-2}) = \log(g(t)^{2\alpha}L(g(t))^{-2}) \leq g(t)^{\alpha\eta}L(g(t))^{\eta}$.

Then by Lemma 3.7.1, uniformly in $h > 0, y > g(h)$, as $g(t)^{1/M} \geq t \to \infty$,

$$\begin{aligned}
(*) \lesssim \quad & tg(t)^{2\alpha\eta}\frac{\overline{\Pi}\left(\frac{g_y^h(t)}{\log(t)}\right)}{\log(t)} \overset{3.7.1}{\lesssim} tg(t)^{2\alpha\eta}\frac{\overline{\Pi}\left(\frac{g(t)}{\log(t)}\right)}{\log(t)} \\
= \quad & tg(t)^{-(1-2\eta)\alpha}\log(t)^{\alpha-1}L\left(\frac{g(t)}{\log(t)}\right) \lesssim tg(t)^{-(1-2\eta)\alpha}\log(t)^{\alpha-1}g(t)^{\frac{(1-2\eta)\alpha}{2}}\log(t)^{-\frac{(1-2\eta)\alpha}{2}} \\
\leq \quad & tg(t)^{-\frac{(1-2\eta)\alpha}{2}}. \qquad (3.96)
\end{aligned}$$

Now, $g(t) \geq t^M$, so $tg(t)^{-(1-2\eta)\alpha} \leq t^{1-(1-2\eta)\alpha M/2}$, so taking $M > 2/((1-2\eta)\alpha)$, $\lim_{t\to\infty}(*) = 0$, uniformly in y, h. Then by (3.95) and Lemma 3.5.10, it follows tha $(2A) \leq \overline{\Pi}(g(t))^2 \lesssim \Phi_y^h(t)u(t)\overline{\Pi}(g(t))/(f(y)-h)$ as $t \to \infty$, for suitable u (i.e. $\overline{\Pi}(g(t)) \leq u(t) = o(1)$ as $t \to \infty$), uniformly in $h > 0, y > g(h)$, as required for (3.86).

Proof for $(2B)$, Case (i) Disintegrating on the value of $\Delta_1^{\left(\frac{g_y^h(t)}{\log(t)}, cg_y^h(t)\right)}$, which is exponentially distributed with parameter $\overline{\Pi}\left(g_y^h(t)/\log(t)\right) - \overline{\Pi}(cg_y^h(t)) \leq \overline{\Pi}\left(g_y^h(t)/\log(t)\right)$, using that $\mathbb{P}(\mathcal{O}_t) \leq \mathbb{P}(\mathcal{O}_s)$ for $s \leq t$,

$$\begin{aligned}
(2B) &= \mathbb{P}\left(\mathcal{O}^{g_y^h}_{t,X^{(0,g_y^h(t))}}; \Delta_m^{\left(\frac{g_y^h(t)}{\log(t)}, cg_y^h(t)\right)} \leq t; \Delta_1^{cg_y^h(t)} > t\right) \\
&\leq \overline{\Pi}\left(\frac{g_y^h(t)}{\log(t)}\right)\int_0^t \mathbb{P}\Big(\mathcal{O}^{g_y^h}_{t,X^{(0,g_y^h(t))}}; \Delta_m^{\left(\frac{g_y^h(t)}{\log(t)}, cg_y^h(t)\right)} \leq t; \Delta_1^{cg_y^h(t)} > t \;\Big|\; \Delta_1^{\left(\frac{g_y^h(t)}{\log(t)}, cg_y^h(t)\right)} = s\Big)ds \\
&\leq \overline{\Pi}\left(\frac{g_y^h(t)}{\log(t)}\right)\int_0^t \mathbb{P}\Big(\mathcal{O}^{g_y^h}_{t,X^{(0,cg_y^h(t))}}; \Delta_m^{\left(\frac{g_y^h(t)}{\log(t)}, cg_y^h(t)\right)} \leq t \;\Big|\; \Delta_1^{\left(\frac{g_y^h(t)}{\log(t)}, cg_y^h(t)\right)} = s\Big)ds \\
&\leq \overline{\Pi}\left(\frac{g_y^h(t)}{\log(t)}\right)\int_0^t \mathbb{P}\Big(\mathcal{O}^{g_y^h}_{s,X^{(0,cg_y^h(t))}}; \Delta_m^{\left(\frac{g_y^h(t)}{\log(t)}, cg_y^h(t)\right)} \leq t \;\Big|\; \Delta_1^{\left(\frac{g_y^h(t)}{\log(t)}, cg_y^h(t)\right)} = s\Big)ds,
\end{aligned}$$

By (3.72), $\mathcal{O}_s^{g_y^h}$ and $\mathcal{O}_{s-}^{g_y^h}$ are interchangeable. Observe that given $\Delta_1^{\left(\frac{g_y^h(t)}{\log(t)}, cg_y^h(t)\right)} = s$, the event $\mathcal{O}^{g_y^h}_{s-, X^{(0,g_y^h(t))}}$ is independent of $\Delta_m^{\left(\frac{g_y^h(t)}{\log(t)}, cg_y^h(t)\right)}$, and moreover $\mathbb{P}(\mathcal{O}_{s,X^{(0,x)}}) \leq \mathbb{P}(\mathcal{O}_s)$, so that

$$\begin{aligned}
(2B) &\leq \overline{\Pi}\left(\frac{g_y^h(t)}{\log(t)}\right)\int_0^t \mathbb{P}\Big(\mathcal{O}^{g_y^h}_{s-, X^{(0,cg_y^h(t))}}; \Delta_m^{\left(\frac{g_y^h(t)}{\log(t)}, cg_y^h(t)\right)} \leq t \mid \Delta_1^{\left(\frac{g_y^h(t)}{\log(t)}, cg_y^h(t)\right)} = s\Big)ds \\
&= \overline{\Pi}\left(\frac{g_y^h(t)}{\log(t)}\right)\int_0^t \mathbb{P}\Big(\mathcal{O}^{g_y^h}_{s-, X^{(0,cg_y^h(t))}} \mid \Delta_1^{\left(\frac{g_y^h(t)}{\log(t)}, cg_y^h(t)\right)} = s\Big)\mathbb{P}\Big(\Delta_m^{\left(\frac{g_y^h(t)}{\log(t)}, cg_y^h(t)\right)} \leq t \mid \Delta_1^{\left(\frac{g_y^h(t)}{\log(t)}, cg_y^h(t)\right)} = s\Big)ds \\
&= \overline{\Pi}\left(\frac{g_y^h(t)}{\log(t)}\right)\int_0^t \mathbb{P}\Big(\mathcal{O}^{g_y^h}_{s-, X^{(0,g_y^h(t)/\log(t))}}\Big)\mathbb{P}\Big(\Delta_m^{\left(\frac{g_y^h(t)}{\log(t)}, cg_y^h(t)\right)} \leq t \mid \Delta_1^{\left(\frac{g_y^h(t)}{\log(t)}, cg_y^h(t)\right)} = s\Big)ds \\
&\leq \overline{\Pi}\left(\frac{g_y^h(t)}{\log(t)}\right)\int_0^t \mathbb{P}\Big(\mathcal{O}^{g_y^h}_{s}\Big)\mathbb{P}\Big(\Delta_m^{\left(\frac{g_y^h(t)}{\log(t)}, cg_y^h(t)\right)} \leq t \mid \Delta_1^{\left(\frac{g_y^h(t)}{\log(t)}, cg_y^h(t)\right)} = s\Big)ds \\
&\leq \overline{\Pi}\left(\frac{g_y^h(t)}{\log(t)}\right)\mathbb{P}\Big(\Delta_{m-1}^{\left(\frac{g_y^h(t)}{\log(t)}, cg_y^h(t)\right)} \leq t\Big)\int_0^t \mathbb{P}\Big(\mathcal{O}^{g_y^h}_{s}\Big)ds \leq \overline{\Pi}\left(\frac{g_y^h(t)}{\log(t)}\right)\mathbb{P}\Big(\Delta_{m-1}^{\frac{g_y^h(t)}{\log(t)}} \leq t\Big)\int_0^t \mathbb{P}\Big(\mathcal{O}^{g_y^h}_{s}\Big)ds.
\end{aligned}$$

Now, since $\Delta_1^{\frac{g_y^h(t)}{\log(t)}}$ is exponentially distributed with parameter $\overline{\Pi}\left(g_y^h(t)/\log(t)\right)$,

$$\mathbb{P}\Big(\Delta_{m-1}^{\frac{g_y^h(t)}{\log(t)}} \leq t\Big) \leq \mathbb{P}\Big(\Delta_1^{\frac{g_y^h(t)}{\log(t)}} \leq t\Big)^{m-1} = \Big(1 - e^{-t\overline{\Pi}\left(\frac{g_y^h(t)}{\log(t)}\right)}\Big)^{m-1} \leq t^{m-1}\overline{\Pi}\left(\frac{g_y^h(t)}{\log(t)}\right)^{m-1},$$

so recalling the notation in (3.14), by Lemma 3.7.1, uniformly in y, h as $t \to \infty$,

$$(2B) \;\leq\; \overline{\Pi}\left(\frac{g_y^h(t)}{\log(t)}\right)^m t^{m-1}\Phi_y^h(t) \overset{3.7.1}{\lesssim} \overline{\Pi}\left(\frac{g(t)}{\log(t)}\right)^m t^{m-1}\Phi_y^h(t). \tag{3.97}$$

Recall $\overline{\Pi}(x) = x^{-\alpha}L(x)$ for L slowly varying at ∞, so by Potter's theorem (Theorem A.4.2), for arbitrarily small $\delta > 0$, as $t \to \infty$,

$$\overline{\Pi}\left(\frac{g(t)}{\log(t)}\right) = \log(t)^\alpha g(t)^{-\alpha}L(g(t))\frac{L\left(\frac{g(t)}{\log(t)}\right)}{L(g(t))} \lesssim \log(t)^{\alpha+\delta}\overline{\Pi}(g(t)). \tag{3.98}$$

Similarly, defining $g_\beta(t) := g(t)/\log(t)^\beta$, for $\beta > (1+\alpha)/(2\alpha+\alpha^2) > 1$ as in (3.3),

$$\overline{\Pi}\left(\frac{g(t)}{\log(t)}\right) = \log(t)^\alpha g(t)^{-\alpha} L\left(g_\beta(t)\right) \frac{L\left(\frac{g(t)}{\log(t)}\right)}{L(g_\beta(t))} \lesssim \log(t)^{\alpha(1-\beta)+\delta\beta}\overline{\Pi}(g_\beta(t)). \tag{3.99}$$

Applying (3.98) to $\overline{\Pi}(g(t))$ and (3.99) to $\overline{\Pi}(g(t))^{m-1}$, then by (3.3), as $t \to \infty$,

$$\begin{aligned}(2B) &\lesssim \log(t)^{m\alpha+m\delta-(m-1)\alpha\beta+\delta(\beta-1)(m-1)}\overline{\Pi}(g(t))\overline{\Pi}(g_\beta(t))^{m-1}t^{m-1}\Phi_y^h(t) \\ &\overset{(3.3)}{\leq} \log(t)^{m\alpha+m\delta-(m-1)\alpha\beta+\delta(\beta-1)(m-1)}\overline{\Pi}(g(t))\Phi_y^h(t).\end{aligned}$$

Now, for β as in (3.3), $m > \beta/(\beta-1)$, so $m\alpha - (m-1)\alpha\beta < 0$, choosing $\delta > 0$ small enough, we conclude $(2B) \leq u(t)\overline{\Pi}(g(t))\Phi_y^h(t)$ as $t \to \infty$, for suitable u (i.e. $\log(t)^{m\alpha+m\delta-(m-1)\alpha\beta+\delta(\beta-1)(m-1)} \leq u(t) = o(1)$ as $t \to \infty$), uniformly among $h > 0, y > g(h)$, as required for (3.86) in case (i).

Now we prove that $(2B) \lesssim t^{-1}\log(t)^{-1-\varepsilon}\Phi_y^h(t)$. With $g_\beta(t) = g(t)/\log(t)^\beta$, by (3.97) and (3.99), for arbitrarily small $\delta > 0$, as $t \to \infty$, uniformly in y, h,

$$(2B) \leq \overline{\Pi}\left(\frac{g(t)}{\log(t)}\right)^m t^{m-1}\Phi_y^h(t) \lesssim \log(t)^{m\alpha(1-\beta)+m\delta}\overline{\Pi}(g_\beta(t))^m t^{m-1}\Phi_y^h(t).$$

By (3.3), $\overline{\Pi}(g_\beta(t)) \leq t^{-1}$, and so $(2B) \lesssim t^{-1}\log(t)^{m\alpha(1-\beta)+m\delta}\Phi_y^h(t)$ as $t \to \infty$. Finally, our choice of m ensures that $m\alpha(1-\beta) < -1$, so choosing δ small enough, we can conclude that for some $\varepsilon > 0$, as $t \to \infty$, $(2B) \lesssim t^{-1}\log(t)^{-1-\varepsilon}\Phi_y^h(t)/(f(y)-h)$ uniformly in $h > 0$, $y > g(h)$, as required for (3.85) in case (i).

Partitioning $(2C) = (b')$ Now we continue with the proof for cases (i) and (ii) in tandem. Recall that $(2C) = (b')$ in case (ii). Define $p^*(t) := 1 - \log(t)^{-\gamma}$ for $\gamma := (1-\alpha)/(2+\alpha)$ in case (i) or $\gamma := 1$ in case (ii).

Let $\Delta_2^{(a,b)}$ denote the time of our subordinator's second jump of size in (a,b). Then we partition:

$$\begin{aligned}(2C) &= \mathbb{P}\left(\mathcal{O}^{g_y^h}_{t,X^{(0,g_y^h(t))}}; \Delta_1^{(cg_y^h(t),g_y^h(t))} \leq t\right)\\ &= \mathbb{P}\left(\mathcal{O}^{g_y^h}_{t,X^{(0,g_y^h(t))}}; \Delta_1^{(cg_y^h(t),g_y^h(t))} \leq t; \Delta_1^{(p^*(t)g_y^h(t),g_y^h(t))} \leq t\right)\\ &+ \mathbb{P}\left(\mathcal{O}^{g_y^h}_{t,X^{(0,g_y^h(t))}}; \Delta_2^{(cg_y^h(t),g_y^h(t))} \leq t; \Delta_1^{(p^*(t)g_y^h(t),g_y^h(t))} > t\right)\\ &+ \mathbb{P}\left(\mathcal{O}^{g_y^h}_{t,X^{(0,g_y^h(t))}}; \Delta_1^{(cg_y^h(t),g_y^h(t))} \leq t; \Delta_2^{(cg_y^h(t),g_y^h(t))} > t; \Delta_1^{(p^*(t)g_y^h(t),g_y^h(t))} > t\right)\\ &=: (2Ca) + (2Cb) + (2Cc). \end{aligned} \tag{3.100}$$

Now we will prove $(2Ca) \leq \Phi_y^h(t)u(t)\overline{\Pi}(g(t))$ for cases (i) and (ii) together, and then we will prove that $(2Ca) \lesssim t^{-1}\log(t)^{-1-\varepsilon}\Phi_y^h(t)$ in case (i) (recall we only need $(2Ca) \leq \Phi_y^h(t)u(t)\overline{\Pi}(g(t))$ in case (ii)).

Proof for $(2Ca)$ As $c \in (0,1)$ is fixed, $c < 1 - \log(t)^{-\gamma} = p^*(t)$ for all large enough t, and so

$$(2Ca) = \mathbb{P}\left(\mathcal{O}^{g_y^h}_{t,X^{(0,g_y^h(t))}}; \Delta_1^{(p^*(t)g_y^h(t),g_y^h(t))} \leq t\right).$$

Disintegrating on the value of $\Delta_1^{(p^*(t)g_y^h(t),g_y^h(t))}$, which is exponentially distributed with rate parameter $\overline{\Pi}(p^*(t)g_y^h(t)) - \overline{\Pi}(g_y^h(t))$, then by (3.73) and the independence as in (3.74), it follows that

$$\begin{aligned}(2Ca) &\leq \left[\overline{\Pi}\left(p^*(t)g_y^h(t)\right) - \overline{\Pi}\left(g_y^h(t)\right)\right]\int_0^t \mathbb{P}\left(\mathcal{O}^{g_y^h}_{t,X^{(0,g_y^h(t))}} \,\middle|\, \Delta_1^{(p^*(t)g_y^h(t),g_y^h(t))} = s\right) ds\\ &\overset{(3.74)}{\leq} \left[\overline{\Pi}\left(p^*(t)g_y^h(t)\right) - \overline{\Pi}\left(g_y^h(t)\right)\right]\int_0^t \mathbb{P}\left(\mathcal{O}^{g_y^h}_{s,X^{(0,p^*(t)g_y^h(t))}}\right) ds\\ &\leq \left[\overline{\Pi}\left(p^*(t)g_y^h(t)\right) - \overline{\Pi}\left(g_y^h(t)\right)\right]\Phi_y^h(t).\end{aligned}$$

By Lemma 3.7.1, $\overline{\Pi}(g_y^h(t)) \lesssim \overline{\Pi}(g(t))$ uniformly across y, h, so we can write

$$(2Ca) \leq \overline{\Pi}\left(g_y^h(t)\right)\left(\frac{\overline{\Pi}\left(p^*(t)g_y^h(t)\right)}{\overline{\Pi}\left(g_y^h(t)\right)} - 1\right)\Phi_y^h(t) \lesssim \overline{\Pi}\left(g(t)\right)\left(\frac{\overline{\Pi}\left(p^*(t)g_y^h(t)\right)}{\overline{\Pi}\left(g_y^h(t)\right)} - 1\right)\Phi_y^h(t). \tag{3.101}$$

As $\lim_{t\to\infty} p^*(t) = 1$ and $\overline{\Pi}$ is CRV at ∞ in both cases, $\lim_{t\to\infty} \overline{\Pi}(p^*(t)g_y^h(t))/\overline{\Pi}(g_y^h(t)) = 1$. Now, $g_y^h(t) \gtrsim g(t)$ uniformly in $h > 0, y > g(h)$, by Lemma 3.7.1, so by (3.101), $(2Ca) \leq \Phi_y^h(t)u(t)\overline{\Pi}(g(t))$ for suitable u satisfying

$$\frac{\overline{\Pi}\left(p^*(t)g_y^h(t)\right)}{\overline{\Pi}\left(g_y^h(t)\right)} - 1 \leq u(t) = o(1) \text{ as } t \to \infty,$$

uniformly in $h > 0$, $y > g(h)$, as required for (3.86) in cases (i) and (ii).

Now we prove $(2Ca) \lesssim t^{-1}\log(t)^{-1-\varepsilon}\Phi_y^h(t)(1 + 1/(f(y) - h))$ in case (i). As $\overline{\Pi}$ is regularly varying at ∞, $\overline{\Pi}(x) = x^{-\alpha}L(x)$ for L slowly varying at ∞, and by (3.101), as $t \to \infty$, uniformly in $h > 0, y > g(h)$,

$$(2Ca) \lesssim \overline{\Pi}\left(g(t)\right)\left(\frac{\overline{\Pi}\left(p^*(t)g_y^h(t)\right)}{\overline{\Pi}\left(g_y^h(t)\right)} - 1\right)\Phi_y^h(t) = \overline{\Pi}\left(g(t)\right)\left(\frac{p^*(t)^{-\alpha}L(p^*(t)g_y^h(t))}{L(g_y^h(t))} - 1\right)\Phi_y^h(t).$$

By Lemma 3.7.1, $g_y^h(t) \gtrsim g(t)$ uniformly in $y > g(h), h > 0$, as $t \to \infty$. Applying Potter's theorem (Theorem A.4.2) to $L(p^*(t)g_y^h(t))/L(g_y^h(t))$, for arbitrarily small $\delta > 0$, uniformly in $y > g(h), h > 0$, as $t \to \infty$,

$$\begin{aligned}(2Ca) &\lesssim \left(p^*(t)^{-\alpha-\delta} - 1\right)\overline{\Pi}(g(t))\Phi_y^h(t) \\ &= \left(\left(1 - \log(t)^{-\gamma}\right)^{-\alpha-\delta} - 1\right)\overline{\Pi}(g(t))\Phi_y^h(t) \leq \frac{\log(t)^{-\gamma}}{1 - \log(t)^{-\gamma}}\overline{\Pi}(g(t))\Phi_y^h(t),\end{aligned}$$

where the last inequality holds since if $\alpha + \delta < 1$ and $x \in (0, 1)$, then $(1 - x)^{-\alpha-\delta} - 1 \leq x/(1 - x)$. Now, for β as in (3.3), $g_\beta(t) := g(t)/\log(t)^\beta$, applying Potter's theorem to $\overline{\Pi}(g(t))/\overline{\Pi}(g_\beta(t))$, we get by (3.3) that for arbitrarily small $\tau > 0$, $\overline{\Pi}(g(t)) \lesssim \overline{\Pi}(g_\beta(t))\log(t)^{-\alpha\beta+\beta\tau} \leq t^{-1}\log(t)^{-\alpha\beta+\beta\tau}$ as $t \to \infty$. Then observing that $\lim_{t\to\infty}(1 - \log(t)^{-\gamma}) = 1$, it follows that uniformly in y, h, as $t \to \infty$,

$$(2Ca) \lesssim t^{-1}\log(t)^{-\gamma-\alpha\beta+\beta\tau}\Phi_y^h(t).$$

Note that $\beta > (1 + 2\alpha)/(2\alpha + \alpha^2)$ implies that $\gamma = (1 - \alpha)/(2 + \alpha) > 1 - \alpha\beta$, so we may choose τ sufficiently small that $-\gamma - \alpha\beta + \beta\tau < -1 - \varepsilon < -1$. Then it follows that as $t \to \infty$, uniformly among $h > 0$, $y > g(h)$, $(2Ca) \lesssim t^{-1}\log(t)^{-1-\varepsilon}\Phi_y^h(t) \leq t^{-1}\log(t)^{-1-\varepsilon}\Phi_y^h(t)(1 + 1/(f(y) - h))$, as required for (3.85) in case (i).

Proof for $(2Cb)$, Case (i) First we prove $(2Cb) \lesssim t^{-1}\log(t)^{-1-\varepsilon}\Phi^h_y(t)$ in case (i). Disintegrating on the value of $\Delta_1^{(cg^h_y(t),g^h_y(t))}$, which is exponentially distributed with rate $\overline{\Pi}(cg^h_y(t)) - \overline{\Pi}(g^h_y(t))$, by independence and by (3.72),

$$\begin{aligned}
(2Cb) &= \mathbb{P}\left(\mathcal{O}^{g^h_y}_{t,X^{(0,g^h_y(t))}};\Delta_2^{(cg^h_y(t),g^h_y(t))}\le t;\Delta_1^{(p^*(t)g^h_y(t),g^h_y(t))}>t\right)\\
&\le \mathbb{P}\left(\mathcal{O}^{g^h_y}_{t,X^{(0,g^h_y(t))}};\Delta_2^{(cg^h_y(t),g^h_y(t))}\le t\right)\\
&\le \overline{\Pi}\left(cg^h_y(t)\right)\int_0^t \mathbb{P}\left(\mathcal{O}^{g^h_y}_{t,X^{(0,g^h_y(t))}};\Delta_2^{(cg^h_y(t),g^h_y(t))}\le t \;\Big|\; \Delta_1^{(cg^h_y(t),g^h_y(t))}=s\right)ds\\
&\le \overline{\Pi}\left(cg^h_y(t)\right)\int_0^t \mathbb{P}\left(\mathcal{O}^{g^h_y}_{s,X^{(0,g^h_y(t))}};\Delta_2^{(cg^h_y(t),g^h_y(t))}\le t \;\Big|\; \Delta_1^{(cg^h_y(t),g^h_y(t))}=s\right)ds\\
&= \overline{\Pi}\left(cg^h_y(t)\right)\int_0^t \mathbb{P}\left(\mathcal{O}^{g^h_y}_{s,X^{(0,g^h_y(t))}}\Big|\Delta_1^{(cg^h_y(t),g^h_y(t))}=s\right)\mathbb{P}\left(\Delta_2^{(cg^h_y(t),g^h_y(t))}\le t \;\Big|\; \Delta_1^{(cg^h_y(t),g^h_y(t))}=s\right)ds\\
&\overset{(3.72)}{\le} \overline{\Pi}\left(cg^h_y(t)\right)\mathbb{P}\left(\Delta_1^{(cg^h_y(t),g^h_y(t))}\le t\right)\int_0^t \mathbb{P}\left(\mathcal{O}^{g^h_y}_s\right)ds \le \overline{\Pi}\left(cg^h_y(t)\right)\mathbb{P}\left(\Delta_1^{cg^h_y(t)}\le t\right)\int_0^t \mathbb{P}\left(\mathcal{O}^{g^h_y}_s\right)ds.
\end{aligned}$$

Recall for L slowly varying at ∞, $\overline{\Pi}(x) = x^{-\alpha}L(x)$. Observe $\mathbb{P}\big(\Delta_1^{cg^h_y(t)}\le t\big) = 1-e^{-t\overline{\Pi}(cg^h_y(t))}\le t\overline{\Pi}(cg^h_y(t))$, so by (3.14) and Lemma 3.7.1, uniformly in $h>0, y>g(h)$, as $t\to\infty$,

$$\begin{aligned}
(2Cb) &\overset{(3.14)}{\le} \overline{\Pi}\left(cg^h_y(t)\right)^2 t\Phi^h_y(t) \overset{3.7.1}{\lesssim} \overline{\Pi}(g(t))^2 t\Phi^h_y(t) &(3.102)\\
&= g(t)^{-2\alpha}L(g(t))^2 t\Phi^h_y(t) = \log(t)^{-2\alpha\beta}\left(\frac{g(t)}{\log(t)^\beta}\right)^{-2\alpha} L\left(\frac{g(t)}{\log(t)^\beta}\right)^2 \frac{L(g(t))^2}{L\left(\frac{g(t)}{\log(t)^\beta}\right)^2} t\Phi^h_y(t)\\
&= \log(t)^{-2\alpha\beta}\overline{\Pi}\left(\frac{g(t)}{\log(t)^\beta}\right)^2 \frac{L(g(t))^2}{L\left(\frac{g(t)}{\log(t)^\beta}\right)^2} t\Phi^h_y(t).
\end{aligned}$$

By Potter's theorem (Theorem A.4.2), for arbitrarily small $\delta>0$, as $t\to\infty$,

$$(2Cb) \lesssim \log(t)^{-2\alpha\beta}\overline{\Pi}\left(\frac{g(t)}{\log(t)^\beta}\right)^2 \log(t)^{2\beta\delta} t\Phi^h_y(t).$$

It follows by (3.3) that $\overline{\Pi}\left(g(t)/\log(t)^{\beta}\right)^2 \lesssim t^{-2}$ as $t \to \infty$, and hence

$$(2Cb) \;\lesssim\; \log(t)^{-2\alpha\beta}\log(t)^{2\beta\delta}t^{-1}\Phi_y^h(t).$$

Now, one can verify that $2\alpha\beta > 1$ by (3.3). Then taking δ small enough that $2\alpha\beta - 2\beta\delta \geq 1+\varepsilon > 1$, it follows that $(2Cb) \lesssim t^{-1}\log(t)^{-1-\varepsilon}\Phi_y^h(t)$, uniformly in y, h, as required for (3.85) in case (i).

Now we show $(2Cb) \leq \Phi_y^h(t)u(t)\overline{\Pi}(g(t))$ in cases (i) and (ii) together. By (3.102), since $\lim_{t\to\infty} t\overline{\Pi}(g(t)) = 0$ by (3.3) or (3.5) for cases (i) and (ii) respectively, uniformly in $h > 0, y > g(h)$, as $t \to \infty$,

$$(2Cb) \lesssim \overline{\Pi}(g(t))^2 t\Phi_y^h(t) = o(1) \times \overline{\Pi}(g(t))\Phi_y^h(t),$$

so $(2Cb) \leq \Phi_y^h(t)u(t)\overline{\Pi}(g(t))$ for suitable u (i.e. $\overline{\Pi}(g(t))t \leq u(t) = o(1)$ as $t \to \infty$), uniformly in y, h, as required for (3.86).

Partitioning $(2Cc)$ By the definition (3.8), disintegrating on the value of $\Delta_1^{c,p^*} := \Delta_1^{(cg_y^h(t),p^*(t)g_y^h(t))}$, which is exponentially distributed with rate $\overline{\Pi}(cg_y^h(t)) - \overline{\Pi}(p^*(t)g_y^h(t)) \leq \overline{\Pi}(cg_y^h(t))$, by (3.72), (3.73), and Lemma 3.7.1, uniformly in $h > 0, y > g(h)$, as $t \to \infty$,

$$\begin{aligned}
(2Cc) \;&=\; \mathbb{P}\left(\mathcal{O}^{g_y^h}_{t,X^{(0,g_y^h(t))}}; \Delta_1^{(cg_y^h(t),g_y^h(t))} \leq t; \Delta_2^{(cg_y^h(t),g_y^h(t))} > t; \Delta_1^{(p^*(t)g_y^h(t),g_y^h(t))} > t\right)\\
&\leq\; \overline{\Pi}(cg_y^h(t))\int_0^t \mathbb{P}\Big(\mathcal{O}^{g_y^h}_{t,X^{(0,g_y^h(t))}}; \Delta_2^{(cg_y^h(t),g_y^h(t))} > t; \Delta_1^{(p^*(t)g_y^h(t),g_y^h(t))} > t\big|\Delta_1^{c,p^*} = s\Big)ds\\
&\overset{(3.73)}{\leq}\; \overline{\Pi}(cg_y^h(t))\int_0^t \mathbb{P}\Big(\mathcal{O}^{g_y^h}_{s,X^{(0,cg_y^h(t))}}; \hat{X}_{t-s}^{(0,cg_y^h(t))} + X_{s-}^{(0,cg_y^h(t))} > (1-p^*(t))g_y^h(t)\big|\Delta_1^{c,p^*} = s\Big)ds\\
&\overset{(3.72)}{=}\; \overline{\Pi}(cg_y^h(t))\int_0^t \mathbb{P}\Big(\mathcal{O}^{g_y^h}_{s,X^{(0,cg_y^h(t))}}; \hat{X}_{t-s}^{(0,cg_y^h(t))} + X_{s-}^{(0,cg_y^h(t))} > (1-p^*(t))g_y^h(t)\Big)ds\\
&\overset{3.7.1}{\lesssim}\; \overline{\Pi}(g(t))\int_0^t \mathbb{P}\Big(\mathcal{O}^{g_y^h}_{s,X^{(0,cg_y^h(t))}}; \hat{X}_{t-s}^{(0,cg_y^h(t))} + X_{s-}^{(0,cg_y^h(t))} > (1-p^*(t))g_y^h(t)\Big)ds,
\end{aligned}$$

where $\hat{X}$ is an independent copy of X, and we use that the jump at time s has size at most $p^*(t)g_y^h(t)$. Recall that $1-p^*(t) = \log(t)^{-\gamma}$. Then partitioning according to the event $\left\{X_{s-}^{(0,cg_y^h(t))} > g_y^h(t)/(2\log(t)^\gamma)\right\}$ and its complement,

$$
\begin{aligned}
(2Cc) &\lesssim \overline{\Pi}(g(t))\int_0^t \mathbb{P}\left(\mathcal{O}^{g_y^h}_{s,X^{(0,cg_y^h(t))}}\ ;\ \hat{X}_{t-s}^{(0,cg_y^h(t))} > \frac{g_y^h(t)}{2\log(t)^\gamma}\right) ds \\
&+ \overline{\Pi}(g(t))\int_0^t \mathbb{P}\left(\mathcal{O}^{g_y^h}_{s,X^{(0,cg_y^h(t))}}\ ;\ X_{s-}^{(0,cg_y^h(t))} > \frac{g_y^h(t)}{2\log(t)^\gamma}\right) ds \\
&=: (S) + (S^*).
\end{aligned} \tag{3.103}
$$

Next we will bound (S), then later we will split up (S^*) into two more pieces.

Proof for (S), Case (i) As $\hat{X}$ is an independent copy of X, we can write

$$
\begin{aligned}
(S) &= \overline{\Pi}(g(t))\int_0^t \mathbb{P}\left(\mathcal{O}^{g_y^h}_{s,X^{(0,cg_y^h(t))}}\right)\mathbb{P}\left(X_{t-s}^{(0,cg_y^h(t))} > \frac{g_y^h(t)}{2\log(t)^\gamma}\right) ds \\
&\le \overline{\Pi}(g(t))\mathbb{P}\left(X_t^{(0,cg_y^h(t))} > \frac{g_y^h(t)}{2\log(t)^\gamma}\right)\Phi_y^h(t).
\end{aligned} \tag{3.104}
$$

Since $\overline{\Pi}$ is regularly varying at ∞, applying Potter's theorem (Theorem A.4.2) to $\overline{\Pi}(g(t))/\overline{\Pi}(g(t)/\log(t)^\beta)$, for β as in (3.3) and for arbitrarily small $\tau > 0$, by (3.3), as $t \to \infty$,

$$
(S) \lesssim \overline{\Pi}\left(\frac{g(t)}{\log(t)^\beta}\right)\log(t)^{-\alpha\beta+\tau\beta}\mathbb{P}\left(X_t^{(0,cg_y^h(t))} > \frac{g_y^h(t)}{2\log(t)^\gamma}\right)\Phi_y^h(t) \tag{3.105}
$$

$$
\lesssim t^{-1}\log(t)^{-\alpha\beta+\tau\beta}\mathbb{P}\left(X_t^{(0,cg_y^h(t))} > \frac{g_y^h(t)}{2\log(t)^\gamma}\right)\Phi_y^h(t). \tag{3.106}
$$

Now, we will show that there exists $\varepsilon > 0$ such that uniformly among $h > 0, y > g(h)$, as $t \to \infty$,

$$
\mathbb{P}\left(X_t^{(0,cg_y^h(t))} > \frac{g_y^h(t)}{2\log(t)^\gamma}\right) \lesssim \log(t)^{-1-\varepsilon+\alpha\beta-\tau\beta}. \tag{3.107}
$$

Then it follows from (3.106) and (3.107) that

$$(S) \lesssim \frac{\Phi_y^h(t)}{t\log(t)^{1+\alpha\beta}} \leq \frac{\Phi_y^h(t)}{t\log(t)^{1+\varepsilon}} \leq \frac{\Phi_y^h(t)}{t\log(t)^{1+\varepsilon}}\left(1+\frac{1}{f(y)-h}\right),$$

as $t \to \infty$, uniformly among $h > 0, y > g(h)$, as required for (3.85) in case (i).

Moreover, it follows by (3.104) and (3.107) that for suitable u (i.e. $\log(t)^{-1-\varepsilon+\alpha\beta-\tau\beta} \leq u(t) = o(1)$, as $t \to \infty$), we have $(S) \leq \Phi_y^h(t)u(t)\overline{\Pi}(g(t))$ as $t \to \infty$, uniformly in y, h, as required for (3.86) in case (i).

Now, to prove (3.107), we set $M := \gamma + 2/(2+\alpha) = (3-\alpha)/(2+\alpha)$, and partition as follows

$$\mathbb{P}\left(X_t^{(0,cg_y^h(t))} > \frac{g_y^h(t)}{2\log(t)^\gamma}\right)$$

$$= \mathbb{P}\Big(X_t^{(0,cg_y^h(t))} > \frac{g_y^h(t)}{2\log(t)^\gamma}; \Delta_1^{\frac{g_y^h(t)}{\log(t)^M}} \leq t\Big) + \mathbb{P}\Big(X_t^{(0,cg_y^h(t))} > \frac{g_y^h(t)}{2\log(t)^\gamma}; \Delta_1^{\frac{g_y^h(t)}{\log(t)^M}} > t\Big) =: (Q) + (Q'). \quad (3.108)$$

Then as $\Delta_1^{\frac{g_y^h(t)}{\log(t)^M}}$ is exponentially distributed with rate $\overline{\Pi}(g_y^h(t)/\log(t)^M)$, we can bound

$$(Q) \leq \mathbb{P}\Big(\Delta_1^{\frac{g_y^h(t)}{\log(t)^M}} \leq t\Big) \leq 1 - e^{-t\overline{\Pi}(g_y^h(t)/\log(t)^M)} \leq t\overline{\Pi}\left(\frac{g_y^h(t)}{\log(t)^M}\right).$$

By Lemma 3.7.1 and (3.3), applying Theorem A.4.2 to $\overline{\Pi}(g(t)/\log(t)^M)/\overline{\Pi}(g(t)/\log(t)^\beta)$, it follows that for arbitrarily small $\kappa > 0$, uniformly in $h > 0, y > g(h)$, as $t \to \infty$,

$$(Q) \overset{3.7.1}{\lesssim} t\overline{\Pi}\left(\frac{g(t)}{\log(t)^M}\right) \overset{A.4.2}{\lesssim} t\overline{\Pi}\left(\frac{g(t)}{\log(t)^\beta}\right)\log(t)^{-\alpha(\beta-M)+\kappa(\beta-M)} \overset{(3.3)}{\lesssim} \log(t)^{-\alpha(\beta-M)+\kappa(\beta-M)},$$

and then in order for (3.107) to hold, we need $-\alpha(\beta-M)+\kappa(\beta-M) \leq -1-\varepsilon+\alpha\beta-\tau\beta$, so taking $\kappa, \tau, \varepsilon$ small enough, we need $-\alpha(\beta-M) < -1+\alpha\beta$, that is, $(3-\alpha)/(2+\alpha) = M < 2\beta - 1/\alpha$. This is indeed true since $\beta > (1+2\alpha)/(2\alpha+\alpha^2)$, from which it follows that

$$2\beta - \frac{1}{\alpha} > 2\frac{1+2\alpha}{2\alpha+\alpha^2} - \frac{1}{\alpha} = \frac{2+4\alpha-2-\alpha}{2\alpha+\alpha^2} = \frac{3}{2+\alpha} > \frac{3-\alpha}{2+\alpha} = M,$$

and the desired bound for (Q) holds. Next we bound (Q'). By Lemma 3.6.1 with $H(t) = \log(t)^{-1-\varepsilon+\alpha\beta-\tau\beta}$,

$$(Q') = \mathbb{P}\Big(X_t^{(0,cg_y^h(t))} > \frac{g_y^h(t)}{2\log(t)^\gamma}; \Delta_1^{\frac{g_y^h(t)}{\log(t)^M}} > t\Big) \le \mathbb{P}\Big(X_t^{\left(0,\frac{g_y^h(t)}{\log(t)^M}\right)} > \frac{g_y^h(t)}{2\log(t)^\gamma}\Big) \overset{3.6.1}{\le} \exp((*))\log(t)^{-1-\varepsilon+\alpha\beta-\tau\beta},$$

$$(*) \lesssim t\log(\log(t)^{1+\varepsilon-\alpha\beta+\tau\beta})\log(t)^{[1+\varepsilon-\alpha\beta+\tau\beta]2\log(t)^{\gamma-M}}\overline{\Pi}\left(\frac{g_y^h(t)}{\log(t)^M}\right)2\log(t)^{\gamma-M}.$$

Now, for $\eta > 0$ small enough that $\eta < M-\gamma$, observe that since $M > \gamma$, as $t \to \infty$,

$$(*) \lesssim t\log(t)^{\frac{\eta}{2}}\log(t)^{\frac{\eta}{2}}\overline{\Pi}\left(\frac{g_y^h(t)}{\log(t)^M}\right)\log(t)^{\gamma-M} \le t\overline{\Pi}\left(\frac{g_y^h(t)}{\log(t)^M}\right).$$

Now, recall $M = (3-\alpha)/(2+\alpha)$, $\beta > (1+2\alpha)/(2\alpha+\alpha^2)$, and $\alpha < 1$. Then

$$\beta > \frac{1+2\alpha}{2\alpha+\alpha^2} = \frac{\frac{1}{\alpha}+2}{2+\alpha} > \frac{3}{2+\alpha} > \frac{3-\alpha}{2+\alpha} = M,$$

so $M < \beta$, and it follows by Lemma 3.7.1 and (3.3) that uniformly in $h > 0, y > g(h)$, as $t \to \infty$,

$$(*) \overset{3.7.1}{\lesssim} t\overline{\Pi}\left(\frac{g(t)}{\log(t)^M}\right) \le t\overline{\Pi}\left(\frac{g(t)}{\log(t)^\beta}\right) \overset{(3.3)}{=} o(1),$$

so the desired bounds for (Q) and (Q') are proven, and the proof of (3.107) is complete.

Proof for (S), Case (ii) To show $(S) \le \Phi_y^h(t)u(t)\overline{\Pi}(g(t))$ in case (ii), we show the probability in (3.104) converges to 0 as $t \to \infty$, uniformly in $h > 0, y > g(h)$. Splitting the probability up as in (3.108) with $M = 2$, we first bound (Q). By Lemma 3.7.1, Theorem A.4.6 and (3.5), uniformly in $h > 0, y > g(h)$, as $t \to \infty$,

$$(Q) = \mathbb{P}\Big(X_t^{(0,cg_y^h(t))} > \frac{g_y^h(t)}{2\log(t)^\gamma}; \Delta_1^{\frac{g_y^h(t)}{\log(t)^M}} \le t\Big) \le \mathbb{P}\Big(\Delta_1^{\frac{g_y^h(t)}{\log(t)^M}} \le t\Big) = 1 - e^{-t\overline{\Pi}\left(\frac{g_y^h(t)}{\log(t)^M}\right)} \le t\overline{\Pi}\left(\frac{g_y^h(t)}{\log(t)^M}\right)$$

$$\overset{3.7.1}{\lesssim} t\overline{\Pi}\left(\frac{g(t)}{\log(t)^M}\right) = t\overline{\Pi}(g(t))\frac{\overline{\Pi}\left(\frac{g(t)}{\log(t)^M}\right)}{\overline{\Pi}(g(t))} \overset{A.4.6}{\lesssim} t\overline{\Pi}(g(t))\frac{\left(\frac{g(t)}{\log(t)^M}\right)^{-1}}{\overline{\Pi}(g(t))^{-1}} = t\overline{\Pi}(g(t))\log(t)^M \lesssim t^{1+\varepsilon}\overline{\Pi}(g(t)),$$

for $\varepsilon > 0$ as in (3.5), so $(Q) \lesssim t^{1+\varepsilon}\overline{\Pi}(g(t)) = o(1)$ as $t \to \infty$ by (3.5). Now, to bound (Q'), by Lemma 3.6.1 with $H(t) = \log(t)^{-1}$, recalling that $M = 2$ and $\gamma = 1$ in case (ii),

$$(Q') = \mathbb{P}\Big(X_t^{(0,cg_y^h(t))} > \frac{g_y^h(t)}{2\log(t)^\gamma}; \Delta_1^{\frac{g_y^h(t)}{\log(t)^M}} > t\Big) \leq \mathbb{P}\Big(X_t^{\left(0,\frac{g_y^h(t)}{\log(t)^M}\right)} > \frac{g_y^h(t)}{2\log(t)^\gamma}\Big) \overset{3.6.1}{\leq} \exp((*))\log(t)^{-1},$$

$$(*) \lesssim t\log(\log(t))\log(t)^{2\log(t)^{-1}}\overline{\Pi}\left(\frac{g_y^h(t)}{\log(t)^M}\right)2\log(t)^{-1}.$$

Now, $\log(\log(t)) \lesssim \log(t)^{1/2}$ and $\log(t)^{2\log(t)^{-1}} \lesssim \log(t)^{1/2}$ for all large enough t, so by Lemma 3.7.1, Theorem A.4.6 and (3.5), uniformly in $h > 0, y > g(h)$, as $t \to \infty$,

$$(*) \lesssim t\overline{\Pi}\left(\frac{g_y^h(t)}{\log(t)^M}\right) \leq t\overline{\Pi}\left(\frac{g_y^h(t)}{\log(t)^M}\right) \overset{3.7.1}{\lesssim} t\overline{\Pi}\left(\frac{g(t)}{\log(t)^M}\right)$$

$$= t\overline{\Pi}(g(t))\frac{\overline{\Pi}\left(\frac{g(t)}{\log(t)^M}\right)}{\overline{\Pi}(g(t))} \overset{A.4.6}{\lesssim} t\overline{\Pi}(g(t))\frac{\left(\frac{g(t)}{\log(t)^M}\right)^{-1}}{\overline{\Pi}(g(t))^{-1}} = t\overline{\Pi}(g(t))\log(t)^M \lesssim t^{1+\varepsilon}\overline{\Pi}(g(t)) \overset{(3.5)}{=} o(1),$$

for $\varepsilon > 0$ as in (3.5), so $(Q') \lesssim \log(t)^{-1} = o(1)$ as $t \to \infty$, uniformly in $y > g(h), h > 0$, and hence the probability in (3.104) converges to 0 as $t \to \infty$, uniformly in y, h, and therefore $(S) \leq \Phi_y^h(t)u(t)\overline{\Pi}(g(t))$ as $t \to \infty$, for suitable u satisfying

$$\mathbb{P}\left(X_t^{(0,cg_y^h(t))} > \frac{g_y^h(t)}{2\log(t)^\gamma}\right) \leq u(t) = o(1) \text{ as } t \to \infty,$$

uniformly in $y > g(h), h > 0$, and so (S) is suitably bounded in case (ii), as required for (3.86).

Partitioning (S^*), Case (i) Now we partition (S^*) in case (i). For $\gamma = (1-\alpha)/(2+\alpha)$ and $\delta := 1+\gamma$, write $g_\delta(t) := g(t)/\log(t)^\delta$. Recall the notation in (3.9). Partitioning according to $\Delta_1^{g_\delta(t)} > s$ and $\Delta_1^{g_\delta(t)} \le s$,

$$\begin{aligned}(S^*) &= \overline{\Pi}(g(t)) \int_0^t \mathbb{P}\left(\mathcal{O}^{g_y^h}_{s,X^{(0,cg_y^h(t))}} \ ; \ X_{s-}^{(0,cg_y^h(t))} > \frac{g_y^h(t)}{2\log(t)^\gamma} \right) ds \\ &= \overline{\Pi}(g(t)) \int_0^t \mathbb{P}\left(\mathcal{O}^{g_y^h}_{s,X^{(0,cg_y^h(t))}} \ ; \ X_{s-}^{(0,cg_y^h(t))} > \frac{g_y^h(t)}{2\log(t)^\gamma} \ ; \ \Delta_1^{g_\delta(t)} \le s \right) ds \\ &+ \overline{\Pi}(g(t)) \int_0^t \mathbb{P}\left(\mathcal{O}^{g_y^h}_{s,X^{(0,cg_y^h(t))}} \ ; \ X_{s-}^{(0,cg_y^h(t))} > \frac{g_y^h(t)}{2\log(t)^\gamma} \ ; \ \Delta_1^{g_\delta(t)} > s \right) ds \\ &=: (S_1^*) + (S_2^*). \end{aligned} \tag{3.109}$$

Proof for (S_1^*), Case (i) Disintegrating on the value of $\Delta_1^{g_\delta(t)}$, which is exponentially distributed with parameter $\overline{\Pi}(g_\delta(t))$, by (3.72) and (3.14),

$$\begin{aligned}(S_1^*) &\le \overline{\Pi}(g(t))\overline{\Pi}(g_\delta(t)) \int_0^t \int_0^s \mathbb{P}\Big(\mathcal{O}^{g_y^h}_{s,X^{(0,cg_y^h(t))}}; X_{s-}^{(0,cg_y^h(t))} > \frac{g_y^h(t)}{2\log(t)^\gamma} \big| \Delta_1^{g_\delta(t)} = v \Big) dv ds \\ &\le \overline{\Pi}(g(t))\overline{\Pi}(g_\delta(t)) \int_0^t \int_0^t \mathbb{P}\Big(\mathcal{O}^{g_y^h}_{v,X^{(0,cg_y^h(t))}}; X_{t}^{(0,cg_y^h(t))} > \frac{g_y^h(t)}{2\log(t)^\gamma} \big| \Delta_1^{g_\delta(t)} = v \Big) dv ds \\ &= \overline{\Pi}(g(t))\overline{\Pi}(g_\delta(t)) \int_0^t \int_0^t \mathbb{P}\Big(\mathcal{O}^{g_y^h}_{v,X^{(0,g_\delta(t))}}; X_{t}^{(0,cg_y^h(t))} > \frac{g_y^h(t)}{2\log(t)^\gamma} \big| \Delta_1^{g_\delta(t)} = v \Big) dv ds \\ &\overset{(3.72)}{\le} t\overline{\Pi}(g(t))\overline{\Pi}(g_\delta(t)) \int_0^t \mathbb{P}\Big(\mathcal{O}^{g_y^h}_{v,X^{(0,g_\delta(t))}} \Big) dv \overset{(3.14)}{\le} t\overline{\Pi}(g(t))\overline{\Pi}(g_\delta(t))\Phi_y^h(t). \end{aligned} \tag{3.110}$$

As $\overline{\Pi}$ is regularly varying at ∞ of index α, applying Potter's theorem (Theorem A.4.2) to $\overline{\Pi}(g(t))/\overline{\Pi}(g_\beta(t))$ and $\overline{\Pi}(g_\delta(t))/\overline{\Pi}(g_\beta(t))$, with $g_\beta(t) := g(t)/\log(t)^\beta$ and β as in (3.3), for arbitrarily small $\tau > 0$, as $t \to \infty$,

$$(S_1^*) \lesssim t\overline{\Pi}(g_\beta(t))^2 \log(t)^{-\alpha\beta - \alpha(\beta-\delta) + \beta\tau + (\beta-\delta)\tau} \Phi_y^h(t),$$

Now, by (3.3), $\lim_{t\to\infty} t\overline{\Pi}(g_\beta(t)) = 0$, so as $t \to \infty$,

$$(S_1^*) \lesssim t^{-1} \log(t)^{-\alpha\beta - \alpha(\beta-\delta) + \beta\tau + (\beta-\delta)\tau} \Phi_y^h(t).$$

Now, $\delta = 1 + \gamma = 1 + (1-\alpha)/(2+\alpha)$, and $\beta > (1+2\alpha)/(2\alpha+\alpha^2)$, from which we can deduce

$$-\alpha\beta - \alpha(\beta - \delta) = -2\alpha\beta + \alpha\left(1 + \frac{1-\alpha}{2+\alpha}\right) < -2\left(\frac{1+2\alpha}{2+\alpha}\right) + \alpha\left(\frac{3}{2+\alpha}\right) = \frac{-2-4\alpha+3\alpha}{2+\alpha} = -1,$$

so taking τ small enough, we conclude that there exists $\varepsilon > 0$ such that $(S_1^*) \lesssim t^{-1}\log(t)^{-1-\varepsilon}\Phi_y^h(t)$ as $t \to \infty$, uniformly among $h > 0, y > g(h)$, as required for (3.85) in case (i).

To show $(S_1^*) \leq \Phi_y^h(t)u(t)\overline{\Pi}(g(t))$, one can verify that for each $\alpha \in (0,1)$ and for $\beta > 1$ as in (3.3),

$$\delta = 1 + \frac{1-\alpha}{2+\alpha} = \frac{3}{2+\alpha} < \frac{\frac{1}{\alpha}+2}{2+\alpha} = \frac{1+2\alpha}{2\alpha+\alpha^2} < \beta.$$

Thus $\lim_{t\to\infty} t\overline{\Pi}(g_\delta(t)) \leq \lim_{t\to\infty} t\overline{\Pi}(g_\beta(t)) = 0$ by (3.3). Then by (3.110), for suitable u (i.e. satisfying $t\overline{\Pi}(g_\delta(t)) \leq u(t) = o(1)$ as $t \to \infty$), uniformly in $y > g(h), h > 0$ as $t \to \infty$, we conclude that $(S_1^*) \lesssim t\overline{\Pi}(g(t))\overline{\Pi}(g_\delta(t))\Phi_y^h(t) \leq u(t)\overline{\Pi}(g(t))\Phi_y^h(t)$, as required for (3.86) in case (i).

Proof for (S_2^*), Case (i) Note for $g_\delta(t) := g(t)/\log(t)^\delta$, by Lemma 3.7.1, for all $h > 0$, $y > g(h)$, and for all large enough t, $g_\delta(t) \leq c g_y^h(t)$, so as $t \to \infty$,

$$\begin{aligned}(S_2^*) &= \overline{\Pi}(g(t)) \int_0^t \mathbb{P}\left(\mathcal{O}^{g_y^h}_{s, X^{(0, c g_y^h(t))}} \;;\; X_{s-}^{(0, c g_y^h(t))} > \frac{g_y^h(t)}{2\log(t)^\gamma} \;;\; \Delta_1^{g_\delta(t)} > s\right) ds \\ &= \overline{\Pi}(g(t)) \int_0^t \mathbb{P}\left(\mathcal{O}^{g_y^h}_{s, X^{(0, g_\delta(t))}} \;;\; X_{s-}^{(0, g_\delta(t))} > \frac{g_y^h(t)}{2\log(t)^\gamma} \;;\; \Delta_1^{g_\delta(t)} > s\right) ds \\ &\leq \overline{\Pi}(g(t)) \int_0^t \mathbb{P}\left(X_{s-}^{(0, g_\delta(t))} > \frac{g_y^h(t)}{2\log(t)^\gamma}\right) ds \leq t\overline{\Pi}(g(t))\mathbb{P}\left(X_t^{(0, g_\delta(t))} > \frac{g_y^h(t)}{2\log(t)^\gamma}\right). \end{aligned} \tag{3.111}$$

For $g_\beta(t) := g(t)/\log(t)^\beta$, with β as in (3.3), applying Potter's theorem (Theorem A.4.2) to $\overline{\Pi}(g(t))/\overline{\Pi}(g_\beta(t))$, for arbitrarily small $\tau > 0$, as $t \to \infty$, by Lemma 3.7.1,

$$\begin{aligned}(S_2^*) &\lesssim t\overline{\Pi}(g_\beta(t))\log(t)^{-\alpha\beta+\tau\beta}\mathbb{P}\left(X_t^{(0, g_\delta(t))} > \frac{g_y^h(t)}{2\log(t)^\gamma}\right) \\ &\overset{3.7.1}{\leq} t\overline{\Pi}(g_\beta(t))\log(t)^{-\alpha\beta+\tau\beta}\mathbb{P}\left(X_t^{(0, g_\delta(t))} > \frac{(1-A^{-1})g(t)}{2\log(t)^\gamma}\right). \end{aligned}$$

Applying Lemma 3.6.1 with $H(t) = 1/(t\log(t)^{1+\varepsilon-\alpha\beta})$, $\varepsilon > \tau\beta$, then by (3.3) and Lemma 3.5.10, uniformly in $h > 0, y > g(h)$ by Lemma 3.7.1, as $t \to \infty$,

$$\begin{aligned}(S_2^*) &\overset{3.6.1}{\lesssim} t\overline{\Pi}(g_\beta(t))\log(t)^{-\alpha\beta+\tau\beta}\exp((*))\frac{1}{t\log(t)^{1+\varepsilon-\alpha\beta}} \overset{(3.3)}{=} o(1)\times\exp((*))\frac{1}{t\log(t)^{1+\varepsilon-\tau\beta}}\\ &\overset{3.5.10}{\leq} o(1)\times\exp((*))\frac{1}{t\log(t)^{1+\varepsilon-\tau\beta}}\frac{\Phi_y^h(t)}{f(y)-h},\\ (*) &\lesssim t\log\left(t\log(t)^{1+\varepsilon-\alpha\beta}\right)\left(t\log(t)^{1+\varepsilon-\alpha\beta}\right)^{\frac{2\log(t)^{\gamma-\delta}}{1-A^{-1}}}\overline{\Pi}(g_\delta(t))\log(t)^{\gamma-\delta}.\end{aligned} \tag{3.112}$$

Now, if $\lim_{t\to\infty}(*) = 0$, then we get $(S_2^*) \lesssim t^{-1}\log(t)^{-1-\varepsilon}\Phi_y^h(t)/(f(y)-h)$. Indeed,

$$\begin{aligned}(*) &\lesssim t\log\left(t\log(t)^{1+\varepsilon-\alpha\beta}\right)\left(t\log(t)^{1+\varepsilon-\alpha\beta}\right)^{\frac{2\log(t)^{\gamma-\delta}}{1-A^{-1}}}\overline{\Pi}(g_\delta(t))\log(t)^{\gamma-\delta}\\ &= t\overline{\Pi}(g_\delta(t))t^{\frac{2\log(t)^{\gamma-\delta}}{1-A^{-1}}}\log\left(t\log(t)^{1+\varepsilon-\alpha\beta}\right)\log(t)^{(1+\varepsilon-\alpha\beta)\frac{2\log(t)^{\gamma-\delta}}{1-A^{-1}}+\gamma-\delta}.\end{aligned}$$

Now, $\delta > \gamma$, so $\lim_{t\to\infty}\log(t)^{\gamma-\delta} = 0$, and for arbitrarily small $\kappa > 0$, as $t \to \infty$,

$$(*) \lesssim t\overline{\Pi}(g_\delta(t))t^{\frac{2\log(t)^{\gamma-\delta}}{1-A^{-1}}}\log\left(t\log(t)^{1+\varepsilon-\alpha\beta}\right)\log(t)^{(1+\varepsilon-\alpha\beta)\kappa+\gamma-\delta}.$$

As $t \to \infty$, $\log(t\log(t)^{1+\varepsilon-\alpha\beta}) \lesssim \log(t)$. Applying Potter's theorem (Theorem A.4.2) to $\overline{\Pi}(g_\delta(t))/\overline{\Pi}(g_\beta(t))$, for β as in (3.3) and arbitrarily small $c > 0$, as $t \to \infty$,

$$(*) \lesssim t\overline{\Pi}(g_\beta(t))t^{\frac{2\log(t)^{\gamma-\delta}}{1-A^{-1}}}\log(t)^{1+(1+\varepsilon-\alpha\beta)\kappa-(\beta-\delta)\alpha+(\beta-\delta)c+\gamma-\delta}.$$

Recalling $\gamma-\delta = -1$, $t^{\frac{2\log(t)^{\gamma-\delta}}{1-A^{-1}}} = e^{\frac{2\log(t)^{1+\gamma-\delta}}{1-A^{-1}}} = e^{\frac{2}{1-A^{-1}}}$. Then since $\lim_{t\to\infty} t\overline{\Pi}(g_\beta(t)) = 0$ by (3.3), using that $1+\gamma-\delta = 0$, as $t \to \infty$,

$$(*) \lesssim \log(t)^{1+(1+\varepsilon-\alpha\beta)\kappa-(\beta-\delta)\alpha+(\beta-\delta)c+\gamma-\delta} = \log(t)^{(1+\varepsilon-\alpha\beta)\kappa-(\beta-\delta)\alpha+(\beta-\delta)c}. \tag{3.113}$$

Now, $\delta < \beta$, so $-(\beta-\delta)\alpha < 0$. Choosing κ, c small enough that the exponent in (3.113) is negative, we get $\lim_{t\to\infty}(*) = 0$. Then by (3.112), it follows that uniformly in $h > 0$ and $y > g(h)$, as $t \to \infty$, $(S_2^*) \lesssim t^{-1}\log(t)^{-1-\varepsilon}\Phi_y^h(t)/(f(y)-h)$, as required for (3.85) in case (i).

To prove $(S_2^*) \leq \Phi_y^h(t)u(t)\overline{\Pi}(g(t))/(f(y)-h)$, applying Lemma 3.7.1 and Lemma 3.6.1 with $H(t) = 1/(t\log(\log(t)))$ to (3.111), as $t\to\infty$, uniformly in $h>0, y>g(h)$,

$$(S_2^*) \overset{(3.111)}{\lesssim} t\overline{\Pi}(g(t))\mathbb{P}\left(X_t^{(0,g_\delta(t))} > \frac{(1-A^{-1})g(t)}{2\log(t)^\gamma}\right) \overset{3.6.1}{\leq} \frac{\overline{\Pi}(g(t))}{\log(\log(t))}\exp\left((*)\right),$$

$$(*) \lesssim t\log\left(t\log(\log(t))\right)\left(t\log(\log(t))\right)^{\frac{2\log(t)^{\gamma-\delta}}{(1-A^{-1})}}\,\overline{\Pi}(g_\delta(t))\log(t)^{\gamma-\delta}.$$

Recall $1+\gamma-\delta = 0$. As $t\log(\log(t)) \lesssim t^2$ for all large enough t, as $t\to\infty$,

$$(*) \lesssim t\log(t)^{1+\gamma-\delta}t^{\frac{4\log(t)^{\gamma-\delta}}{1-A^{-1}}}\,\overline{\Pi}(g_\delta(t)) = te^{\frac{4(1-A^{-1})\log(t)^{1+\gamma-\delta}}{1-A^{-1}}}\,\overline{\Pi}(g_\delta(t)) = te^{\frac{4(1-A^{-1})}{1-A^{-1}}}\,\overline{\Pi}(g_\delta(t)).$$

Now, since $\delta<\beta$, by (3.3), $\lim_{t\to\infty} t\overline{\Pi}(g_\delta(t)) \leq \lim_{t\to\infty} t\overline{\Pi}(g_\beta(t)) = 0$, and hence $\lim_{t\to\infty}(*) = 0$, so that for suitable u (i.e. $1/\log(\log(t)) \leq u(t) = o(1)$ as $t\to\infty$), $(S_2^*) \leq \Phi_y^h(t)u(t)\overline{\Pi}(g(t))/(f(y)-h)$, uniformly among $h>0, y>g(h)$ as $t\to\infty$, as required for (3.86), and the proof of Lemma 3.5.3 is complete in case (i).

Partitioning (S^*), Case (ii) In case (ii), we partition (S^*) differently. Recall the notation in (3.9). Partitioning according to $\Delta_1^{\frac{1}{2}cg_y^h(t)} > s$ and $\Delta_1^{\frac{1}{2}cg_y^h(t)} \leq s$,

$$\begin{aligned}
(S^*) &= \overline{\Pi}(g(t))\int_0^t \mathbb{P}\left(\mathcal{O}^{g_y^h}_{s,X^{(0,cg_y^h(t))}}\ ;\ X_{s-}^{(0,cg_y^h(t))} > \frac{g_y^h(t)}{2\log(t)^\gamma}\right)ds\\
&= \overline{\Pi}(g(t))\int_0^t \mathbb{P}\left(\mathcal{O}^{g_y^h}_{s,X^{(0,cg_y^h(t))}}\ ;\ X_{s-}^{(0,cg_y^h(t))} > \frac{g_y^h(t)}{2\log(t)^\gamma}\ ;\ \Delta_1^{\frac{1}{2}cg_y^h(t)} \leq s\right)ds\\
&+ \overline{\Pi}(g(t))\int_0^t \mathbb{P}\left(\mathcal{O}^{g_y^h}_{s,X^{(0,cg_y^h(t))}}\ ;\ X_{s-}^{(0,cg_y^h(t))} > \frac{g_y^h(t)}{2\log(t)^\gamma}\ ;\ \Delta_1^{\frac{1}{2}cg_y^h(t)} > s\right)ds\\
&=: (S_1^{**}) + (S_2^{**}).
\end{aligned} \tag{3.114}$$

Proof for (S_1^{**}), Case (ii) Disintegrating on the value of $\Delta_1^{\frac{1}{2}cg_y^h(t)}$, by Lemma 3.7.1 and (3.72), as $t \to \infty$,

$$\begin{aligned}
(S_1^{**}) &\leq \overline{\Pi}(g(t))\overline{\Pi}\left(\frac{1}{2}cg_y^h(t)\right)\int_0^t\int_0^s \mathbb{P}\Big(\mathcal{O}^{g_y^h}_{s,X^{(0,cg_y^h(t))}}; X_{s-}^{(0,cg_y^h(t))} > \frac{g_y^h(t)}{2\log(t)^\gamma}\big|\Delta_1^{\frac{1}{2}cg_y^h(t)} = v\Big)dvds \\
&\overset{3.7.1}{\lesssim} \overline{\Pi}(g(t))^2\int_0^t\int_0^s \mathbb{P}\Big(\mathcal{O}^{g_y^h}_{v,X^{(0,cg_y^h(t))}}\big|\Delta_1^{\frac{1}{2}cg_y^h(t)} = v\Big)dvds \\
&\overset{(3.72)}{=} \overline{\Pi}(g(t))^2\int_0^t\int_0^s \mathbb{P}\Big(\mathcal{O}^{g_y^h}_{v,X^{(0,\frac{1}{2}cg_y^h(t))}}\Big)dvds \\
&\leq t\overline{\Pi}(g(t))^2\int_0^t \mathbb{P}\big(\mathcal{O}_v^{g_y^h}\big)dv \\
&\lesssim t\overline{\Pi}(g(t))^2\Phi_y^h(t).
\end{aligned}$$

Now, $\lim_{t\to\infty} t\overline{\Pi}(g(t))=0$ by (3.5), and so $(S_1^{**}) \leq \Phi_y^h(t)u(t)\overline{\Pi}(g(t))$ for suitable choice of the function u (i.e. $t\overline{\Pi}(g(t)) \leq u(t) = o(1)$ as $t \to \infty$), uniformly in $h > 0, y > g(h)$ as $t \to \infty$, as required for (3.86) in case (ii).

Proof for (S_2^{**}), Case (ii) Recalling the notation (3.9), (3.13), and the definition in (3.8),

$$\begin{aligned}
(S_2^{**}) &= \overline{\Pi}(g(t))\int_0^t \mathbb{P}\left(\mathcal{O}^{g_y^h}_{s,X^{(0,cg_y^h(t))}}; X_{s-}^{(0,cg_y^h(t))} > \frac{g_y^h(t)}{2\log(t)^\gamma}; \Delta_1^{\frac{1}{2}cg_y^h(t)} > s\right)ds \\
&\leq t\overline{\Pi}(g(t))\mathbb{P}\left(X_t^{(0,\frac{1}{2}cg_y^h(t))} > \frac{g_y^h(t)}{2\log(t)^\gamma}\right).
\end{aligned}$$

By Lemma 3.6.1 with $H(t) = 1/(t\log(\log(t)))$, uniformly in y, h by Lemma 3.7.1,

$$\begin{aligned}
(S_2^{**}) &\lesssim \frac{\overline{\Pi}(g(t))}{\log(\log(t))}\exp\left((*)\right), \\
(*) &\lesssim t\log(t\log(\log(t)))\ \ (t\log(\log(t)))^{\frac{c}{4\log(t)^\gamma}}\,\overline{\Pi}\left(\frac{1}{2}cg_y^h(t)\right)\frac{c}{4\log(t)^\gamma}.
\end{aligned}$$

For each arbitrarily small $\eta > 0$, as $t \to \infty$, $\log(t\log(\log(t))) \leq t^{\eta}$ and $(t\log(\log(t)))^{c/(4\log(t)^{\gamma})} \leq t^{\eta}$. As $\overline{\Pi}$ is $\mathcal{O}$-regularly varying at ∞, by Lemma 3.7.1,

$$(*) \lesssim t^{1+2\eta}\overline{\Pi}\left(\frac{cg_y^h(t)}{2}\right)\frac{1}{\log(t)^{\gamma}}$$
$$\lesssim t^{1+2\eta}\overline{\Pi}\left(g(t)\right).$$

Now, $\lim_{t\to\infty} t^{1+\varepsilon}\overline{\Pi}(g(t)) = 0$ by (3.5), so taking $\eta < \varepsilon/2$, it follows by Lemma 3.5.10 that $(S_2^{**}) \leq \Phi_y^h(t)u(t)\overline{\Pi}(g(t))/(f(y)-h)$ as $t \to \infty$, for suitable u (i.e. $1/\log(\log(t)) \leq u(t) = o(1)$ as $t \to \infty$), uniformly in $h > 0, y > g(h)$, as required for (3.86) in case (ii), and the proof of Lemma 3.5.3 is complete.

□

Remark 3.7.2. *One can verify that the above choices of γ and δ are chosen in an optimal way to ensure β can be as small as possible while the proof of Lemma 3.5.3 still holds, thus giving as much generality as possible in case (i).*

3.8 Proof of Lemma 3.5.7

Proof of Lemma 3.5.7. Let $T_{g(h)}$ denote the time when X first passes above the level $g(h)$, and let $S_{\Delta_1^{g(h)}}$ be the size of X's first jump of size larger than $g(h)$. For each $y > K$, where $K > 0$ is a large, fixed constant,

$$\begin{aligned}\mathbb{P}\left(X_h \in g(h)dy; \mathcal{O}_h\right) &= \mathbb{P}\left(X_h \in g(h)dy; X_{T_{g(h)}} \leq \frac{g(h)y}{2}; \mathcal{O}_h\right)\\ &+ \mathbb{P}\left(X_h \in g(h)dy; X_{T_{g(h)}} > \frac{g(h)y}{2}; S_{\Delta_1^{g(h)}} < \frac{g(h)y}{2}; \mathcal{O}_h\right)\\ &+ \mathbb{P}\left(X_h \in g(h)dy; X_{T_{g(h)}} > \frac{g(h)y}{2}; S_{\Delta_1^{g(h)}} \geq \frac{g(h)y}{2}; \mathcal{O}_h\right)\\ &=: \sigma_h^1(dy) + \sigma_h^2(dy) + \sigma_h^3(dy). \end{aligned} \tag{3.115}$$

We will bound $\mathbb{P}\left(X_h \in g(h)dy; \mathcal{O}_h\right)$ by bounding these 3 terms separately.

Upper Bound for $\sigma_h^1(dy)$ We shall disintegrate on the values of $T_{g(h)}$ and $X_{T_{g(h)}}$. Observe by (3.6) that $\mathbb{P}(\mathcal{O}_h; T_{g(h)} \in ds) = \mathbb{P}(\mathcal{O}_s; T_{g(h)} \in ds)$, so that we can apply (3.72) and the independent increments property, with the notation $\mathbb{P}(X_t \in dx) = f_t(x)dx$, to yield

$$\begin{aligned}
\sigma_h^1(dy) &= \int_{s=0}^{h} \int_{w=1}^{\frac{y}{2}} \mathbb{P}\left(X_h \in g(h)dy; X_{T_{g(h)}} \in g(h)dw; T_{g(h)} \in ds; \mathcal{O}_h\right) \\
&= \int_0^h \int_1^{\frac{y}{2}} f_{h-s}(g(h)(y-w))g(h)dy\mathbb{P}\left(X_{T_{g(h)}} \in g(h)dw; T_{g(h)} \in ds; \mathcal{O}_h\right).
\end{aligned}$$

Now, $g(h)(y-w) > g(h)y/2 > g(h) + x_0 \geq g(h-s) + x_0$, for all large enough h, with x_0 as in Definition 3.4.5, so (3.4) applies to $f_{h-s}(g(h)(y-w))$. Applying (3.59), since $y - w \geq y/2$ and L is slowly varying at ∞, uniformly in $y > K$ by (Theorem A.4.1), as $h \to \infty$,

$$\begin{aligned}
\sigma_h^1(dy) &\overset{(3.4)}{\lesssim} \int_0^h \int_1^{\frac{y}{2}} (h-s)u(g(h)(y-w))g(h)dy\mathbb{P}\left(X_{T_{g(h)}} \in g(h)dw; T_{g(h)} \in ds; \mathcal{O}_h\right) \\
&\overset{(3.59)}{\lesssim} \int_0^h \int_1^{\frac{y}{2}} \frac{(h-s)}{g(h)^\alpha}(y-w)^{-1-\alpha} L(g(h)(y-w))dy\mathbb{P}\left(X_{T_{g(h)}} \in g(h)dw; T_{g(h)} \in ds; \mathcal{O}_h\right) \\
&\lesssim \int_0^h \int_1^{\frac{y}{2}} \frac{h}{g(h)^\alpha}\frac{L(g(h))}{L(g(h))} y^{-1-\alpha} L\left(\frac{g(h)y}{2}\right) dy\mathbb{P}\left(X_{T_{g(h)}} \in g(h)dw; T_{g(h)} \in ds; \mathcal{O}_h\right) \\
&\lesssim y^{-1-\alpha}\frac{L(g(h)y)}{L(g(h))} dy \frac{hL(g(h))}{g(h)^\alpha} \int_0^h \int_1^{\frac{y}{2}} \mathbb{P}\left(X_{T_{g(h)}} \in g(h)dw; T_{g(h)} \in ds; \mathcal{O}_h\right) \\
&\leq y^{-1-\alpha}\frac{L(g(h)y)}{L(g(h))} dy\ h\overline{\Pi}(g(h))\ \mathbb{P}(\mathcal{O}_h) \overset{(3.3)}{=} o(1) \times y^{-1-\alpha}\frac{L(g(h)y)}{L(g(h))}\mathbb{P}(\mathcal{O}_h)dy, \qquad (3.116)
\end{aligned}$$

where, recalling $g(h)^{-\alpha}L(g(h)) = \overline{\Pi}(g(h))$, the last step follows by (3.3).

Simplifying the Expressions for $\sigma_h^2(dy)$ and $\sigma_h^3(dy)$ Recall the notation (3.9), and that

$$\sigma_h^2(dy) + \sigma_h^3(dy) = \mathbb{P}(X_h \in g(h)dy; X_{T_{g(h)}} > g(h)y/2; \mathcal{O}_h).$$

Choosing $K > 4$, we have $g(h)y/2 > 2g(h)$ for $y > K$. As $T_{g(h)}$ is the first passage time above $g(h)$, if $X_{T_{g(h)}} > 2g(h)$, then X crosses $g(h)$ by a jump larger than $g(h)$, so since $T_{g(h)} \leq \Delta_1^{g(h)}$, $T_{g(h)} = \Delta_1^{g(h)}$. Then since $X_t < g(h)$ for all $t < T_{g(h)}$, it follows that $X_{T_{g(h)}-} = X_{\Delta_1^{g(h)}-} < g(h)$, as X has càdlàg sample paths, almost surely. Moreover, if $\mathcal{O}_h$ holds, then X crosses $g(h)$ by time h, so $\Delta_1^{g(h)} = T_{g(h)} \leq h$. Thus:

$$\left\{X_{T_{g(h)}} > \frac{g(h)y}{2}; S_{\Delta_1^{g(h)}} < \frac{g(h)y}{2}; \mathcal{O}_h\right\} \subseteq \left\{\Delta_1^{g(h)} \leq h; X_{\Delta_1^{g(h)}-} < g(h); S_{\Delta_1^{g(h)}} < \frac{g(h)y}{2}; \mathcal{O}_h\right\}, \tag{3.117}$$

and therefore we can bound $\sigma_h^2(dy)$ by

$$\begin{aligned}\sigma_h^2(dy) &= \mathbb{P}\left(X_h \in g(h)dy; X_{T_{g(h)}} > \frac{g(h)y}{2}; S_{\Delta_1^{g(h)}} < \frac{g(h)y}{2}; \mathcal{O}_h\right) \\ &\leq \mathbb{P}\left(X_h \in g(h)dy; \Delta_1^{g(h)} \leq h; X_{\Delta_1^{g(h)}-} < g(h); S_{\Delta_1^{g(h)}} < \frac{g(h)y}{2}; \mathcal{O}_h\right).\end{aligned} \tag{3.118}$$

For the event in $\sigma_h^3(dy)$, we have $S_{\Delta_1^{g(h)}} \geq g(h)y/2$, and so the reverse analogous inclusion to (3.117) holds too, that is, if $\Delta_1^{g(h)} \leq h$, $X_{\Delta_1^{g(h)}-} < g(h)$, $S_{\Delta_1^{g(h)}} \geq g(h)y/2$, and $\mathcal{O}_h$ hold, then we have

$$X_{T_{g(h)}} = X_{\Delta_1^{g(h)}} \geq X_{\Delta_1^{g(h)}} - X_{\Delta_1^{g(h)}-} = S_{\Delta_1^{g(h)}} > g(h)y/2,$$

and therefore $\sigma_h^3(dy)$ satisfies

$$\begin{aligned}\sigma_h^3(dy) &= \mathbb{P}\left(X_h \in g(h)dy; X_{T_{g(h)}} > \frac{g(h)y}{2}; S_{\Delta_1^{g(h)}} \geq \frac{g(h)y}{2}; \mathcal{O}_h\right) \\ &= \mathbb{P}\left(X_h \in g(h)dy; \Delta_1^{g(h)} \leq h; X_{\Delta_1^{g(h)}-} < g(h); S_{\Delta_1^{g(h)}} \geq \frac{g(h)y}{2}; \mathcal{O}_h\right).\end{aligned} \tag{3.119}$$

Upper Bound for $\sigma_h^2(dy)$ By (3.118) and (3.74), disintegrating on the values of $\Delta_1^{g(h)}$, $X_{\Delta_1^{g(h)}-}$, and $S_{\Delta_1^{g(h)}}$, by independence of increments and the Markov property, with $\mathbb{P}(X_t \in dx) = f_t(x)dx$,

$$\begin{aligned}
\sigma_h^2(dy) &\overset{(3.118)}{\leq} \mathbb{P}\left(X_h \in g(h)dy; \Delta_1^{g(h)} \leq h; X_{\Delta_1^{g(h)}-} < g(h); S_{\Delta_1^{g(h)}} < \frac{g(h)y}{2}; \mathcal{O}_h\right) \\
&\overset{(3.74)}{=} \int_{s=0}^{h}\int_{w=0}^{1}\int_{v=0}^{\frac{y}{2}} \mathbb{P}\big(X_h \in g(h)dy; \Delta_1^{g(h)} \in ds; X_{\Delta_1^{g(h)}-} \in g(h)dw; \\
&\qquad\qquad S_{\Delta_1^{g(h)}} \in g(h)dv; \mathcal{O}_s\big) \\
&= \int_0^h \int_0^1 \int_0^{\frac{y}{2}} f_{h-s}(g(h)(y-w-v))g(h)dy \\
&\qquad \times \mathbb{P}\left(\Delta_1^{g(h)} \in ds; X_{\Delta_1^{g(h)}-} \in g(h)dw; S_{\Delta_1^{g(h)}} \in g(h)dv; \mathcal{O}_s\right) \\
&\overset{(3.74)}{=} \int_0^h \int_0^1 \int_0^{\frac{y}{2}} f_{h-s}(g(h)(y-w-v))g(h)dy \\
&\qquad \times \mathbb{P}\left(\Delta_1^{g(h)} \in ds; X_{\Delta_1^{g(h)}-} \in g(h)dw; S_{\Delta_1^{g(h)}} \in g(h)dv; \mathcal{O}_h\right).
\end{aligned}$$

Note $y - w - v > y/3 > K/3$ for $w \leq 1$, $v \leq y/2$. So as $h \to \infty$, $g(h)(y-w-v) \geq g(h-s) + x_0$, so we can apply (3.4), and we can apply (3.59) too. Now, $g(h)^{-\alpha}L(g(h)) = \overline{\Pi}(g(h))$ for L slowly varying at ∞, so by

(3.3), uniformly in $y > K$ by the uniform convergence theorem (Theorem A.4.1), as $h \to \infty$,

$$\begin{aligned}
\sigma_h^2(dy) &\overset{(3.4)}{\lesssim} \int_0^h \int_0^1 \int_0^{\frac{y}{2}} (h-s)u(g(h)(y-w-v))g(h)dy \\
&\qquad \times \mathbb{P}\left(\Delta_1^{g(h)} \in ds; X_{\Delta_1^{g(h)}-} \in g(h)dw; S_{\Delta_1^{g(h)}} \in g(h)dv; \mathcal{O}_h\right) \\
&\overset{(3.59)}{\lesssim} \int_0^h \int_0^1 \int_0^{\frac{y}{2}} \frac{(h-s)L(g(h)(y-w-v))}{g(h)^\alpha (y-w-v)^{1+\alpha}} dy \\
&\qquad \times \mathbb{P}\left(\Delta_1^{g(h)} \in ds; X_{\Delta_1^{g(h)}-} \in g(h)dw; S_{\Delta_1^{g(h)}} \in g(h)dv; \mathcal{O}_h\right) \\
&\lesssim \int_0^h \int_0^1 \int_0^{\frac{y}{2}} \frac{(h-s)}{g(h)^\alpha y^{1+\alpha}} L(g(h)y)dy \\
&\qquad \times \mathbb{P}\left(\Delta_1^{g(h)} \in ds; X_{\Delta_1^{g(h)}-} \in g(h)dw; S_{\Delta_1^{g(h)}} \in g(h)dv; \mathcal{O}_h\right) \\
&\leq \frac{hL(g(h))}{g(h)^\alpha} \int_0^h \int_0^1 \int_0^{\frac{y}{2}} y^{-1-\alpha} \frac{L(g(h)y)}{L(g(h)} dy \\
&\qquad \times \mathbb{P}\left(\Delta_1^{g(h)} \in ds; X_{\Delta_1^{g(h)}-} \in g(h)dw; S_{\Delta_1^{g(h)}} \in g(h)dv; \mathcal{O}_h\right) \\
&\overset{(3.3)}{=} o(1) \times \int_0^h \int_0^1 \int_0^{\frac{y}{2}} y^{-1-\alpha} \frac{L(g(h)y)}{L(g(h)} dy \\
&\qquad \times \mathbb{P}\left(\Delta_1^{g(h)} \in ds; X_{\Delta_1^{g(h)}-} \in g(h)dw; S_{\Delta_1^{g(h)}} \in g(h)dv; \mathcal{O}_h\right) \\
&\leq o(1) \times y^{-1-\alpha} \frac{L(g(h)y)}{L(g(h))} \mathbb{P}(\mathcal{O}_h)dy. \qquad (3.120)
\end{aligned}$$

Upper Bound for $\sigma_h^3(dy)$ Disintegrating on the values of $\Delta_1^{g(h)}$, $X_{\Delta_1^{g(h)}-}$ and $S_{\Delta_1^{g(h)}}$, observing that $X_h \in g(h)dy$ implies $X_{\Delta_1^{g(h)}-} + S_{\Delta_1^{g(h)}} \leq g(h)y$, then applying (3.74), independence of increments, the Markov property, and Lemma 3.5.9, it follows that uniformly among $y > K$ as $h \to \infty$, with $\mathbb{P}(X_t \in dx) = f_t(x)dx$,

$$\begin{aligned}
\sigma_h^3(dy) &= \int_{s=0}^h \int_{w=0}^1 \int_{v=\frac{y}{2}}^{y-w} \mathbb{P}\Big(X_h \in g(h)dy; \Delta_1^{g(h)} \in ds; \\
&\qquad X_{\Delta_1^{g(h)}-} \in g(h)dw; S_{\Delta_1^{g(h)}} \in g(h)dv; \mathcal{O}_h\Big) \\
&\overset{3.74}{=} \int_0^h \int_0^1 \int_{\frac{y}{2}}^{y-w} f_{h-s}(g(h)(y-w-v))g(h)dy \mathbb{P}\left(S_{\Delta_1^{g(h)}} \in g(h)dv\right) \\
&\qquad \times \mathbb{P}\left(\Delta_1^{g(h)} \in ds; X_{\Delta_1^{g(h)}-} \in g(h)dw; \mathcal{O}_s\right)
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^h \int_0^1 \int_{\frac{y}{2}}^{y-w} f_{h-s}(g(h)(y-w-v))g(h)dy\mathbb{P}\left(S_{\Delta_1^{g(h)}} \in g(h)dv\right) \\
&\qquad \times \mathbb{P}\left(\Delta_1^{g(h)} \in ds; X_{\Delta_1^{g(h)}-} \in g(h)dw; \mathcal{O}_s\right) \\
&\overset{3.5.9}{\lesssim} \int_0^h \int_0^1 \int_{\frac{y}{2}}^{y-w} f_{h-s}(g(h)(y-w-v))g(h)dy\; v^{-1-\alpha}\frac{L(g(h)v)}{L(g(h))}dv \\
&\qquad \times \mathbb{P}\left(\Delta_1^{g(h)} \in ds; X_{\Delta_1^{g(h)}-} \in g(h)dw; \mathcal{O}_s\right) \\
&\le \int_0^h \int_0^1 \int_{\frac{y}{2}-w}^{y-w} f_{h-s}(g(h)(y-w-v))g(h)dy\; v^{-1-\alpha}\frac{L(g(h)v)}{L(g(h))}dv \\
&\qquad \times \mathbb{P}\left(\Delta_1^{g(h)} \in ds; X_{\Delta_1^{g(h)}-} \in g(h)dw; \mathcal{O}_s\right) \\
&\overset{(3.74)}{=} \int_0^h \int_0^1 \int_{\frac{y}{2}-w}^{y-w} f_{h-s}(g(h)(y-w-v))g(h)dy\; v^{-1-\alpha}\frac{L(g(h)v)}{L(g(h))}dv \\
&\qquad \times \mathbb{P}\left(\Delta_1^{g(h)} \in ds; X_{\Delta_1^{g(h)}-} \in g(h)dw; \mathcal{O}_h\right).
\end{aligned}
$$

Now, as $y/3 \le y/2 - 1 \le y/2 - w \le v \le y$, applying the uniform convergence theorem (Theorem A.4.1) to $L(g(h)v)/L(g(h)y)$, uniformly in $y > K$ as $h \to \infty$,

$$
\begin{aligned}
\sigma_h^3(dy) \lesssim y^{-1-\alpha}\frac{L(g(h)y)}{L(g(h))}dy \int_0^h \int_0^1 \int_{\frac{y}{2}-w}^{y-w} f_{h-s}(g(h)(y-w-v))g(h)dv \\
\times \mathbb{P}\left(\Delta_1^{g(h)} \in ds; X_{\Delta_1^{g(h)}-} \in g(h)dw; \mathcal{O}_h\right).
\end{aligned}
\tag{3.121}
$$

Changing variables to $u = g(h)(y-w-v)$, uniformly in $y > K$, as $h \to \infty$,

$$
\begin{aligned}
\sigma_h^3(dy) &\lesssim y^{-1-\alpha}\frac{L(g(h)y)}{L(g(h))}dy \int_0^h \int_0^1 \int_0^{\frac{g(h)y}{2}} f_{h-s}(u)du \\
&\qquad \times \mathbb{P}\left(\Delta_1^{g(h)} \in ds; X_{\Delta_1^{g(h)}-} \in g(h)dw; \mathcal{O}_h\right) \\
&= y^{-1-\alpha}\frac{L(g(h)y)}{L(g(h))}dy \int_0^h \int_0^1 \mathbb{P}\left(X_{h-s} \le \frac{g(h)y}{2}\right) \\
&\qquad \times \mathbb{P}\left(\Delta_1^{g(h)} \in ds; X_{\Delta_1^{g(h)}-} \in g(h)dw; \mathcal{O}_h\right) \\
&\le y^{-1-\alpha}\frac{L(g(h)y)}{L(g(h))}dy \int_0^h \int_0^1 \mathbb{P}\left(\Delta_1^{g(h)} \in ds; X_{\Delta_1^{g(h)}-} \in g(h)dw; \mathcal{O}_h\right)
\end{aligned}
$$

$$= y^{-1-\alpha}\frac{L(g(h)y)}{L(g(h))}dy\mathbb{P}\left(\Delta_1^{g(h)} \leq h; X_{\Delta_1^{g(h)}-} < g(h); \mathcal{O}_h\right) \tag{3.122}$$

$$\leq y^{-1-\alpha}\frac{L(g(h)y)}{L(g(h))}\mathbb{P}\left(\mathcal{O}_h\right)dy. \tag{3.123}$$

Conclusion of Upper Bound By (3.115), (3.116), (3.120), and (3.123), we conclude, as required for the upper bound in (3.58), that uniformly in $y > K$, as $h \to \infty$,

$$\mathbb{P}(X_h \in g(h)dy; \mathcal{O}_h) \lesssim y^{-1-\alpha}\frac{L(g(h)y)}{L(g(h))}\mathbb{P}(\mathcal{O}_h)dy. \tag{3.124}$$

Now we will prove the lower bound on $\mathbb{P}\left(X_h \in g(h)dy; \mathcal{O}_h\right)$.

Proof of Lower Bound Now, fixing $y_0 > 0$, for all $y > K$, as $h \to \infty$,

$$\mathbb{P}\left(X_h \in g(h)dy; \mathcal{O}_h\right) \geq \mathbb{P}\Big(X_h \in g(h)dy; \Delta_1^{g(h)} \leq h-1; X_{\Delta_1^{g(h)}-} < g(h); \frac{g(h)y}{2} \leq S_{\Delta_1^{g(h)}} \leq g(h)y - y_0; \mathcal{O}_h\Big). \tag{3.125}$$

Disintegrating on the values of $\Delta_1^{g(h)}$, $X_{\Delta_1^{g(h)}-}$, and $S_{\Delta_1^{g(h)}}$, applying the Markov property, noting that by (3.74), the measures $\mathbb{P}(\Delta_1^{g(h)} \in ds; \mathcal{O}_h)$ and $\mathbb{P}(\Delta_1^{g(h)} \in ds; \mathcal{O}_{\Delta_1^{g(h)}})$ are equivalent for each $s \leq h$, with $\mathbb{P}(X_t \in dx) = f_t(x)dx$,

$$\begin{aligned}
(3.125) &\overset{(3.74)}{=} \int_{s=0}^{h-1}\int_{w=0}^{1}\int_{v=\frac{y}{2}}^{y-\frac{y_0}{g(h)}-w} \mathbb{P}\Big(X_h \in g(h)dy; \Delta_1^{g(h)} \in ds; X_{\Delta_1^{g(h)}-} \in g(h)dw; \\
&\qquad S_{\Delta_1^{g(h)}} \in g(h)dv; \mathcal{O}_{\Delta_1^{g(h)}}\Big) \\
&\overset{(3.74)}{=} \int_{0}^{h-1}\int_{0}^{1}\int_{v=\frac{y}{2}}^{y-\frac{y_0}{g(h)}-w} f_{h-s}(g(h)(y-w-v))g(h)dy\mathbb{P}\left(S_{\Delta_1^{g(h)}} \in g(h)dv\right) \\
&\qquad \times \mathbb{P}\left(\Delta_1^{g(h)} \in ds; X_{\Delta_1^{g(h)}-} \in g(h)dw; \mathcal{O}_h\right).
\end{aligned}$$

Applying Lemma 3.5.9, noting $h - s \geq 1$, $y/2 < 2y/3 - w$, $v \asymp y$, and $L(g(h)v) \asymp L(g(h)y)$ uniformly in $y > K$ as $h \to \infty$ by the uniform convergence theorem (Theorem A.4.1), it follows that uniformly in $y > K$ as $h \to \infty$,

$$\begin{aligned}(3.125) &\overset{3.5.9}{\gtrsim} \int_0^{h-1} \int_0^1 \int_{v=\frac{y}{2}}^{y-\frac{y_0}{g(h)}-w} f_{h-s}(g(h)(y-w-v))g(h)dyv^{-1-\alpha}\frac{L(g(h)v)}{L(g(h))}dv \\ &\qquad\qquad \times \mathbb{P}\left(\Delta_1^{g(h)} \in ds; X_{\Delta_1^{g(h)}-} \in g(h)dw; \mathcal{O}_h\right) \\ &\gtrsim \int_0^{h-1} \int_0^1 \int_{v=\frac{2y}{3}-w}^{y-\frac{y_0}{g(h)}-w} f_{h-s}(g(h)(y-w-v))g(h)dyy^{-1-\alpha}\frac{L(g(h)y)}{L(g(h))}dv \\ &\qquad\qquad \times \mathbb{P}\left(\Delta_1^{g(h)} \in ds; X_{\Delta_1^{g(h)}-} \in g(h)dw; \mathcal{O}_h\right).\end{aligned}$$

Changing variables to $u = g(h)(y - w - v)$, noting that $y/3 > 1$ for all $y > K$ and that $h - s \geq 1$,

$$\begin{aligned}(3.125) &\gtrsim y^{-1-\alpha}\frac{L(g(h)y)}{L(g(h))}dy \int_0^{h-1} \int_0^1 \int_{u=y_0}^{\frac{g(h)y}{3}} f_{h-s}(u)du \\ &\qquad\qquad \times \mathbb{P}\left(\Delta_1^{g(h)} \in ds; X_{\Delta_1^{g(h)}-} \in g(h)dw; \mathcal{O}_h\right) \\ &= y^{-1-\alpha}\frac{L(g(h)y)}{L(g(h))}dy \int_0^{h-1} \int_0^1 \left[\mathbb{P}\left(X_{h-s} \leq \frac{g(h)y}{3}\right) - \mathbb{P}\left(X_{h-s} \leq y_0\right)\right] \\ &\qquad\qquad \times \mathbb{P}\left(\Delta_1^{g(h)} \in ds; X_{\Delta_1^{g(h)}-} \in g(h)dw; \mathcal{O}_h\right) \\ &\geq y^{-1-\alpha}\frac{L(g(h)y)}{L(g(h))}dy \int_0^{h-1} \int_0^1 \left[\mathbb{P}\left(X_h \leq g(h)\right) - \mathbb{P}\left(X_1 \leq y_0\right)\right] \\ &\qquad\qquad \times \mathbb{P}\left(\Delta_1^{g(h)} \in ds; X_{\Delta_1^{g(h)}-} \in g(h)dw; \mathcal{O}_h\right).\end{aligned}$$

Now, with $X^{(0,g(h))}$ again denoting the process with no jumps bigger than $g(h)$ as in (3.8),

$$\mathbb{P}\left(X_h \leq g(h)\right) = \mathbb{P}(X_h^{(0,g(h))} \leq g(h))\mathbb{P}(\Delta_1^{g(h)} > h) = \mathbb{P}(X_h^{(0,g(h))} \leq g(h))e^{-h\overline{\Pi}(g(h))},$$

and since $\lim_{h\to\infty} h\overline{\Pi}(g(h)) = 0$ by (3.3), by Markov's inequality (Theorem A.2.1), as $h \to \infty$,

$$\mathbb{P}\left(X_h \leq g(h)\right) \overset{(3.3)}{\sim} \mathbb{P}(X_h^{(0,g(h))} \leq g(h)) \geq 1 - \frac{\mathbb{E}[X_h^{(0,g(h))}]}{g(h)} \geq 1 - \frac{h\int_0^{g(h)} \overline{\Pi}(x)dx}{g(h)}.$$

Now, by (3.3) and Karamata's theorem (Theorem A.4.3), as $h \to \infty$, we have

$$\mathbb{P}\left(X_h \leq g(h)\right) \gtrsim 1 - \frac{hg(h)\overline{\Pi}(g(h))}{g(h)} = 1 - h\overline{\Pi}(g(h)) \overset{(3.3)}{\sim} 1.$$

Then since $\mathbb{P}(X_1 \leq y_0) = \text{constant} < 1$, taking y_0 sufficiently large that $\mathbb{P}\left(X_h \leq g(h)\right) - \mathbb{P}(X_1 \leq y_0) \gtrsim 1$ uniformly, we get that uniformly in $y > K$ as $h \to \infty$,

$$\begin{aligned}(3.125) &\gtrsim \frac{L(g(h)y)}{L(g(h))} y^{-1-\alpha} dy \int_{s=0}^{h-1} \int_{w=0}^{1} \mathbb{P}\left(\Delta_1^{g(h)} \in ds; X_{\Delta_1^{g(h)}-} \in g(h)dw; \mathcal{O}_h\right) \\ &= \frac{L(g(h)y)}{L(g(h))} y^{-1-\alpha} dy \mathbb{P}\left(\Delta_1^{g(h)} \leq h-1; X_{\Delta_1^{g(h)}-} < g(h); \mathcal{O}_h\right). \qquad (3.126)\end{aligned}$$

Proof by Contradiction Step Now we assume for a contradiction that

$$\liminf_{h\to\infty} \frac{\mathbb{P}\left(\Delta_1^{g(h)} \leq h-1; X_{\Delta_1^{g(h)}-} < g(h); \mathcal{O}_h\right)}{\mathbb{P}(\mathcal{O}_h)} = 0. \qquad (3.127)$$

As $\Delta_1^{g(h)}$ is exponentially distributed with rate $\overline{\Pi}(g(h))$, $\Delta_1^{g(h)}$ must have a decreasing probability density function, and so by Lemma 3.5.1, as $h \to \infty$,

$$\begin{aligned}&\mathbb{P}\left(\Delta_1^{g(h)} \in [h-1,h]; X_{\Delta_1^{g(h)}-} < g(h); \mathcal{O}_h\right) \quad \leq \mathbb{P}\left(\Delta_1^{g(h)} \in [h-1,h]\right) \\ &\leq \mathbb{P}\left(\Delta_1^{g(h)} \leq 1\right) = 1 - e^{-\overline{\Pi}(g(h))} \leq \overline{\Pi}(g(h)) \overset{3.5.1}{\sim} \frac{\mathbb{P}(\mathcal{O}_h)}{\Phi(h)} \overset{(3.34)}{=} o(1) \times \mathbb{P}\left(\mathcal{O}_h\right),\end{aligned}$$

since $\lim_{h\to\infty} \Phi(h) = \infty$ by (3.34), so it follows that (3.127) holds if and only if

$$\liminf_{h\to\infty} \frac{\mathbb{P}\left(\Delta_1^{g(h)} \leq h; X_{\Delta_1^{g(h)}-} < g(h); \mathcal{O}_h\right)}{\mathbb{P}(\mathcal{O}_h)} = 0. \qquad (3.128)$$

By (3.115), (3.116), (3.120), and (3.122), we get that (3.128) implies, along a subsequence of h, as $h \to \infty$,

$$\begin{aligned}\mathbb{P}\left(X_h \geq Kg(h); \mathcal{O}_h\right) &= \int_K^\infty \mathbb{P}(X_h \in g(h)dy; \mathcal{O}_h) \\ &= \left[\left(o(1) \times \mathbb{P}(\mathcal{O}_h)\right) + \mathbb{P}\left(\Delta_1^{g(h)} \leq h; X_{\Delta_1^{g(h)}-} < g(h); \mathcal{O}_h\right)\right] \int_K^\infty y^{-1-\alpha} \frac{L(g(h)y)}{L(g(h))} dy \\ &\overset{(3.128)}{=} o(1) \times \mathbb{P}(\mathcal{O}_h) \int_K^\infty y^{-1-\alpha} \frac{L(g(h)y)}{L(g(h))} dy.\end{aligned}$$

Now, changing variables from y to $u = g(h)y$,

$$\mathbb{P}\left(X_h \geq Kg(h); \mathcal{O}_h\right) = o(1) \times \frac{g(h)^\alpha \mathbb{P}(\mathcal{O}_h)}{L(g(h))} \int_{Kg(h)}^\infty u^{-1-\alpha} L(u) du.$$

As L is slowly varying at ∞, applying Theorem A.4.4 to $\int_{Kg(h)}^\infty u^{-1-\alpha} L(u) du$, as $h \to \infty$,

$$\mathbb{P}\left(X_h \geq Kg(h); \mathcal{O}_h\right) \lesssim o(1) \times \frac{g(h)^\alpha \mathbb{P}(\mathcal{O}_h)}{L(g(h))} (Kg(h))^{-\alpha} L(Kg(h)) = o(1) \times \mathbb{P}(\mathcal{O}_h). \tag{3.129}$$

But considering the subevent $\{\Delta_1^{g(h)} = \Delta_1^{Kg(h)} \leq h; \mathcal{O}_h\} \subseteq \{X_h \geq Kg(h); \mathcal{O}_h\}$, disintegrating on the value of $\Delta_1^{g(h)}$, and applying the Markov property,

$$\begin{aligned}\mathbb{P}\left(X_h \geq Kg(h); \mathcal{O}_h\right) \geq \mathbb{P}\left(\Delta_1^{g(h)} = \Delta_1^{Kg(h)} \leq h; \mathcal{O}_h\right) &= \int_0^h \mathbb{P}\left(\Delta_1^{g(h)} = \Delta_1^{Kg(h)} \in ds; \mathcal{O}_h\right) \\ &= \int_0^h \mathbb{P}\left(\Delta_1^{g(h)} \in ds; S_{\Delta_1^{g(h)}} \geq Kg(h); \mathcal{O}_{s, X^{(0,g(h))}}\right) \\ &= \int_0^h \mathbb{P}\left(\mathcal{O}_{s, X^{(0,g(h))}}; \Delta_1^{g(h)} \in ds\right) \mathbb{P}\left(S_{\Delta_1^{g(h)}} \geq Kg(h)\right) \\ &= \frac{\overline{\Pi}(Kg(h))}{\overline{\Pi}(g(h))} \int_0^h \mathbb{P}\left(\mathcal{O}_{s, X^{(0,g(h))}}; \Delta_1^{g(h)} \in ds\right) \\ &= \frac{\overline{\Pi}(Kg(h))}{\overline{\Pi}(g(h))} \int_0^h \mathbb{P}\left(\mathcal{O}_h; \Delta_1^{g(h)} \in ds\right) = \frac{\overline{\Pi}(Kg(h))}{\overline{\Pi}(g(h))} \mathbb{P}\left(\mathcal{O}_h; \Delta_1^{g(h)} \leq h\right).\end{aligned}$$

Now, disintegrating on the value of $\Delta_1^{g(h)}$, by (3.74), (3.72), and Lemma 3.5.1, as $h \to \infty$,

$$\mathbb{P}(\mathcal{O}_h; \Delta_1^{g(h)} \le h) = \int_0^h \mathbb{P}(\mathcal{O}_h; \Delta_1^{g(h)} \in ds) \overset{(3.74)}{=} \int_0^h \mathbb{P}(\mathcal{O}_s; \Delta_1^{g(h)} \in ds)$$

$$\overset{(3.72)}{=} \int_0^h \mathbb{P}(\mathcal{O}_s)\overline{\Pi}(g(h))e^{-\overline{\Pi}(g(h))s} ds \sim \overline{\Pi}(g(h))\Phi(h) \overset{3.5.1}{\sim} \mathbb{P}(\mathcal{O}_h),$$

so as $h \to \infty$,

$$\mathbb{P}\left(X_h \ge Kg(h); \mathcal{O}_h\right) \ge \frac{\overline{\Pi}(Kg(h))}{\overline{\Pi}(g(h))}\mathbb{P}\left(\mathcal{O}_h\right) \sim K^{-\alpha}\mathbb{P}(\mathcal{O}_h), \tag{3.130}$$

because $\overline{\Pi}$ is regularly varying at ∞, so the inequality (3.129) contradicts (3.130), and therefore it follows that $\liminf_{h\to\infty} \mathbb{P}\big(\Delta_1^{g(h)} \le h; X_{\Delta_1^{g(h)}-} < g(h); \mathcal{O}_h\big)/\mathbb{P}(\mathcal{O}_h) > 0$. Then by (3.126), uniformly in $y > K$ as $h \to \infty$,

$$\mathbb{P}(X_h \in g(h)dy; \mathcal{O}_h) \gtrsim y^{-1-\alpha}\frac{L(g(h)y)}{L(g(h))}\mathbb{P}\left(\mathcal{O}_h\right) dy,$$

as required for the lower bound in (3.58), so the proof of Lemma 3.5.7 is complete.

□

3.9 Proofs of Auxiliary Lemmas

Lemma 3.5.5 *In case (i), for the function ρ as defined in* (3.15)*,* $\liminf_{t\to\infty} \rho(t) \ge 0$.

Proof of Lemma 3.5.5. Recall from (3.15) and (3.77) that

$$\rho(t) = \frac{1}{\Phi(t)}\left[\mathbb{P}(\mathcal{O}_t; \Delta_1^{g(t)} > t) - \overline{\Pi}(g(t))^2 \int_0^t \int_0^s \mathbb{P}(\mathcal{O}_v)e^{-\overline{\Pi}(g(t))v} dvds\right].$$

If $\rho(t) \le 0$, then as for all $t > 1$, $\Phi(t) \ge \Phi(1) = \text{constant} > 0$, for all $t > 1$,

$$\begin{aligned}|\rho(t)| &\le \frac{\overline{\Pi}(g(t))^2}{\Phi(t)} \int_0^t \int_0^s \mathbb{P}(\mathcal{O}_v)e^{-\overline{\Pi}(g(t))v} dvds \\ &\le \frac{t\overline{\Pi}(g(t))^2}{\Phi(t)} \int_0^t \mathbb{P}(\mathcal{O}_v)e^{-\overline{\Pi}(g(t))v} dv \le \frac{t^2\overline{\Pi}(g(t))^2}{\Phi(t)} \lesssim t^2\overline{\Pi}(g(t))^2.\end{aligned}$$

Now, $\lim_{t\to\infty} t\overline{\Pi}(g(t)) = 0$ by (3.3), so it must be the case that $\liminf_{t\to\infty} \rho(t) \geq 0$, as required.

□

Lemma 3.5.6 *In case (i), for $t_0(y)$ as defined in (3.22), uniformly in $h > 0, y > g(h)$, and $t \in (t_0(y), \infty]$,*

$$\int_{t_0(y)}^{t} \left(\overline{\Pi}(g(s+h)-y) - \overline{\Pi}(g(s))\right) ds \lesssim y f'(y)\overline{\Pi}(y).$$

Proof of Lemma 3.5.6. Recall that $\overline{\Pi}(x) = x^{-\alpha}L(x)$ for L slowly varying at ∞, so as $\overline{\Pi}$ is non-increasing, for large $N > 0$, using that $t_0(y) \geq f(Ay)$,

$$\begin{aligned}
\int_{t_0(y)}^{t} (\overline{\Pi}(g(s+h)-y) - \overline{\Pi}(g(s)))ds &\leq \int_{t_0(y)}^{t} (\overline{\Pi}(g(s)-y) - \overline{\Pi}(g(s)))ds \\
\leq \int_{f(Ay)}^{\infty} (\overline{\Pi}(g(s)-y) - \overline{\Pi}(g(s)))ds &= \int_{f(Ay)}^{\infty} \left(\frac{L(g(s)-y)}{(g(s)-y)^{\alpha}} - \frac{L(g(s))}{g(s)^{\alpha}}\right) ds \\
= \int_{f(Ay)}^{\infty} \frac{L(g(s)-y)}{(g(s)-y)^{\alpha}} &\left(1 - \left(\frac{g(s)-y}{g(s)}\right)^{\alpha} \frac{L(g(s))}{L(g(s)-y)}\right) ds \\
= \int_{f(Ay)}^{\infty} \frac{L(g(s)-y)}{(g(s)-y)^{\alpha}} &\left(1 - \left(\frac{g(s)-y}{g(s)}\right)^{\alpha+N} \frac{g(s)^N L(g(s))}{(g(s)-y)^N L(g(s)-y)}\right) ds.
\end{aligned} \tag{3.131}$$

Now, $A > B - 1$, and $x^N L(x)$ is non-decreasing in x for $x > B$ in case (i), so it follows that

$$(3.131) \leq \int_{f(Ay)}^{\infty} \frac{L(g(s)-y)}{(g(s)-y)^{\alpha}} \left(1 - \left(\frac{g(s)-y}{g(s)}\right)^{\alpha+N}\right) ds.$$

One can verify $1 - (1 - y/g(s))^{\alpha+N} \lesssim y/g(s)$, uniformly in $y > 0, s > f(Ay)$, and therefore

$$(3.131) \lesssim \int_{f(Ay)}^{\infty} \frac{L(g(s)-y)}{(g(s)-y)^{\alpha}} \frac{y}{g(s)} ds.$$

As $g(s) - y \geq (1 - A^{-1})g(s)$ for $s > f(Ay)$, and $\overline{\Pi}(x) = x^{-\alpha}L(x)$ is non-increasing,

$$(3.131) \lesssim y \int_{f(Ay)}^{\infty} \frac{L((1-A^{-1})g(s))}{g(s)^{1+\alpha}} ds. \tag{3.132}$$

Applying the uniform convergence theorem (Theorem A.4.1) to the slowly varying function L, substituting $u = g(s)$, as $uf'(u)\overline{\Pi}(u)$ is decreasing, we conclude that uniformly in $y > 0$ (and so also uniformly in $h > 0, y > g(h)$),

$$\begin{aligned}(3.131) &\lesssim y \int_{f(Ay)}^{\infty} \frac{L(g(s))}{g(s)^{1+\alpha}} ds = y \int_{Ay}^{\infty} \frac{L(u)}{u^{1+\alpha}} f'(u) du = y \int_{Ay}^{\infty} u^{-2} u f'(u)\overline{\Pi}(u) du \\ &\leq Ay^2 f'(Ay)\overline{\Pi}(Ay) \int_{Ay}^{\infty} u^{-2} du = \frac{1}{2} y f'(Ay)\overline{\Pi}(Ay) \lesssim y f'(y)\overline{\Pi}(y). \qquad \square\end{aligned}$$

Lemma 3.5.8 *In case (i), for $\delta > 0$ small enough that $0 < f(0) < f(\delta) < 1$, uniformly for all $h > 0$ and $y > g(h + f(\delta))$,*

$$q_h(y) \asymp \Phi_y^h(f(Ay)) \exp\left(-\int_1^{f(Ay)} \overline{\Pi}(g(s)) ds\right).$$

Proof of Lemma 3.5.8. First recall that by Theorem 3.4.18,

$$q_h(y) = \frac{\Phi_y^h(t_0(y))}{\Phi(1)} \lim_{t\to\infty} \exp\left(\int_{t_0(y)}^{t} \left(\overline{\Pi}(g_y^h(s)) + \rho_y^h(s)\right) ds - \int_1^t \left(\overline{\Pi}(g(s)) + \rho(s)\right) ds\right).$$

Now, by (3.30) in Lemma 3.5.3, uniformly in $h > 0, y > g(h)$,

$$\left|\int_{t_0(y)}^{\infty} \rho_y^h(s) ds\right| \lesssim \int_{t_0(y)}^{\infty} \frac{1}{s \log(s)^{1+\varepsilon}} \left(1 + \frac{1}{f(y) - h}\right) ds,$$

and $1/(f(y) - h) \leq 1/f(\delta) < \infty$ since $y \geq g(h + f(\delta))$, and thus $\int_{t_0(y)}^{\infty} \rho_y^h(s) ds$ is bounded uniformly in $h > 0, y > g(h + f(\delta))$. Moreover, by Remark 3.5.4, we have $\int_1^\infty \rho(s) ds < \infty$. For $y > g(h + f(\delta)) > \delta$, $f(Ay) > f(A\delta)$, so taking A sufficiently large if necessary, $t_0(y) := f(Ay) \vee f(1 + 2/A) = f(Ay)$, then by Lemma 3.5.6, $\limsup_{t\to\infty} \int_{f(Ay)}^t (\overline{\Pi}(g_y^h(s)) - \overline{\Pi}(g(s))) ds < \infty$, and so we have uniformly in $h > 0, y > g(h)$,

$$q_h(y) \lesssim \Phi_y^h(f(Ay)) \exp\left(-\int_1^{f(Ay)} \overline{\Pi}(g(s)) ds\right).$$

For the converse inequality, as $\overline{\Pi}$ is non-increasing, for $y > g(h)$ (so $f(y) > h$),

$$\begin{aligned} \int_{f(Ay)}^{t} (\overline{\Pi}(g(s)) - \overline{\Pi}(g(s+h)-y))ds \quad &\leq \int_{f(Ay)}^{t} (\overline{\Pi}(g(s)) - \overline{\Pi}(g(s+h)))ds \\ = \int_{f(Ay)}^{t} \overline{\Pi}(g(s))ds - \int_{f(Ay)+h}^{t+h} \overline{\Pi}(g(s))ds \quad &\leq \int_{f(Ay)}^{f(Ay)+h} \overline{\Pi}(g(s))ds \\ &\leq h\overline{\Pi}(g(f(Ay))) \leq h\overline{\Pi}(y) \leq f(y)\overline{\Pi}(y). \end{aligned}$$

Then as $y > g(h + f(\delta)) > \delta$ and $\lim_{y\to\infty} f(y)\overline{\Pi}(y) = 0$ by (3.3) (recall $f^{-1} = g$), we conclude that

$$q_h(y) \asymp \Phi_y^h(f(Ay)) \exp\left(-\int_1^{f(Ay)} \overline{\Pi}(g(s))ds\right).$$

□

Lemma 3.5.9 *For a subordinator and a function $g = f^{-1}$ as in case (ia), let $S_{\Delta_1^{g(h)}}$ denote the size of its first jump of size greater than $g(h)$. Then there exists $h_0 > 0$ such that uniformly for all $h > h_0$ and $v > 1$,*

$$\mathbb{P}\left(S_{\Delta_1^{g(h)}} \in g(h)dv\right) = \frac{\Pi(g(h)dv)}{\overline{\Pi}(g(h))} \asymp \frac{L(g(h)v)}{L(g(h))} v^{-1-\alpha} dv.$$

In particular there is $x_0 \in (0, \infty)$ so that for all $x > x_0$, with $\Pi(dx) = u(x)dx$,

$$u(x) \asymp x^{-1}\overline{\Pi}(x) = L(x)x^{-1-\alpha}. \tag{3.133}$$

Proof of Lemma 3.5.9. In case (ia), with $\Pi(dx) = u(x)dx$, $u(x)$ has bounded decrease and bounded increase (see Definition A.4.5), and as $\overline{\Pi}$ is regularly varying at ∞ with index $-\alpha \in (-1, 0)$ in case (i), it follows that $\overline{\Pi}$ has positive increase and bounded increase (see Definition A.4.5). Thus we can apply Theorem A.4.9, yielding that $xu(x) \asymp \overline{\Pi}(x)$ for all sufficiently large x, so

$$\frac{\Pi(g(h)dv)}{\overline{\Pi}(g(h))} = \frac{u(g(h)v)g(h)dv}{\overline{\Pi}(g(h))} \asymp \frac{\overline{\Pi}(g(h)v)g(h)dv}{g(h)v\overline{\Pi}(g(h))} = v^{-1-\alpha}\frac{L(g(h)v)}{L(g(h))}dv.$$

□

Lemma 3.5.10 *Recall the notation* (3.14), (3.22)*. If* $h > 0$*,* $y > g(h)$*, and* $t \geq f(Ay)$*, for* $A > 3 \vee (B-1)$*, then* $\Phi_y^h(t) \geq f(y) - h$*.*

Proof of Lemma 3.5.10. For $t \geq f(Ay)$, $A > 3 \vee (B-1)$, as f is increasing,

$$t \geq f(Ay) \geq f(y) \geq f(y) - h.$$

For $y > 0$, $y > g(h)$, and $s \leq f(y) - h$, we have $g_y^h(s) = g(s+h) - y \leq g(f(y)) - y = 0$, so

$$\mathbb{P}\Big(\mathcal{O}_s^{g_y^h}\Big) = \mathbb{P}\left(X_u \geq g_y^h(u), \forall u \leq s\right) \geq \mathbb{P}\left(X_u \geq 0, \forall u \leq s\right) = 1,$$

and we conclude, as required, that

$$\Phi_y^h(t) = \int_0^t \mathbb{P}\Big(\mathcal{O}_s^{g_y^h}\Big) ds \geq \int_0^{f(y)-h} \mathbb{P}\Big(\mathcal{O}_s^{g_y^h}\Big) ds = f(y) - h.$$

□

Lemma 3.6.1 *Let* $(X_t)_{t\geq 0}$ *be a subordinator satisfying the assumptions in case (i) or (ii). Then there exists a constant* $C > 0$*, which depends only on the law of* X*, such that for all* $t > 0$*,* $A(t) \in (1, \infty)$*,* $B(t) > 0$*, and* $H(t) \in (0,1)$*,*

$$\mathbb{P}\left(X_t^{(0,A(t))} > B(t)\right) \leq \exp\left(Ct \log\left(\frac{1}{H(t)}\right) H(t)^{-\frac{A(t)}{B(t)}} \overline{\Pi}(A(t)) \frac{A(t)}{B(t)}\right) H(t). \tag{3.134}$$

Proof of Lemma 3.6.1. By Markov's inequality (Theorem A.2.1), with $\lambda = \log(1/H(t))/B(t)$,

$$\begin{aligned}
\mathbb{P}\left(X_t^{(0,A(t))} > B(t)\right) \quad &= \mathbb{P}\left(e^{\lambda X_t^{(0,A(t))}} \geq e^{\lambda B(t)}\right) \\
\leq \mathbb{E}\left[e^{\lambda X_t^{(0,A(t))}}\right] e^{-\lambda B(t)} \quad &= \exp\left(t\int_0^{A(t)} \lambda e^{\lambda x}(\overline{\Pi}(x) - \overline{\Pi}(A(t)))dx\right) H(t) \\
\leq \exp\left(t\frac{\log(1/H(t))}{B(t)} e^{\lambda A(t)} \int_0^{A(t)} \overline{\Pi}(x)dx\right) H(t) \\
= \exp\left(t\frac{\log(1/H(t))}{B(t)} H(t)^{-\frac{A(t)}{B(t)}} \int_0^{A(t)} \overline{\Pi}(x)dx\right) H(t). \quad (3.135)
\end{aligned}$$

Now, by Theorem A.4.8, which applies as $\overline{\Pi}$ has lower index $\beta(\overline{\Pi}) > -1$ in cases (i) and (ii), there exists $C' > 0$ such that for all $A(t) > 1$,

$$\int_0^{A(t)} \overline{\Pi}(x)dx \leq \int_0^1 \overline{\Pi}(x)dx + C'\overline{\Pi}(A(t))A(t). \quad (3.136)$$

Now, consider the *lower order* $\mu(\overline{\Pi}) := \liminf_{x\to\infty} \log(\overline{\Pi}(x))/\log(x)$, which by Theorem A.4.7 satisfies $\mu(\overline{\Pi}) \geq \beta(\overline{\Pi}) > -1$, so that $\liminf_{x\to\infty} \log(\overline{\Pi}(x))/\log(x) > 1$, which implies that $\liminf_{x\to\infty} x\overline{\Pi}(x) > 0$, so that uniformly among $A(t) > 1$, we have $\int_0^1 \overline{\Pi}(x)dx \lesssim A(t)\overline{\Pi}(A(t))$, and (3.134) follows immediately from (3.135) and (3.136), as required. □

Lemma 3.7.1 *Recalling* (3.12) *and* (3.22)*, if* $t > t_0(y)$*, then for all* $y, h > 0$*, we have* $g_y^h(t) \geq (1 - 1/A)\, g(t)$.

Proof of Lemma 3.7.1. By (3.22), $t > t_0(y) \geq f(Ay)$. As $g = f^{-1}$ is increasing, we conclude that

$$g_y^h(t) = \left(\frac{g(t+h)}{g(t)} - \frac{y}{g(t)}\right) g(t) \geq \left(1 - \frac{y}{g(t)}\right) g(t) \geq \left(1 - \frac{y}{g(f(Ay))}\right) g(t) = (1 - A^{-1})g(t).$$

□

3.10 Extensions

In case (i), the boundary at which the conditioned process is transient/recurrent is found in Proposition 3.4.15, but we only cover a small class of functions for which the conditioned process is recurrent. In cases (i) and (ii), the tail function $\overline{\Pi}$ satisfies $\lim_{t\to\infty} t\overline{\Pi}(g(t)) = 0$, but Proposition 3.10.1 applies in the case where $\lim_{t\to\infty} t\overline{\Pi}(g(t)) = \infty$. This lies on the recurrent side of the boundary between $I(f) < \infty$ and $I(f) = \infty$, so the Markov process should be recurrent under the law $\mathbb{Q}$.

The equation (3.137) is much harder to handle than the corresponding equation (3.18), but if one is able to replicate the steps between (3.18) and (3.20) to find a suitably tractable expression analogous to (3.20), then this should lead to a broad set of results for the recurrent case. Moreover, it can be verified that the precise asymptotic result [47, Theorem 2.2] for the density $\mathbb{P}(X_t \in dx)$ can be utilised in this case, but not when $\lim_{t\to\infty} t\overline{\Pi}(g(t)) = 0$.

Proposition 3.10.1. *For a subordinator, if* $\lim_{t\to\infty} t\overline{\Pi}(g(t)) = \infty$, *then as* $t \to \infty$,

$$\mathbb{P}(\mathcal{O}_t) \sim \mathbb{P}\left(\mathcal{O}_t; \Delta_1^{g(t)} \leq t\right) = \overline{\Pi}(g(t)) \int_0^t \mathbb{P}(\mathcal{O}_s) e^{-(t-s)\overline{\Pi}(g(t))} ds. \tag{3.137}$$

Proof of Proposition 3.10.1. As $\Delta_1^{g(t)}$ is exponentially distributed with rate $\overline{\Pi}(g(t))$,

$$\mathbb{P}(\mathcal{O}_t) = \mathbb{P}\left(\mathcal{O}_t; \Delta_1^{g(t)} \leq t\right) + \mathbb{P}\left(\mathcal{O}_t | \Delta_1^{g(t)} > t\right) e^{-t\overline{\Pi}(g(t))}.$$

Now, $\mathbb{P}\left(\mathcal{O}_t | \Delta_1^{g(t)} > t\right) \leq \mathbb{P}(\mathcal{O}_t)$, and $\lim_{t\to\infty} e^{-t\overline{\Pi}(g(t))} = 0$, so as $t \to \infty$, $\mathbb{P}(\mathcal{O}_t) \sim \mathbb{P}\left(\mathcal{O}_t; \Delta_1^{g(t)} \leq t\right)$, as required for the first claim of Proposition 3.10.1.

For the second claim, recall the notation introduced in (3.8). Disintegrating on the value $\Delta_1^{g(t)}$,

$$\begin{aligned}\mathbb{P}\left(\mathcal{O}_t; \Delta_1^{g(t)} \leq t\right) &= \int_0^t \mathbb{P}\left(\mathcal{O}_t | \Delta_1^{g(t)} = t_1\right) \overline{\Pi}(g(t)) e^{-t_1\overline{\Pi}(g(t))} dt_1 \\ &= \overline{\Pi}(g(t)) \int_0^t \mathbb{P}\left(\mathcal{O}_{t_1, X^{(0,g(t))}}\right) e^{-t_1\overline{\Pi}(g(t))} dt_1.\end{aligned}$$

Now, partitioning according to the size of $\Delta_1^{g(t)}$, we can rewrite the above integrand as

$$\mathbb{P}\left(\mathcal{O}_{t_1,X^{(0,g(t))}}\right)e^{-t_1\overline{\Pi}(g(t))} = \mathbb{P}(\mathcal{O}_{t_1};\Delta_1^{g(t)} > t_1) = \mathbb{P}\left(\mathcal{O}_{t_1}\right) - \mathbb{P}\left(\mathcal{O}_{t_1};\Delta_1^{g(t)} \leq t_1\right).$$

Disintegrating again on the value of $\Delta_1^{g(t)}$, it follows that

$$\begin{aligned}\mathbb{P}\left(\mathcal{O}_t;\Delta_1^{g(t)} \leq t\right) &= \overline{\Pi}(g(t))\int_0^t \mathbb{P}\left(\mathcal{O}_{t_1}\right)dt_1 - \overline{\Pi}(g(t))\int_0^t \mathbb{P}\left(\mathcal{O}_{t_1};\Delta_1^{g(t)} \leq t_1\right)dt_1 \\ &= \overline{\Pi}(g(t))\int_0^t \mathbb{P}\left(\mathcal{O}_{t_1}\right)dt_1 - \overline{\Pi}(g(t))^2\int_0^t\int_0^{t_1} \mathbb{P}\left(\mathcal{O}_{t_2,X^{(0,g(t))}}\right)e^{-t_2\overline{\Pi}(g(t))}dt_1dt_2.\end{aligned}$$

One can repeat this procedure to deduce that for all $n \in \mathbb{N}$,

$$\mathbb{P}\left(\mathcal{O}_t;\Delta_1^{g(t)} \leq t\right) = \sum_{k=1}^{n-1}(-1)^{k-1}\overline{\Pi}(g(t))^k\int_0^t\int_0^{t_1}\cdots\int_0^{t_{k-1}}\mathbb{P}(\mathcal{O}_{t_k})dt_k\cdots dt_2dt_1 \tag{3.138}$$

$$+(-1)^{n-1}\overline{\Pi}(g(t))^n\int_0^t\cdots\int_0^{t_{n-1}}\mathbb{P}(\mathcal{O}_{t_k,X^{(0,g(t))}})e^{-t_n\overline{\Pi}(g(t))}dt_n\cdots dt_1. \tag{3.139}$$

The contribution in (3.139) tends to 0 as $n \to \infty$ since

$$\lim_{n\to\infty}|\,(3.139)\,| \leq \lim_{n\to\infty}\overline{\Pi}(g(t))^n\int_0^t\cdots\int_0^{t_{n-1}}dt_n\cdots dt_1 = \lim_{n\to\infty}\frac{(t\overline{\Pi}(g(t)))^n}{n!} = 0.$$

It follows by the same argument that the sum in (3.138) is absolutely convergent as $n \to \infty$, and so

$$\begin{aligned}\mathbb{P}\left(\mathcal{O}_t;\Delta_1^{g(t)} \leq t\right) &= \sum_{k=1}^{\infty}(-1)^{k-1}\overline{\Pi}(g(t))^k\int_0^t\cdots\int_0^{t_{k-1}}\mathbb{P}(\mathcal{O}_{t_k})dt_k\cdots dt_1 \\ &= \sum_{k=1}^{\infty}(-1)^{k-1}\overline{\Pi}(g(t))^k\int_0^t\mathbb{P}(\mathcal{O}_s)\frac{(t-s)^{k-1}}{(k-1)!}ds.\end{aligned}$$

Interchanging the order of summation and integration, we conclude that

$$\begin{aligned}\mathbb{P}\left(\mathcal{O}_t; \Delta_1^{g(t)} \leq t\right) &= \overline{\Pi}(g(t)) \int_0^t \mathbb{P}(\mathcal{O}_s) \sum_{k=1}^{\infty} (-1)^{k-1} \overline{\Pi}(g(t))^{k-1} \frac{(t-s)^{k-1}}{(k-1)!} ds \\ &= \overline{\Pi}(g(t)) \int_0^t \mathbb{P}(\mathcal{O}_s) e^{-(t-s)\overline{\Pi}(g(t))} ds.\end{aligned}$$

□

If one can replicate the steps between (3.18) and (3.20) to help understand the new equation (3.137), then for any process satisfying $\lim_{t\to\infty} t\overline{\Pi}(g(t)) = \infty$, the law of the conditioned process can be studied using the argument as in (3.17), under some regularity conditions. This would allow us to study a large class of cases in which the conditioned process is recurrent, complementing our results in case (i) and case (ii).

Appendix

A.1 Supplementary Figures

$$\frac{N(t,\delta) - ta(\delta)}{t^{\frac{1}{2}} b(\delta)} \xrightarrow{d} \mathcal{N}(0,1), \quad \text{the CLT result in Theorem 2.2.1}$$

$$\Uparrow \qquad \text{(Lemma 2.3.1)}$$

$$\lim_{\delta \to 0} \frac{U(\delta)^{\frac{7}{3}}}{\sigma_\delta^2} = 0$$

$$\Uparrow \qquad \text{(Lemma 2.3.9)}$$

$$\liminf_{\delta \to 0} \left[\mathbb{P}\left(\tilde{X}^\delta_{(1+\alpha)U(\delta)} \leq \delta \right) + \mathbb{P}\left(\tilde{X}^\delta_{(1-\alpha)U(\delta)} \geq \delta \right) \right] > 0$$

$$\Uparrow \qquad \text{(Lemma 2.3.12)}$$

$$\limsup_{\delta \to 0} \; tR(\lambda_\delta) < \infty$$

$$\Uparrow \qquad \text{(Lemma 2.3.5)}$$

$$\limsup_{\delta \to 0} \; \delta\lambda_\delta < \infty$$

$$\Uparrow \qquad \text{(Proof of Theorem 2.2.1)}$$

$$\liminf_{\delta \to 0} \frac{I(2\delta)}{I(\delta)} > 1, \quad \text{the imposed regularity condition}$$

Figure A.1: Chain of sufficient conditions used to prove Theorem 2.2.1

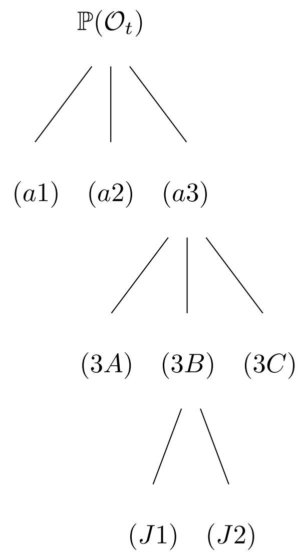

Figure A.2: Structure of the proof of Lemma 3.5.1

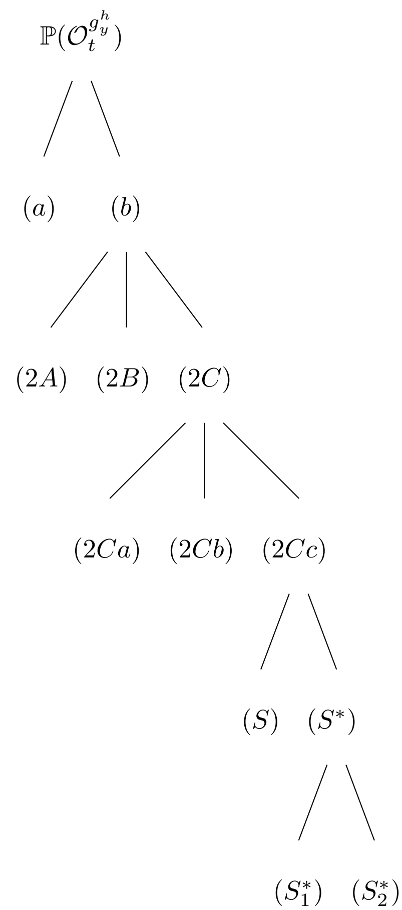

Figure A.3: Structure of the proof of Lemma 3.5.3, case (i)

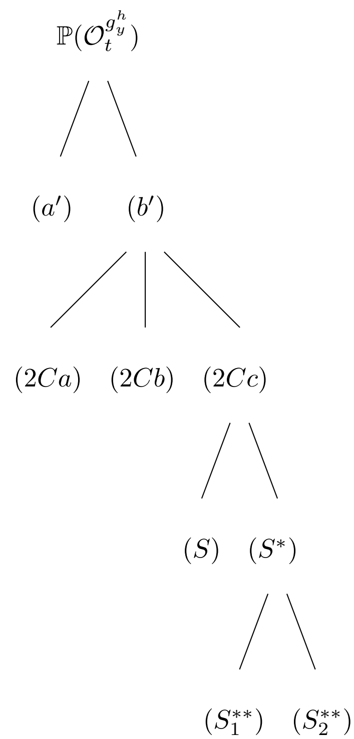

Figure A.4: Structure of the proof of Lemma 3.5.3, case (ii)

A.2 Key Inequalities

Theorem A.2.1. (Markov's Inequality, [69, Lemma 3.1])

For each random variable X *taking values in* $[0,\infty)$*, such that* $\mathbb{E}[X]>0$*, for all* $r>0$*, we have*

$$\mathbb{P}(X>r)\leq\frac{\mathbb{E}[X]}{r}.$$

Theorem A.2.2. (Chebyshev's Inequality, [69, Lemma 3.1])

For each random variable X *taking values in* $[0,\infty)$*, such that* $\mathbb{E}[X^2]<\infty$*, for all* $r>0$*, we have*

$$\mathbb{P}(|X-\mathbb{E}[X]|>r)\leq\frac{\mathrm{Var}(X)}{r^2}.$$

A.3 Key Theorems in Measure Theory

Theorem A.3.1. (Dominated Convergence Theorem, [69, Theorem 1.21])

Consider measurable functions f,g,f_n,g_n*,* $n\in\mathbb{N}$*, on a measure space* $(\Omega,\mathcal{A},\mu)$*. Let* $|f_n(x)|\leq g_n(x)$ *for all* $x\in\Omega$*,* $n\in\mathbb{N}$*, and moreover let* $\lim_{n\to\infty}f_n(x)=f(x)$ *and* $\lim_{n\to\infty}g_n(x)=g(x)$ *for all* $x\in\Omega$*. If* $\lim_{n\to\infty}\int_\Omega g_n(x)\mu(dx)=\int_\Omega g(x)\mu(dx)$*, then*

$$\lim_{n\to\infty}\int_\Omega f_n(x)\mu(dx)=\int_\Omega f(x)\mu(dx).$$

Theorem A.3.2. (Fubini's Theorem, [69, Theorem 1.27])

Consider sigma-finite measure spaces $(S,\mathcal{S},\mu)$ *and* $(T,\mathcal{T},\nu)$ *with product space* $(S\times T,\mathcal{S}\times\mathcal{T},\mu\otimes\nu)$*. For each measurable function* $f:S\times T\to[0,\infty)$ *such that* $\int_{S\times T}|f(s,t)|(\mu\otimes\nu)(ds,dt)<\infty$*, we have*

$$\int_S\int_T f(s,t)\mu(ds)\nu(dt)=\int_T\int_S f(s,t)\nu(ds)\mu(dt)$$

Lemma A.3.3. (Moments and Tails Lemma, [69, Lemma 2.4])

For each random variable X, and for all $m > 0$, we have

$$\mathbb{E}[X^m] = \int_0^\infty m y^{m-1} \mathbb{P}(X > y) dy.$$

Lemma A.3.4. (Borel-Cantelli Lemma, [69, Theorem 2.18])

Consider measurable events $A_n, n \in \mathbb{N}$ on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. If $\sum_{n=1}^\infty \mathbb{P}(A_n) < \infty$ then

$$\mathbb{P}\,(\text{infinitely many of the events } A_n, n \in \mathbb{N} \text{ occur}) = 0.$$

A.4 Results on Regularly Varying Functions and Related Functions

Theorem A.4.1. (Uniform Convergence Theorem, [21, Theorem 1.2.1])

If L is a slowly varying function, then $\lim_{x\to\infty} L(\lambda x)/L(x) = 1$, uniformly among λ in each compact subset of $(0, \infty)$.

Theorem A.4.2. (Potter's Theorem [21, Theorem 1.5.6])

If L is a slowly varying function, then for each $A > 1$ and $\delta > 0$, there exists $K > 0$ such that for all $x, y \geq K$,

$$\frac{L(x)}{L(y)} \leq A \max\left\{\frac{x^\delta}{y^\delta}, \frac{y^\delta}{x^\delta}\right\}.$$

Theorem A.4.3. (Karamata's Theorem, [21, Prop. 1.5.8])

For each slowly varying function L, with $K > 0$ large enough that L is locally bounded on $[K, \infty)$, for all $\alpha > -1$, as $x \to \infty$,

$$\int_K^x y^\alpha L(y) dy \sim \frac{x^{\alpha+1} L(x)}{\alpha + 1}.$$

Theorem A.4.4. (Alternative Version of Karamata's Theorem, [21, Prop 1.5.10])

If L is a slowly varying function, then for all $p < -1$, the integral $\int_x^\infty y^p L(y) dy$ converges (for $x > 0$), and

$$\lim_{x\to\infty} \frac{x^{p+1} L(x)}{\int_x^\infty y^p L(y) dy} = -p - 1.$$

Definition A.4.5. (Bounded Increase, Bounded Decrease, and Positive Increase, [21, p71])

The lower index, $\beta(h)$, of a function $h : \mathbb{R} \to \mathbb{R}$ is the supremum of $\beta \in \mathbb{R}$ for which there exists $C > 0$ so that for all $\Lambda > 1$, $h(\lambda x)/h(x) \geq (1 + o(1))C\lambda^{\beta}$, uniformly in $\lambda \in [1, \Lambda]$, as $x \to \infty$, see [21, p68].

The upper index, $\alpha(h)$, of a function $h : \mathbb{R} \to \mathbb{R}$ is the infimum of $\alpha \in \mathbb{R}$ for which there exists $C' > 0$ so that for all $\Lambda > 1$, $h(\lambda x)/h(x) \leq (1 + o(1))C'\lambda^{\alpha}$, uniformly in $\lambda \in [1, \Lambda]$, as $x \to \infty$, see [21, p68].

The function h has bounded increase if $\alpha(h) < \infty$.

The function h has bounded decrease if $\beta(h) > -\infty$.

The function h has positive increase if $\beta(h) > 0$.

Theorem A.4.6. (Potter's Theorem for Bounded Increase/Decrease, [21, Prop 2.2.1])
If the function u has bounded increase, then for each $p > \alpha(u)$, there exist constants $C_1, C_2 > 0$ such that for all $y \geq x \geq C_1$,

$$\frac{u(y)}{u(x)} \leq C_2 \frac{y^p}{x^p},$$

and if the function u has bounded decrease, then for each $q < \beta(u)$, there exist constants $C_1', C_2' > 0$ such that for all $y \geq x \geq C_1'$,

$$\frac{u(y)}{u(x)} \geq C_2' \frac{y^q}{x^q}.$$

Theorem A.4.7. (Relationship Between Lower Order and Lower Index, [21, Prop 2.2.5])
For a function $f : \mathbb{R} \to [0, \infty)$, the lower order, defined as $\mu(f) := \liminf_{x\to\infty} \log(f(x))/\log(x)$, satisfies

$$\mu(f) \geq \beta(f).$$

Theorem A.4.8. (Karamata's Theorem for Positive Increase, [21, Prop 2.6.1(b)])
If f is positive, locally integrable on $[K, \infty)$ for $K > 0$, and has positive increase (see Definition A.4.5), then

$$\liminf_{x\to\infty} \frac{f(x)}{\int_K^x s^{-1} f(s)ds} > 0.$$

Theorem A.4.9. ($\mathcal{O}$-Version of Monotone Density Theorem, [21, Prop 2.10.3])

Let $u : [0, \infty) \to \mathbb{R}$ *be eventually positive, with bounded increase or bounded decrease. Defining the function* $U(x) := \int_0^x u(y)dy$*, if* U *has bounded increase and positive increase, then as* $x \to \infty$*,*

$$U(x) \asymp xu(x).$$

Bibliography

[1] Applebaum, D. *Lévy processes and stochastic calculus*, volume 116. Cambridge university press, 2009.

[2] Aurzada, F. and Kramm, T. First exit of Brownian motion from a one-sided moving boundary. In *High Dimensional Probability VI*, pages 213–217. Springer, 2013.

[3] Aurzada, F. and Kramm, T. The first passage time problem over a moving boundary for asymptotically stable Lévy processes. *J. Theoret Probab.*, 29(3):737–760, 2016.

[4] Aurzada, F., Kramm, T., and Savov, M. First passage times of Lévy processes over a one-sided moving boundary. *Markov Process. Related Fields*, 21(1):1–38, 2015.

[5] Barker, A. Fractal-dimensional properties of subordinators. *J. Theoret Probab.*, pages 1–18.

[6] Barker, A. Transience and recurrence of Markov processes with constrained local time. *arXiv preprint arXiv:1806.05965*, 2018.

[7] Barndorff-Nielsen, O. Processes of normal inverse Gaussian type. *Finance Stoch.*, 2(1):41–68, 1997.

[8] Benjamini, I. and Berestycki, N. An integral test for the transience of a Brownian path with limited local time. *Ann. Inst. H. Poincaré Probab. Statist.*, 47(2):539–558, 2011.

[9] Berger, Q., den Hollander, F., and Poisat, J. Annealed scaling for a charged polymer in dimensions two and higher. *J. Phys. A*, 51(5):054002, 2018.

[10] Bernoulli, J. *Ars conjectandi.* Impensis Thurnisiorum, fratrum, 1713.

[11] Bertoin, J. An extension of Pitman's theorem for spectrally positive lévy processes. *Ann. Probab.*, pages 1464–1483, 1992.

[12] Bertoin, J. Splitting at the infimum and excursions in half-lines for random walks and Lévy processes. *Stochastic Process. Appl.*, 47(1):17–35, 1993.

[13] Bertoin, J. Cauchy's principal value of local times of Lévy processes with no negative jumps via continuous branching processes. *Electron. J. Probab.*, 2(6):1–12, 1997.

[14] Bertoin, J. *Lévy processes*, volume 121. Cambridge university press, 1998.

[15] Bertoin, J. Subordinators: examples and applications. *Lectures on probability theory and statistics (Saint-Flour, 1997)*, 1717:1–91, 1999.

[16] Bertoin, J. Homogeneous fragmentation processes. *Probab. Theory Related Fields*, 121(3):301–318, 2001.

[17] Bertoin, J. *Random fragmentation and coagulation processes*, volume 102. Cambridge University Press, 2006.

[18] Bertoin, J. and Martínez, S. Fragmentation energy. *Adv. in Appl. Probab.*, 37(2):553–570, 2005.

[19] P. Billingsley. *Convergence of probability measures.* John Wiley & Sons, 2013.

[20] Bingham, N. Continuous branching processes and spectral positivity. *Stochastic Process. Appl.*, 4(3):217–242, 1976.

[21] Bingham, N., Goldie, C., and Teugels, J. *Regular variation*, volume 27. Cambridge university press, 1989.

[22] Biskup, M. and Louidor, O. Full extremal process, cluster law and freezing for two-dimensional discrete gaussian free field. *Adv. Math.*, 330:589 – 687, 2018.

[23] Blumenthal, R. and Getoor, R. Some theorems on stable processes. *Trans. Amer. Math. Soc.*, 95(2):263–273, 1960.

[24] Bogdan, K., Burdzy, K., and Chen, Z. Censored stable processes. *Probab. Theory Related Fields*, 127(1):89–152, 2003.

[25] Bolthausen, E. Part i: Large deviations and interacting random walks. In *Lectures on Probability Theory and Statistics*, pages 1–124. Springer, 2002.

[26] Boyarchenko, S. and Levendorskii, S. *Non-Gaussian Merton-Black-Scholes Theory*, volume 9. World Scientific, 2002.

[27] Boyarchenko, S. and Levendorskii, S. Perpetual American options under Lévy processes. *SIAM J. Control Optim.*, 40(6):1663–1696, 2002.

[28] Caravenna, F., den Hollander, F., and Pétrélis, N. Lectures on random polymers. *Clay Math. Proc.*, 15:319–393, 2012.

[29] Carr, P., Geman, H., Madan, D., and Yor, M. The fine structure of asset returns: An empirical investigation. *J. Bus.*, 75(2):305–332, 2002.

[30] Chan, T., Kyprianou, A., and Savov, M. Smoothness of scale functions for spectrally negative Lévy processes. *Probab. Theory Related Fields*, 150(3):691–708, 2011.

[31] Charmoy, P., Peres, Y., and Sousi, P. Minkowski dimension of Brownian motion with drift. *J. Fractal Geom.*, 1(2):153–176, 2014.

[32] Chaumont, L. Sur certains processus de Lévy conditionnés à rester positifs. *Stochastics*, 47(1-2):1–20, 1994.

[33] Chaumont, L. Conditionings and path decompositions for Lévy processes. *Stochastic Process. Appl.*, 64(1):39–54, 1996.

[34] Chaumont, L. and Doney, R. On Lévy processes conditioned to stay positive. *Electron. J. Probab.*, 10(289):948–961, 2005.

[35] Chen, X. and Khoshnevisan, D. From charged polymers to random walk in random scenery. *Lect. Notes Monogr. Ser.*, pages 237–251, 2009.

[36] Clayfield, E. and Lumb, E. A theoretical approach to polymeric dispersant action I. calculation of entropic repulsion exerted by random polymer chains terminally adsorbed on plane surfaces and spherical particles. *J. Colloid Interface Sci.*, 22(3):269–284, 1966.

[37] Cont, R. *Empirical properties of asset returns: stylized facts and statistical issues*, volume 1. Taylor & Francis, 2001.

[38] Cont, R. and Tankov, P. Financial modelling with jump processes, Chapman & Hall/CRC financ. *Math. Ser*, 2004.

[39] Crepel, P., Fienberg, S., and Gani, J. *Statisticians of the Centuries.* Springer Science & Business Media, 2013.

[40] Denisov, D., Sakhanenko, A., and Wachtel, V. First-passage times over moving boundaries for asymptotically stable walks. *Theory Probab. Appl.*, 63(4):613–633, 2019.

[41] Denisov, D. and Wachtel, V. Conditional limit theorems for ordered random walks. *Electron. J. Probab.*, 15:292–322, 2010.

[42] Denisov, D. and Wachtel, V. Exact asymptotics for the instant of crossing a curved boundary by an asymptotically stable random walk. *Teor. Veroyatn. Primen.*, 60(3):459–481, 2015.

[43] Denisov, D. and Wachtel, V. Random walks in cones. *Ann. Probab.*, 43(3):992–1044, 2015.

[44] Dragan Djurčić. O-regularly varying functions and strong asymptotic equivalence. *J. Math. Anal. Appl.*, 1998.

[45] Doney, R. *Fluctuation Theory for Lévy Processes: Ecole D'Eté de Probabilités de Saint-Flour XXXV-2005.* Springer, 2007.

[46] Doney, R. and Maller, R. Random walks crossing curved boundaries: A functional limit theorem, stability and asymptotic distributions for exit times and positions. *Adv. in Appl. Probab.*, 32(4):1117–1149, 2000.

[47] Doney, R. and Rivero, V. Asymptotic behaviour of first passage time distributions for subordinators. *Electron. J. Probab.*, 20(91):1–28, 2015.

[48] Döring, L., Kyprianou, A., and Weissman, P. Stable processes conditioned to avoid an interval. *arXiv preprint arXiv:1802.07223*, 2018.

[49] Döring, L. and Savov, M. (Non) Differentiability and asymptotics for potential densities of subordinators. *Electron. J. Probab.*, 16(17):470–503, 2011.

[50] Durrett, R. *Probability: theory and examples*, volume 49. Cambridge university press, 2010.

[51] Eberlein, E. and Keller, U. Hyperbolic distributions in finance. *Bernoulli*, 1(3):281–299, 1995.

[52] Embrechts, P., Goldie, C., and Veraverbeke, N. Subexponentiality and infinite divisibility. *Probab. Theory Related Fields*, 49(3):335–347, 1979.

[53] Erlang, A. The theory of probabilities and telephone conversations. *Nyt. Tidsskr. Mat. Ser. B*, 20:33–39, 1909.

[54] Falconer, K. *Fractal geometry: mathematical foundations and applications.* John Wiley & Sons, 2004.

[55] Feller, W. *An introduction to probability theory and its applications*, volume 2. John Wiley & Sons, 2008.

[56] Feng, J. and Sha, Z. Packing measure functions of subordinators sample paths. *Sci. China Ser. A*, 41(5):505–509, 1998.

[57] Feng, R., Kuznetsov, A., and Yang, F. Exponential functionals of levy processes and variable annuity guaranteed benefits. *Stochastic Process. Appl.*, 2018.

[58] Fontbona, J., Krell, N., and Martínez, S. Energy efficiency of consecutive fragmentation processes. *J. Appl. Probab.*, 47(2):543–561, 2010.

[59] Fristedt, E. and Taylor, S. The packing measure of a general subordinator. *Probab. Theory Related Fields*, 92(4):493–510, 1992.

[60] Garbit, R. Brownian motion conditioned to stay in a cone. *J. Math. Kyoto Univ.*, 49(3):573–592, 2009.

[61] Geman, H., Madan, D., and Yor, M. Asset prices are Brownian motion: only in business time. In *Quantitative Analysis In Financial Markets: Collected Papers of the New York University Mathematical Finance Seminar (Volume II)*, pages 103–146. World Scientific, 2001.

[62] Golding, I. and Kantor, Y. Two-dimensional polymers with random short-range interactions. *Phys. Rev. E*, 56(2):R1318, 1997.

[63] Hardy, G. H. *A mathematician's apology.* Cambridge University Press, 1940.

[64] Hawkes, J. Image and intersection sets for subordinators. *J. London Math. Soc.*, 2(3):567–576, 1978.

[65] Heyde, C. and Seneta, E. Studies in the history of probability and statistics. xxxi. the simple branching process, a turning point test and a fundamental inequality: A historical note on IJ Bienaymé. *Biometrika*, 59(3):680–683, 1972.

[66] Hu, Y., Khoshnevisan, D., and Wouts, M. Charged polymers in the attractive regime: a first-order transition from Brownian scaling to four-point localization. *J. Stat. Phys.*, 144(5):948, 2011.

[67] Jain, N. and Pruitt, W. Lower tail probability estimates for subordinators and nondecreasing random walks. *Ann. Probab.*, pages 75–101, 1987.

[68] Jiřina, M. Stochastic branching processes with continuous state space. *Czechoslovak Math. J.*, 8(2):292–313, 1958.

[69] Kallenberg, O. *Foundations of modern probability.* Springer Science & Business Media, 2006.

[70] Kerkhoff, U and Lerche, H. Boundary crossing distributions of random walks related to the law of the iterated logarithm. *Statist. Sinica*, pages 1697–1715, 2013.

[71] Kesten, H. and Maller, R. Random walks crossing high level curved boundaries. *J. Theoret. Probab.*, 11(4):1019–1074, 1998.

[72] Khintchine, A. A new derivation of one formula by Lévy P. *Bull. Moscow State University*, 1(1):1, 1937.

[73] Khoshnevisan, D. From fractals and probability to Lévy processes and stochastic PDEs. In *Fractal Geometry and Stochastics IV*, pages 111–141. Springer, 2009.

[74] Khoshnevisan, D., Schilling, R., and Xiao, Y. Packing dimension profiles and Lévy processes. *Bull. Lond. Math. Soc.*, page bds022, 2012.

[75] Khoshnevisan, D. and Xiao, Y. Lévy processes: capacity and Hausdorff dimension. *Ann. Probab.*, 33(3):841–878, 05 2005.

[76] Khoshnevisan, D. and Xiao, Y. Packing dimension of the range of a Lévy process. *Proc. Amer. Math. Soc.*, 136(7):2597–2607, 2008.

[77] Khoshnevisan, D. and Xiao, Y. On the macroscopic fractal geometry of some random sets. *arXiv preprint arXiv:1605.01365*, 2016.

[78] Kolb, M. and Savov, M. Transience and recurrence of a brownian path with limited local time. *Ann. Probab.*, 44(6):4083–4132, 11 2016.

[79] Kolb, M. and Savov, M. Conditional survival distributions of Brownian trajectories in a one dimensional Poissonian environment in the critical case. *Electron. J. Probab.*, 22, 2017.

[80] König, W. and Schmid, P. Random walks conditioned to stay in weyl chambers of type c and d. *Electron. Comm. Probab.*, 15:286–296, 2010.

[81] Kou, S. A jump-diffusion model for option pricing. *Management science*, 48(8):1086–1101, 2002.

[82] Kyprianou, A. *Introductory lectures on fluctuations of Lévy processes with applications.* Springer Science & Business Media, 2006.

[83] Kyprianou, A., Rivero, V., and Satitkanitkul, W. Stable Lévy processes in a cone. *arXiv preprint arXiv:1804.08393v2*, 2018.

[84] Kyprianou, A., Schoutens, W., and Wilmott, P. *Exotic option pricing and advanced Lévy models.* John Wiley & Sons, 2006.

[85] Lamperti, J. Continuous state branching processes. *Bull. Amer. Math. Soc.*, 73(3):382–386, 1967.

[86] Lamperti, J. Semi-stable Markov processes. I. *Z. Wahrscheinlichkeit.*, 22(3):205–225, 1972.

[87] Le Gall, J. *Spatial branching processes, random snakes and partial differential equations.* Springer Science & Business Media, 1999.

[88] Lerche, H. *Boundary crossing of Brownian motion: Its relation to the law of the iterated logarithm and to sequential analysis*, volume 40. Springer Science & Business Media, 2013.

[89] Lévy, P. Sur les intégrales dont les éléments sont des variables aléatoires indépendantes. *Annali della Scuola Normale Superiore di Pisa-Classe di Scienze*, 3(3-4):337–366, 1934.

[90] Li, Z. *Measure-valued branching Markov processes.* Springer Science & Business Media, 2010.

[91] Lundberg, F. Approximations of the probability function/reinsurance of collective risks. *PhD diss., University of Uppsala*, 1903.

[92] Madan, D., Carr, P., and Chang, E. The variance gamma process and option pricing. *Review of Finance*, 2(1):79–105, 1998.

[93] Madan, D. and Seneta, E. The variance gamma (VG) model for share market returns. *J. Bus.*, pages 511–524, 1990.

[94] Mallein, B. Asymptotic of the maximal displacement in a branching random walk. *Grad. Journ. Math.*, 1(2):92–104, 2016.

[95] Millan, J. On the rate of growth of lévy processes with no positive jumps conditioned to stay positive. *Electron. Commun. Probab.*, 13:494–506, 2008.

[96] Nolan, J. *Stable Distributions - Models for Heavy Tailed Data.* Birkhauser, Boston, 2018. In progress, Chapter 1 online at http://fs2.american.edu/jpnolan/www/stable/stable.html.

[97] Pantí, H. On Lévy processes conditioned to avoid zero. *Lat. Am. J. Probab. Math. Stat.*, 14:657–690, 2017.

[98] Pitman, J. One-dimensional Brownian motion and the three-dimensional Bessel process. *Adv. Appl. Probab.*, 7(3):511–526, 1975.

[99] Pitman, J. and Rogers, L. Markov functions. *Ann. Probab.*, pages 573–582, 1981.

[100] Pötzelberger, K. and Wang, L. Boundary crossing probability for Brownian motion. *J. Appl. Probab.*, 38(1):152–164, 2001.

[101] Pruitt, W. The Hausdorff dimension of the range of a process with stationary independent increments. *J. Math. Mech.*, 19(4):371–378, 1969.

[102] Pruitt, W. The growth of random walks and Lévy processes. *Ann. Probab.*, 9(6):948–956, 1981.

[103] Razumovskii, N. Metal distributions in ore deposites. In *Dokl. Akad. Nauk SSSR*, volume 28, pages 815–817, 1940.

[104] Revuz, D. and Yor, M. *Continuous martingales and Brownian motion*, volume 293. Springer Science & Business Media, 2013.

[105] Roynette, B., Vallois, P., and Yor, M. Some penalisations of the Wiener measure. *Jpn. J. Math.*, 1(1):263–290, 2006.

[106] Roynette, B., Vallois, P., and Yor, M. Penalizing a BES(d) process (0¡ d¡ 2) with a function of its local time, V. *Studia Sci. Math. Hungar.*, 45(1):67–124, 2007.

[107] Roynette, B., Vallois, P., and Yor, M. Penalisations of Brownian motion with its maximum and minimum processes as weak forms of Skorokhod embedding, X. *Theory Stoch. Process.*, 14(2):116–138, 2008.

[108] Rutherford, E., Geiger, H., and Bateman, H. LXXVI. the probability variations in the distribution of α particles. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, 20(118):698–707, 1910.

[109] Salminen, P. and Vallois, P. On subexponentiality of the Lévy measure of the diffusion inverse local time; with applications to penalizations. *Electron. J. Probab*, 14:1963–1991, 2009.

[110] Sato, K. *Lévy processes and infinitely divisible distributions.* Cambridge university press, 1999.

[111] Savov, M. On the range of subordinators. *Electron. Commun. Probab.*, 19:1–10, 2014.

[112] Schoutens, W. *Lévy processes in finance.* Wiley, 2003.

[113] Seshadri, V. and West, B. Fractal dimensionality of Lévy processes. *Proc. Natl. Acad. Sci. USA*, 79(14):4501, 1982.

[114] Shiryayev, A. On the logarithmic normal distribution of particle sizes under grinding. In *Selected works of AN Kolmogorov*, pages 281–284. Springer, 1992.

[115] Stoll, A. Invariance principles for Brownian intersection local time and polymer measures. *Math. Scand.*, pages 133–160, 1989.

[116] van der Hofstad, R., den Hollander, F., and König, W. Large deviations for the one-dimensional Edwards model. *Ann. Probab.*, pages 2003–2039, 2003.

[117] van der Hofstad, R. and Klenke, A. Self-attractive random polymers. *Ann. Appl. Probab.*, pages 1079–1115, 2001.

[118] van der Hofstad, R. and König, W. A survey of one-dimensional random polymers. *J. Stat. Phys.*, 103(5):915–944, 2001.

[119] Velenik, Y. Localization and delocalization of random interfaces. *Probab. Surv.*, 3:112–169, 2006.

[120] Wang, L. and Pötzelberger, K. Boundary crossing probability for Brownian motion and general boundaries. *J. Appl. Probab.*, 34(1):54–65, 1997.

[121] Watson, H. and Galton, F. On the probability of the extinction of families. *The Journal of the Anthropological Institute of Great Britain and Ireland*, 4:138–144, 1875.

[122] Westwater, M. On Edwards' model for long polymer chains. *Comm. Math. Phys.*, 72(2):131–174, 1980.

[123] Xiao, Y. Random fractals and Markov processes. *Fractal Geometry and Applications: A Jubilee of Benoit Mandelbrot*, 2:261–338, 2004.

[124] Yakymiv, A. On the asymptotics of the density of an infinitely divisible distribution at infinity. *Theory Probab. Appl.*, 47(1):114–122, 2003.

[125] Yano, K., Yano, Y., and Yor, M. Penalising symmetric stable Lévy paths. *J. Math. Soc. Japan*, 61(3):757–798, 2009.

[126] Yano, K., Yano, Y., and Yor, M. Penalisation of a stable Lévy process involving its one-sided supremum. *Ann. Inst. H. Poincaré Probab. Statist.*, 46(4):1042–1054, 2010.

Index

Applications: Branching processes 5-6;
Financial mathematics 4-5;
Fragmentation processes 6;
Polymer models 6, 46-49
Berry-Esseen theorem 27-30
Box-counting dimension 16-22, 42-43
Brownian motion 2, 4, 10, 55-58
Central limit theorems 6, 23, 25
Conditioned/constrained processes 46, 49, 53-59, 65-70
Entropic repulsion 57-58, 70
Fractal dimension 16-23
Graph of a subordinator 42-43
Inverse local time 51-53
Lévy-Khintchine formula 7-8, 13-14, 22, 60
Lévy processes 1-14, 17-19, 21-22, 51-56, 60-63, 68
Local time 49-53
Localisation/delocalisation 46-49
Markov processes 12, 49-53
Markov property 12, 49, 59
Poisson process 2-3, 11
Random walks 3-4
Range of a subordinator 17-19, 23-27
Regularly varying functions 43-44, 60-63, 151-153
Regularity conditions 22-23, 61-63, 126
Strong limit theorems 19, 25, 43-44, 58-59, 70
Subordinators 12-14, 21-22, 51-53, 60
Transience/recurrence 46-48, 56-60, 64-70

Glossary of Notation

Real Numbers

$\lfloor x \rfloor$	Floor function
$\lceil x \rceil$	Ceiling function
$x \vee y$	$\max\{x, y\}$
$x \wedge y$	$\min\{x, y\}$
$\sim$	Strong asymptotic equivalence, $f(x) \sim g(x)$ if $\lim_{x\to\infty} f(x)/g(x) = 1$
$\lesssim$	Weak asymptotic bound, $f(x) \lesssim g(x)$ if $\limsup_{x\to\infty} f(x)/g(x) < \infty$
$\gtrsim$	Weak asymptotic bound, $f(x) \gtrsim g(x)$ if $\limsup_{x\to\infty} g(x)/f(x) < \infty$
$\asymp$	Weak asymptotic equivalence, $f(x) \asymp g(x)$ if $f(x) \lesssim g(x)$ and $f(x) \gtrsim g(x)$
$\Delta_\delta(dx)$	Dirac measure, unit point mass at δ
$\beta(f)$	Lower index of f, see Definition 3.3.3(3)

Lévy Processes

Ψ	Characteristic exponent
ϕ	Laplace exponent
d	Linear drift of the process
$\Pi(dx)$	Lévy measure
$\overline{\Pi}(x)$	Tail of the Lévy measure, $\int_x^\infty \Pi(dx)$
$I(\delta)$	Integrated tail function, $I(\delta) := \int_0^\delta \overline{\Pi}(x)dx$
X_{t-}	Left limit, $\lim_{s\uparrow t} X_s$
Δ_1^x	Time of the first jump of size greater than x
$\Delta_1^{(a,b)}$	Time of the first jump of size between a and b

Functions

$\Gamma(x)$	Gamma function
$g_y^h(t)$	Augmented version of $g(t)$, $g_y^h(t) := g(t+h) - y$
$g_\delta(t)$	Rescaled version of $g(t)$, $g(t)/\log(t)^\delta$

Lévy Processes - Chapter 2

T_δ	First passage time above the level δ
$T_{(\delta,\infty)}$	First passage time above the level δ
$U(\delta)$	Renewal function, $U(\delta) := \mathbb{E}[T_\delta]$
$N(t,\delta)$	Minimal number of boxes of length at most δ needed to cover the range up to time t
$M(t,\delta)$	Number of boxes in a mesh of side length δ which intersect with the range up to time t
$(\tilde{X}_t^\delta)_{t\geq 0}$	Process with δ-shortened jumps
$\tilde{\phi}^\delta(dx)$	Laplace exponent of $\tilde{X}_t^\delta$
$\tilde{\Pi}^\delta(dx)$	Lévy measure of $\tilde{X}_t^\delta$
$L(t,\delta)$	New box-counting scheme, $L(t,\delta) := \frac{1}{\delta}\tilde{X}_t^\delta$

Lévy Processes - Chapter 3

$\mathcal{O}_t$	Event upon which we condition, $\mathcal{O}_t := \{X_s \geq g(s),\ 0 \leq s \leq t\}$
$\mathcal{O}_t^{g_y^h}$	Augmented version of $\mathcal{O}_t$, $\mathcal{O}_t^{g_y^h} := \{X_s \geq g_y^h(s),\ 0 \leq s \leq t\}$
$\mathbb{P}(\cdot)$	Original probability measure
$\mathbb{Q}(\cdot)$	Conditioned weak limit measure, $\mathbb{Q}(\cdot) := \lim_{t\to\infty} \mathbb{P}(\cdot\|\mathcal{O}_t)$
$\rho(t)$	Error term, $\rho(t) := \mathbb{P}(\mathcal{O}_t)/\Phi(t) - \overline{\Pi}(g(t))$
$\rho_y^h(t)$	Error term, $\rho_y^h(t) := \mathbb{P}(\mathcal{O}_t^{g_y^h})/\Phi_y^h(t) - \overline{\Pi}(g_y^h(t))$
$\Phi(t)$	Integral of $\mathbb{P}(\mathcal{O}_s)$, $\Phi(t) := \int_0^t \mathbb{P}(\mathcal{O}_s)ds$
$\Phi_y^h(t)$	Integral of $\mathbb{P}(\mathcal{O}_s^{g_y^h})$, $\Phi_y^h(t) := \int_0^t \mathbb{P}(\mathcal{O}_s^{g_y^h})ds$