Spectral theory of ordinary and partial linear differential operators on finite intervals

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Abstract

A new, unified transform method for boundary value problems on linear and integrable nonlinear partial differential equations was recently introduced by Fokas. We consider initial-boundary value problems for linear, constant-coefficient evolution equations of arbitrary order on a finite domain. We use Fokas' method to fully characterise well-posed problems. For odd order problems with non-Robin boundary conditions we identify sufficient conditions that may be checked using a simple combinatorial argument without the need for any analysis. We derive similar conditions for the existence of a series representation for the solution to a well-posed problem.

We also discuss the spectral theory of the associated linear two-point ordinary differential operator. We give new conditions for the eigenfunctions to form a complete system, characterised in terms of initial-boundary value problems.

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Declaration

I confirm that this is my own work and all material from other sources has been fully and properly acknowledged.

Contents

CHAPTER 1

Introduction

1.1. Background and motivation

This thesis is concerned with the theory of linear two-point initial-boundary value problems, the spectral theory of linear differential operators and the connections between the two fields. The boundary value problems we study are posed for linear, constant-coefficient, evolution partial differential equations in one space and one time variable. One of the best known examples of such a problem is the heat equation for a finite rod,

$$q_t = q_{xx}, \quad x \in [0,1], \quad t \in [0,T].$$

The primary interest in this work is not second order partial differential equations, such as the heat equation, but third and higher odd order equations. Indeed we study equations of the form

$$\partial_t q \pm (\partial_x)^n q = 0, \quad x \in [0,1], \quad t \in [0,T], \tag{1.1.1}$$

for any $n \geqslant 3$, n an odd integer.

To define an initial-boundary value problem for the partial differential equation (1.1.1) one must specify the initial state of the system, by prescribing $q(x,0)$ to be equal to some known function, and impose some conditions on the value of q and its x-derivatives at the left and right ends of the space interval. The problem is then to find a sufficiently smooth function $q : [0,1] \times [0,T] \to \mathbb{C}$ which satisfies the partial differential equation (1.1.1), the initial condition and the boundary conditions. It is reasonable to ask two questions relating to such problems:

(1) Does a solution exist and is that solution unique?
(2) If the answer is yes, how can the solution be expressed explicitly and unambiguously in terms of the known data of the problem?

For the case of second-order partial differential equations these questions are fully resolved, at least when the solution and the data both satisfy some differentiability conditions. Indeed, Cauchy not only posed the problem but solved it for analytic data [**8, 6**].[1] Hadamard [**35**] examined question (2) in particular for second order problems. However, when the partial differential equation is of a higher order, the application of their techniques works only with very specific types of boundary conditions.

In this work, we characterise the boundary conditions that ensure the problem has a unique solution. For such problems, we find the solution and discuss its representation by contour integrals and discrete series. Finally we use the solution to derive new results on the spectral theory of linear differential operators acting on one variable defined on a finite interval.

1.1.1. Methods for initial-boundary value problems

One may attempt to solve an initial-boundary value problem for the partial differential equation (1.1.1) using a wide array of techniques. In what follows, we give a description of three methods in order to highlight the similarites and differences between them: the separation of variables and formalised Fourier series method, the Laplace transfrom method and the more recent unified transform method of Fokas.

[1] See also [**5, 7, 12, 13, 33, 37, 65**].

The wave equation was introduced and solved by d'Alembert [**11**], albeit under strict restrictions on the boundary conditions. The method was refined by Euler [**21**]. Bernoulli [**2**] introduced the idea that a solution of the wave equation might be expressed as an infinite series and Fourier [**30**] studied the heat equation similarly.

A form of Laplace transform method for partial differential equations was introduced by Euler in a paper [**22**], first presented in 1779 but not published until 1813. The integral Euler used had indefinite limits. Lagrange [**38**], originally published in 1759, used a Fourier transform method with definite integrals to solve the wave equation. Laplace himself solved a linear evolution partial differential equation using his eponymous transform with definite limits in Section V of [**43**], originally published in 1810, where he also derived an inverse transform. A survey of the history of the Laplace transform is given in [**17, 18**].

Fokas' transform method was originally developed for solving boundary value problems for non-linear partial differential equations [**23**] but has been successfully applied to elliptic [**60**] as well as evolution [**24**] linear partial differential equations. A good introduction to the significance of Fokas' method is given in [**23**] but it should be noted that the method was not fully refined at this stage. Sections 1.1–1.3 of [**24**] give a good overview of the method for linear, constant-coefficient boundary value problems.

Separation of variables

We aim to find a solution to a partial differential equation subject to an initial condition and some boundary conditions. To solve such an initial-boundary value problem using the method of separation of variables [**30**] one must make two assumptions: that a solution exists and that a solution is separable, in the sense that there exist sequences of functions $\xi_k(x)$, $\tau_k(t)$, whose products $\xi_k(x)\tau_k(t)$ satisfy the partial differential equation and boundary conditions, such that the solution may be expressed as a series with uniform convergence,

$$q(x,t) = \sum_{k\in\mathbb{N}} \alpha_k \xi_k(x)\tau_k(t), \tag{1.1.2}$$

for some sequence of constants α_k. The former assumption can be justified a posteriori, as one may verify that any solution thus obtained does indeed satisfy the partial differential equation and the initial and boundary conditions after the solution has been derived. The latter assumption is more troublesome, as this method cannot be used to find unseparable solutions.

Separating the partial differential equation (1.1.1),[2]

$$q(x,t) = \xi(x)\tau(t),$$

we rewrite it as a pair of uncoupled linear ordinary differential equations,

$$\frac{\tau'(t)}{\tau(t)} = \frac{\mp(-i)^n \xi^{(n)}(x)}{\xi(x)} = \sigma^n.$$

[2]Equivalently, one may take the Fourier, sine or cosine transform of the partial differential equation in the spatial variable. This method requires that the eigenfunctions of the differential operator form a basis which is equivalent to convergence of the series (1.1.2).

It is trivial to find the general solutions of these equations in terms of the common spectral parameter, $\sigma \in \mathbb{C}$. The boundary conditions then restrict σ to a sequence of discrete points σ_k, defining the ξ_k, τ_k. Under the assumption that the series (1.1.2) converges uniformly, Fourier transform methods are used to determine the constants α_k in terms of the initial datum. It is well-known that the family of solutions ξ_k obtained from particular spectral problems forms an eigenfunction basis for the x-differential operator, with eigenvalues σ_k^n, but for partial differential equations of third or higher order with any but the simplest boundary conditions this is not always true. This connection is critical in our work.

Laplace transform

In the Laplace transform method, separability of the solution is not assumed directly but it is necessary to assume that the Laplace transform can be inverted. The first step is to apply the time Laplace transform to the partial differential equation (1.1.1). Using the properties of this transform and the initial datum, this yields an inhomogeneous ordinary differential equation of order n in the Laplace transform of q. Solving this equation subject to the boundary conditions yields an expression for the Laplace transform of the solution.

The final step is to reconstruct the solution from its Laplace transform. If the domain is semi-infinite in time, if $T = \infty$, and the boundary data have sufficient decay then the transform may be invertible. An example is given in Appendix C of [**28**]. However, we study initial-boundary value problems on a finite domain so the solution at final time appears in the representation. To remove the effects of this function, it is necessary to make arguments similar to those we make for Fokas' method. However these arguments are more complex than their equivalents below because of the presence of fractional powers in the integrands.

Fokas' unified transform method

The first step of Fokas' method is to construct a Lax pair for the partial differential equation. The term 'Lax pair' is usually reserved for nonlinear partial differential equations, following the introduction of the concept in [**44**]. The existence of a Lax pair is essential in solving nonlinear equations using the inverse spectral method[3] but a Lax pair always exists for linear equations [**26**]. The advantage of the Lax pair formulation is that it allows one to express a linear partial differential equation of any order, and even many nonlinear integrable[4] partial differential equations, as a pair of first order, linear partial differential equations. In contrast to the nonlinear case, for constant-coefficient linear evolution equations the Lax pair is scalar, and has been derived in general, see equations (2.1) of [**27**].

The next step is to perform the simultaneous spectral analysis of the Lax pair. This is essentially different from both classical and traditional inverse spectral methods in which only the spatial part of the Lax pair is used. As this Lax pair has a particularly simple form, it

[3]The inverse spectral method was introduced in [**31, 32**] for the Korteweg-de Vries equation but has since been applied to many other nonlinear partial differntial equations. See [**29**] for more recent developments in the field.

[4]The term integrable is often defined by the existence of a Lax pair.

is trivial to find an integral solution with lower limit at an arbitrary point in the domain of the original partial differential equation. In Proposition 3.1 of [**24**] it is argued that, by taking the lower limit at each corner of the domain, a sectionally analytic function in the auxiliary parameter ρ is defined in the whole complex ρ-plane, which decays as $\rho \to \infty$. The resulting inhomogeneous Riemann-Hilbert problem is scalar and therefore its general solution is known. The spatial part of the Lax pair is now used to find an expression for the general solution of the partial differential equation. For a particular third order example, this expression is

$$q(x,t) = \frac{1}{2\pi}\left(\int_{-\infty}^{\infty} e^{i\rho x + i\rho^3 t}\hat{q}_0(\rho)\,\mathrm{d}\rho - \int_{\Gamma^+} e^{i\rho x + i\rho^3 t}Q(0,\rho)\,\mathrm{d}\rho \right.$$
$$\left. - \int_{\Gamma^-} e^{i\rho(x-1) + i\rho^3 t}Q(1,\rho)\,\mathrm{d}\rho\right), \quad (1.1.3)$$

where $\hat{q}_0$ is the usual Fourier transform of the initial datum, $Q(0,\lambda)$, $Q(1,\lambda)$ are transforms of the boundary functions and $\Gamma^{\pm}$ are the rays in the upper and lower half-plane on which $\mathrm{Im}(\rho^n) = 0$.

To derive an equivalent integral representation to (1.1.3) for a nonlinear evolution equation the Lax pair and Riemann-Hilbert formalism is necessary but that is not the case for linear problems. One could instead use a Fourier transform and deform the contour of integration from the real line onto the required contours. The advantage of the above method is that those contours are automatically determined by the solution procedure.

The penultimate step is to write the transformed boundary functions $Q(0,\lambda)$ and $Q(1,\lambda)$ in terms of the boundary data. To do this one must provide a Dirichlet to Neumann map in the form of the 'global relation'. This exploits the rotational symmetry of the contours $\Gamma^{\pm}$ to determine the unknown boundary functions in terms of the initial datum and the solution at final time. For example, a single homogeneous Neumann condition and a pair of homogeneous Dirichlet conditions applied to equation (1.1.3) yield the representation

$$q(x,t) = \frac{1}{2\pi}\left(\int_{-\infty}^{\infty} e^{i\rho x + i\rho^3 t}\hat{q}_0(\rho)\,\mathrm{d}\rho - \int_{\Gamma^+} e^{i\rho x + i\rho^3 t}\frac{\zeta_1(\rho) + \zeta_3(\rho) - e^{-i\rho^3 T}(\eta_1(\rho) + \eta_3(\rho))}{\Delta_{\mathrm{PDE}}(\rho)}\,\mathrm{d}\rho \right.$$
$$\left. - \int_{\Gamma^-} e^{i\rho(x-1) + i\rho^3 t}\frac{\zeta_2(\rho) - e^{-i\rho^3 T}\eta_2(\rho)}{\Delta_{\mathrm{PDE}}(\rho)}\,\mathrm{d}\rho\right), \quad (1.1.4)$$

where Δ_{PDE} is an exponential polynomial, each ζ_j is a function depending upon the Fourier transform of the initial datum and each η_j is a function depending upon the Fourier transform of the solution at final time.

Finally, analyticity properties and residue computations are used to remove the contribution of the functions η_j.

Comparison

It would be inaccurate to assert that separation of variables is the only classical method available for the analysis of this kind of initial-boundary value problem but it is representative of other such methods in that it highlights the essential difficulties one faces when applying them. It employs a transform in only the spatial variable to solve the partial differential equation;

the Laplace transform method uses a transform in only the time variable. Different partial differential equations and different boundary conditions require different transforms and finding a transform that will work for a particular initial-boundary value problem is not a simple task. It is particularly problematic when the partial differential equation is of third or higher order, particularly odd order, or the boundary conditions are complex.

In Fokas' method, as a simultaneous spectral analysis in both the space and the time variable is performed, a different type of transform is used. This simplifies the process of choosing the relevant transform as it may be immediately deduced from the Lax pair and is independent of the boundary conditions. It is therefore unsurprising that such a method should yield novel results, not only for nonlinear but also for linear partial differential equations.

One great advantage of the universal applicability of Fokas' method in the linear, constant-coefficient context is that for it to produce a solution one only has to guarantee that the problem is well posed, whereas separation of variables requires an extra assumption on the solution, that it be separable or that the x-differential operator admits a suitable basis of eigenfunctions. This means that, armed with Fokas' method, question (2) on page 2 may be considered fully resolved for any initial-boundary value problem posed for the partial differential equation (1.1.1). Question (1) may be expressed as the question *Is the problem well-posed?* This is one of the major topics of the present work.

Another great difference between the methods presented above is the representation of the result. Separation of variables yields a discrete series representation of the solution whereas Fokas' method gives the solution as a contour integral. The use of the definite article to describe '*the* solution' in the previous sentence is intentional as both of the methods are applied to problems known to be well-posed. This means that for separable, well-posed problems we now have two methods which yield two different representations of the same solution.

A method for converting the integral representation to a series representation for third order problems with particular boundary conditions is discussed in [**9, 54**]. In any attempt to generalise this argument to higher order problems and those with more exotic boundary conditions it is certainly necessary to consider another question, supplementary to the two questions on page 2: *Which well-posed initial-boundary value problems have the property that their solutions may also be expressed as discrete series?* The answer to this question is the second major topic of this thesis.

It is shown in [**54**] that there is no series representation of the solution for a particular example. Algebraic methods are used in [**36**] to show that some linear partial differential equations are inseparable for *any* boundary conditions but this requires either non-constant coefficients or systems of constant-coefficient equations. There is an important distinction between the work of Johnson et al. and our work—the partial differential equations we study are all separable because separation of variables always yields a solution for periodic boundary conditions, it is particular sets of boundary conditions that may make the initial-boundary value problem inseparable by preventing the eigenfunctions of the differential operator from forming a basis.

1.1.2. Spectral theory of two-point ordinary differential operators

Birkhoff [**3, 4**] systematically developed the spectral theory of two-point differential operators. On pages 382–383 of the latter the concept of regularity was first defined. Birkhoff proved that the eigenfunctions of the operator and its adjoint are mutually orthogonal systems and used this to give an integral representation of the solution to a boundary value problem posed for a linear ordinary differential equation. This could be considered as an extention both to non-self-adjoint and to arbitrary order differential operators of Liouville's much earlier work [**46**]. Stone [**61**] extended Birkhoff's theorems on continuous functions onto the more modern Sobolev space. The principal references for regular problems are [**10**], Chapter XIX of [**20**] and the more recent [**47**] which uses the theory of Fredholm operators to improve upon the treatment of Dunford & Schwartz in several aspects.

Irregular boundary conditions are comparatively less studied. Second order problems were first investigated by Stone [**62**], who derived their characteristic determinant. The completeness of the eigenfunctions of many such operators was established by Yakubov [**66**] but Lang and Locker [**39, 40**] showed that it does not hold for general second order irregular operators. Locker's more recent monograph [**48**] concentrates on simply irregular operators, finding eigenvalues and their multiplicites, and showing that the eigenfunctions are a complete system in L^2. The third class, the degenerate irregular operators, is largely unstudied.

One of the most fundamental theorems[5] in the spectral theory of two-point ordinary differential operators is that the eigenvalues are precisely the zeros of the characteristic determinant, a function defined in terms of the boundary conditions. As the characteristic determinant is an exponential polynomial the theory of the distribution of the zeros of such functions is of great importance. We prefer Langer's papers [**41, 42**] to the more general, but considerably more dense, book of Levin [**45**] as the former focus on finite sums instead of infinite series.[6]

1.2. Thesis overview

Aims

In this thesis we aim to contribute to the theory of initial-boundary value problems for linear constant-coefficient evolution equations and relations between these problems and the spectral theory of the associated ordinary differential operator. Specifically, we aim to

- Improve upon Fokas' transform method for linear evolution partial differential equations by making it fully algorithmic.
- Investigate well-posedness of initial-boundary value problems in general, giving both necessary and sufficient conditions.

[5] Given as Result II on pages 376–377 of [**4**].

[6] The author wishes to express his thanks to Brian Davies and Jim Langley for recommending Langer's work.

- Investigate the existence of a series representation of the solution to well-posed problems in general, giving both necessary and sufficient conditions.
- Investigate inseparable boundary conditions by linking the initial-boundary value problem to the study of the ordinary differential operator.
- Contribute to the spectral theory of degenerate irregular non-self-adjoint two-point linear ordinary differential operators.

Chapter 2

As noted above, it is known that one may use Fokas' transform method to find a solution to any well-posed initial-boundary value problem on a linear, constant-coefficient evolution partial differential equation on a rectangular domain. In view of this it is perhaps surprising that any improvement may be made to the means of derivation of a solution but we have some small contributions in this area beyond the overview of the established method in Section 2.1.

Chapter 2 provides a modest development upon the method in the following way. While it is established that a system of linear equations for the boundary functions must exist in the method as presented in [**27**], we derive that system explicitly and in general. The *reduced global relation* is given in Lemma 2.17. Further, we explicitly solve the system to yield, in Theorem 2.20, the general expression for the solution in terms of the initial and boundary data and the solution at final time. Mathematically this is elementary linear algebra but the explicit determination of these functions is necessary to support the remainder of the thesis.

Chapter 3

Chapter 3 contains a discussion of well-posedness of initial-boundary value problems and the existence of a series representation of their solutions using only analytic techniques.

We make a pair of assumptions on the decay of certain meromorphic functions, which are the general analogues of the functions appearing in the integrands of equation (1.1.4). In Section 3.1 we work under those assumptions, removing the effects of the solution at final time and obtaining a series representation for the the solution. The second and third sections are devoted to discussing those assumptions.

In Section 3.2 one of the aforementioned assumptions is shown to be equivalent to well-posedness of the initial-boundary value problem. This new condition of well-posedness is at once much simpler to check than the characterisation by admissible functions of [**27**] and more general than the result for simple, uncoupled boundary conditions of [**53**] and [**55**]. We also give the final result of Fokas' method in Theorem 3.29, an integral representation for the solution involving only the initial and boundary data. In the case of odd-order problems with non-Robin boundary conditions, we give a pair of conditions sufficient for well-posedness and demonstrate their use for a variety of examples.

For well-posed problems, it is shown that the other decay assumption is equivalent to the existence of a series representation of the solution in Section 3.3. We also give a pair of sufficient conditions for a well-posed odd-order problem with non-Robin boundary conditions to have a solution that admits representation by a series. These conditions mirror those in the previous section.

Chapter 3 forms a considerable volume of the work. It is presented together, rather than split into two or more chapters, to emphasise the parallels between the questions of well-posedness and existence of a series representation. To highlight this further, we also discuss well-posedness of final-boundary value problems in Section 3.3.

Chapter 4

In this chapter we investigate the spectral theory of the ordinary differential operator that forms the spatial part of the initial-boundary value problem. The operators we study in Chapter 4 have been investigated extensively by Locker but we study them from the fresh perspective of their associated initial-boundary value problems. The concept of regularity is important in this field, as demonstrated in the following text, appearing in the preface of Locker's most recent monograph, [**48**].

> The regular class has been studied extensively, and has a more or less complete spectral theory; the simply irregular class is a new and unexplored class, and its spectral theory, together with the regular class, is the main subject of this book; the degenerate irregular class has never been studied, and is a topic for future work.

In the present work we focus on the degenerate irregular class for odd order differential operators. Some progress can be made by investigating the links between the spectral theory of a differential operator with the well-posedness of its associated initial-boundary value problems.

We also present two theorems directly linking the study of initial-boundary value problems to the study of the associated ordinary differential operator. The first result is that the zeros of Δ_{PDE}, which are of central importance to the representation of the solution to the initial-boundary value problem, are precisely the eigenvalues of the associated ordinary differential operator. This theorem is only proven for non-Robin boundary conditions satisfying a technical symmetry requirement but it is shown that the symmetry condition is unnecessary in the third order. Although for Robin boundary conditions the structure of the determinant function Δ_{PDE} appears to be very different from the structure of the characteristic determinant, it appears that the non-Robin condition may be weakened, at least for third order.

Our main result, Theorem 4.18, provides another link between the theory of ordinary differential operators and the associated initial-boundary value problems. Indeed, we show that when a series representation of the solution to such a problem can be obtained by the methods discussed in Chapter 3, and if the zeros of Δ_{PDE} are simple, then that series is an expansion in the eigenfunctions of the operator. Hence, by evaluating the solution at initial time, we conclude in Theorem 4.19 that the eigenfunctions of the associated operator form a complete system.

The results of this work suggest that an operator is degenerate irregular if and only if at least one of the associated initial-boundary value problem and final-boundary value problem is ill-posed. This assertion is not proven but is supported by a body of examples. A proof would require a significant strengthening of the result relating the zeros of the two determinant functions.

In order to discuss complete, biorthogonal and basic systems of eigenfunctions it is necessary to understand the established theory of these concepts in Banach spaces. We give an overview of the essential definitions and a few theorems in Section 4.4, following the construction in [**15**]. More complete treatments of the subject are given in the excellent two-part survey article [**56, 57**] and the lecture notes [**58**]; these sources have large bibliographies containing the original research upon which they draw.

Chapter 5

In Chapter 5 we present two examples, one of which has degenerate irregular boundary conditions. We prove that the eigenfunctions of this operator do not form a basis, following a method of Davies [**14, 15**]. Indeed, we show that certain projection operators, defined in terms of the eigenfunctions, are not uniformly bounded in norm. The exponential blow-up of these norms is of the same rate as the divergence of the meromorphic function from the initial-boundary value problem.

Chapter 6

In the final chapter we draw together some conclusions and present some directions for further work.

Appendices and additional material

Appendix A contains tables of results for third order and arbitrary odd order initial-boundary value problems and ordinary differential operators. In each case, the well-posedness and spectral theory are investigated under different types of boundary conditions.

Appendix B contains some standard theorems that are used extensively in this work and some of the more technical proofs of the thesis.

After the appendices we present, for the convenience of the reader, a list of the numbered theorems, definitions etc. that appear in the thesis with page references. Finally, a bibliography is given.

CHAPTER 2

Initial-boundary value problems

In this chapter we give an account of Fokas' unified transform method for solving initial-boundary value problems on evolution equations posed on a rectangular (1 time and 1 space variable) domain. We also present our contribution to developing the method further. The method, as described in [**27, 24, 25, 54**], is not fully algorithmic. One step of the established method takes the form

> There exists a system of $2n$ linear equations in $2n$ unknowns. By solving that system, the unknowns may be determined.

This system has been derived and solved in many specific cases for second and third order initial-boundary value problems and even some fourth order problems [**9, 27, 24, 25, 54**] but not in general. In this chapter we make two contributions:

- We determine the system explicitly, Lemma 2.17.
- We solve the system explicitly, Theorem 2.20.

These results are theoretically modest, they require only elementary linear algebra, but are significant because they allow Fokas' method to be expressed in a single theorem; they remove the necessity of doing any analysis to solve initial-boundary value problems. Theorem 2.20 is not the final result of Fokas' method but it is only one step away. Theorem 3.29 gives the final result.

The caveat is that in order to achieve this it is necessary to develop a great volume of notation. For solving a third-order initial-boundary value problem with simple boundary conditions, such as a single Dirichlet condition and two Neumann conditions, the results of this chapter offer no benefit over the established method. Indeed, in such a case it would take one considerably longer to work through the definitions presented here, defining all the required index sets, than to perform the relatively simple direct calculation required to determine the unknown quantities.

Thus the usefulness of the results in this chapter in solving particular initial-boundary value problems is restricted to high-order problems with complex, Robin-type boundary conditions. However, the precise form of the system of equations given in Lemma 2.17 is essential in the proof of results in Chapters 3 and 4 that are of greater, immediate utility.

In this chapter we assume throughout that the initial-boundary value problem being studied is well-posed in the sense that it admits a unique, smooth solution. Fokas' method does yield a way of checking well-posedness of a problem, and we investigate this in greater detail in Chapter 3, but that is not the focus of the present chapter.

2.1. Fokas' transform method

In this section we develop the major steps of Fokas' unified transform method for solving initial-boundary value problems on linear evolution partial differential equations. The principal results are Theorem 2.1, which gives an implicit representation of the solution of a well-posed initial-boundary value problem, and Lemma 2.3, which is the tool that may be used to turn the implicit representation into an explicit representation of the solution.

We do not attempt to give a full proof of the validity of the method in Subsection 2.1.2 or discuss how it may be applied in the more general settings of nonlinear or non-evolution equations.[1] The results of this section are not new and, to avoid devoting a large amount of space to an established result, we give only an outline proof of Theorem 2.1, aimed at highlighting the important steps of the argument.

The global relation, proved in Subsection 2.1.3, is used to make the penultimate step in Fokas' method. By exploiting the rotational symmetry of the transforms, it complements the boundary conditions, a system of n simultaneous linear equations, making it possible to solve for $2n$ unknowns.

2.1.1. The IBVP

We consider the partial differential equation

$$\partial_t q(x,t) + a(-i\partial_x)^n q(x,t) = 0 \text{ for } (x,t) \in \Omega = [0,1] \times [0,T], \tag{2.1.1}$$

where $n \geqslant 2$ and $a = e^{i\theta}$ for some $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ if n is even or $a = \pm i$ if n is odd. This choice of a is motivated by our interest in well-posed problems; the reverse heat equation exhibits instantaneous blow-up, for example. We study the initial-boundary value problem on this partial differential equation with initial condition

$$q(x,0) = q_0(x), \tag{2.1.2}$$

where $q_0 : [0,1] \to \mathbb{R}$ is a given, sufficiently smooth initial datum, and n linearly independent boundary conditions,

$$\sum_{j=0}^{n-1} \alpha_{k\,j} \partial_x^j q(0,t) + \sum_{j=0}^{n-1} \beta_{k\,j} \partial_x^j q(1,t) = h_k(t), \tag{2.1.3}$$

indexed by $k \in \{1,2,\ldots,n\}$ where the $h_k : [0,T] \to \mathbb{R}$ are given, sufficiently smooth boundary data and the boundary coefficients, $\alpha_{k\,j}$ and $\beta_{k\,j}$, are given real constants. We also require that the boundary data are compatible with the initial datum in the sense that

$$\sum_{j=0}^{n-1} \alpha_{k\,j} \frac{\mathrm{d}^j}{\mathrm{d}x^j} q_0(0) + \sum_{j=0}^{n-1} \beta_{k\,j} \frac{\mathrm{d}^j}{\mathrm{d}x^j} q_0(1) = h_k(0). \tag{2.1.4}$$

This class of initial-boundary value problems includes many problems of physical importance. The partial differential equation (2.1.1) is the heat equation for a finite uniform rod when $n = 2$, $a = 1$. A linearization of the Schrödinger equation is given by $n = 2$, $a = i$ and a linearized Korteweg-de Vries equation appears for $n = 3$, $a = -i$.

It is possible to weaken the smoothness requirements on our initial and boundary data provided we weaken the smoothness requirements on the solution accordingly. Such weaker smoothness conditions are considered in [**63**] for problems whose spatial domain is the half-line.

[1] See [**24**] and [**25**] for the proof and the other applications, respectively.

2.1.2. An implicit solution to the IBVP

The initial-boundary value problem (2.1.1)–(2.1.4) may, under the assumption that it is well-posed, be solved using Fokas' unified transform method. Theorem 2.1, below, does not give the full solution but only the first steps. To obtain the full solution the transformed boundary functions $\widetilde{f}_j$ and $\widetilde{g}_j$ must be determined from the boundary conditions. The power of Theorem 2.1 is that it gives a representation of the solution q on the whole of Ω in terms of the value of q and its normal derivatives on three sides of the boundary of Ω.

THEOREM 2.1. *Let the initial-boundary value problem* (2.1.1)–(2.1.4) *be well-posed in the sense that it admits a unique smooth solution. Then its solution q may be expressed as the sum of three integrals,*

$$q(x,t) = \frac{1}{2\pi}\left(\int_{\mathbb{R}} e^{i\rho x - a\rho^n t}\hat{q}_0(\rho)\,\mathrm{d}\rho - \int_{\partial D^+} e^{i\rho x - a\rho^n t}\sum_{j=0}^{n-1} c_j(\rho)\widetilde{f}_j(\rho)\,\mathrm{d}\rho \right.$$
$$\left. - \int_{\partial D^-} e^{i\rho(x-1) - a\rho^n t}\sum_{j=0}^{n-1} c_j(\rho)\widetilde{g}_j(\rho)\,\mathrm{d}\rho\right), \quad (2.1.5)$$

where the integrands are given in terms of the initial datum and the boundary functions by

$$\begin{aligned} \widetilde{f}_j(\rho) &= \int_0^T e^{a\rho^n s} f_j(s)\,\mathrm{d}s, & \widetilde{g}_j(\rho) &= \int_0^T e^{a\rho^n s} g_j(s)\,\mathrm{d}s, \\ f_j(t) &= \partial_x^j q(0,t), & g_j(t) &= \partial_x^j q(1,t), \\ \hat{q}_0(\rho) &= \int_0^1 e^{-i\rho y} q_0(y)\,\mathrm{d}y, & c_j(\rho) &= -a\rho^n (i\rho)^{-(j+1)}, \\ D^{\pm} &= D \cap \mathbb{C}^{\pm}, & D &= \{\rho \in \mathbb{C} : \mathrm{Re}(a\rho^n) < 0\}. \end{aligned} \quad (2.1.6)$$

A full proof of Theorem 2.1 involves some technicality and considerable care with the smoothness of the data. We present below an outline of the proof to give the essential argument of Fokas' unified transform method, as applied to linear two-point initial-boundary value problems such as these. The full proof may be found in [**27**] with the general argument appearing in [**24**]. There are also treatments in [**9**] and [**25**].

The first step of the proof of Theorem 2.1 is the expression of the partial differential equation (2.1.1) in the form of a Lax pair. This is accomplished in Lemma 2.2. After this, particular solutions of the Lax pair may be used to specify a Riemann-Hilbert problem. The solution of that Riemann-Hilbert problem yields the result.

LEMMA 2.2. *The partial differential equation* (2.1.1) *is equivalent to the compatibility condition,*

$$\partial_t \partial_x \mu = \partial_x \partial_t \mu, \quad (2.1.7)$$

of the Lax pair,

$$\partial_t \mu + a\rho^n \mu = \sum_{j=0}^{n-1} c_j(\rho)\partial_x^j q, \quad (2.1.8)$$

$$\partial_x \mu - i\rho\mu = q, \quad (2.1.9)$$

for the function $\mu(x,t,\rho)$, where $c_j(\rho)$ are the functions defined in equations (2.1.6).

PROOF. We take the x partial derivative of equation (2.1.8),

$$\begin{aligned}\partial_x\partial_t\mu &= -a\rho^n\left[\partial_x\mu + \sum_{j=1}^{n}(i\rho)^{-j}\partial_x^j q\right]\\ &= -a\rho^n\left[q + i\rho\mu + \sum_{j=1}^{n}(i\rho)^{-j}\partial_x^j q\right],\end{aligned} \tag{2.1.10}$$

the latter equality being justified by equation (2.1.9). Similarly, we take the t partial derivative of equation (2.1.9),

$$\partial_t\partial_x\mu = \partial_t q + i\rho\partial_t\mu \tag{2.1.11}$$

$$= \partial_t q - a\rho^n\left[i\rho\mu + \sum_{j=0}^{n-1}(i\rho)^{-j}\partial_x^j q\right], \tag{2.1.12}$$

by equation (2.1.8). The compatibility condition (2.1.7) is equivalent to the right hand sides of equations (2.1.10) and (2.1.12) being equal, which is equivalent to the partial differential equation (2.1.1). □

OUTLINE PROOF OF THEOREM 2.1. We break the derivation into three steps.

Solutions of the Lax Pair: The first step is to find particular integrals of the Lax pair (2.1.8)–(2.1.9). We rewrite equation (2.1.9) as

$$\partial_x(e^{-i\rho x}\mu) = e^{-i\rho x}q$$

which has particular integral

$$\mu(x,t,\rho) = \int_{x^\star}^{x} e^{i\rho(x-y)}q(y,t)\,\mathrm{d}y + e^{i\rho(x-x^\star)}\mu(x^\star,t,\rho), \tag{2.1.13}$$

where $\mu(x^\star,t,\rho)$ is a solution of equation (2.1.8) at $x = x^\star$. Taking a particular integral of equation (2.1.8) in the same way we obtain

$$\mu(x^\star,t,\rho) = \int_{t^\star}^{t} e^{-a\rho^n(t-s)}\sum_{j=0}^{n-1}c_j(\rho)\partial_x^j q(x^\star,s)\,\mathrm{d}s. \tag{2.1.14}$$

Substituting equation (2.1.14) into equation (2.1.13) yields

$$\mu^\star(x,t,\rho;x^\star,t^\star) = \int_{x^\star}^{x} e^{i\rho(x-y)}q(y,t)\,\mathrm{d}y + e^{i\rho(x-x^\star)}\int_{t^\star}^{t} e^{-a\rho^n(t-s)}\sum_{j=0}^{n-1}c_j(\rho)\partial_x^j q(x^\star,s)\,\mathrm{d}s.$$

The function $\mu^\star$ is a solution of the Lax pair (2.1.8)–(2.1.9) for any particular choice of the pair $(x^\star,t^\star)\in\Omega$. The open sets of complex numbers $D^\pm$ are defined in (2.1.6). We also define the open sets

$$E^\pm = \mathbb{C}^\pm \cap E, \quad E = \{\rho\in\mathbb{C} : \mathrm{Re}(a\rho^n) > 0\}.$$

Following Proposition 3.1 of [**24**] we choose the the points $(x^\star, t^\star)$ to be the four corners of Ω, defining the functions $\mu_Y(x,t,\rho)$ for $Y \in \{D^\pm, E^\pm\}$:

$$\mu_{E^+}(x,t,\rho) = \int_0^x e^{i\rho(x-y)} q(y,t)\,\mathrm{d}y + e^{i\rho x}\int_0^t e^{-a\rho^n(t-s)} \sum_{j=0}^{n-1} c_j(\rho)\partial_x^j q(0,s)\,\mathrm{d}s,$$

$$\mu_{D^+}(x,t,\rho) = \int_0^x e^{i\rho(x-y)} q(y,t)\,\mathrm{d}y - e^{i\rho x}\int_t^T e^{a\rho^n(s-t)} \sum_{j=0}^{n-1} c_j(\rho)\partial_x^j q(0,s)\,\mathrm{d}s,$$

$$\mu_{D^-}(x,t,\rho) = -\int_x^1 e^{-i\rho(y-x)} q(y,t)\,\mathrm{d}y - e^{i\rho(x-1)}\int_t^T e^{a\rho^n(s-t)} \sum_{j=0}^{n-1} c_j(\rho)\partial_x^j q(1,s)\,\mathrm{d}s,$$

$$\mu_{E^-}(x,t,\rho) = -\int_x^1 e^{-i\rho(y-x)} q(y,t)\,\mathrm{d}y + e^{i\rho(x-1)}\int_0^t e^{-a\rho^n(t-s)} \sum_{j=0}^{n-1} c_j(\rho)\partial_x^j q(1,s)\,\mathrm{d}s.$$

These functions are all entire in ρ and are all particular solutions of the Lax pair. The μ_Y are indexed with the sets $D^\pm$, $E^\pm$ because property

$$\mu_Y(x,t,\rho) \to 0 \text{ as } \rho \to \infty \text{ from within } Y$$

holds.

Riemann-Hilbert Problem: For any $Y, Z \in \{D^\pm, E^\pm\}$ let the function

$$\mu_{Y\,Z} = \mu_j - \mu_k.$$

Then $\mu_{j\,k}$ is a solution of the homogeneous Lax pair

$$\partial_t \mu + a\rho^n \mu = 0,$$
$$\partial_x \mu - i\rho\mu = 0.$$

Taking particular integrals of the homogeneous Lax pair, in the same way as is done above for the inhomogeneous Lax pair, we deduce that

$$\mu_{j\,k}(x,t,\rho) = e^{i\rho x - a\rho^n t} X_{j\,k}(\rho), \tag{2.1.15}$$

where the function $X_{j\,k}$ may be easily obtained by evaluating equation (2.1.15) at $x = t = 0$.[2] This yields, in particular,

$$\mu_{D^+\,E^+}(x,t,\rho) = -e^{i\rho x - a\rho^n t} \sum_{j=0}^{n-1} c_j(\rho)\widetilde{f}_j(\rho),$$

$$\mu_{D^-\,E^-}(x,t,\rho) = -e^{i\rho(x-1) - a\rho^n t} \sum_{j=0}^{n-1} c_j(\rho)\widetilde{g}_j(\rho),$$

$$\mu_{E^+\,E^-}(x,t,\rho) = e^{i\rho x - a\rho^n t}\hat{q}_0(\rho).$$

As the sets $D^\pm$, $E^\pm$ are each comprised of finitely many simply connected components, are disjoint and have union $\mathbb{C} \setminus (\partial D^+ \cup \partial D^- \cup \mathbb{R})$, it makes sense to define a 'jump function'

[2] If we were instead solving a final-boundary value problem we would evaluate equation (2.1.15) at $x = 0$, $t = T$, so that $\mu_{E^+\,E^-}$ is defined in terms of the Fourier transform of the final datum.

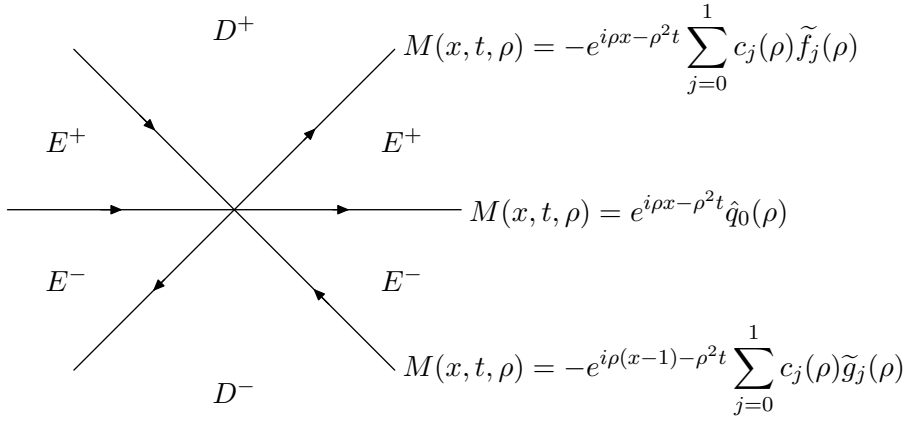

FIGURE 2.1. The Riemann-Hilbert problem for the heat equation

$M : \Omega \times (\partial D^+ \cup \partial D^- \cup \mathbb{R}) \to \mathbb{C}$ by

$$M(x,t,\rho) = \mu_{D^+\,E^+}(x,t,\rho)\chi_{\partial D^+} + \mu_{D^-\,E^-}(x,t,\rho)\chi_{\partial D^-} + \mu_{E^+\,E^-}(x,t,\rho)\chi_{\mathbb{R}},$$

where each χ_j is the indicator function for the set j. Then M represents the jumps on the boundaries of the domains of definition of the μ_Y. Figure 2.1 shows the positions of $D^\pm$ and $E^\pm$ and the oriented boundaries that separate them for the heat equation, $q_t = q_{xx}$, in which $n = 2$, $a = 1$. The figure also shows the value the jump function M takes on each of the boundaries.

By partial integration in the definitions of $\widetilde{f}_j$, $\widetilde{g}_j$ and $\hat{q}_0$ it may be shown that $M(x,t,\rho) = O(1/\rho)$ as $\rho \to \infty$. This specifies an inhomogeneous, scalar Riemann-Hilbert problem on the contours $\partial D^\pm$, oriented in the usual direction, and $\mathbb{R}$, oriented in the positive direction.

Solution: The Reimann-Hilbert problem is solved [**1**] by the sectionally analytic function

$$\mu(x,t,\rho) = \frac{1}{2\pi i}\left\{\int_{\mathbb{R}} + \int_{\partial D^+} + \int_{\partial D^-}\right\}\frac{M(x,t,\lambda)}{\lambda - \rho}\,\mathrm{d}\lambda.$$

As μ satisfies the Lax pair, we may use equation (2.1.9) to obtain the expression (2.1.5) for q. □

The integral representation (2.1.5) is only a formal solution to the initial-boundary value problem (2.1.1)–(2.1.3) because it depends upon all n of the left-hand boundary functions, f_j, and all n of the right-hand boundary functions, g_j. There are only n boundary conditions in (2.1.3) so the boundary conditions may explicitly specify at most n of these $2n$ boundary functions in terms of boundary data. Hence the fundamental problem is the determination of the (at least n) unknown boundary functions. We address this issue by considering the global relation.

2.1.3. The global relation

The *global relation* is derived from Green's Theorem B.1 and represents an equation relating the t-transforms of the boundary functions, defined in (2.1.6) to the Fourier transforms of the

initial datum, $q_0(x)$, and *final function*, $q_T(x) = q(x,T)$. We define these Fourier transforms as

$$\hat{q}_0(\rho) = \int_0^1 e^{-i\rho x} q_0(x)\,\mathrm{d}x = \int_{\mathbb{R}} e^{-i\rho x} q_0(x)\chi_{[0,1]}\,\mathrm{d}x \qquad \rho \in \mathbb{C},$$
$$\hat{q}_T(\rho) = \int_0^1 e^{-i\rho x} q(x,T)\,\mathrm{d}x, \qquad \rho \in \mathbb{C}.$$

Now we derive the global relation.

LEMMA 2.3 (Global relation). *Let $q : \Omega \to \mathbb{R}$ be a formal solution to an initial-boundary value problem specified by the partial differential equation* (2.1.1) *and initial condition* (2.1.2). *Then the functions $\hat{q}_0$, $\hat{q}_T$ defined above and the functions $\widetilde{f}_j$ and $\widetilde{g}_j$, given by* (2.1.6) *satisfy*

$$\sum_{j=0}^{n-1} c_j(\rho)\left(\tilde{f}_j(\rho) - e^{-i\rho}\tilde{g}_j(\rho)\right) = \hat{q}_0(\rho) - e^{a\rho^n T}\hat{q}_T(\rho), \qquad \rho \in \mathbb{C}. \tag{2.1.16}$$

PROOF. For $(x,t) \in \Omega$ and $\rho \in \mathbb{C}$ let

$$X(x,t,\rho) = e^{-i\rho x + a\rho^n t} q(x,t), \qquad Y(x,t,\rho) = e^{-i\rho x + a\rho^n t} \sum_{j=0}^{n-1} c_j(\rho)\partial_x^j q(x,t).$$

Then

$$\partial_t X(x,t,\rho) = e^{-i\rho x + a\rho^n t}(a\rho^n + \partial_t)q(x,t),$$
$$\partial_x Y(x,t,\rho) = e^{-i\rho x + a\rho^n t}(-i\rho + \partial_x)\sum_{j=0}^{n-1} c_j(\rho)\partial_x^j q(x,t)$$

hence

$$(\partial_t X - \partial_x Y)(x,t,\rho) = e^{-i\rho x + a\rho^n t}\left[(a\rho^n + \partial_t) + (i\rho - \partial_x)\sum_{j=0}^{n-1} c_j(\rho)\partial_x^j\right] q(x,t)$$
$$= e^{-i\rho x + a\rho^n t}\left[(a\rho^n - a(-i\partial_x)^n) - a\rho^n(i\rho - \partial_x)\sum_{j=0}^{n-1}(i\rho)^{-(j+1)}\partial_x^j\right] q(x,t),$$

using the differential equation (2.1.1) and the definition of the polynomials c_j,

$$= e^{-i\rho x + a\rho^n t} a\left[(\rho^n - (-i\partial_x)^n) - \rho^n(1 - (i\rho)^{-n}\partial_x^n)\right] q(x,t)$$
$$= 0.$$

If we apply Green's Theorem B.1 to Ω then we see that

$$\int_\Omega (\partial_t X - \partial_x Y)(x,t,\rho)\,\mathrm{d}x\,\mathrm{d}t = \int_{\partial\Omega}(Y\,\mathrm{d}t + X\,\mathrm{d}x)$$
$$\Rightarrow \quad 0 = \int_0^1 X(x,0,\rho)\,\mathrm{d}x + \int_0^T Y(1,t,\rho)\,\mathrm{d}t$$
$$- \int_0^1 X(x,T,\rho)\,\mathrm{d}x - \int_0^T Y(0,t,\rho)\,\mathrm{d}t$$
$$= \hat{q}_0(\rho) + e^{-i\rho}\sum_{j=0}^{n-1} c_j(\rho)\widetilde{g}_j(\rho) - e^{a\rho^n T}\hat{q}_T(\rho) - \sum_{j=0}^{n-1} c_j(\rho)\widetilde{f}_j(\rho),$$

where $\widetilde{f}_j$, $\widetilde{g}_j$ are defined in (2.1.6), from which the result follows. □

The global relation is useful because of the particular form of the spectral transforms of the boundary functions. The transformed boundary functions may be considered as functions not of ρ but of ρ^n. This means that the transforms are invariant under the map $\rho \mapsto \omega^j \rho$, for $\omega = e^{\frac{2\pi i}{n}}$, $j \in \mathbb{Z}$. This, together with the identity

$$c_j(\omega^k \rho) = -a(\omega^k \rho)^n (i\omega^k \rho)^{-(j+1)} = \omega^{k(n-1-j)} c_j(\rho), \tag{2.1.17}$$

for $j \in \{0, 1, \ldots, n-1\}$ and $k \in \mathbb{Z}$, establishes the following corollary.

Corollary 2.4 (Global relation—matrix form). *Suppose the function $q : \Omega \to \mathbb{R}$ satisfies the partial differential equation* (2.1.1) *and initial condition* (2.1.2). *Then the t-transforms of the boundary functions of q, defined in* (2.1.6), *satisfy*

$$\mathcal{B}(\rho) \begin{pmatrix} c_{n-1}(\rho)\widetilde{f}_{n-1}(\rho) \\ c_{n-1}(\rho)\widetilde{g}_{n-1}(\rho) \\ c_{n-2}(\rho)\widetilde{f}_{n-2}(\rho) \\ c_{n-2}(\rho)\widetilde{g}_{n-2}(\rho) \\ \vdots \\ c_0(\rho)\widetilde{f}_0(\rho) \\ c_0(\rho)\widetilde{g}_0(\rho) \end{pmatrix} = \begin{pmatrix} \hat{q}_0(\rho) \\ \hat{q}_0(\omega\rho) \\ \vdots \\ \hat{q}_0(\omega^{n-1}\rho) \end{pmatrix} - e^{a\rho^n T} \begin{pmatrix} \hat{q}_T(\rho) \\ \hat{q}_T(\omega\rho) \\ \vdots \\ \hat{q}_T(\omega^{n-1}\rho) \end{pmatrix}, \tag{2.1.18}$$

for $\rho \in \mathbb{C}$, where

$$\mathcal{B}(\rho) = \begin{pmatrix} 1 & -e^{-i\rho} & 1 & -e^{-i\rho} & \ldots & 1 & -e^{-i\rho} \\ 1 & -e^{-i\omega\rho} & \omega & -\omega e^{-i\omega\rho} & \ldots & \omega^{n-1} & -\omega^{n-1} e^{-i\omega\rho} \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & -e^{-i\omega^{n-1}\rho} & \omega^{n-1} & -\omega^{n-1} e^{-i\omega^{n-1}\rho} & \ldots & \omega^{(n-1)(n-1)} & -\omega^{(n-1)(n-1)} e^{-i\omega^{n-1}\rho} \end{pmatrix}. \tag{2.1.19}$$

All maximal square submatrices of $\mathcal{B}$ have non-zero determinant so the matrix has rank n. We derive $\mathcal{B}$ in two examples.

Example 2.5. Let q be a function which satisfies the heat equation,

$$q_t(x,t) = q_{xx}(x,t) \text{ for } (x,t) \in \Omega = [0,1] \times [0,T]$$

and the initial condition $q(x,0) = q_0(x)$. Let $q_T(x) = q(x,T)$. In this case $n = 2$ and $a = 1$ so

$$c_0(\rho) = i\rho, \qquad c_1(\rho) = 1, \qquad \rho \in \mathbb{C}.$$

The global relation Lemma 2.3 yields the equation

$$i\rho \left(\widetilde{f}_0(\rho) - e^{-i\rho}\widetilde{g}_0(\rho) \right) + \left(\widetilde{f}_1(\rho) - e^{-i\rho}\widetilde{g}_1(\rho) \right) = \hat{q}_0(\rho) - e^{\rho^2 T}\hat{q}_T(\rho), \tag{2.1.20}$$

for $\rho \in \mathbb{C}$, where the transformed boundary functions are defined for $\rho \in \mathbb{C}$ by

$$\begin{aligned} \widetilde{f}_0(\rho) &= \int_0^T e^{\rho^2 t} q(0,t)\,\mathrm{d}t, & \widetilde{g}_0(\rho) &= \int_0^T e^{\rho^2 t} q(1,t)\,\mathrm{d}t, \\ \widetilde{f}_1(\rho) &= \int_0^T e^{\rho^2 t} \partial_x q(0,t)\,\mathrm{d}t, & \widetilde{g}_1(\rho) &= \int_0^T e^{\rho^2 t} \partial_x q(1,t)\,\mathrm{d}t. \end{aligned} \tag{2.1.21}$$

By examining the definitions (2.1.21) we see that the transformed boundary functions are functions of ρ^2. This means that they are invariant under the map $\rho \mapsto -\rho$, that is

$$\begin{aligned} \widetilde{f}_0(-\rho) &= \widetilde{f}_0(\rho), & \widetilde{g}_0(-\rho) &= \widetilde{g}_0(\rho), \\ \widetilde{f}_1(-\rho) &= \widetilde{f}_1(\rho), & \widetilde{g}_1(-\rho) &= \widetilde{g}_1(\rho). \end{aligned} \tag{2.1.22}$$

Since the global relation (2.1.20) is valid for any $\rho \in \mathbb{C}$, evaluating it at $-\rho$ we obtain

$$-i\rho\left(\widetilde{f}_0(-\rho) - e^{i\rho}\widetilde{g}_0(-\rho)\right) + \left(\widetilde{f}_1(-\rho) - e^{i\rho}\widetilde{g}_1(-\rho)\right) = \hat{q}_0(-\rho) - e^{(-\rho)^2 T}\hat{q}_T(-\rho),$$

which, by equations (2.1.22), is

$$-i\rho\left(\widetilde{f}_0(\rho) - e^{i\rho}\widetilde{g}_0(\rho)\right) + \left(\widetilde{f}_1(\rho) - e^{i\rho}\widetilde{g}_1(\rho)\right) = \hat{q}_0(-\rho) - e^{\rho^2 T}\hat{q}_T(-\rho). \tag{2.1.23}$$

The global relation equations (2.1.20) and (2.1.23) may now be written in matrix form

$$\mathcal{B}(\rho)\begin{pmatrix} \widetilde{f}_1(\rho) \\ \widetilde{g}_1(\rho) \\ i\rho\widetilde{f}_0(\rho) \\ i\rho\widetilde{g}_0(\rho) \end{pmatrix} = \begin{pmatrix} \hat{q}_0(\rho) \\ \hat{q}_0(-\rho) \end{pmatrix} - e^{\rho^2 T}\begin{pmatrix} \hat{q}_T(\rho) \\ \hat{q}_T(-\rho) \end{pmatrix} \tag{2.1.24}$$

where

$$\mathcal{B}(\rho) = \begin{pmatrix} 1 & -e^{-i\rho} & 1 & -e^{-i\rho} \\ 1 & -e^{i\rho} & -1 & e^{i\rho} \end{pmatrix}.$$

Equation (2.1.24) corresponds to Corollary 2.4.

Example 2.6. Let q be a function which satisfies the partial differential equation

$$q_t(x,t) = q_{xxx}(x,t) \text{ for } (x,t) \in \Omega = [0,1] \times [0,T]$$

and the initial condition $q(x,0) = q_0(x)$. Let $q_T(x) = q(x,T)$. In this case $n = 3$ and $a = i$ so

$$c_0(\rho) = -\rho^2, \qquad c_1(\rho) = i\rho, \qquad c_2(\rho) = 1, \qquad \rho \in \mathbb{C}.$$

The global relation Lemma 2.3 yields the relation

$$-\rho^2\left(\widetilde{f}_0(\rho) - e^{-i\rho}\widetilde{g}_0(\rho)\right) + i\rho\left(\widetilde{f}_1(\rho) - e^{-i\rho}\widetilde{g}_1(\rho)\right) + \left(\widetilde{f}_2(\rho) - e^{-i\rho}\widetilde{g}_2(\rho)\right) = \hat{q}_0(\rho) - e^{\rho^3 T}\hat{q}_T(\rho), \tag{2.1.25}$$

for $\rho \in \mathbb{C}$, where the transformed boundary functions are defined for $\rho \in \mathbb{C}$ by

$$\widetilde{f}_j(\rho) = \int_0^T e^{\rho^3 t}\partial_x^j q(0,t)\,\mathrm{d}t, \qquad \widetilde{g}_j(\rho) = \int_0^T e^{\rho^3 t}\partial_x^j q(1,t)\,\mathrm{d}t, \qquad j \in \{0,1,2\}. \tag{2.1.26}$$

By examining the definitions (2.1.26) we see that the transformed boundary functions are functions of ρ^3. This means that, for $\omega = e^{\frac{2\pi i}{3}}$ they are invariant under the maps $\rho \mapsto \omega\rho$ and $\rho \mapsto \omega^2\rho$, that is

$$\widetilde{f}_j(\omega^k\rho) = \widetilde{f}_j(\rho), \qquad \widetilde{g}_j(\omega^k\rho) = \widetilde{g}_j(\rho), \qquad j \in \{0,1,2\}, \quad k \in \mathbb{Z} \tag{2.1.27}$$

Since the global relation (2.1.25) is valid for any $\rho \in \mathbb{C}$, evaluating it at $\omega^k\rho$ and applying equations (2.1.27) we obtain

$$\begin{aligned} -\omega^{2k}\rho^2\left(\widetilde{f}_0(\rho) - e^{-i\omega^k\rho}\widetilde{g}_0(\rho)\right) + i\omega^k\rho\left(\widetilde{f}_1(\rho) - e^{-i\omega^k\rho}\widetilde{g}_1(\rho)\right) + \left(\widetilde{f}_2(\rho) - e^{-i\omega^k\rho}\widetilde{g}_2(\rho)\right) \\ = \hat{q}_0(\omega^k\rho) - e^{\rho^3 T}\hat{q}_T(\omega^k\rho), \end{aligned} \tag{2.1.28}$$

for each $k = 0, 1, 2$. The global relation equations (2.1.28) may now be written in matrix form

$$\mathcal{B}(\rho)\begin{pmatrix} \widetilde{f}_2(\rho) \\ \widetilde{g}_2(\rho) \\ i\rho\widetilde{f}_1(\rho) \\ i\rho\widetilde{g}_1(\rho) \\ -\rho^2\widetilde{f}_0(\rho) \\ -\rho^2\widetilde{g}_0(\rho) \end{pmatrix} = \begin{pmatrix} \hat{q}_0(\rho) \\ \hat{q}_0(\omega\rho) \\ \hat{q}_0(\omega^2\rho) \end{pmatrix} - e^{\rho^3 T}\begin{pmatrix} \hat{q}_T(\rho) \\ \hat{q}_T(\omega\rho) \\ \hat{q}_T(\omega^2\rho) \end{pmatrix} \tag{2.1.29}$$

where

$$\mathcal{B}(\rho) = \begin{pmatrix} 1 & -e^{-i\rho} & 1 & -e^{-i\rho} & 1 & -e^{-i\rho} \\ 1 & -e^{-i\omega\rho} & \omega & -\omega e^{-i\omega\rho} & \omega^2 & -\omega^2 e^{-i\omega\rho} \\ 1 & -e^{-i\omega^2\rho} & \omega^2 & -\omega^2 e^{-i\omega^2\rho} & \omega & -\omega e^{-i\omega^2\rho} \end{pmatrix}.$$

Equation (2.1.29) corresponds to Corollary 2.4.

2.1.4. Finding the boundary functions

Corollary 2.4 defines a system of n linear equations for the transforms of the $2n$ boundary functions, where we treat the terms on the right hand side of equation (2.1.18), Fourier transforms of the initial datum and the final function, as known quantities. In order to solve for the boundary functions we require another n equations. These may be derived from the boundary conditions (2.1.3). Indeed, as the t-transform $X \mapsto \widetilde{X}$ defined by

$$\widetilde{X}(\rho) = \int_0^T e^{a\rho^n s} X(s)\,\mathrm{d}s \tag{2.1.30}$$

is linear, we may transform the boundary conditions to give

$$\sum_{j=0}^{n-1} \alpha_{j\,r}\widetilde{f}_j(\rho) + \sum_{j=0}^{n-1} \beta_{j\,r}\widetilde{g}_j(\rho) = \widetilde{h}_r(\rho), \quad r \in \{1, 2, \ldots, n\} \tag{2.1.31}$$

where the functions $\widetilde{h}_r$ are the transforms defined by equation (2.1.30) of the boundary data h_r.

Now Corollary 2.4 and equation (2.1.31) together give a system of $2n$ linear equations relating the transforms of the $2n$ boundary functions to the Fourier transforms of the initial datum and the final function and the transforms of the boundary data. If we could assume that the final function is a known quantity then this would be a system of $2n$ equations in $2n$ unknowns. If it can be guaranteed that the system is full rank then this system may be solved using Cramer's rule, Theorem B.2, to give expressions for the boundary functions in terms of the initial and boundary data and the final function.

Once the transforms of the boundary functions are known they may be substituted into equation (2.1.5) to give an integral representation of the solution in terms of the initial and boundary data and the final function. It will be shown in Section 3.2 of Chapter 3 that, provided the initial-boundary value problem is well-posed, terms containing the final functions make no contribution to expression (2.1.5) and this gives an integral representation for the solution that may be evaluated in terms of the known data.

2.1.5. A classification of boundary conditions

In Definition 2.7 we provide a rough classification of boundary values. We classify the boundary conditions in terms of the representation used in Locker's work [**47**] on differential operators.

Definition 2.7 (Classification of boundary conditions). *We rewrite the boundary conditions* (2.1.3) *in the form of a matrix equation:*

$$A(f_{n-1}, g_{n-1}, f_{n-2}, g_{n-2}, \ldots, f_0, g_0)^{\mathrm{T}} = (h_1, h_2, \ldots, h_n)^{\mathrm{T}}, \tag{2.1.32}$$

where the boundary functions f_j are defined in equations (2.1.6) *and the* boundary coefficient matrix*, A, is in reduced row-echelon form and is given by*

$$A = \begin{pmatrix} \alpha_{1\,n-1} & \beta_{1\,n-1} & \alpha_{1\,n-2} & \beta_{1\,n-2} & \cdots & \alpha_{1\,0} & \beta_{1\,0} \\ \alpha_{2\,n-1} & \beta_{2\,n-1} & \alpha_{2\,n-2} & \beta_{2\,n-2} & \cdots & \alpha_{2\,0} & \beta_{2\,0} \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \alpha_{n\,n-1} & \beta_{n\,n-1} & \alpha_{n\,n-2} & \beta_{n\,n-2} & \cdots & \alpha_{n\,0} & \beta_{n\,0} \end{pmatrix}, \tag{2.1.33}$$

hence the left $n \times n$ block of A is upper triangular. Note that any set of n linearly independent boundary conditions has a unique expression in this form, so this is indeed equivalent to the boundary conditions (2.1.3)*.*

- *The boundary conditions of an initial-boundary value problem are said to be* homogeneous *if the boundary data h_k are identically zero on $[0, T]$ for all $k \in \{1, 2, \ldots, n\}$. Otherwise the boundary conditions are* inhomogeneous*.*
- *If each boundary condition has involves only a single order of spatial derivitive (though possibly at both ends) then we call the boundary conditions* non-Robin*. Boundary conditions are non-Robin if each contains only one order of partial derivative. Otherwise we say that boundary condition is* of Robin type*.*
- *Boundary conditions with the property*

  Every non-zero entry in the boundary coefficient matrix is a pivot.

  are called simple*.*
- *A set of boundary conditions is* uncoupled *(or* does not couple the ends of the interval*) if*

  $$\text{If } \alpha_{k\,j} \text{ is a pivot in } A \text{ then } \beta_{k\,r} = 0 \;\forall\, r \text{ and}$$
  $$\text{If } \beta_{k\,j} \text{ is a pivot in } A \text{ then } \alpha_{k\,r} = 0 \;\forall\, r.$$

  Otherwise we say that the boundary conditions are coupled *(or that they* couple the ends of the interval*).*

Note that a set of boundary conditions is uncoupled and non-Robin if and only if it is simple. We present several examples of sets of boundary conditions to illustrate Definition 2.7. It may be shown that each specify a well-posed problem on the same partial differential equation (2.1.1) with $a = i$, $n = 3$.

EXAMPLE 2.8. The boundary conditions

$$q_x(0,t) = q_x(1,t) \qquad q(0,t) = q(1,t) = 0$$

may be expressed by specifying the boundary data $h_1 = h_2 = h_3 = 0$ and boundary coefficient matrix

$$A = \begin{pmatrix} 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Hence these boundary conditions are homogeneous and non-Robin but coupled.

EXAMPLE 2.9. The boundary conditions

$$q_x(0,t) = t(T-t) \qquad q(0,t) = q(1,t) = 0$$

may be expressed by specifying the boundary data $h_1 = t(T-t)$, $h_2 = h_3 = 0$ and boundary coefficient matrix

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Hence these boundary conditions are non-Robin and uncoupled, hence simple, but inhomogeneous.

EXAMPLE 2.10. The boundary conditions

$$q_{xx}(0,t) = q_x(0,t) \qquad q(0,t) = q(1,t) = 0$$

may be expressed by specifying the boundary data $h_1 = h_2 = h_3 = 0$ and boundary coefficient matrix

$$A = \begin{pmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Hence these boundary conditions are homogeneous and do not couple the ends of the interval but one of them is of Robin type.

EXAMPLE 2.11. The boundary conditions

$$q_{xx}(0,t) + q_{xx}(1,t) + q_x(1,t) = 0 \qquad q_x(0,t) + q_x(1,t) + q(0,t) = 0 \qquad q(1,t) = 0$$

may be expressed by specifying the boundary data $h_1 = h_2 = h_3 = 0$ and boundary coefficient matrix

$$A = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Hence these boundary conditions are homogeneous but two of them couple the ends of the interval and are of Robin type.

2.2. The reduced global relation

In this section we state and prove the main lemma of this chapter. It uses the boundary conditions and global relation associated to an initial-boundary value problem to yield a system of equations relating the boundary functions to the boundary data, initial datum and final function. This system may be solved using Cramer's rule, Theorem B.2. We assume (without loss of generality) in this and subsequent sections the boundary conditions to be expressed in the form of equation (2.1.32) with the matrix A, given by equation (2.1.33), in reduced row-echelon form. This will be important in simplifying the calculation that follows.

We break the analysis into two cases: in Subsection 2.2.1 we consider homogeneous, non-Robin boundary conditions and in Subsection 2.2.2 we consider any set of n linearly independent boundary conditions. The reason for this is that, although the proof is similar, the lemma we wish to prove is considerably easier to state under the more restrictive conditions of Subsection 2.2.1 than in general. Another motivation for giving the two separate statements of the main lemma is that in later chapters homogeneous, non-Robin boundary values will be of particular significance.

For each case, the main lemma is stated in such a way as to break the system of $2n$ equations discussed in Subsection 2.1.4 into two systems, each containing n equations. When compared to solving the original system of $2n$ equations directly, this has the disadvantage of requiring more cumbersome notation but also has two advantages. The first is that it best follows the natural way one would choose to solve a particular example of the system by hand or with a computer, exploiting the fact that the boundary coefficient matrix is in reduced row-echelon form. The second, and more significant, reason is to aid comparison with the results of [**47**]. Indeed, in Chapter 4 we compare an $n \times n$ matrix of Locker with the matrix $\mathcal{A}$ defined in the main lemmata of this section. If our $\mathcal{A}$ were a $2n \times 2n$ matrix this would be considerably more difficult.

2.2.1. Homogeneous, non-Robin

In this subsection we assume the boundary conditions to be homogeneous and non-Robin (see Definition 2.7). It should be noted that, in contrast with classical approaches to solving this kind of initial-boundary value problem, the homogeneity of the boundary conditions does not simplify the proof, only the statement of the result.

The calculation involves elementary linear algebra but notationally it is somewhat complex even under these restrictions. Indeed we must develop some notation to state the result. The aim of Notation 2.12 is to ensure that we may split the vector

$$\begin{pmatrix} \widetilde{f}_{n-1}(\rho) \\ \widetilde{g}_{n-1}(\rho) \\ \widetilde{f}_{n-2}(\rho) \\ \widetilde{g}_{n-2}(\rho) \\ \vdots \\ \widetilde{f}_{0}(\rho) \\ \widetilde{g}_{0}(\rho) \end{pmatrix} \tag{2.2.1}$$

into the two vectors V and W. The entries in W are the transform, $\widetilde{f}_j$ or $\widetilde{g}_j$, of a boundary function that is, in equation (2.1.32), multiplied by a pivot of A, where the entries in V are the other entries in the vector (2.2.1) and overall we preserve the order of the entries in the original vector (2.2.1).

2.2.1.1. Developing some notation

NOTATION 2.12. Given boundary conditions defined by equations (2.1.32) and (2.1.33) such that A is in reduced row-echelon form, we define the following index sets and functions.

- $\widehat{J}^+ = \{j \in \{0, 1, \ldots, n-1\}$ such that $\alpha_{k\,j}$ is a pivot in A for some $k \in \{1, 2, \ldots, n\}\}$, the greatest order of each boundary condition whose leading term is a left-hand boundary function.
- $\widehat{J}^- = \{j \in \{0, 1, \ldots, n-1\}$ such that $\beta_{k\,j}$ is a pivot in A for some $k \in \{1, 2, \ldots, n\}\}$, the greatest order of each boundary condition whose leading term is a right-hand boundary function.
- $\widetilde{J}^+ = \{0, 1, \ldots, n-1\} \setminus \widehat{J}^+$, the order of each left-hand boundary function that does not lead any boundary condition.
- $\widetilde{J}^- = \{0, 1, \ldots, n-1\} \setminus \widehat{J}^-$, the order of each right-hand boundary function that does not lead any boundary condition.
- $J = \{2j+1$ such that $j \in \widetilde{J}^+\} \cup \{2j$ such that $j \in \widetilde{J}^-\}$, an index set for the boundary functions that do not lead any boundary condition. Also, the decreasing sequence $(J_j)_{j=1}^n$ of elements of J.
- $J' = \{2j+1$ such that $j \in \widehat{J}^+\} \cup \{2j$ such that $j \in \widehat{J}^-\} = \{0, 1, \ldots, 2n-1\} \setminus J$, an index set for the boundary functions that lead boundary conditions. Also, the decreasing sequence $(J'_j)_{j=1}^n$ of elements of J'.
- The functions

$$V(\rho) = (V_1(\rho), V_2(\rho), \ldots, V_n(\rho))^{\mathrm{T}}, \qquad V_j(\rho) = \begin{cases} \widetilde{f}_{(J_j-1)/2}(\rho) & J_j \text{ odd}, \\ \widetilde{g}_{J_j/2}(\rho) & J_j \text{ even}, \end{cases}$$

  the boundary functions that do not lead any boundary condition.
- The functions

$$W(\rho) = (W_1(\rho), W_2(\rho), \ldots, W_n(\rho))^{\mathrm{T}}, \qquad W_j(\rho) = \begin{cases} \widetilde{f}_{(J'_j-1)/2}(\rho) & J'_j \text{ odd}, \\ \widetilde{g}_{J'_j/2}(\rho) & J'_j \text{ even}, \end{cases}$$

  the boundary functions that do lead boundary conditions.
- $(\widehat{J}^+_j)_{j\in\widehat{J}^+}$, a sequence such that $\alpha_{\widehat{J}^+_j\,j}$ is a pivot in A when $j \in \widehat{J}^+$. Note that this sequence may have no terms.
- $(\widehat{J}^-_j)_{j\in\widehat{J}^-}$, a sequence such that $\beta_{\widehat{J}^-_j\,j}$ is a pivot in A when $j \in \widehat{J}^-$. This sequence may have no terms.
- For each $j \in \widetilde{J}^+$ define $\beta_{\widehat{J}^+_j\,j} = 0$. This is done to simplify the statement of the simpler version of the lemma. It is not required for the general version.

EXAMPLE 2.13. If $n = 3$ and the boundary conditions are specified by equation (2.1.32) where

$$A = \begin{pmatrix} 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.2.2}$$

then $c_2(\rho) = -ai$, $c_1(\rho) = a\rho$, $c_0(\rho) = ai\rho^2$,

$$W(\rho) = \begin{pmatrix} \widetilde{f}_1(\rho) \\ \widetilde{f}_0(\rho) \\ \widetilde{g}_0(\rho) \end{pmatrix} \text{ and } V(\rho) = \begin{pmatrix} \widetilde{f}_2(\rho) \\ \widetilde{g}_2(\rho) \\ \widetilde{g}_1(\rho) \end{pmatrix}.$$

Indeed, comparing equations (2.1.33) and (2.2.2) we see that

$$\begin{pmatrix} \alpha_{1\,2} & \beta_{1\,2} & \alpha_{1\,1} & \beta_{1\,1} & \alpha_{1\,0} & \beta_{1\,0} \\ \alpha_{2\,2} & \beta_{2\,2} & \alpha_{2\,1} & \beta_{2\,1} & \alpha_{2\,0} & \beta_{2\,0} \\ \alpha_{3\,2} & \beta_{3\,2} & \alpha_{3\,1} & \beta_{3\,1} & \alpha_{3\,0} & \beta_{3\,0} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The pivots in this boundary coefficient matrix are $\alpha_{1\,1}$, $\alpha_{2\,0}$ and $\beta_{3\,0}$ so

$$\widehat{J}^+ = \{0, 1\}, \qquad \widehat{J}^- = \{0\},$$
$$\widetilde{J}^+ = \{2\} \text{ and} \qquad \widetilde{J}^- = \{1, 2\}.$$

Following through Notation 2.12 in order we see that

$$J = \{2, 4, 5\}, \qquad J' = \{0, 1, 3\},$$
$$(J_j)_{j=1}^3 = (5, 4, 2), \qquad (J'_j)_{j=1}^3 = (3, 1, 0),$$
$$V(\rho) = \begin{pmatrix} \widetilde{f}_2(\rho) \\ \widetilde{g}_2(\rho) \\ \widetilde{g}_1(\rho) \end{pmatrix} \text{ and} \qquad W(\rho) = \begin{pmatrix} \widetilde{f}_1(\rho) \\ \widetilde{f}_0(\rho) \\ \widetilde{g}_0(\rho) \end{pmatrix}.$$

We also note that, defining the sequences

$$\widehat{J}^+_j = \begin{cases} 2 & \text{if } j = 0, \\ 1 & \text{if } j = 1, \end{cases} \qquad \widehat{J}^-_j = 3 \text{ for } j = 0,$$

the pivots in A are

$$\alpha_{\widehat{J}^+_{(J'_1-1)/2}\,(J_1-1)/2} = \alpha_{1\,1},\ \alpha_{\widehat{J}^+_{(J'_2-1)/2}\,(J_2-1)/2} = \alpha_{2\,0} \text{ and } \beta_{\widehat{J}^-_{J'_3/2}\,J_3/2} = \beta_{3\,0}.$$

Indeed the aim of the definition of sequences $(\widehat{J}^+_j)_{j\in\widehat{J}^+}$ and $(\widehat{J}^-_j)_{j\in\widehat{J}^-}$ is to select the row of A containing the pivot corresponding to f_j or g_j.

2.2.1.2. The main lemma

We may now state the result.

LEMMA 2.14. *Let $q : [0,1] \times [0,T] \to \mathbb{R}$ be a solution of the initial-boundary value problem specified by the partial differential equation* (2.1.1)*, the initial condition* (2.1.2) *and the homogeneous, non-Robin boundary conditions* (2.1.32)*. Assume the matrix A, whose entries are defined by equation* (2.1.33)*, is in reduced row-echelon form. Then the vectors V and W from Notation 2.12 satisfy*

$$\mathcal{A}(\rho)\begin{pmatrix} V_1(\rho) \\ V_2(\rho) \\ \vdots \\ V_n(\rho) \end{pmatrix} = \begin{pmatrix} \hat{q}_0(\rho) \\ \vdots \\ \hat{q}_0(\omega^{n-1}\rho) \end{pmatrix} - e^{a\rho^n T}\begin{pmatrix} \hat{q}_T(\rho) \\ \vdots \\ \hat{q}_T(\omega^{n-1}\rho) \end{pmatrix} \quad \textit{and} \tag{2.2.3}$$

$$\begin{pmatrix} W_1(\rho) \\ W_2(\rho) \\ \vdots \\ W_n(\rho) \end{pmatrix} = -\widehat{A}\begin{pmatrix} V_1(\rho) \\ V_2(\rho) \\ \vdots \\ V_n(\rho) \end{pmatrix}, \tag{2.2.4}$$

where

$$\mathcal{A}_{k\,j}(\rho) = \begin{cases} \omega^{(n-1-[J_j-1]/2)(k-1)} c_{(J_j-1)/2}(\rho) & J_j \textit{ odd,} \\ -\omega^{(n-1-J_j/2)(k-1)} c_{J_j/2}(\rho)\left(e^{-i\omega^{k-1}\rho} + \beta_{\widehat{J}^+_{J_j/2}\,J_j/2}\right) & J_j \textit{ even,} \end{cases} \tag{2.2.5}$$

$$\widehat{A}_{k\,j} = \begin{cases} \beta_{k\,J_j/2} & J_j \textit{ even and } k = \widehat{J}^+_{J_j/2} \\ 0 & \textit{otherwise.} \end{cases} \tag{2.2.6}$$

Further, $\mathcal{A}$ is full rank.

2.2.1.3. A sketch proof of the main lemma in the form of an example

Before giving the full proof of Lemma 2.14 we work through the derivation for the particular Example 2.16 of simple boundary conditions. This example also motivates the following nomenclature:

DEFINITION 2.15. *When Lemma 2.14 applies we call equation* (2.2.3) *the* reduced global relation *and equation* (2.2.4) *the* reduced boundary conditions*, where the matrix $\mathcal{A}$ given by equation* (2.2.5) *is called the* reduced global relation matrix *and the matrix $\widehat{A}$ defined by equation* (2.2.6) *is called the* reduced boundary coefficient matrix.

EXAMPLE 2.16. Consider the following initial-boundary value problem:

$$q_t(x,t) - q_{xxx}(x,t) = 0, \qquad (x,t) \in \Omega = [0,1] \times [0,T], \tag{2.2.7}$$
$$q(x,0) = q_0(x)$$

with the simple, homogeneous boundary conditions

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} f_2 \\ g_2 \\ f_1 \\ g_1 \\ f_0 \\ g_0 \end{pmatrix} = 0. \tag{2.2.8}$$

Then, as in Example 2.13,

$$W(\rho) = \begin{pmatrix} \widetilde{f}_1(\rho) \\ \widetilde{f}_0(\rho) \\ \widetilde{g}_0(\rho) \end{pmatrix} \text{ and } V(\rho) = \begin{pmatrix} \widetilde{f}_2(\rho) \\ \widetilde{g}_2(\rho) \\ \widetilde{g}_1(\rho) \end{pmatrix}.$$

The boundary conditions (2.2.8) may be rewritten

$$I_3 \begin{pmatrix} f_1 \\ f_0 \\ g_0 \end{pmatrix} + 0_3 \begin{pmatrix} f_2 \\ g_2 \\ g_1 \end{pmatrix} = 0,$$

where I_3 is the 3×3 identity matrix and 0_3 is the 3×3 zero matrix, which yields

$$\begin{pmatrix} f_1(t) \\ f_0(t) \\ g_0(t) \end{pmatrix} = 0 \qquad t \in [0, T].$$

Applying the t-transform (2.1.30) entrywise we see that

$$W(\rho) = \begin{pmatrix} \widetilde{f}_1(\rho) \\ \widetilde{f}_0(\rho) \\ \widetilde{g}_0(\rho) \end{pmatrix} = 0, \qquad \rho \in \mathbb{C} \tag{2.2.9}$$

This corresponds to the reduced boundary conditions (2.2.4) in the lemma.

The fact we have exploited here is that, because it is in reduced row-echelon form, the boundary coefficient matrix has I_3 as a maximal square submatrix. This allows us to break the boundary coefficient matrix into two parts: the identity and the rest of it, which we call the reduced boundary coefficient matrix. In this example the reduced boundary coefficient matrix is the zero matrix. This need not be the case but, provided the boundary conditions are non-Robin, this matrix must be diagonal. Of course, this process will work for any regularised boundary coefficient matrix, the only requirement being that the boundary coefficient matrix has the identity as a maximal square submatrix, which is guaranteed by the reduced row-echelon form it is assumed to take.

We still have to find the other three boundary functions, those that appear in the vector V. To do this we will make use of the global relation in the form of Corollary 2.4. The partial differential equation (2.2.7) studied in this example defines $n = 3$ and $a = i$ so the corollary may

be written

$$\begin{pmatrix} 1 & -e^{-i\rho} & i\rho & -i\rho e^{-i\rho} & -\rho^2 & \rho^2 e^{-i\rho} \\ 1 & -e^{-i\omega\rho} & \omega i\rho & -\omega i\rho e^{-i\omega\rho} & -\omega^2\rho^2 & \omega^2\rho^2 e^{-i\omega\rho} \\ 1 & -e^{-i\omega^2\rho} & \omega^2 i\rho & -\omega^2 i\rho e^{-i\omega^2\rho} & -\omega\rho^2 & \omega\rho^2 e^{-i\omega^2\rho} \end{pmatrix} \begin{pmatrix} \widetilde{f}_2(\rho) \\ \widetilde{g}_2(\rho) \\ \widetilde{f}_1(\rho) \\ \widetilde{g}_1(\rho) \\ \widetilde{f}_0(\rho) \\ \widetilde{g}_0(\rho) \end{pmatrix}$$
$$= \begin{pmatrix} \hat{q}_0(\rho) \\ \hat{q}_0(\omega\rho) \\ \hat{q}_0(\omega^2\rho) \end{pmatrix} - e^{i\rho^3 T} \begin{pmatrix} \hat{q}_T(\rho) \\ \hat{q}_T(\omega\rho) \\ \hat{q}_T(\omega^2\rho) \end{pmatrix},$$

the right hand side of which is the right hand side of the reduced global relation (2.2.3) from the lemma. The left hand side must be simplified. Substituting the reduced boundary conditions (2.2.9) into the global relation gives

$$\begin{pmatrix} 1 & -e^{-i\rho} & i\rho & -i\rho e^{-i\rho} & -\rho^2 & \rho^2 e^{-i\rho} \\ 1 & -e^{-i\omega\rho} & \omega i\rho & -\omega i\rho e^{-i\omega\rho} & -\omega^2\rho^2 & \omega^2\rho^2 e^{-i\omega\rho} \\ 1 & -e^{-i\omega^2\rho} & \omega^2 i\rho & -\omega^2 i\rho e^{-i\omega^2\rho} & -\omega\rho^2 & \omega\rho^2 e^{-i\omega^2\rho} \end{pmatrix} \begin{pmatrix} \widetilde{f}_2(\rho) \\ \widetilde{g}_2(\rho) \\ 0 \\ \widetilde{g}_1(\rho) \\ 0 \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} \hat{q}_0(\rho) \\ \hat{q}_0(\omega\rho) \\ \hat{q}_0(\omega^2\rho) \end{pmatrix} - e^{i\rho^3 T} \begin{pmatrix} \hat{q}_T(\rho) \\ \hat{q}_T(\omega\rho) \\ \hat{q}_T(\omega^2\rho) \end{pmatrix},$$

hence

$$\begin{pmatrix} 1 & -e^{-i\rho} & -i\rho e^{-i\rho} \\ 1 & -e^{-i\omega\rho} & -\omega i\rho e^{-i\omega\rho} \\ 1 & -e^{-i\omega^2\rho} & -\omega^2 i\rho e^{-i\omega^2\rho} \end{pmatrix} \begin{pmatrix} \widetilde{f}_2(\rho) \\ \widetilde{g}_2(\rho) \\ \widetilde{g}_1(\rho) \end{pmatrix} = \begin{pmatrix} \hat{q}_0(\rho) \\ \hat{q}_0(\omega\rho) \\ \hat{q}_0(\omega^2\rho) \end{pmatrix} - e^{i\rho^3 T} \begin{pmatrix} \hat{q}_T(\rho) \\ \hat{q}_T(\omega\rho) \\ \hat{q}_T(\omega^2\rho) \end{pmatrix}, \tag{2.2.10}$$

which is the reduced global relation (2.2.3) from the lemma.

In this example we were able to simply discard three columns of the global relation matrix (2.1.19). This is because the boundary conditions are simple, hence the reduced boundary coefficient matrix is the zero matrix. This will not always happen but the reduced boundary conditions allow us to express n of the boundary functions in terms of the other n. This means we can incorporate the information from n columns of the global relation matrix into the other n columns, defining the reduced global relation matrix.

The reduced global relation matrix is a $n \times n$, rank n matrix so the reduced global relation is a system of n linear equations that may be solved using Cramer's rule (Theorem B.2) to give expressions for the entries in V in terms of the Fourier transforms of the intial datum and final function. In this example the entries in W have already been determined to be identically zero but if the reduced boundary coefficient matrix was not the zero matrix we could now write the

entries in W in terms of the entries in V hence in terms of the Fourier transforms of the initial datum and final function.

2.2.1.4. Proof of the main lemma

Using Example 2.16 as a model, we give the full proof of Lemma 2.14.

PROOF. Because A is in reduced row-echelon form it has the $n \times n$ identity matrix, I_n, as a submatrix. That submatrix is the one obtained by taking all n rows of A but only the columns which contain pivots. These are the columns of A indexed by $2n - j$, where $j \in J'$. Any such column multiplies the boundary function $f_{(j-1)/2}$ or $g_{j/2}$, for j odd or j even respectively, in the boundary conditions (2.1.32). The columns of A not appearing in the identity submatrix are those indexed by $2n - k$ for $k \in J$. Any such column multiplies the boundary function $f_{(k-1)/2}$ or $g_{k/2}$, for k odd or k even respectively, in the boundary conditions (2.1.32). The sequences $(J_j)_{j=1}^n$ and $(J'_j)_{j=1}^n$ simply ensure the entries in the vectors V and W appear in the correct order. We may now break the $n \times 2n$ matrix A into two square matrices, rewriting the boundary conditions in the form

$$I_n \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{pmatrix} + \widehat{A} \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix} = 0, \tag{2.2.11}$$

where

$$X_j = \begin{cases} f_{(J_j-1)/2} & J_j \text{ odd}, \\ g_{J_j/2} & J_j \text{ even}, \end{cases} \qquad Y_j = \begin{cases} f_{(J'_j-1)/2} & J'_j \text{ odd}, \\ g_{J'_j/2} & J'_j \text{ even}, \end{cases} \tag{2.2.12}$$

and $\widehat{A}$ is initially defined as the square matrix given by

$$\widehat{A}_{k\,j} = \begin{cases} \alpha_{k\,(J_j-1)/2} & J_j \text{ odd}, \\ \beta_{k\,J_j/2} & J_j \text{ even}. \end{cases}$$

If J_j is odd then there does not exist $k \in \{1, 2, \ldots, n\}$ such that $\alpha_{k\,(J_j-1)/2}$ is a pivot of A. Because the boundary conditions are non-Robin, this implies

$$\alpha_{k\,(J_j-1)/2} = 0 \quad \forall\, k \in \{1, 2, \ldots, n\}, \quad \forall\, j \text{ odd}.$$

If J_j is even then there does not exist $k \in \{1, 2, \ldots, n\}$ such that $\beta_{k\,J_j/2}$ is a pivot. If it happens that there does exist some $k \in \{1, 2, \ldots, n\}$ such that $\alpha_{k\,J_j/2}$ is a pivot, that is $J_j + 1 \in J'$ hence $J_j/2 \in \widehat{J}^+$, then $\beta_{k\,J_j/2}$ may be nonzero. In this case $k = \widehat{J}^+_{J_j/2}$. If $J_j/2 \in \widetilde{J}^+$ then, by Notation 2.12, $\beta_{k\,J_j/2} = 0$. Hence if J_j is even then

$$\widehat{A}_{k\,j} = \begin{cases} \beta_{k\,J_j/2} & k = \widehat{J}^+_{J_j/2} \\ 0 & \text{otherwise}. \end{cases}$$

Thus we may write

$$\widehat{A}_{k\,j} = \begin{cases} \beta_{k\,J_j/2} & J_j \text{ even and } k = \widehat{J}^+_{J_j/2} \\ 0 & \text{otherwise}, \end{cases}$$

the reduced boundary coefficient matrix defined by equation (2.2.6). By applying the transform (2.1.30) to each row of equation (2.2.11), making use of its linearity and observing that the transformation maps

$$X_j \mapsto V_j, \qquad Y_j \mapsto W_j,$$

we obtain the reduced boundary conditions (2.2.4).

The reduced boundary coefficient matrix has at most one non-zero entry on each row so the reduced boundary conditions (2.2.4) may be written as

$$\widetilde{g}_j(\rho) = 0 \qquad j \in \widehat{J}^-, \tag{2.2.13}$$

$$\widetilde{f}_j(\rho) = -\beta_{\widehat{J}_j^+ j}\widetilde{g}_j(\rho) \qquad j \in \widehat{J}^+. \tag{2.2.14}$$

Corollary 2.4 may be rewritten as a system of linear equations

$$\sum_{j=0}^{n-1} \omega^{(n-1-j)r} c_j(\rho)\widetilde{f}_j(\rho) - \sum_{j=0}^{n-1} \omega^{(n-1-j)r} e^{-i\omega^r\rho} c_j(\rho)\widetilde{g}_j(\rho) = \hat{q}_0(\omega^r\rho) - e^{a\rho^n T}\hat{q}_T(\omega^r\rho),$$

for $r \in \{0, 1, \ldots, n-1\}$. Using the fact $\widehat{J}^+ \cup \widetilde{J}^+ = \widehat{J}^- \cup \widetilde{J}^- = \{0, 1, \ldots, n-1\}$ we may split the sums on the left hand side to give

$$\begin{aligned}\sum_{j\in\widehat{J}^+} \omega^{(n-1-j)r} c_j(\rho)\widetilde{f}_j(\rho) + \sum_{j\in\widetilde{J}^+} \omega^{(n-1-j)r} c_j(\rho)\widetilde{f}_j(\rho) \\ - \sum_{j\in\widehat{J}^-} \omega^{(n-1-j)r} e^{-i\omega^r\rho} c_j(\rho)\widetilde{g}_j(\rho) - \sum_{j\in\widetilde{J}^-} \omega^{(n-1-j)r} e^{-i\omega^r\rho} c_j(\rho)\widetilde{g}_j(\rho) \\ = \hat{q}_0(\omega^r\rho) - e^{a\rho^n T}\hat{q}_T(\omega^r\rho),\end{aligned}$$

for $r \in \{0, 1, \ldots, n-1\}$. We may use equations (2.2.13) and (2.2.14) to simplify this to

$$\begin{aligned}\sum_{j\in\widehat{J}^+} \omega^{(n-1-j)r}(-\beta_{\widehat{J}_j^+ j}) c_j(\rho)\widetilde{g}_j(\rho) + \sum_{j\in\widetilde{J}^+} \omega^{(n-1-j)r} c_j(\rho)\widetilde{f}_j(\rho) \\ - 0 - \sum_{j\in\widetilde{J}^-} \omega^{(n-1-j)r} e^{-i\omega^r\rho} c_j(\rho)\widetilde{g}_j(\rho) = \hat{q}_0(\omega^r\rho) - e^{a\rho^n T}\hat{q}_T(\omega^r\rho),\end{aligned}$$

If $j \in \widehat{J}^+ \cap \widehat{J}^-$ then $\beta_{\widehat{J}_j^+ j} = 0$ as A is in reduced row-echelon form. If $j \in \widetilde{J}^- \setminus \widehat{J}^+ = \widetilde{J}^- \cap \widetilde{J}^+ \subseteq \widetilde{J}^+$ then $\beta_{\widehat{J}_j^+ j} = 0$ by Definition 2.12 so we may rewrite this as

$$\begin{aligned}\sum_{j\in\widetilde{J}^+} \omega^{(n-1-j)r} c_j(\rho)\widetilde{f}_j(\rho) - \sum_{j\in\widetilde{J}^-} \omega^{(n-1-j)r} \left(\beta_{\widehat{J}_j^+ j} + e^{-i\omega^r\rho}\right) c_j(\rho)\widetilde{g}_j(\rho) \\ = \hat{q}_0(\omega^r\rho) - e^{ai\rho^n T}\hat{q}_T(\omega^r\rho),\end{aligned}$$

for $r \in \{0, 1, \ldots, n-1\}$. Putting this system of linear equations into matrix form, we obtain the reduced global relation (2.2.3).

Because A is in reduced row-echelon form there cannot be two identical columns of $\mathcal{A}$. Further, if the same powers of ω appear in two columns of $\mathcal{A}$ then in the first column these powers of ω do not multiply exponential functions of ρ (type (1)) and in the second column they only multiply exponential functions of ρ without a constant (type (2)). Hence if there is a

column in $\mathcal{A}$ whose entries are given by powers of ω multiplied by the sum of exponential powers of ρ and a constant (type (3)) then that is the only column with those powers of ω.

Consider boundary conditions that are all specified at the end $x = 1$, that is the boundary coefficient matrix has the form

$$A' = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

Then $\widetilde{J}^+ = \{0, 1, \dots, n-1\}$ and $\widetilde{J}^+ = \emptyset$ so $\mathcal{A}'$ is a Vandermonde matrix which has rank n, as is shown in Section 1.4 of [**50**]. This matrix contains all columns of type (1) that may appear in any $\mathcal{A}$, so given any A the columns of the corresponding $\mathcal{A}$ of type (1) are linearly independent.

If instead the boundary conditions are all specified at $x = 0$, that is

$$A'' = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix},$$

then the determinant of $\mathcal{A}''$ is equal to the determinant of the same Vandermonde matrix. This matrix contains all columns of type (2) that may appear in any $\mathcal{A}$, so given any A the columns of the corresponding $\mathcal{A}$ of type (2) are linearly independent.

Other columns of any $\mathcal{A}$, that is a column of type (3), can be written as the sum of two columns: one of type (1) and one of type (2). But we have already established that neither of these may appear in $\mathcal{A}$ and neither may be written as a linear combination of columns that do appear in $\mathcal{A}$. This establishes that the column rank of any reduced global relation matrix is n. □

2.2.2. General boundary conditions

In this subsection we state and prove the form of Lemma 2.14 generalised to any set of linearly independent boundary conditions. We also show that, when the boundary conditions are homogeneous and non-Robin, Lemma 2.17 reduces to Lemma 2.14 so the notation is consistent between the lemmata.

LEMMA 2.17. *Let $q : [0, 1] \times [0, T] \to \mathbb{R}$ be a solution of the initial-boundary value problem specified by the partial differential equation* (2.1.1)*, the initial condition* (2.1.2) *and the boundary conditions* (2.1.32)*. Assume the matrix A, whose entries are defined by equation* (2.1.33)*, is in*

reduced row-echelon form. Then the vectors V and W from Notation 2.12 satisfy

$$\mathcal{A}(\rho)\begin{pmatrix} V_1(\rho) \\ V_2(\rho) \\ \vdots \\ V_n(\rho) \end{pmatrix} = U(\rho) - e^{a\rho^n T}\begin{pmatrix} \hat{q}_T(\rho) \\ \vdots \\ \hat{q}_T(\omega^{n-1}\rho) \end{pmatrix} \text{ and} \tag{2.2.15}$$

$$\begin{pmatrix} W_1(\rho) \\ W_2(\rho) \\ \vdots \\ W_n(\rho) \end{pmatrix} = \begin{pmatrix} \tilde{h}_1(\rho) \\ \tilde{h}_2(\rho) \\ \vdots \\ \tilde{h}_n(\rho) \end{pmatrix} - \widehat{A}\begin{pmatrix} V_1(\rho) \\ V_2(\rho) \\ \vdots \\ V_n(\rho) \end{pmatrix}, \tag{2.2.16}$$

where $\tilde{h}_j$ is the t-transform (2.1.30) *applied to the boundary data h_j,*

$$U(\rho) = (u(\rho,1), u(\rho,2), \dots, u(\rho,n))^{\mathrm{T}}, \tag{2.2.17}$$

$$u(\rho,k) = \hat{q}_0(\omega^{k-1}\rho) - \sum_{l\in\widehat{J}^+} c_l(\omega^{k-1}\rho)\tilde{h}_{\widehat{J}_l^+}(\rho) + e^{-i\omega^{k-1}\rho}\sum_{l\in\widehat{J}^-} c_l(\omega^{k-1}\rho)\tilde{h}_{\widehat{J}_l^-}(\rho), \tag{2.2.18}$$

$$\mathcal{A}_{k\,j}(\rho) = \begin{cases} c_{(J_j-1)/2}(\rho)\Bigg(\omega^{(n-1-[J_j-1]/2)(k-1)} & \\ \quad - \displaystyle\sum_{r\in\widehat{J}^+} \alpha_{\widehat{J}_r^+\,(J_j-1)/2}\omega^{(n-1-r)(k-1)}(i\rho)^{(J_j-1)/2-r} & J_j \text{ odd}, \\ \quad + e^{-i\omega^{k-1}\rho}\displaystyle\sum_{r\in\widehat{J}^-} \alpha_{\widehat{J}_r^-\,(J_j-1)/2}\omega^{(n-1-r)(k-1)}(i\rho)^{(J_j-1)/2-r}\Bigg) & \\ c_{J_j/2}(\rho)\Bigg(-\omega^{(n-1-J_j/2)(k-1)}e^{-i\omega^{k-1}\rho} & \\ \quad - \displaystyle\sum_{r\in\widehat{J}^+} \beta_{\widehat{J}_r^+\,J_j/2}\omega^{(n-1-r)(k-1)}(i\rho)^{J_j/2-r} & J_j \text{ even}, \\ \quad + e^{-i\omega^{k-1}\rho}\displaystyle\sum_{r\in\widehat{J}^-} \beta_{\widehat{J}_r^-\,J_j/2}\omega^{(n-1-r)(k-1)}(i\rho)^{J_j/2-r}\Bigg) & \end{cases} \tag{2.2.19}$$

$$\widehat{A}_{k\,j} = \begin{cases} \alpha_{k\,(J_j-1)/2} & J_j \text{ odd}, \\ \beta_{k\,J_j/2} & J_j \text{ even}. \end{cases} \tag{2.2.20}$$

Further, $\mathcal{A}$ is full rank.

Proof. The proof of the equation (2.2.16) begins in the same way as the proof of the reduced boundary conditions (2.2.4) in Lemma 2.14 but equation (2.2.11) must be replaced with

$$I_n\begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{pmatrix} + \widehat{A}\begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix} = \begin{pmatrix} h_1(t) \\ h_2(t) \\ \vdots \\ h_n(t) \end{pmatrix}, \tag{2.2.21}$$

where X and Y are given by equations (2.2.12) in the previous proof and the reduced boundary coefficient matrix, $\widehat{A}$, is defined by equation (2.2.20). Now the t-transform (2.1.30) may be applied to each line of equation (2.2.21) and by the linearity of the transform we obtain equation (2.2.16).

We may rewrite equation (2.2.16) in the form

$$\widetilde{f}_j(\rho) = \widetilde{h}_{\widehat{J}_j^+}(\rho) - \sum_{r\in\widetilde{J}^+} \alpha_{\widehat{J}_j^+\, r}\widetilde{f}_r(\rho) - \sum_{r\in\widetilde{J}^-} \beta_{\widehat{J}_j^+\, r}\widetilde{g}_r(\rho), \qquad \text{for } j \in \widehat{J}^+ \text{ and} \tag{2.2.22}$$

$$\widetilde{g}_j(\rho) = \widetilde{h}_{\widehat{J}_j^-}(\rho) - \sum_{r\in\widetilde{J}^+} \alpha_{\widehat{J}_j^-\, r}\widetilde{f}_r(\rho) - \sum_{r\in\widetilde{J}^-} \beta_{\widehat{J}_j^-\, r}\widetilde{g}_r(\rho), \qquad \text{for } j \in \widehat{J}^-. \tag{2.2.23}$$

Corollary 2.4 may be rewritten as the system of linear equations

$$\sum_{j=0}^{n-1} c_j(\rho)\omega^{(n-1-j)r}\widetilde{f}_j(\rho) - \sum_{j=0}^{n-1} e^{-i\omega^r\rho}c_j(\rho)\omega^{(n-1-j)r}\widetilde{g}_j(\rho) = \hat{q}_0(\omega^r\rho) - e^{a\rho^n T}\hat{q}_T(\omega^r\rho),$$

for $r \in \{0, 1, \ldots, n-1\}$. Using the fact $\widehat{J}^+ \cup \widetilde{J}^+ = \widehat{J}^- \cup \widetilde{J}^- = \{0, 1, \ldots, n-1\}$ we may split the sums on the left hand side to give

$$\begin{aligned}\sum_{j\in\widehat{J}^+} c_j(\rho)\omega^{(n-1-j)r}\widetilde{f}_j(\rho) + \sum_{j\in\widetilde{J}^+} c_j(\rho)\omega^{(n-1-j)r}\widetilde{f}_j(\rho)& \\ - \sum_{j\in\widehat{J}^-} e^{-i\omega^r\rho}c_j(\rho)\omega^{(n-1-j)r}\widetilde{g}_j(\rho) - \sum_{j\in\widetilde{J}^-} e^{-i\omega^r\rho}c_j(\rho)\omega^{(n-1-j)r}\widetilde{g}_j(\rho)& \\ = \hat{q}_0(\omega^r\rho) - e^{a\rho^n T}\hat{q}_T(\omega^r\rho),&\end{aligned}$$

for $r \in \{0, 1, \ldots, n-1\}$. Using equations (2.2.22) and (2.2.23) we obtain

$$\begin{aligned}\sum_{j\in\widehat{J}^+} c_j(\rho)\omega^{(n-1-j)r}\left(\widetilde{h}_{\widehat{J}_j^+}(\rho) - \sum_{k\in\widetilde{J}^+}\alpha_{\widehat{J}_j^+\, k}\widetilde{f}_k(\rho) - \sum_{k\in\widetilde{J}^-}\beta_{\widehat{J}_j^+\, k}\widetilde{g}_k(\rho)\right) + \sum_{j\in\widetilde{J}^+} c_j(\rho)\omega^{(n-1-j)r}\widetilde{f}_j(\rho)& \\ - \sum_{j\in\widehat{J}^-} e^{-i\omega^r\rho}c_j(\rho)\omega^{(n-1-j)r}\left(\widetilde{h}_{\widehat{J}_j^-}(\rho) - \sum_{k\in\widetilde{J}^+}\alpha_{\widehat{J}_j^-\, k}\widetilde{f}_k(\rho) - \sum_{k\in\widetilde{J}^-}\beta_{\widehat{J}_j^-\, k}\widetilde{g}_k(\rho)\right)& \\ - \sum_{j\in\widetilde{J}^-} e^{-i\omega^r\rho}c_j(\rho)\omega^{(n-1-j)r}\widetilde{g}_j(\rho) = \hat{q}_0(\omega^r\rho) - e^{a\rho^n T}\hat{q}_T(\omega^r\rho),&\end{aligned}$$

hence

$$\sum_{j\in\widetilde{J}^+}\widetilde{f}_j(\rho)\left[c_j(\rho)\omega^{(n-1-j)r}-\sum_{k\in\widehat{J}^+}\alpha_{\widehat{J}_k^+\,j}c_k(\rho)\omega^{(n-1-k)r}+e^{-i\omega^r\rho}\sum_{k\in\widehat{J}^-}\alpha_{\widehat{J}_k^-\,j}c_k(\rho)\omega^{(n-1-k)r}\right]$$
$$-\sum_{j\in\widetilde{J}^-}\widetilde{g}_j(\rho)\left[e^{-i\omega^r\rho}c_j(\rho)\omega^{(n-1-j)r}+\sum_{k\in\widehat{J}^+}\beta_{\widehat{J}_k^+\,j}c_k(\rho)\omega^{(n-1-k)r}\right.$$
$$\left.-e^{-i\omega^r\rho}\sum_{k\in\widehat{J}^-}\beta_{\widehat{J}_k^-\,j}c_k(\rho)\omega^{(n-1-k)r}\right]$$
$$=\hat{q}_0(\omega^r\rho)-\sum_{j\in\widehat{J}^+}c_j(\omega^r\rho)\widetilde{h}_{\widehat{J}_j^+}(\rho)+\sum_{j\in\widehat{J}^-}e^{-i\omega^r\rho}c_j(\omega^r\rho)\widetilde{h}_{\widehat{J}_j^-}(\rho)-e^{a\rho^nT}\hat{q}_T(\omega^r\rho),$$

for $r\in\{0,1,\ldots,n-1\}$. Taking a factor of $c_j(\rho)$ out of each square bracket and using the identity

$$\frac{c_k(\rho)}{c_j(\rho)}=(i\rho)^{j-k},$$

we establish

$$\sum_{j\in\widetilde{J}^+}\widetilde{f}_j(\rho)c_j(\rho)\left[\omega^{(n-1-j)r}-\sum_{k\in\widehat{J}^+}\alpha_{\widehat{J}_k^+\,j}(i\rho)^{j-k}\omega^{(n-1-k)r}+e^{-i\omega^r\rho}\sum_{k\in\widehat{J}^-}\alpha_{\widehat{J}_k^-\,j}(i\rho)^{j-k}\omega^{(n-1-k)r}\right]$$
$$-\sum_{j\in\widetilde{J}^-}\widetilde{g}_j(\rho)c_j(\rho)\left[e^{-i\omega^r\rho}\omega^{(n-1-j)r}+\sum_{k\in\widehat{J}^+}\beta_{\widehat{J}_k^+\,j}(i\rho)^{j-k}\omega^{(n-1-k)r}\right.$$
$$\left.-e^{-i\omega^r\rho}\sum_{k\in\widehat{J}^-}\beta_{\widehat{J}_k^-\,j}(i\rho)^{j-k}\omega^{(n-1-k)r}\right]$$
$$=\hat{q}_0(\omega^r\rho)-\sum_{j\in\widehat{J}^+}c_j(\omega^r\rho)\widetilde{h}_{\widehat{J}_j^+}(\rho)+\sum_{j\in\widehat{J}^-}e^{-i\omega^r\rho}c_j(\omega^r\rho)\widetilde{h}_{\widehat{J}_j^-}(\rho)-e^{a\rho^nT}\hat{q}_T(\omega^r\rho),$$

for $r\in\{0,1,\ldots,n-1\}$. When put into matrix form this gives equation (2.2.15).

We now turn our attention to showing that the reduced global relation matrix is full rank. Each entry of $\mathcal{A}$ is made up of a sum of two terms: a polynomial and an exponential multiplied by a polynomial. If we were to try to write one entry as a linear combination of other entries from the same row then the exponential part would have to come from only the exponential parts of other cells or, by multiplying by an exponential, it could come from the polynomial parts. However, if another column was multiplied by a particular exponential then only one entry would have a purely polynomial term because the exponentials that appear in $\mathcal{A}$ depend upon the row, rather than the column. This means that we can split the reduced global matrix

into two matrices

$$X_{k\,j} = \begin{cases} c_{(J_j-1)/2}(\rho)\omega^{(n-1-[J_j-1]/2)(k-1)} - \displaystyle\sum_{r\in\widehat{J}^+} \alpha_{\widehat{J}_r^+\,(J_j-1)/2} c_r(\rho)\omega^{(n-1-r)(k-1)} & J_j \text{ odd}, \\ -\displaystyle\sum_{r\in\widehat{J}^+} \beta_{\widehat{J}_r^+\,J_j/2} c_r(\rho)\omega^{(n-1-r)(k-1)} & J_j \text{ even}, \end{cases}$$

$$Y_{k\,j} = \begin{cases} \displaystyle\sum_{r\in\widehat{J}^-} \alpha_{\widehat{J}_r^-\,(J_j-1)/2} c_r(\rho)\omega^{(n-1-r)(k-1)} & J_j \text{ odd}, \\ c_{J_j/2}(\rho)\omega^{(n-1-J_j/2)(k-1)} + \displaystyle\sum_{r\in\widehat{J}^-} \beta_{\widehat{J}_r^-\,J_j/2} c_r(\rho)\omega^{(n-1-r)(k-1)} & J_j \text{ even}, \end{cases}$$

and observe that if a column of $\mathcal{A}$ may be written as a linear combination of other columns of $\mathcal{A}$ then either

- there exists j with J_j odd such that the j^{th} column of X can be written as a linear combination of other columns of X or
- there exists j with J_j even such that the j^{th} column of Y can be written as a linear combination of other columns of Y.[3]

We concentrate on matrix X noting that the argument is the same for Y. In X, for each $j \in \widetilde{J}^+$ there is precisely one column of the form

$$c_j(\rho)\omega^{(n-1-j)(k-1)} - \sum_{r\in\widehat{J}^+} \alpha_{\widehat{J}_r^+\,j} c_r(\rho)\omega^{(n-1-r)(k-1)}, \quad k \in \{1,2,\ldots,n\} \tag{2.2.24}$$

and for each $j \in \widetilde{J}^-$ there is precisely one column of the form

$$-\sum_{r\in\widehat{J}^+} \beta_{\widehat{J}_r^+\,j} c_r(\rho)\omega^{(n-1-r)(k-1)}, \quad k \in \{1,2,\ldots,n\}. \tag{2.2.25}$$

We choose a particular $p \in \widetilde{J}^+$ and show that the column

$$c_p(\rho)\omega^{(n-1-p)(k-1)} - \sum_{r\in\widehat{J}^+} \alpha_{\widehat{J}_r^+\,p} c_r(\rho)\omega^{(n-1-r)(k-1)} \quad k \in \{1,2,\ldots,n\} \tag{2.2.26}$$

cannot be expressed as a linear combination of columns of type (2.2.24) for $j \in \widetilde{J}^+ \setminus \{p\}$ and columns of type (2.2.25) for $j \in \widetilde{J}^-$. As column (2.2.26) appears only once in X the term $c_p(\rho)\omega^{(n-1-p)(k-1)}$ cannot be expressed as a linear combination of the terms $c_j(\rho)\omega^{(n-1-j)(k-1)}$. As $p \in \widetilde{J}^+$, $p \notin \widehat{J}^+$ so the term $c_p(\rho)\omega^{(n-1-p)(k-1)}$ cannot be expressed as as a linear combination of the sum terms in columns (2.2.24) and (2.2.25). This is because to do so would require multiplying columns by some factor of the form $\omega^{(j-p)(k-1)}$ which is impossible as k is the row index.

This establishes that there does not exist a j with J_j odd such that the j^{th} column of X can be written as a linear combination of other columns of X. The same argument proves that there does not exist a j with J_j even such that the j^{th} column of Y can be written as a linear combination of other columns of Y. Hence the column rank of $\mathcal{A}$ is n. □

[3]Note we only claim these conditions are necessary, not sufficient.

Note that the proof of Lemma 2.14 initially defines $\widehat{A}$ in the same way as Lemma 2.17 but then establishes the simpler representation of the boundary coefficient matrix (2.2.6) so these definitions are equivalent for non-Robin boundary conditions. Similarly, if J_j is odd then the column of A whose entries are $\alpha_{r\,(J_j-1)/2}$ for $r \in \{1, 2, \ldots, n\}$ contains no pivots. Hence, if the boundary conditions are non-Robin and J_j is odd, the definition of $\mathcal{A}_{k\,j}$ agrees between the two lemmata. If J_j is even then the column of A whose entries are $\beta_{r\,J_j/2}$ for $r \in \{1, 2, \ldots, n\}$ contains no pivots. Hence, if the boundary conditions are non-Robin and J_j is odd at most one of the entries in this column can be non-zero, that is the one in the (unique and possibly existent) row r for which $\alpha_{r\,J_j/2}$ is a pivot, that is the entry $\beta_{\widehat{J}^+_{J_j/2}\,J_j/2}$. This establishes that the two definitions (2.2.5) and (2.2.19) are equivalent for non-Robin boundary conditions. Clearly the vector $U(\rho)$ reduces to the vector $(\hat{q}_0(\rho), \hat{q}_0(\omega\rho), \ldots, \hat{q}_0(\omega^{n-1}\rho))^T$ and equation (2.2.16) simplifies to the reduced boundary conditions (2.2.4) for homogeneous boundary conditions.

This proves that if the boundary conditions are homogeneous and non-Robin then the two lemmata in this chapter are equivalent. Thus we extend the use of the terms *reduced boundary conditions*, *reduced global relation* and their associated matrices from Definition 2.15 to their equivalents in Lemma 2.17.

2.3. An explicit integral representation

The achievement of Lemmata 2.14 and 2.17 is that we may now write the solution to an initial-boundary value problem in terms of the boundary data and initial data with only one unknown, the final function. We formalise this expression in Theorem 2.20. Once again, this is a simple application of linear algebra but requires the development of some notation. To help illustrate the notation and derivation we give an initial example.

Example 2.18. We continue with Example 2.16. The reduced global relation (2.2.10) is a system of 3 equations in the 3 boundary functions so it may be solved using Cramer's rule B.2. In order to do so, we define

$$\widehat{\zeta}_1(\rho) = \det \begin{pmatrix} \hat{q}_0(\rho) & -e^{-i\rho} & -i\rho e^{-i\rho} \\ \hat{q}_0(\omega\rho) & -e^{-i\omega\rho} & -\omega i\rho e^{-i\omega\rho} \\ \hat{q}_0(\omega^2\rho) & -e^{-i\omega^2\rho} & -\omega^2 i\rho e^{-i\omega^2\rho} \end{pmatrix},$$

$$\widehat{\zeta}_2(\rho) = \det \begin{pmatrix} 1 & \hat{q}_0(\rho) & -i\rho e^{-i\rho} \\ 1 & \hat{q}_0(\omega\rho) & -\omega i\rho e^{-i\omega\rho} \\ 1 & \hat{q}_0(\omega^2\rho) & -\omega^2 i\rho e^{-i\omega^2\rho} \end{pmatrix}, \qquad \widehat{\zeta}_3(\rho) = \det \begin{pmatrix} 1 & -e^{-i\rho} & \hat{q}_0(\rho) \\ 1 & -e^{-i\omega\rho} & \hat{q}_0(\omega\rho) \\ 1 & -e^{-i\omega^2\rho} & \hat{q}_0(\omega^2\rho) \end{pmatrix},$$

$$\widehat{\eta}_1(\rho) = \det \begin{pmatrix} \hat{q}_T(\rho) & -e^{-i\rho} & -i\rho e^{-i\rho} \\ \hat{q}_T(\omega\rho) & -e^{-i\omega\rho} & -\omega i\rho e^{-i\omega\rho} \\ \hat{q}_T(\omega^2\rho) & -e^{-i\omega^2\rho} & -\omega^2 i\rho e^{-i\omega^2\rho} \end{pmatrix},$$

$$\widehat{\eta}_2(\rho) = \det \begin{pmatrix} 1 & \hat{q}_T(\rho) & -i\rho e^{-i\rho} \\ 1 & \hat{q}_T(\omega\rho) & -\omega i\rho e^{-i\omega\rho} \\ 1 & \hat{q}_T(\omega^2\rho) & -\omega^2 i\rho e^{-i\omega^2\rho} \end{pmatrix}, \qquad \widehat{\eta}_3(\rho) = \det \begin{pmatrix} 1 & -e^{-i\rho} & \hat{q}_T(\rho) \\ 1 & -e^{-i\omega\rho} & \hat{q}_T(\omega\rho) \\ 1 & -e^{-i\omega^2\rho} & \hat{q}_T(\omega^2\rho) \end{pmatrix}$$

and

$$\Delta_{\mathrm{PDE}}(\rho) = \det \begin{pmatrix} 1 & -e^{-i\rho} & -i\rho e^{-i\rho} \\ 1 & -e^{-i\omega\rho} & -\omega i\rho e^{-i\omega\rho} \\ 1 & -e^{-i\omega^2\rho} & -\omega^2 i\rho e^{-i\omega^2\rho} \end{pmatrix}$$

in accordance with the following Definition 2.19. Indeed η_j may be found from ζ_j by replacing $\hat{q}_0$ with $\hat{q}_T$. Applying Theorem B.2 to the reduced global relation (2.2.10) and observing that the reduced boundary coefficient matrix is 0_3 we obtain

$$\begin{aligned} \widetilde{f}_2(\rho) &= \frac{\widehat{\zeta}_1(\rho) - e^{i\rho^3 T}\widehat{\eta}_1(\rho)}{\Delta_{\mathrm{PDE}}(\rho)}, \\ \widetilde{g}_2(\rho) &= \frac{\widehat{\zeta}_2(\rho) - e^{i\rho^3 T}\widehat{\eta}_2(\rho)}{\Delta_{\mathrm{PDE}}(\rho)}, \\ \widetilde{g}_1(\rho) &= i\rho\frac{\widehat{\zeta}_3(\rho) - e^{i\rho^3 T}\widehat{\eta}_3(\rho)}{\Delta_{\mathrm{PDE}}(\rho)}, \\ \widetilde{f}_1(\rho) &= \widetilde{f}_0(\rho) = \widetilde{g}_0(\rho) = 0. \end{aligned}$$

Substituting the above expressions and equation (2.2.9) into equation (2.1.5) we obtain

$$\begin{aligned} 2\pi q(x,t) = \int_{\mathbb{R}} e^{i\rho x - i\rho^3 t}\hat{q}_0(\rho)\,\mathrm{d}\rho - \int_{\partial D^+} e^{i\rho x - i\rho^3 t}\frac{\widehat{\zeta}_1(\rho) - e^{i\rho^3 T}\widehat{\eta}_1(\rho)}{\Delta_{\mathrm{PDE}}(\rho)}\,\mathrm{d}\rho \\ - \int_{\partial D^-} e^{i\rho(x-1) - i\rho^3 t}\frac{\left(\widehat{\zeta}_2(\rho) - e^{i\rho^3 T}\widehat{\eta}_2(\rho)\right) + i\rho\left(\widehat{\zeta}_3(\rho) - e^{i\rho^3 T}\widehat{\eta}_3(\rho)\right)}{\Delta_{\mathrm{PDE}}(\rho)}\,\mathrm{d}\rho. \end{aligned}$$

The right hand side of this equation is a linear combination of contour integrals of transforms of the inital datum and the final function. The initial datum is a known quantity, specified in the initial-boundary value problem; the final function is unknown. It will be shown in Chapter 3 using analyticity considerations that the final function does not contribute to this representation.

Following Example 2.18, we define the functions ζ_j and η_j and then derive and state the main Theorem 2.20 of the present section. Finally, Example 2.21 shows how the boundary functions may be calculated for a particular case.

Definition 2.19. *For $j \in \{1, 2, \ldots, n\}$, let $\widehat{\zeta}_j(\rho)$ be the determinant of the matrix obtained by replacing the j^{th} column of the reduced global relation matrix with the vector $U(\rho)$ and $\widehat{\eta}_j(\rho)$ be the determinant of the matrix obtained by replacing the j^{th} column of the reduced global relation matrix with the vector $(\hat{q}_T(\rho), \hat{q}_T(\omega\rho), \ldots, \hat{q}_T(\omega^{n-1}\rho))^{\mathrm{T}}$ for $j \in \{1, 2, \ldots, n\}$ and $\rho \in \mathbb{C}$. For $j \in \{n+1, n+2, \ldots, 2n\}$ Let*

$$\begin{aligned} \widehat{\zeta}_j(\rho) &= \widetilde{h}_{j-n}(\rho) - \sum_{k=1}^{n} \widehat{A}_{j-n\,k}\widehat{\zeta}_k(\rho), \\ \widehat{\eta}_j(\rho) &= \widetilde{h}_{j-n}(\rho) - \sum_{k=1}^{n} \widehat{A}_{j-n\,k}\widehat{\eta}_k(\rho). \end{aligned} \tag{2.3.1}$$

We define further

$$\zeta_j(\rho) = \begin{cases} c_{(J_j-1)/2}(\rho)\widehat{\zeta}_j(\rho) & J_j \text{ odd}, \\ c_{J_j/2}(\rho)\widehat{\zeta}_j(\rho) & J_j \text{ even}, \\ c_{(J'_{j-n}-1)/2}(\rho)\widehat{\zeta}_j(\rho) & J'_{j-n} \text{ odd}, \\ c_{J'_{j-n}/2}(\rho)\widehat{\zeta}_j(\rho) & J'_{j-n} \text{ even}, \end{cases} \qquad \eta_j(\rho) = \begin{cases} c_{(J_j-1)/2}(\rho)\widehat{\eta}_j(\rho) & J_j \text{ odd}, \\ c_{J_j/2}(\rho)\widehat{\eta}_j(\rho) & J_j \text{ even}, \\ c_{(J'_{j-n}-1)/2}(\rho)\widehat{\eta}_j(\rho) & J'_{j-n} \text{ odd}, \\ c_{J'_{j-n}/2}(\rho)\widehat{\eta}_j(\rho) & J'_{j-n} \text{ even}, \end{cases} \tag{2.3.2}$$

for $\rho \in \mathbb{C}$, where the monomials c_k are defined in equations (2.1.5)*. Define the index sets*

$$J^+ = \{j : J_j \text{ odd}\} \cup \{n + j : J'_j \text{ odd}\},$$
$$J^- = \{j : J_j \text{ even}\} \cup \{n + j : J'_j \text{ even}\}.$$

Also let

$$\Delta_{\mathrm{PDE}}(\rho) = \det \mathcal{A}(\rho), \qquad \rho \in \mathbb{C}. \tag{2.3.3}$$

Note that, for homogeneous boundary conditions, the η_j are simply the ζ_j with $\hat{q}_T$ replacing with $\hat{q}_0$.

Now by Lemma 2.17 and Cramer's rule, Theorem B.2, we may obtain expressions for the boundary functions:

$$\frac{\widehat{\zeta}_j(\rho) - e^{a\rho^n T}\widehat{\eta}_j(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} = \begin{cases} \widetilde{f}_{(J_j-1)/2}(\rho) & J_j \text{ odd}, \\ \widetilde{g}_{J_j/2}(\rho) & J_j \text{ even}, \\ \widetilde{f}_{(J'_{j-n}-1)/2}(\rho) & J'_{j-n} \text{ odd}, \\ \widetilde{g}_{J'_{j-n}/2}(\rho) & J'_{j-n} \text{ even}, \end{cases}$$

hence

$$\frac{\zeta_j(\rho) - e^{a\rho^n T}\eta_j(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} = \begin{cases} c_{(J_j-1)/2}(\rho)\widetilde{f}_{(J_j-1)/2}(\rho) & J_j \text{ odd}, \\ c_{J_j/2}(\rho)\widetilde{g}_{J_j/2}(\rho) & J_j \text{ even}, \\ c_{(J'_{j-n}-1)/2}(\rho)\widetilde{f}_{(J'_{j-n}-1)/2}(\rho) & J'_{j-n} \text{ odd}, \\ c_{J'_{j-n}/2}(\rho)\widetilde{g}_{J'_{j-n}/2}(\rho) & J'_{j-n} \text{ even}, \end{cases} \tag{2.3.4}$$

and

$$\sum_{j=0}^{n-1} c_j(\rho)\widetilde{f}_j(\rho) = \sum_{j\in J^+} \frac{\zeta_j(\rho) - e^{a\rho^n T}\eta_j(\rho)}{\Delta_{\mathrm{PDE}}(\rho)},$$
$$\sum_{j=0}^{n-1} c_j(\rho)\widetilde{g}_j(\rho) = \sum_{j\in J^-} \frac{\zeta_j(\rho) - e^{a\rho^n T}\eta_j(\rho)}{\Delta_{\mathrm{PDE}}(\rho)}.$$

This establishes the following theorem, the main result of this chapter.

THEOREM 2.20. *Assume that there exists a unique $q : [0,1] \times [0,T] \to \mathbb{R}$ solving the initial-boundary value problem specified by the partial differential equation* (2.1.1)*, the initial condition* (2.1.2) *and the boundary conditions* (2.1.32)*. Then $q(x,t)$ may be expressed in terms of*

contour integrals of transforms of the boundary data, initial datum and final function as follows:

$$2\pi q(x,t) = \int_{\mathbb{R}} e^{i\rho x - a\rho^n t}\hat{q}_0(\rho)\,\mathrm{d}\rho - \int_{\partial D^+} e^{i\rho x - a\rho^n t}\sum_{j\in J^+}\frac{\zeta_j(\rho) - e^{a\rho^n T}\eta_j(\rho)}{\Delta_{\mathrm{PDE}}(\rho)}\,\mathrm{d}\rho$$
$$- \int_{\partial D^-} e^{i\rho(x-1) - a\rho^n t}\sum_{j\in J^-}\frac{\zeta_j(\rho) - e^{a\rho^n T}\eta_j(\rho)}{\Delta_{\mathrm{PDE}}(\rho)}\,\mathrm{d}\rho, \quad (2.3.5)$$

where $D^{\pm} = \mathbb{C}^{\pm} \cap \{\rho \in \mathbb{C} : \mathrm{Re}(a\rho^n) < 0\}$.

Example 2.21. We give another example to illustrate Definition 2.19 and Theorem 2.20. The boundary value problem we consider is the same is in Example 2.13; $n = 3$, $a = i$ and the boundary conditions are given by equation (2.1.32) with $h_j = 0$ and a boundary coefficient matrix

$$A = \begin{pmatrix} 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

This establishes:

$$c_2(\rho) = 1, \qquad c_1(\rho) = i\rho, \qquad c_0(\rho) = -\rho^2,$$
$$W(\rho) = \begin{pmatrix} \widetilde{f}_1(\rho) \\ \widetilde{f}_0(\rho) \\ \widetilde{g}_0(\rho) \end{pmatrix}, \qquad V(\rho) = \begin{pmatrix} \widetilde{f}_2(\rho) \\ \widetilde{g}_2(\rho) \\ \widetilde{g}_1(\rho) \end{pmatrix},$$
$$\widehat{J}^+ = \{0,1\}, \qquad \widehat{J}^- = \{0\}, \qquad \widetilde{J}^+ = \{2\}, \qquad \widetilde{J}^- = \{1,2\},$$
$$\mathcal{A}(\rho) = \begin{pmatrix} 1 & -e^{-i\rho} & -i\rho(e^{-i\rho}-1) \\ 1 & -e^{-i\omega\rho} & -i\omega\rho(e^{-i\omega\rho}-1) \\ 1 & -e^{-i\omega^2\rho} & -i\omega^2\rho(e^{-i\omega^2\rho}-1) \end{pmatrix} \quad \text{and} \quad \widehat{A} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence, by Definition 2.19,

$$\begin{aligned}
\Delta_{\mathrm{PDE}}(\rho) &= \det \mathcal{A}(\rho) \\
&= \sqrt{3}\rho[(e^{i\rho} + e^{-i\rho}) + \omega(e^{i\omega\rho} + e^{-i\omega\rho}) + \omega^2(e^{i\omega^2\rho} + e^{-i\omega^2\rho})], \\
\widehat{\zeta}_1(\rho) &= i\rho[\hat{q}_0(\rho)(\omega^2 e^{-i\omega\rho}(e^{-i\omega^2\rho}-1) - \omega e^{-i\omega^2\rho}(e^{-i\omega\rho}-1)) \\
&\qquad \hat{q}_0(\omega\rho)(e^{-i\omega^2\rho}(e^{-i\rho}-1) - \omega^2 e^{-i\rho}(e^{-i\omega^2\rho}-1)) \\
&\qquad \hat{q}_0(\omega^2\rho)(\omega e^{-i\rho}(e^{-i\omega\rho}-1) - e^{-i\omega\rho}(e^{-i\rho}-1))], \\
\widehat{\zeta}_2(\rho) &= -i\rho[\omega^2\hat{q}_0(\omega\rho)(e^{-i\omega^2\rho}-1) - \omega\hat{q}_0(\omega^2\rho)(e^{-i\omega\rho}-1) \\
&\qquad \hat{q}_0(\omega^2\rho)(e^{-i\rho}-1) - \omega^2\hat{q}_0(\rho)(e^{-i\omega^2\rho}-1) \\
&\qquad \omega\hat{q}_0(\rho)(e^{-i\omega\rho}-1) - \hat{q}_0(\omega\rho)(e^{-i\rho}-1)], \\
\widehat{\zeta}_3(\rho) &= -[e^{-i\omega\rho}\hat{q}_0(\omega^2\rho) - e^{-i\omega^2\rho}\hat{q}_0(\omega\rho) \\
&\qquad e^{-i\omega^2\rho}\hat{q}_0(\rho) - e^{-i\rho}\hat{q}_0(\omega^2\rho) \\
&\qquad e^{-i\rho}\hat{q}_0(\omega\rho) - e^{-i\omega\rho}\hat{q}_0(\rho)], \\
\widehat{\zeta}_4 &= \widehat{\zeta}_3 \qquad \text{and} \qquad \widehat{\zeta}_5 = \widehat{\zeta}_6 = 0.
\end{aligned}$$

Further,

$$\begin{aligned} \zeta_1(\rho) &= \widehat{\zeta_1}(\rho), & \zeta_3(\rho) &= \zeta_4(\rho) = i\rho\widehat{\zeta_3}(\rho), \\ \zeta_2(\rho) &= \widehat{\zeta_2}(\rho), & \zeta_5(\rho) &= \zeta_6(\rho) = 0, \\ J^+ &= \{1,4,5\}, & J^- &= \{2,3,6\}, \end{aligned}$$

so that

$$\sum_{j\in J^+} \zeta_j(\rho) = \rho[\hat{q}_0(\rho)(e^{i\rho} + \omega e^{-i\omega\rho} + \omega^2 e^{-i\omega^2\rho}) + \omega\hat{q}_0(\omega\rho)(e^{i\omega\rho} - e^{-i\rho}) + \omega^2\hat{q}_0(\omega^2\rho)(e^{i\omega^2\rho} - e^{-i\rho})],$$

$$\sum_{j\in J^-} \zeta_j(\rho) = \rho[\hat{q}_0(\rho)(1 + \omega^2 e^{-i\omega\rho} + \omega e^{-i\omega^2\rho}) + \omega\hat{q}_0(\omega\rho)(1 - e^{-i\omega^2\rho}) + \omega^2\hat{q}_0(\omega^2\rho)(1 - e^{-i\omega\rho})].$$

It should be noted that Theorem 2.20 does not give the solution in terms of only the known data. The functions η_j are defined in terms of $q(x,T)$, the solution at final time. Section 3.2 contains a proof that these functions do not make a contribution to expression (2.3.5) for well-posed problems.

CHAPTER 3

Series representations and well-posedness

While Chapter 2 is concerned with deriving an integral representation for the solution to a well-posed initial-boundary value problem, the present chapter is devoted to investigating well-posedness of such a problem and the related question of finding a discrete series representation of its solution. We continue in the general setting of Chapter 2 with a partial differential equation (2.1.1) specified by its order $n \geqslant 2$ and the parameter a. The form of our results depends upon the value of a; we present them in the three cases $a = i$, $a = -i$ and $\mathrm{Re}(a) > 0$.

THEOREM 3.1. *Let the homogeneous initial-boundary value problem* (2.1.1)–(2.1.3) *obey Assumptions 3.2 and 3.3. Then the solution to the problem may be written in series form as follows:*

$$q(x,t) = \frac{i}{2} \sum_{\substack{k \in K^+ \\ \cup K^{D^+} \cup K^{E^+} \\ \cup K^{\mathbb{R}} \cup \{0\}}} \operatorname*{Res}_{\rho=\sigma_k} \frac{P(\rho;x,t)}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j \in J^+} \zeta_j(\rho) + \frac{i}{2} \sum_{\substack{k \in K^- \\ \cup K^{D^-} \cup K^{E^-}}} \operatorname*{Res}_{\rho=\sigma_k} \frac{\widehat{P}(\rho;x,t)}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j \in J^-} \zeta_j(\rho). \tag{3.0.6}$$

If n is odd and $a = -i$,

$$q(x,t) = \frac{i}{2} \sum_{\substack{k \in K^+ \\ \cup K^{D^+} \cup K^{E^+}}} \operatorname*{Res}_{\rho=\sigma_k} \frac{P(\rho;x,t)}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j \in J^+} \zeta_j(\rho) + \frac{i}{2} \sum_{\substack{k \in K^- \\ \cup K^{D^-} \cup K^{E^-} \\ \cup K^{\mathbb{R}} \cup \{0\}}} \operatorname*{Res}_{\rho=\sigma_k} \frac{\widehat{P}(\rho;x,t)}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j \in J^-} \zeta_j(\rho). \tag{3.0.7}$$

If n is even and $a = \pm i$,

$$q(x,t) = \frac{i}{2} \sum_{\substack{k \in K^+ \\ \cup K^{D^+} \cup K^{E^+} \\ \cup K_E^D}} \operatorname*{Res}_{\rho=\sigma_k} \frac{P(\rho;x,t)}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j \in J^+} \zeta_j(\rho) + \frac{i}{2} \sum_{\substack{k \in K^- \\ \cup K^{D^-} \cup K^{E^-} \\ \cup K_D^E}} \operatorname*{Res}_{\rho=\sigma_k} \frac{\widehat{P}(\rho;x,t)}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j \in J^-} \zeta_j(\rho) + \frac{i}{4} \operatorname*{Res}_{\rho=0} \frac{1}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j \in J^+ \cup J^-} \zeta_j(\rho). \tag{3.0.8}$$

If n is even and $a = e^{i\theta}$ for some $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$,

$$q(x,t) = \frac{i}{2} \sum_{\substack{k \in K^+ \\ \cup K^{D^+} \cup K^{E^+} \\ \cup K^{\mathbb{R}} \cup \{0\}}} \operatorname*{Res}_{\rho=\sigma_k} \frac{P(\rho;x,t)}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j \in J^+} \zeta_j(\rho) + \frac{i}{2} \sum_{\substack{k \in K^- \\ \cup K^{D^-} \cup K^{E^-}}} \operatorname*{Res}_{\rho=\sigma_k} \frac{\widehat{P}(\rho;x,t)}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j \in J^-} \zeta_j(\rho). \tag{3.0.9}$$

The functions $P, \widehat{P}$ are given in Definition 3.4. The functions $\Delta_{\mathrm{PDE}}, \zeta_j$ and the index sets $J^{\pm}$ are given in Definition 2.19 and the points σ_k are the zeros of Δ_{PDE}.

The multiple representations in Theorem 3.1 are equivalent for any permissible choice of n and a. This is shown in [**59**].

Theorem 3.1 is the result of the first section of this chapter. Indeed Section 3.1 is concerned entirely with the definition of the index sets, functions and assumptions used to state Theorem 3.1 and the derivation of this result, with one computational section of the proof presented in the Appendix. Well-posedness of the initial-boundary value problem is not discussed and the

assumptions, once stated, are considered to hold throughout Section 3.1. Particular examples are not discussed as, for any but the most trivial examples, a lengthy calculation of bounds on zeros of certain exponential polynomials is required in order to perform any meaningful simplification of the general definitions or argument. Instead, the definitions and subsequent derivation are broken up depending upon the value of the parameter a.

The theme of the two assumptions is that certain meromorphic functions, given in Definition 2.19, decay as $\rho \to \infty$ from within certain sectors of the complex plane. The domains of interest are formally defined in Definition 3.9 but they are $D^{\pm}$ and $E^{\pm}$ with neighbourhoods of the zeros of Δ_{PDE} removed.

ASSUMPTION 3.2. We assume the initial-boundary value problem is such that η_j is entire and

$$\frac{\eta_j(\rho)}{\Delta_{\text{PDE}}(\rho)} \to 0 \text{ as } \rho \to \infty \text{ within } \widetilde{D}^{\pm},$$

for each $j \in \{1, 2, \ldots, n\}$.

ASSUMPTION 3.3. We assume the initial-boundary value problem is such that ζ_j is entire and

$$\frac{\zeta_j(\rho)}{\Delta_{\text{PDE}}(\rho)} \to 0 \text{ as } \rho \to \infty \text{ within } \widetilde{E}^{\pm},$$

for each $j \in \{1, 2, \ldots, n\}$.

The meromorphic functions, $\frac{\eta_j(\rho)}{\Delta_{\text{PDE}}(\rho)}$ and $\frac{\zeta_j(\rho)}{\Delta_{\text{PDE}}(\rho)}$, encode t-transforms of the boundary functions in terms of Fourier transforms of the initial datum and the solution at final time respectively. They originate from an application of Cramer's rule Theorem B.2 to the reduced global relation (2.2.15).

The assumptions are of interest as they permit the deformation of certain contours of integration that pass through infinity. It is not true that these assumptions always hold and it is by no means trivial to show that they do hold for an arbitrary initial-boundary value problem. Sections 3.2 and 3.3 are given over to the discussion of these assumptions.

The main result of Section 3.2 is that one of the assumptions is equivalent to the initial-boundary value problem being well-posed. This transforms Assumption 3.2 into a condition that is at once easier to check than that given in [**27**] and much more general than the condition in [**53**]. Section 3.2 also gives a pair of sufficient conditions for the assumption to hold that are still easier to check for certain types of boundary conditions. This minimizes the weight of calculation required to check for well-posedness and permits the investigation of some high order examples.

Finally, in Section 3.3, we show that Assumption 3.3 is equivalent to the existence of a series representation of the solution to a well-posed initial-boundary value problem. We also show a duality between initial- and final-boundary value problems and changing the sign of the parameter a in the partial differential equation. Section 3.3 also gives a pair of sufficient conditions for Assumption 3.3 to hold, parallel to those given in Section 3.2 for the other assumption.

3.1. Derivation of a series representation

In this section we apply Jordan's Lemma B.3 to deform the contours of integration in the integral representation given by Theorem 2.20. We do not investigate whether the conditions of Jordan's Lemma are met, instead we assume that they are met and show that this implies we may perform a residue calculation, obtaining a series representation of the solution. Sections 3.2 and 3.3 are concerned with investigating the validity of these assumptions.

Consider the same initial-boundary value problem studied in Chapter 2. That is, we wish to find q which satisfies the partial differential equation (2.1.1) subject to initial condition (2.1.2) and boundary conditions (2.1.32) where the boundary coefficient matrix A, given by equation (2.1.33), is in reduced row-echelon form. We assume throughout this section that such a function q exists and is unique hence the initial-boundary value problem is well-posed. The criteria for Theorem 2.20 are now met.

DEFINITION 3.4. *Let the functions $P, \widehat{P} : \mathbb{C} \times \Omega \to \mathbb{C}$ be defined by*

$$P(\rho; x, t) = e^{i\rho x - a\rho^n t} \qquad \text{and} \qquad \widehat{P}(\rho; x, t) = e^{-i\rho(1-x) - a\rho^n t}$$

We shall usually omit the x and t dependence of these functions, writing simply $P(\rho)$ and $\widehat{P}(\rho)$.

The aim of Definition 3.4 is that we may write the result of Theorem 2.20 in a way that emphasises the ρ-dependence of the integrands, instead of their dependence on x and t. Indeed as x and t are both bounded real numbers they are treated as parameters in what follows.

We also define the five integrals

$$I_1 = \int_{\mathbb{R}} P(\rho)\hat{q}_0(\rho)\,\mathrm{d}\rho, \tag{3.1.1}$$

$$I_2 = \int_{-\partial D^+} P(\rho) \sum_{j\in J^+} \frac{\zeta_j(\rho)}{\Delta_{\mathrm{PDE}}(\rho)}\,\mathrm{d}\rho, \qquad I_3 = \int_{\partial D^+} P(\rho)e^{a\rho^n T} \sum_{j\in J^+} \frac{\eta_j(\rho)}{\Delta_{\mathrm{PDE}}(\rho)}\,\mathrm{d}\rho,$$

$$I_4 = \int_{-\partial D^-} \widehat{P}(\rho) \sum_{j\in J^-} \frac{\zeta_j(\rho)}{\Delta_{\mathrm{PDE}}(\rho)}\,\mathrm{d}\rho, \qquad I_5 = \int_{\partial D^-} \widehat{P}(\rho)e^{a\rho^n T} \sum_{j\in J^-} \frac{\eta_j(\rho)}{\Delta_{\mathrm{PDE}}(\rho)}\,\mathrm{d}\rho,$$

where ζ_j, η_j, Δ_{PDE}, J^+ and J^- are given in Definition 2.19. We may now rewrite the result of Theorem 2.20 in the form

$$2\pi q = \sum_{k=1}^{5} I_k. \tag{3.1.2}$$

3.1.1. The behaviour of the integrands

We put aside I_1 for this subsection and investigate the behaviour of the integrands in the other four integrals in the regions to the left of the contours of integration. The results of this subsection are summarised in the following lemma.

LEMMA 3.5. *Let q be the solution of the well-posed initial-boundary value problem studied in this section. Under Assumptions 3.2 and 3.3 the following hold:*

- *The integrand of I_2 is analytic within $\widetilde{E}^+$ and decays as $\rho \to \infty$ within $\widetilde{E}^+$.*
- *The integrand of I_3 is analytic within $\widetilde{D}^+$ and decays as $\rho \to \infty$ within $\widetilde{D}^+$.*

- *The integrand of I_4 is analytic within $\widetilde{E}^-$ and decays as $\rho \to \infty$ within $\widetilde{E}^-$.*
- *The integrand of I_5 is analytic within $\widetilde{D}^-$ and decays as $\rho \to \infty$ within $\widetilde{D}^-$.*

The open sets $\widetilde{D}^{\pm}, \widetilde{E}^{\pm}$ are given in Definition 3.9.

The integral I_2 follows a contour that has E^+ to its left and for $\rho \in E^+$ the function $P(\rho)$ is an analytic, bounded function which decays as $\rho \to \infty$. Similarly, I_4 is an integral along the contour that has E^- to its left and $\widehat{P}(\rho)$ is bounded, analytic in this region and decays as $\rho \to \infty$. To the left of the contours of I_3 and I_5 are D^+ and D^-; $P(\rho)e^{a\rho^n T}$ and $\widehat{P}(\rho)e^{a\rho^n T}$ are bounded analytic functions in these regions respectively, each decaying as $\rho \to \infty$ from within the respective region.

For a well-posed initial-boundary value problem, $\hat{q}_0$ and $\hat{q}_T$ are entire hence ζ_j and η_j are also entire. The analyticity of these functions is included in Assumptions 3.2 and 3.3 in order to ensure that the Assumption 3.2 is equivalent to well-posedness; see Section 3.2.

As Δ_{PDE} is an entire function it has only countably many zeros and the only accumulation point of its zeros is infinity. The functions ζ_j and η_j are all entire so the ratios in Assumptions 3.2 and 3.3 yield meromorphic functions on $\mathbb{C}$. This means we need only investigate behaviour at zeros of Δ_{PDE} and the behaviour of both the numerator and denominator as $\rho \to \infty$. In this subsection we are interested in avoiding the effects of the zeros of the denominator. These effects will be accounted for in following subsection.

The aim of the following definitions is to allow the deformation of the contours of integration so that the singularities of $\frac{\zeta_j}{\Delta_{\text{PDE}}}$ and $\frac{\eta_j}{\Delta_{\text{PDE}}}$ caused by the zeros of Δ_{PDE} may be considered separately. We define new regions $\widetilde{D}^{\pm}$ and $\widetilde{E}^{\pm}$ that exclude neighbourhoods of these zeros from $D^{\pm}$ and $E^{\pm}$ respectively.

Definition 3.6 (PDE Discrete Spectrum). *Let $(\sigma_k)_{k\in\mathbb{N}}$ be a sequence containing each non-zero zero of Δ_{PDE} precisely once. The* PDE discrete spectrum *is the set*

$$\{\sigma_k : k \in \mathbb{N}\}.$$

We also define $\sigma_0 = 0$ for notational convenience, though 0 may or may not be a zero of Δ_{PDE}.

We wish to classify points of the PDE discrete spectrum based upon their location relative to ∂D and $\mathbb{R}$.

Definition 3.7. *We define the following index sets:*

- *For $X \subset \mathbb{C}$ an open subset let $K^X \subset \mathbb{N}$ be defined such that $k \in K^X$ if and only if $\sigma_k \in X$. That is, K^X is the set of indices of zeros of Δ_{PDE} that lie in X.*
- *Define $K^{\mathbb{R}} \subset \mathbb{N}$ such that $k \in K^{\mathbb{R}}$ if and only if $\sigma_k \in \mathbb{R} \setminus \{0\}$. That is, $K^{\mathbb{R}}$ is the set of indices of nonzero, real zeros of Δ_{PDE}.*
- *Define $K^{\pm} \subset \mathbb{N}$ such that $k \in K^{\pm}$ if and only if $\sigma_k \in \partial D^{\pm} \setminus \mathbb{R}$. That is, $K^{\pm}$ is the set of indices of nonreal zeros of Δ_{PDE} that lie in $\partial D^{\pm}$.*

These index sets are disjoint with union $\mathbb{N}$.

The next definition associates a distance ε_k with each point σ_k of the PDE discrete spectrum. The ε_k have been chosen small enough to ensure that the pairwise intersection of the closed discs

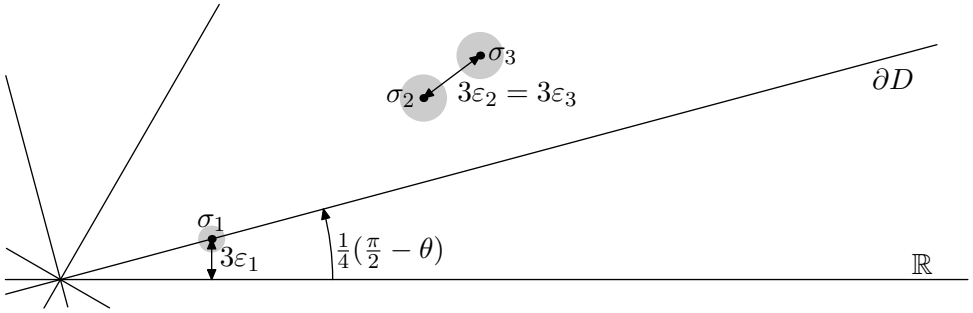

FIGURE 3.1. The bounds on ε_k

$\overline{B}(\sigma_k, \varepsilon_k) \cap \overline{B}(\sigma_j, \varepsilon_j)$ is empty for $j \neq k$. Also, for $k > 0$, the closed disc $\overline{B}(\sigma_k, \varepsilon_k)$ does not touch any part of ∂D except, when $k \in K^+ \cup K^- \cup K^{\mathbb{R}}$, the half line on which σ_k lies. Choosing such small ε_k is not necessary for this subsection but it is useful for simplifying the residue calculations of Subsection 3.1.3. Figure 3.1 shows the suprema of ε_k given some particular σ_k; the shaded regions are the discs $B(\sigma_k, \varepsilon_k)$.

The definition must be split into two cases, depending upon the value of a. In either case it is justified as we know that Δ_{PDE} is holomorphic on $\mathbb{C}$ so its zeros are isolated. For each $k \in K^X$ we define a small disc around σ_k that is wholly contained within X. This disc is labeled $B(\sigma_k, \varepsilon_k)$, using the "ball" notation to avoid confusion with the notation D, representing the subset of the complex plane for which $\text{Re}(a\rho^n) < 0$.

DEFINITION 3.8 (ε_k). *Let $a = \pm i$. For each $k \in \mathbb{N}$ we define $\varepsilon_k > 0$ as follows:*

- *For each $k \in K^+ \cup K^- \cup K^{\mathbb{R}}$, we select $\varepsilon_k > 0$ such that*

$$3\varepsilon_k < |\sigma_k| \sin(\tfrac{\pi}{n}) \text{ and } \overline{B}(\sigma_k, 3\varepsilon_k) \cap \{\sigma_j : j \in \mathbb{N}^0\} = \{\sigma_k\}.$$

- *For each $k \in K^{D^+} \cup K^{D^-} \cup K^{E^+} \cup K^{E^-}$ we select $\varepsilon_k > 0$ such that*

$$3\varepsilon_k < \text{dist}(\sigma_k, \partial D) \text{ and } \overline{B}(\sigma_k, 3\varepsilon_k) \cap \{\sigma_j : j \in \mathbb{N}^0\} = \{\sigma_k\}.$$

- *We define $\varepsilon_0 > 0$ such that*

$$\overline{B}(0, 3\varepsilon_0) \cap \{\sigma_j : j \in \mathbb{N}^0\} = \{\sigma_0\}.$$

Let $a = e^{i\theta}$ for some $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$. For each $k \in \mathbb{N}$ we define $\varepsilon_k > 0$ as follows:

- *For each $k \in K^+ \cup K^- \cup K^{\mathbb{R}}$, we select $\varepsilon_k > 0$ such that*

$$3\varepsilon_k < |\sigma_k| \sin(\tfrac{1}{n}(\tfrac{\pi}{2} - |\theta|)) \text{ and } \overline{B}(\sigma_k, 3\varepsilon_k) \cap \{\sigma_j : j \in \mathbb{N}^0\} = \{\sigma_k\}.$$

- *For each $k \in K^{D^+} \cup K^{D^-} \cup K^{E^+} \cup K^{E^-}$ we select $\varepsilon_k > 0$ such that*

$$3\varepsilon_k < \text{dist}(\sigma_k, \partial D \cup \mathbb{R}) \text{ and } \overline{B}(\sigma_k, 3\varepsilon_k) \cap \{\sigma_j : j \in \mathbb{N}^0\} = \{\sigma_k\}.$$

- *We define $\varepsilon_0 > 0$ such that*

$$\overline{B}(0, 3\varepsilon_0) \cap \{\sigma_j : j \in \mathbb{N}^0\} = \{\sigma_0\}.$$

The next definition uses Definitions 3.6, 3.7 and 3.8 to define subsets of $D^{\pm}$ and $E^{\pm}$ on which the functions $\frac{\zeta_j}{\Delta_{\text{PDE}}}$ and $\frac{\eta_j}{\Delta_{\text{PDE}}}$ are not just meromorphic but holomorphic.

DEFINITION 3.9. *We define the sets of complex numbers*

$$\widetilde{D}^{\pm} = D^{\pm} \setminus \bigcup_{k\in\mathbb{N}^0} \overline{B}(\sigma_k, \varepsilon_k) \text{ and } \widetilde{E}^{\pm} = E^{\pm} \setminus \bigcup_{k\in\mathbb{N}^0} \overline{B}(\sigma_k, \varepsilon_k),$$

and observe that $\frac{\zeta_j}{\Delta_{\mathrm{PDE}}}$ *is analytic on* $\widetilde{E}^{\pm}$ *and* $\frac{\eta_j}{\Delta_{\mathrm{PDE}}}$ *is analytic on* $\widetilde{D}^{\pm}$.

Because the positions of the zeros of Δ_{PDE} are affected by the boundary conditions, the sets $\widetilde{D}^{\pm}$, $\widetilde{E}^{\pm}$ depend upon the boundary conditions. This is in contrast to the sets $D^{\pm}$ and $E^{\pm}$ which depend only upon the partial differential equation (that is upon n and a) and are independent of the boundary conditions.

To complete this subsection, we give an example for which Assumptions 3.2 and 3.3 hold.

EXAMPLE 3.10. Consider the initial-boundary value problem of Example 2.21; $n = 3$, $a = i$ and the boundary conditions are given by equation (2.1.32) with $h_j = 0$ and a boundary coefficient matrix

$$A = \begin{pmatrix} 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.1.3}$$

From Example 2.21 we know that $\omega = e^{\frac{2\pi i}{3}}$

$$\begin{aligned}
\Delta_{\mathrm{PDE}}(\rho) &= -\rho[(e^{i\rho} + e^{-i\rho}) + \omega(e^{i\omega\rho} + e^{-i\omega\rho}) + \omega^2(e^{i\omega^2\rho} + e^{-i\omega^2\rho})], \\
\sum_{j\in J^+} \zeta_j(\rho) &= \rho[\hat{q}_0(\rho)(e^{i\rho} + \omega e^{-i\omega\rho} + \omega^2 e^{-i\omega^2\rho}) \\
&\qquad + \omega\hat{q}_0(\omega\rho)(e^{i\omega\rho} - e^{-i\rho}) + \omega^2\hat{q}_0(\omega^2\rho)(e^{i\omega^2\rho} - e^{-i\rho})], \\
\sum_{j\in J^-} \zeta_j(\rho) &= \rho[\hat{q}_0(\rho)(1 + \omega^2 e^{-i\omega\rho} + \omega e^{-i\omega^2\rho}) + \omega\hat{q}_0(\omega\rho)(1 - e^{-i\omega^2\rho}) + \omega^2\hat{q}_0(\omega^2\rho)(1 - e^{-i\omega\rho})]
\end{aligned}$$

and the η_j are given by replacing $\hat{q}_0$ with $\hat{q}_T$ in the corresponding ζ_j.

It is possible to exploit the symmetry of Δ_{PDE} with a simple geometric argument similar to that found in the Appendix of [**54**] to find the location of the zeros of Δ_{PDE}. The zeros are all points on ∂D, distributed identically on each ray, and are asymptotically separated by 2π along each ray. This means that $\widetilde{D}$ is simply D with semicircles around each of these points removed and similarly for $\widetilde{E}$, as shown in Figure 3.2.

To avoid presenting very similar arguments multiple times we show that

$$\lim_{\rho\to\infty} \frac{\sum_{j\in J^+} \zeta_j(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} = 0, \quad \rho \in \widetilde{E}^+, \tag{3.1.4}$$

and observe that the remaining requirements,

$$\begin{aligned}
\lim_{\rho\to\infty} \frac{\sum_{j\in J^-} \zeta_j(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} &= 0, \quad \rho \in \widetilde{E}^-, \\
\lim_{\rho\to\infty} \frac{\sum_{j\in J^+} \eta_j(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} &= 0, \quad \rho \in \widetilde{D}^+ \text{ and} \\
\lim_{\rho\to\infty} \frac{\sum_{j\in J^-} \eta_j(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} &= 0, \quad \rho \in \widetilde{D}^-,
\end{aligned}$$

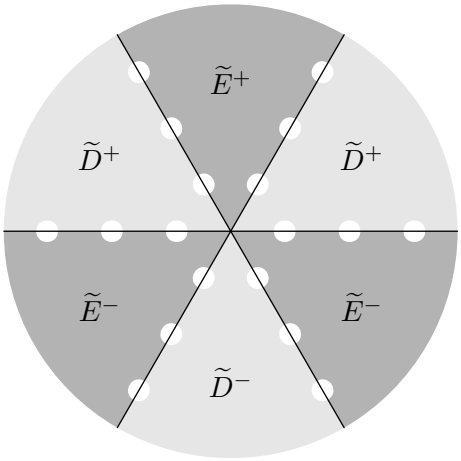

FIGURE 3.2. The positions of the sets $\widetilde{D}^{\pm}$ and $\widetilde{E}^{\pm}$ for $n = 3$, $a = i$ and boundary conditions with one first-order coupling

of Assumptions 3.2 and 3.3 may be checked in the same way.

For $\rho \in \widetilde{E}^+$, $\arg(\rho) \in (\frac{\pi}{3}, \frac{2\pi}{3})$ so $\hat{q}_0(\rho)e^{i\rho}$, $e^{-i\omega\rho}$ and $e^{-i\omega^2\rho}$ decay as $\rho \to \infty$ from within $\widetilde{E}^+$ so we may write the fraction as

$$-\frac{\hat{q}_0(\rho)(\omega e^{-i\omega\rho} + \omega^2 e^{-i\omega^2\rho}) + \omega\hat{q}_0(\omega\rho)(e^{i\omega\rho} - e^{-i\rho}) + \omega^2\hat{q}_0(\omega^2\rho)(e^{i\omega^2\rho} - e^{-i\rho}) + \text{decaying}}{e^{-i\rho} + \omega e^{i\omega\rho} + \omega^2 e^{i\omega^2\rho} + \text{decaying}}$$

but, as $\arg(\rho) \in (\frac{\pi}{3}, \frac{2\pi}{3})$, $e^{-i\rho} = O(e^{\operatorname{Im}\rho})$, $e^{i\omega\rho} = o(e^{\operatorname{Im}\rho})$ and $e^{i\omega^2\rho} = o(e^{\operatorname{Im}\rho})$ so the dominant term in the denominator is $e^{-i\rho}$. Further

$$\frac{\hat{q}_0(\rho)e^{-i\omega\rho}}{e^{-i\rho}} = \int_0^1 e^{i\rho(1-y-\omega)} q_0(y)\,\mathrm{d}y$$

and $\operatorname{Re}(i\rho(1-y-\omega)) = \operatorname{Im}(\omega\rho) - (1-y)\operatorname{Im}(\rho) < 0$ for all $y \in (0,1)$ and $\rho \in \widetilde{E}$ hence the integral decays as $\rho \to \infty$ from within $\widetilde{E}$. Similarly

$$\frac{\hat{q}_0(\rho)e^{-i\omega^2\rho}}{e^{-i\rho}}, \qquad \frac{\hat{q}_0(\omega\rho)e^{-i\omega\rho}}{e^{-i\rho}}, \qquad \frac{\hat{q}_0(\omega^2\rho)e^{-i\omega^2\rho}}{e^{-i\rho}}, \qquad \hat{q}_0(\omega\rho), \qquad \hat{q}_0(\omega^2\rho)$$

all decay as $\rho \to \infty$ from within $\widetilde{E}$. This establishes that expression (3.1.4) evaluates to 0.

3.1.2. Deforming the contours of integration

In what follows we regard Assumptions 3.2 and 3.3 as given. We use Lemma 3.5 to deform the contours of integration in each of I_2, I_3, I_4 and I_5 so that we are left only with integrals along the real axis and along finite contours around or through each singularity of Δ_{PDE} and through 0. In order to do this we must further refine the index set $K^{\mathbb{R}}$ from Definition 3.7 and define several new contours. This refinement is necessary as we need to know not only which points of the PDE discrete spectrum lie on $\mathbb{R}^{\pm}$ but where $\mathbb{R}^+$ and $\mathbb{R}^-$ lie relative to D and E. Figures 3.4, 3.5 and 3.6 show how this can vary.

DEFINITION 3.11. *We define the index sets*

$$K_E^D = \{k \in \mathbb{N} \text{ such that } \sigma_k \in \mathbb{R} \cap \partial D^+ \cap \partial E^-\},$$
$$K_D^E = \{k \in \mathbb{N} \text{ such that } \sigma_k \in \mathbb{R} \cap \partial E^+ \cap \partial D^-\},$$
$$K_E^E = \{k \in \mathbb{N} \text{ such that } \sigma_k \in \mathbb{R} \cap E\}.$$

Note that in Definition 3.11 we do not define a set K_D^D as such a set is guaranteed to be empty. This is because $a \neq e^{i\theta}$ for $\theta \in (\frac{\pi}{2}, \frac{3\pi}{2})$.[1] It is also clear from the definition and the fact that $D^\pm$ and $E^\pm$ are open sets that the index sets K_E^D, K_D^E and K_E^E are disjoint with union $K^{\mathbb{R}}$.

DEFINITION 3.12. *Let $(\sigma_k)_{k\in\mathbb{N}}$ be the PDE discrete spectrum of an initial-boundary value problem, and ε_k be the associated radii from Definition 3.8. We define the following contours, whose traces are circles or the boundaries of semicircles or circular sectors. Each is oriented such that the corresponding σ_k lies to the left of the circular arc which forms part of the contour; so that they enclose a finite region.*

- *For $k \in K^{D^+} \cup K^{E^+} \cup K^{D^-} \cup K^{E^-}$ we define the contour*
  - : $\Gamma_k = \partial D(\sigma_k, \varepsilon_k)$.
- *For $k \in K^+ \cup K^- \cup K_E^D \cup K_D^E$ we define the contours*
  - : $\Gamma_k = \partial D(\sigma_k, \varepsilon_k)$,
  - : $\Gamma_k^D = \partial(D(\sigma_k, \varepsilon_k) \cap D)$ *and*
  - : $\Gamma_k^E = \partial(D(\sigma_k, \varepsilon_k) \cap E)$.
- *For $k \in K_E^E$ we define the contours*
  - : $\Gamma_k = \partial D(\sigma_k, \varepsilon_k)$,
  - : $\Gamma_k^+ = \partial(D(\sigma_k, \varepsilon_k) \cap \mathbb{C}^+)$ *and*
  - : $\Gamma_k^- = \partial(D(\sigma_k, \varepsilon_k) \cap \mathbb{C}^-)$.
- *We define the contours*
  - : $\Gamma_0 = \partial D(0, \varepsilon_0)$,
  - : $\Gamma_0^{D^+} = \partial(D(0, \varepsilon_0) \cap D^+)$,
  - : $\Gamma_0^{E^+} = \partial(D(0, \varepsilon_0) \cap E^+)$,
  - : $\Gamma_0^{D^-} = \partial(D(0, \varepsilon_0) \cap D^-)$,
  - : $\Gamma_0^{E^-} = \partial(D(0, \varepsilon_0) \cap E^-)$,
  - : $\Gamma_0^+ = \partial(D(0, \varepsilon_0) \cap \mathbb{C}^+)$ *and*
  - : $\Gamma_0^- = \partial(D(0, \varepsilon_0) \cap \mathbb{C}^-)$.

Some of the contours in Definition 3.12 are shown in Figure 3.3. In this example $1 \in K^{D^+}$ and $2 \in K_E^E$ and the partial differential equation is the heat equation, $q_t = q_{xx}$. We do not claim that there exists any particular set of boundary conditions for the heat equation such that these particular σ_1 and σ_2 are in the PDE discrete spectrum; the figure is purely to illustrate Definition 3.12. The contours associated with 0 and σ_2 are shown slightly away from these points for clarity on the figure but they do pass through the points. Indeed $\Gamma_0^{E^+}$ and $\Gamma_0^{E^-}$ each self-intersect at 0.

[1] See Figures 3.4, 3.5 and 3.6.

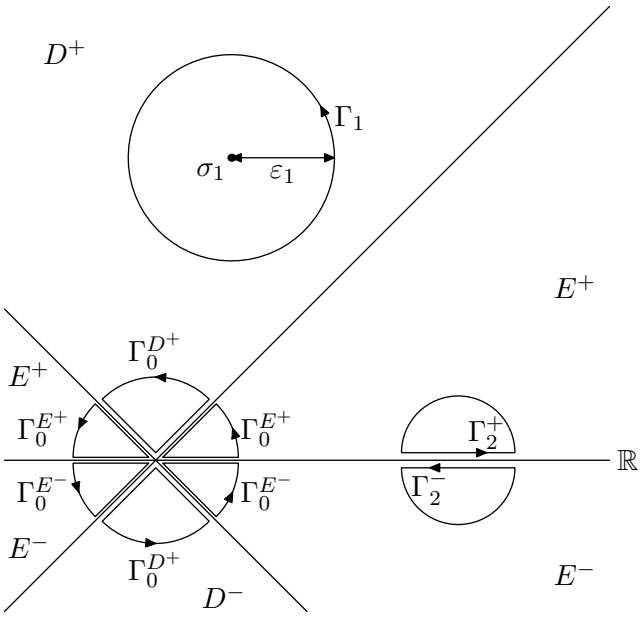

FIGURE 3.3. Some contours from Definition 3.12

The first step is to rewrite the integrals I_k for $k \in \{2, 4\}$ found in equations (3.1) as

$$I_2 = \left\{ \int_{-\mathbb{R}} + \int_{\partial E^+} \right\} P(\rho) \sum_{j \in J^+} \frac{\zeta_j(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \, \mathrm{d}\rho,$$
$$I_4 = \left\{ \int_{\mathbb{R}} + \int_{\partial E^-} \right\} \widehat{P}(\rho) \sum_{j \in J^-} \frac{\zeta_j(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \, \mathrm{d}\rho.$$

Now each I_k contains an integral along the boundary of $D^\pm$ or $E^\pm$ in the positive direction and the terms I_2 and I_4 also contain an integral along the real line. We apply Lemma 3.5 together with Jordan's Lemma B.3 to rewrite the integrals I_k for $k \geqslant 2$ in the following way:

$$I_2 = \left\{ \int_{-\mathbb{R}} + \int_{\partial(E^+ \setminus \widetilde{E}^+)} \right\} P(\rho) \sum_{j \in J^+} \frac{\zeta_j(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \, \mathrm{d}\rho, \tag{3.1.5}$$
$$I_3 = \int_{\partial(D^+ \setminus \widetilde{D}^+)} P(\rho) e^{a\rho^n T} \sum_{j \in J^+} \frac{\eta_j(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \, \mathrm{d}\rho, \tag{3.1.6}$$
$$I_4 = \left\{ \int_{\mathbb{R}} + \int_{\partial(E^- \setminus \widetilde{E}^-)} \right\} \widehat{P}(\rho) \sum_{j \in J^-} \frac{\zeta_j(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \, \mathrm{d}\rho, \tag{3.1.7}$$
$$I_5 = \int_{\partial(D^- \setminus \widetilde{D}^-)} \widehat{P}(\rho) e^{a\rho^n T} \sum_{j \in J^-} \frac{\eta_j(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \, \mathrm{d}\rho. \tag{3.1.8}$$

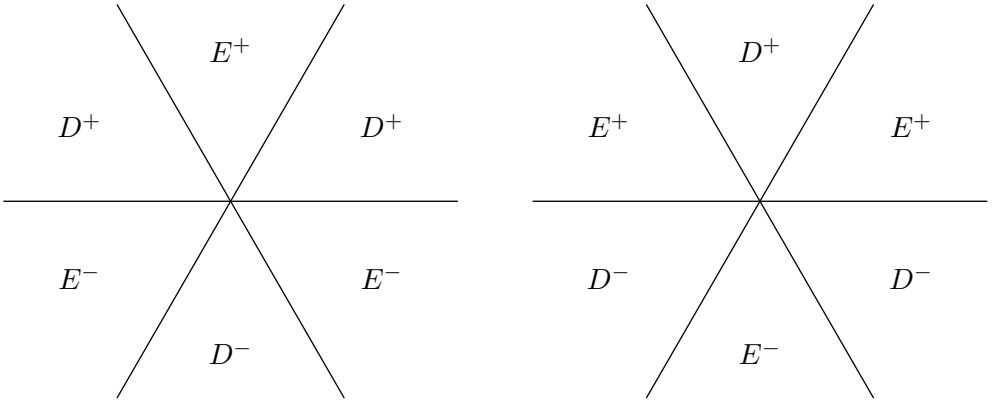

FIGURE 3.4. The regions $D^{\pm}$ and $E^{\pm}$ for n odd and $a = \pm i$

Using Definition 3.12 we may rewrite equations (3.1.5)–(3.1.8) as

$$I_2 = \left\{ \int_{-\mathbb{R}} + \int_{\Gamma_0^{E^+}} + \sum_{k \in K^{E^+}} \int_{\Gamma_k} + \sum_{\substack{k \in \\ K^+ \cup K_D^E}} \int_{\Gamma_k^E} + \sum_{k \in K_E^E} \int_{\Gamma_k^+} \right\} P(\rho) \sum_{j \in J^+} \frac{\zeta_j(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \, \mathrm{d}\rho, \tag{3.1.9}$$

$$I_3 = \left\{ \int_{\Gamma_0^{D^+}} + \sum_{k \in K^{D^+}} \int_{\Gamma_k} + \sum_{\substack{k \in \\ K^+ \cup K_E^D}} \int_{\Gamma_k^D} \right\} P(\rho) e^{a\rho^n T} \sum_{j \in J^+} \frac{\eta_j(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \, \mathrm{d}\rho, \tag{3.1.10}$$

$$I_4 = \left\{ \int_{\mathbb{R}} + \int_{\Gamma_0^{E^-}} + \sum_{k \in K^{E^-}} \int_{\Gamma_k} + \sum_{\substack{k \in \\ K^- \cup K_E^D}} \int_{\Gamma_k^E} + \sum_{k \in K_E^E} \int_{\Gamma_k^-} \right\} \widehat{P}(\rho) \sum_{j \in J^-} \frac{\zeta_j(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \, \mathrm{d}\rho, \tag{3.1.11}$$

$$I_5 = \left\{ \int_{\Gamma_0^{D^-}} + \sum_{k \in K^{D^-}} \int_{\Gamma_k} + \sum_{\substack{k \in \\ K^- \cup K_E^D}} \int_{\Gamma_k^D} \right\} \widehat{P}(\rho) e^{a\rho^n T} \sum_{j \in J^-} \frac{\eta_j(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \, \mathrm{d}\rho. \tag{3.1.12}$$

It should be noted that for any particular values of n and a at least one of the index sets K_E^D, K_D^E and K_E^E is empty. In the following paragraphs we investigate this in more detail.

1. n odd, a = i. Then $\mathrm{Re}(a\rho^n) = -\mathrm{Im}(\rho^n)$. Thus $\rho \in D$ if and only if $\mathrm{Im}(\rho^n) > 0$. This establishes that

$$\rho \in D \Leftrightarrow \arg \rho \in \bigcup_{j=0}^{n-1} \left(\frac{2j\pi}{n}, \frac{(2j+1)\pi}{n} \right) \tag{3.1.13}$$

Hence $K_E^E = K_D^E = \emptyset$ and $K_E^D = K^{\mathbb{R}}$. The left of Figure 3.4 shows the positions of $D^{\pm}$ and $E^{\pm}$ for $a = i$ when $n = 3$.

2. n odd, a = −i. Then $\mathrm{Re}(a\rho^n) = \mathrm{Im}(\rho^n)$. Thus $\rho \in D$ if and only if $\mathrm{Im}(\rho^n) < 0$. This establishes that

$$\rho \in D \Leftrightarrow \arg \rho \in \bigcup_{j=0}^{n-1} \left(\frac{(2j+1)\pi}{n}, \frac{(2j+2)\pi}{n} \right) \tag{3.1.14}$$

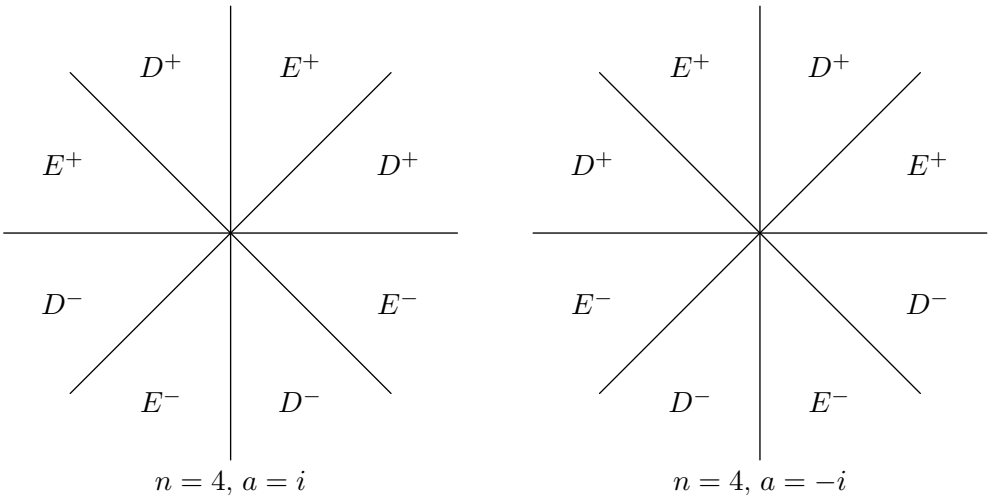

FIGURE 3.5. The regions $D^{\pm}$ and $E^{\pm}$ for $n = 4$ and $a = \pm i$

Hence $K_E^E = K_E^D = \emptyset$ and $K_E^D = K^{\mathbb{R}}$. The right of Figure 3.4 shows the positions of $D^{\pm}$ and $E^{\pm}$ for $a = -i$ when $n = 3$.

3. n even, a = i. Then statement (3.1.13) holds hence $K_E^E = \emptyset$, $K_E^D = \{k \in K^{\mathbb{R}} \text{ such that } \sigma_k > 0\}$ and $K_D^E = \{k \in K^{\mathbb{R}} \text{ such that } \sigma_k < 0\}$. The left of Figure 3.5 shows the positions of $D^{\pm}$ and $E^{\pm}$ for $a = i$ when $n = 4$.

4. n even, a = −i. Then statement (3.1.14) holds hence $K_E^E = \emptyset$, $K_E^E = \{k \in K^{\mathbb{R}} \text{ such that } \sigma_k > 0\}$ and $K_E^D = \{k \in K^{\mathbb{R}} \text{ such that } \sigma_k < 0\}$. The right of Figure 3.5 shows the positions of $D^{\pm}$ and $E^{\pm}$ for $a = -i$ when $n = 4$.

5. n even, $\mathbf{a = e^{i\theta}}$, $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$. We know that $\rho \in D$ if and only if $\text{Re}(a\rho^n) < 0$ hence

$$\rho \in D \Leftrightarrow \arg(\rho) \in \bigcup_{j=0}^{2n-1} \left(\frac{1}{n}\left[\frac{\pi}{2}(4j+1) - \theta\right], \frac{1}{n}\left[\frac{\pi}{2}(4j+3) - \theta\right] \right). \tag{3.1.15}$$

Figure 3.6 shows the positions of $D^{\pm}$ and $E^{\pm}$ for $a = e^{i\theta}$ when $n = 4$.

3.1.3. Formulation of integrals as series

The main result of this section, Theorem 3.1, gives a series representation of the solution to the initial-boundary value problem that is independent of $\hat{q}_T$. The proof involves residue calculations on equations (3.1.9)–(3.1.12).

If the boundary conditions are inhomogeneous an integral term remains due to the inhomogeneities. The equivalent of Theorem 3.1 for inhomogeneous boundary conditions is Theorem 3.13.

THEOREM 3.13. *The solution to the inhomogeneous initial-boundary value problem* (2.1.1)–(2.1.3) *obeying Assumptions 3.2 and 3.3 may be written in series form as follows:*

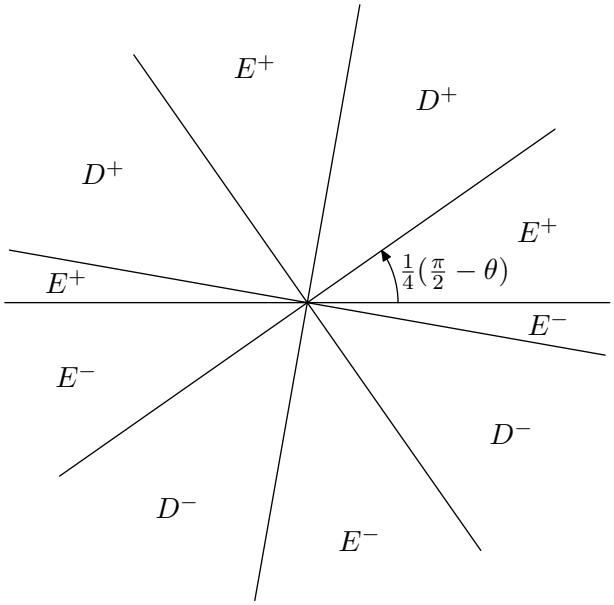

FIGURE 3.6. The regions $D^{\pm}$ and $E^{\pm}$ for n even

If n is odd and $a = i$,

$$q(x,t) = \frac{i}{2} \sum_{\substack{k\in K^+ \\ \cup K^{D^+}\cup K^{E^+} \\ \cup K^{\mathbb{R}}\cup\{0\}}} \operatorname*{Res}_{\rho=\sigma_k} \frac{P(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^+} \zeta_j(\rho) + \frac{i}{2} \sum_{\substack{k\in K^- \\ \cup K^{D^-}\cup K^{E^-}}} \operatorname*{Res}_{\rho=\sigma_k} \frac{\widehat{P}(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^-} \zeta_j(\rho)$$
$$- \frac{1}{2\pi} \left\{ \sum_{k\in K^{\mathbb{R}}} \int_{\Gamma_k^E} + \int_{\Gamma_0^-} + \int_{\mathbb{R}} \right\} P(\rho) \left(\frac{1}{\Delta_{\mathrm{PDE}}(\rho)} - 1 \right) H(\rho)\, \mathrm{d}\rho, \quad (3.1.16)$$

If n is odd and $a = -i$,

$$q(x,t) = \frac{i}{2} \sum_{\substack{k\in K^+ \\ \cup K^{D^+}\cup K^{E^+}}} \operatorname*{Res}_{\rho=\sigma_k} \frac{P(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^+} \zeta_j(\rho) + \frac{i}{2} \sum_{\substack{k\in K^- \\ \cup K^{D^-}\cup K^{E^-} \\ \cup K^{\mathbb{R}}\cup\{0\}}} \operatorname*{Res}_{\rho=\sigma_k} \frac{\widehat{P}(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^-} \zeta_j(\rho)$$
$$+ \frac{1}{2\pi} \left\{ \sum_{k\in K^{\mathbb{R}}} \int_{\Gamma_k^E} + \int_{\Gamma_0^+} - \int_{\mathbb{R}} \right\} P(\rho) \left(\frac{1}{\Delta_{\mathrm{PDE}}(\rho)} - 1 \right) H(\rho)\, \mathrm{d}\rho, \quad (3.1.17)$$

If n is even and $a = \pm i$,

$$q(x,t) = \frac{i}{2} \sum_{\substack{k\in K^+ \\ \cup K^{D^+}\cup K^{E^+} \\ \cup K_E^D}} \operatorname*{Res}_{\rho=\sigma_k} \frac{P(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^+} \zeta_j(\rho) + \frac{i}{2} \sum_{\substack{k\in K^- \\ \cup K^{D^-}\cup K^{E^-} \\ \cup K_D^E}} \operatorname*{Res}_{\rho=\sigma_k} \frac{\widehat{P}(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^-} \zeta_j(\rho)$$
$$+ \frac{i}{4} \frac{1}{\Delta_{\mathrm{PDE}}{}'(0)} \mathrm{d}\rho \sum_{j\in J^+\cup J^-} \zeta_j(\rho)$$
$$+ \frac{1}{2\pi} \left\{ \sum_{k\in K_E^D} \int_{\Gamma_k^E} + \frac{1}{2}\int_{\Gamma_0^-} - \int_0^{-\infty} - \sum_{k\in K_D^E} \int_{\Gamma_k^E} - \frac{1}{2}\int_{\Gamma_0^+} + \int_0^{\infty} \right\}$$
$$P(\rho)\left(\frac{1}{\Delta_{\mathrm{PDE}}(\rho)} - 1\right) H(\rho)\,\mathrm{d}\rho \quad (3.1.18)$$

If n is even and $a = e^{i\theta}$ for some $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$,

$$q(x,t) = \frac{i}{2} \sum_{\substack{k\in K^+ \\ \cup K^{D^+}\cup K^{E^+} \\ \cup K^{\mathbb{R}}\cup\{0\}}} \operatorname*{Res}_{\rho=\sigma_k} \frac{P(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^+} \zeta_j(\rho) + \frac{i}{2} \sum_{\substack{k\in K^- \\ \cup K^{D^-}\cup K^{E^-}}} \operatorname*{Res}_{\rho=\sigma_k} \frac{\widehat{P}(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^-} \zeta_j(\rho)$$
$$- \frac{1}{2\pi} \left\{ \sum_{k\in K^{\mathbb{R}}} \int_{\Gamma_k^-} + \int_{\Gamma_0^-} + \int_{\mathbb{R}} \right\} P(\rho)\left(\frac{1}{\Delta_{\mathrm{PDE}}(\rho)} - 1\right) H(\rho)\,\mathrm{d}\rho, \quad (3.1.19)$$

The proofs of these theorems are mathematically simple but, partly due to the range of values of a, take a large amount of space. For this reason, they are relegated to the Appendix Section B.2.

3.2. Well-posed IBVP

In this section we investigate Assumption 3.2. Specifically, we give a sufficient condition for this assumption to hold in Theorem 3.23 and show that it is equivalent to the initial-boundary value problem being well-posed in the sense of admitting a unique solution.

To help motivate the present section we give an example for which Assumption 3.2 holds. We also consider an example for which Assumption 3.2 does not to hold.

Example 3.14. We consider the 3rd order initial-boundary value problem with boundary conditions specified by the boundary coefficient matrix

$$A = \begin{pmatrix} 0 & 0 & 1 & \beta & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

This gives reduced global relation matrix

$$\mathcal{A}(\rho) = \begin{pmatrix} c_2(\rho) & -c_2(\rho)e^{-i\rho} & -(e^{-i\rho}+\beta)c_1(\rho) \\ c_2(\rho) & -c_2(\rho)e^{-i\omega\rho} & -\omega(e^{-i\omega\rho}+\beta)c_1(\rho) \\ c_2(\rho) & -c_2(\rho)e^{-i\omega^2\rho} & -\omega^2(e^{-i\omega^2\rho}+\beta)c_1(\rho) \end{pmatrix}$$

and reduced boundary coefficient matrix

$$\widehat{A} = \begin{pmatrix} 0 & 0 & \beta \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Following Definition 2.19 we calculate

$$\begin{aligned}
\Delta_{\text{PDE}}(\rho) &= (\omega^2-\omega)c_2^2(\rho)c_1(\rho)\left[(e^{i\rho}-\beta e^{-i\rho})+\omega(e^{i\omega\rho}-\beta e^{-i\omega\rho})+\omega^2(e^{i\omega^2\rho}-\beta e^{-i\omega^2\rho})\right],\\
\eta_1(\rho) &= c_2^2(\rho)c_1(\rho)\Big\{\hat{q}_T(\rho)\left[\omega^2 e^{-i\omega\rho}\left(e^{-i\omega^2\rho}+\beta\right)-\omega e^{-i\omega^2\rho}\left(e^{-i\omega\rho}+\beta\right)\right]\\
&\qquad +\hat{q}_T(\omega\rho)\left[e^{-i\omega^2\rho}\left(e^{-i\rho}+\beta\right)-\omega^2 e^{-i\rho}\left(e^{-i\omega^2\rho}+\beta\right)\right]\\
&\qquad +\hat{q}_T(\omega^2\rho)\left[\omega e^{-i\rho}\left(e^{-i\omega\rho}+\beta\right)-e^{-i\omega\rho}\left(e^{-i\rho}+\beta\right)\right]\Big\},\\
\eta_2(\rho) &= -c_2^2(\rho)c_1(\rho)\Big\{\hat{q}_T(\rho)\left[\omega\left(e^{-i\omega\rho}+\beta\right)-\omega^2\left(e^{-i\omega^2\rho}+\beta\right)\right]\\
&\qquad +\hat{q}_T(\omega\rho)\left[\omega^2\left(e^{-i\omega^2\rho}+\beta\right)-\left(e^{-i\rho}+\beta\right)\right]\\
&\qquad +\hat{q}_T(\omega^2\rho)\left[\left(e^{-i\rho}+\beta\right)-\omega\left(e^{-i\omega\rho}+\beta\right)\right]\Big\},\\
\eta_3(\rho) &= -c_2^2(\rho)c_1(\rho)\Big\{\hat{q}_T(\rho)\left[e^{-i\omega^2\rho}-e^{-i\omega\rho}\right]+\hat{q}_T(\omega\rho)\left[e^{-i\rho}-e^{-i\omega^2\rho}\right]\\
&\qquad\qquad \hat{q}_T(\omega^2\rho)\left[e^{-i\omega\rho}-e^{-i\rho}\right]\Big\},\\
\eta_4(\rho) &= -\beta\eta_3(\rho),\\
\eta_5(\rho) &= 0 \text{ and}\\
\eta_6(\rho) &= 0.
\end{aligned}$$

Let $a = i$. Then $\widetilde{D}_1 \subseteq \{\rho \in \mathbb{C} \text{ such that } 0 < \arg\rho < \frac{\pi}{n}\}$. As $\rho \to \infty$ from within $\widetilde{D}_1$ the exponentials $e^{-i\rho}$, $e^{-i\omega\rho}$ and $e^{i\omega^2\rho}$ grow while the exponentials $e^{i\rho}$, $e^{i\omega\rho}$ and $e^{-i\omega^2\rho}$ decay. Hence the functions $\hat{q}_T(\rho)$ and $\hat{q}_T(\omega\rho)$ are also growing but $\hat{q}_T(\omega^2\rho)$ decays.

The dominant term in

$$\begin{aligned}
\Delta_{\text{PDE}}(\rho) &\text{ is } (\omega^2-\omega)c_2^2(\rho)c_1(\rho)\omega^2 e^{i\omega^2\rho},\\
\eta_1(\rho) &\quad \text{is } \beta\omega^2 c_2^2(\rho)c_1(\rho)\left\{\hat{q}_T(\rho)e^{-i\omega\rho}-\hat{q}_T(\omega\rho)e^{-i\rho}\right\},\\
\eta_2(\rho) &\quad \text{is } -c_2^2(\rho)c_1(\rho)\left\{\hat{q}_T(\rho)\omega e^{-i\omega\rho}-\hat{q}_T(\omega\rho)e^{-i\rho}\right\},\\
\eta_3(\rho) &\quad \text{is } c_2^2(\rho)c_1(\rho)\left\{\hat{q}_T(\rho)e^{-i\omega\rho}-\hat{q}_T(\omega\rho)e^{-i\rho}\right\},\\
\eta_4(\rho) &\quad \text{is } -\beta c_2^2 c_1(\rho)(\rho)\left\{\hat{q}_T(\rho)e^{-i\omega\rho}-\hat{q}_T(\omega\rho)e^{-i\rho}\right\},\\
\eta_5(\rho) &\quad \text{is 0 and}\\
\eta_6(\rho) &\quad \text{is 0.}
\end{aligned}$$

The ratio

$$\frac{\beta\omega^2 c_2^2(\rho)c_1(\rho)\left\{\hat{q}_T(\rho)e^{-i\omega\rho}-\hat{q}_T(\omega\rho)e^{-i\rho}\right\}}{(\omega^2-\omega)c_2^2(\rho)c_1(\rho)\omega^2 e^{i\omega^2\rho}} = \frac{\beta}{(\omega^2-\omega)}\int_0^1 \left\{e^{i\rho(1-x)}-e^{i\omega\rho(1-x)}\right\} q_T(\rho)(x)\,\mathrm{d}x$$

is decaying as $\rho \to \infty$ from within $\widetilde{D}_1$ because, as noted above, the exponentials $e^{i\rho(1-x)}$ and $e^{i\omega\rho(1-x)}$ are decaying for $x \in (0,1)$. Hence the ratio

$$\frac{\eta_1(\rho)}{\Delta_{\mathrm{PDE}}(\rho)}$$

also decays as $\rho \to \infty$ from within $\widetilde{D}_1$. The same calculation can be performed to check the other η_j. Indeed the dominant terms in the ratio $\frac{\eta_2(\rho)}{\Delta_{\mathrm{PDE}}(\rho)}$ have ratio

$$\frac{1}{\omega^2-\omega}\int_0^1 \left\{\omega e^{i\rho(1-x)} - e^{i\omega\rho(1-x)}\right\} q_T(\rho)(x)\,\mathrm{d}x$$

and the dominant terms in the ratio $\frac{\eta_3(\rho)}{\Delta_{\mathrm{PDE}}(\rho)}$ have ratio

$$\frac{1}{\omega^2-\omega}\int_0^1 \left\{e^{i\rho(1-x)} - e^{i\omega\rho(1-x)}\right\} q_T(\rho)(x)\,\mathrm{d}x,$$

both of which decay as $\rho \to \infty$ from within $\widetilde{D}_1$.

We do not present the calculation for $\widetilde{D}_2$ or $\widetilde{D}_3$ or for $a = -i$ but it may be checked in the same way, case-by-case.

Remark 3.15. Although in Example 3.14 the full calculation is not presented for each case it is not true that

$$\frac{\eta_j(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \to 0 \text{ as } \rho \to \infty \text{ from within } \widetilde{D}_p \Rightarrow \frac{\eta_k(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \to 0 \text{ as } \rho \to \infty \text{ from within } \widetilde{D}_r$$

for any j, k, p, r and it is not true that if Assumption 3.2 holds for a particular initial-boundary value problem then it holds for the initial-boundary value problem with the same boundary conditions but with a different value of a. Specific counterexamples are given in Example 3.16 (see Remark 3.17) and the uncoupled example of Chapter 5.

Example 3.16. We consider the 3rd order initial-boundary value problem with $a = i$ and boundary conditions specified by the boundary coefficient matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

This gives reduced global relation matrix

$$\mathcal{A}(\rho) = \begin{pmatrix} c_2(\rho) & -c_2(\rho)e^{-i\rho} & c_1(\rho) \\ c_2(\rho) & -c_2(\rho)e^{-i\omega\rho} & \omega c_1(\rho) \\ c_2(\rho) & -c_2(\rho)e^{-i\omega^2\rho} & \omega^2 c_1(\rho) \end{pmatrix}$$

and reduced boundary coefficient matrix

$$\widehat{A} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Following Definition 2.19 we calculate

$$\begin{aligned}
\Delta_{\text{PDE}}(\rho) &= (\omega^2-\omega)c_2^2(\rho)c_1(\rho)\sum_{k=0}^{2}\omega^k e^{-i\omega^k\rho},\\
\eta_1(\rho) &= c_2^2(\rho)c_1(\rho)\sum_{k=0}^{2}\omega^{k+2}\left(e^{-i\omega^k\rho}\hat{q}_T(\omega^{k+1}\rho)-e^{-i\omega^{k+1}\rho}\hat{q}_T(\omega^k\rho)\right),\\
\eta_2(\rho) &= -(\omega^2-\omega)c_2^2(\rho)c_1(\rho)\sum_{k=0}^{2}\omega^k\hat{q}_T(\omega^k\rho),\\
\eta_3(\rho) &= -c_2^2(\rho)c_1(\rho)\sum_{k=0}^{2}\left(e^{-i\omega^k\rho}\hat{q}_T(\omega^{k+1}\rho)-e^{-i\omega^{k+1}\rho}\hat{q}_T(\omega^k\rho)\right) \text{ and}\\
\eta_4(\rho) &= \eta_5(\rho)=\eta_6(\rho)=0.
\end{aligned}$$

As $a=i$, $\widetilde{D}_1\subseteq\{\rho\in\mathbb{C}$ such that $0<\arg\rho<\frac{\pi}{n}\}$. Hence as $\rho\to\infty$ from within $\widetilde{D}_1$ the exponentials $e^{-i\rho}$, $e^{-i\omega\rho}$ and $e^{i\omega^2\rho}$ grow while the exponentials $e^{i\rho}$, $e^{i\omega\rho}$ and $e^{-i\omega^2\rho}$ decay. Hence the functions $\hat{q}_T(\rho)$ and $\hat{q}_T(\omega\rho)$ are also growing but $\hat{q}_T(\omega^2\rho)$ decays.

We show that the ratio

$$\frac{\eta_3(\rho)}{\Delta_{\text{PDE}}(\rho)} \tag{3.2.1}$$

is unbounded for $\rho\in\widetilde{D}_1$ by choosing a particular sequence $(\rho_j)_{j\in\mathbb{N}}$ and showing that the ratio (3.2.1) approaches infinity as $j\to\infty$. Clearly the ray $\arg\rho=\frac{\pi}{12}$ is wholly contained within D_1 and, by the definition of $\widetilde{D}_1$, we may choose an increasing, sequence $(R_j)_{j\in\mathbb{N}}$ of positive, real numbers such that $R_j\to\infty$ as $j\to\infty$ and

$$\rho_j=R_je^{i\frac{\pi}{12}}\in\widetilde{D}_1 \text{ for each } j\in\mathbb{N}.$$

The ratio (3.2.1) may be evaluated to

$$\frac{\hat{q}_T(\omega\rho_j)e^{-i(1-\omega)\rho_j}-\hat{q}_T(\rho_j)+O(1)}{(\omega^2-\omega)(e^{-i(1-\omega)\rho_j}+\omega+\omega^2e^{-i(\omega^2-\omega)\rho_j})}.$$

But

$$\begin{aligned}
\text{Re}(-i(\omega^2-\omega)\rho_j) &= -R_j\sqrt{3}\cos(\tfrac{\pi}{12})<0\\
\text{Re}(-i(1-\omega)\rho_j) &= -R_j\sqrt{3}\sin(\tfrac{\pi}{12})<0
\end{aligned}$$

so as $j\to\infty$ the two exponentials in the denominator approach 0 so the denominator approaches a constant in the limit $j\to\infty$. The dominant terms in the numerator may be evaluated:

$$\hat{q}_T(\omega\rho_j)e^{-i(1-\omega)\rho_j}-\hat{q}_T(\rho_j)=\int_0^1\left(e^{i\omega R_je^{i\frac{\pi}{12}}(1-x)}e^{-iR_je^{i\frac{\pi}{12}}}-e^{-iR_je^{i\frac{\pi}{12}}x}\right)q_T(x)\,\mathrm{d}x$$

but

$$\text{Re}\left(i\omega R_je^{i\frac{\pi}{12}}(1-x)-iR_je^{i\frac{\pi}{12}}\right)=R_j\left(\sin(\tfrac{\pi}{12})-(1-x)\sin(\tfrac{5\pi}{12})\right)$$

and

$$\text{Re}\left(-iR_je^{i\frac{\pi}{12}}x\right)=R_jx\sin(\tfrac{\pi}{12})\to+\infty \text{ as } j\to\infty \text{ for } x\in(0,1)$$

so, provided q_T is not identically zero, the numerator approaches infinity hence

$$\frac{\eta_3(\rho_j)}{\Delta_{\mathrm{PDE}}(\rho_j)} \to \infty \text{ as } j \to \infty.$$

Hence the ratio (3.2.1) is unbounded for $\rho \in \widetilde{D}_1$.

This establishes that Assumption 3.2 does not hold.

REMARK 3.17. Although Assumption 3.2 does not hold in Example 3.16, it may be seen that the ratio

$$\frac{\eta_2(\rho)}{\Delta_{\mathrm{PDE}}(\rho)}$$

is bounded within $\widetilde{D}_3$ and decaying as $\rho \to \infty$ from within $\widetilde{D}_3$. Clearly the ratios

$$\frac{\eta_4(\rho)}{\Delta_{\mathrm{PDE}}(\rho)}, \quad \frac{\eta_6(\rho)}{\Delta_{\mathrm{PDE}}(\rho)}$$

both evaluate to 0 and $J^- = \{2, 4, 6\}$ so

$$\frac{\sum_{j\in J^-} \eta_j(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} = \frac{\eta_2(\rho)}{\Delta_{\mathrm{PDE}}(\rho)}$$

so it is possible to make the necessary contour deformations in the lower half plane, that is in $\widetilde{D}_3$, just not in the upper half plane, that is $\widetilde{D}_1$ and $\widetilde{D}_2$. This is not particularly interesting in this example, except to give one of the counterexamples for Remark 3.15, as the problem is still ill-posed but a similar fact may be exploited in the uncoupled example of Chapter 5 to give a partial series representation of a solution to a well-posed problem; see Remark 5.9.

3.2.1. n odd, homogeneous, non-Robin

A sufficient condition for homogeneous, non-Robin boundary conditions to specify a problem that satisfies Assumption 3.2 may be written as two conditions of the form:

(1) There are enough boundary conditions that couple of the ends of the interval and, of the remaining boundary conditions, roughly the same number are specified at the left hand side of the interval as are specified at the right hand side.
(2) Certain coefficients are non-zero.

More precise formulations of these conditions are given below.

3.2.1.1. The first condition

To formally give the first condition we require the following:

NOTATION 3.18. Define

$$L = |\{j : \alpha_{r\,j} = 0 \,\forall\, r\}| \qquad \text{The number of left-hand boundary functions that do not appear in the boundary conditions} \tag{3.2.2}$$

$$R = |\{j : \beta_{r\,j} = 0 \,\forall\, r\}| \qquad \text{The number of right-hand boundary functions that do not appear in the boundary conditions} \tag{3.2.3}$$

$$C = |\{j : \exists\, r : \beta_{r\,j}, \alpha_{r\,j} \neq 0\}| \qquad \text{The number of boundary conditions that couple the ends of the } x \text{ interval} \tag{3.2.4}$$

Indeed, there are C boundary conditions that couple the ends of the interval, L boundary conditions prescribed at the right end of the interval and R boundary conditions prescribed at the left end of the interval. Clearly $n = L + R + C$. We now state the first condition.

CONDITION 3.19. If $a = i$ then the $2\nu - 1$ boundary conditons are such that

$$R \leqslant \nu \leqslant R + C$$

and if $a = -i$ then the $2\nu - 1$ boundary conditons are such that

$$R \leqslant \nu - 1 \leqslant R + C$$

where R and C are defined by (3.2.3) and (3.2.4).

The remainder of this subsubsection is devoted to showing the relevance of the above condition. Consider the ratio

$$\frac{\eta_j(\rho)}{\Delta_{\mathrm{PDE}}(\rho)}. \tag{3.2.5}$$

The denominator is an exponential polynomial, hence it is a sum of terms of the form

$$Z(\rho)e^{-i\rho\sum_{r=1}^{k}\omega^{\sigma(r)}}$$

where Z is some polynomial, $k \in \{1, 2, \ldots, n\}$ and $\sigma \in S_n$. Similarly, the numerator of expression (3.2.5) is a sum of terms of the form

$$Z(\rho)\int_0^1 e^{-i\rho\sum_{r=1}^{k}\omega^{\sigma(r)}}e^{-i\rho x\omega^{\sigma(k+1)}}q_T(x)\,\mathrm{d}x$$

where Z is some polynomial, $q_0 \in C^\infty[0,1]$, $k \in \{1, 2, \ldots, n\}$ and $\sigma \in S_n$. To formulate the first condition we assume that none of the polynomials Z are identically zero and concentrate on how the boundary conditions affect which values σ and k may take. The second condition is used to ensure that certain particular polynomials Z are not identically zero.

NOTATION 3.20. For any $\sigma \in S_n$ we extend its domain of definition to $\mathbb{Z}$ so that $\sigma(j) = \sigma(k)$ if and only if $j \equiv k \pmod{n}$.

As $n \geqslant 3$ is odd we define the integer $\nu \geqslant 2$ such that $n = 2\nu - 1$.

The open set D is the union of n open sectors of $\mathbb{C}$. We define the regions D_j as follows:

- If $a = i$ then $D = \bigcup_{j=1}^{n} D_j$ where

$$D_j = \{\rho \in \mathbb{C} : \tfrac{\pi}{n}(2j-2) < \arg(\rho) < \tfrac{\pi}{n}(2j-1)\}.$$

- If $a = -i$ then $D = \bigcup_{j=1}^{n} D_j$ where

$$D_j = \{\rho \in \mathbb{C} : \tfrac{\pi}{n}(2j-1) < \arg(\rho) < \tfrac{\pi}{n}2j\}.$$

For concreteness, let $a = i$. If $\rho \in D_j$ then, for all $\sigma \in S_n$ and for all $k \in \{1, 2, \ldots, n\}$,

$$\operatorname{Re}\left(\sum_{r=1-j}^{\nu-j} \omega^r\right) \geqslant \operatorname{Re}\left(\sum_{r=1}^{k} \omega^{\sigma(r)}\right)$$

with equality if and only if $k = \nu$ and the first ν entries in σ are some permutation of $(1-j, 2-j, \ldots, \nu-j)$ (modulo n). Hence the exponential

$$e^{-i\rho\sum_{r=1-j}^{\nu-j}\omega^r} \tag{3.2.6}$$

dominates all other exponentials of the form

$$e^{-i\rho\sum_{r=1}^{k}\omega^{\sigma(r)}}$$

and all functions of the form

$$Z(\rho)\int_0^1 e^{-i\rho\sum_{r=1}^{k}\omega^{\sigma(r)}} e^{-i\rho x\omega^{\sigma(k+1)}} q_T(x)\,\mathrm{d}x.$$

Hence, if the exponential (3.2.6) multiplied by some polynomial appears in $\Delta_{\mathrm{PDE}}(\rho)$ then Assumption 3.2 must hold. The conditions are necessary and sufficient for this exponential to appear in $\Delta_{\mathrm{PDE}}(\rho)$.

By Lemma 2.14, we know that we may express the matrix $\mathcal{A}$ in the form

$$\mathcal{A}_{k\,j}(\rho) = \begin{cases} \omega^{(n-1-[J_j-1]/2)(k-1)} c_{(J_j-1)/2}(\rho) & J_j \text{ odd}, \\ -\omega^{(n-1-J_j/2)(k-1)} c_{J_j/2}(\rho)\left(e^{-i\omega^{k-1}\rho} + \beta_{\widehat{J}^{+}_{J_j/2}\,J_j/2}\right) & J_j \text{ even}, \end{cases}$$

but we may express this in terms of the three possible kinds of columns that $\mathcal{A}$ may contain. Indeed, using Notation 3.18, $\mathcal{A}$ has L columns of the form

$$c_{(n-1-j)}(\rho)(1, \omega^j, \ldots, \omega^{j(n-1)})^{\mathrm{T}},$$

R columns of the form

$$c_{(n-1-j)}(\rho)(-e^{-i\rho}, -\omega^j e^{-i\omega\rho}, \ldots, -\omega^{j(n-1)} e^{-i\omega^{n-1}\rho})^{\mathrm{T}}$$

and C columns of the form

$$c_{(n-1-j)}(\rho)(-(e^{-i\rho} + \beta_{r\,j}), -\omega^j(e^{-i\omega\rho} + \beta_{r\,j}), \ldots, -\omega^{j(n-1)}(e^{-i\omega^{n-1}\rho} + \beta_{r\,j}))^{\mathrm{T}},$$

where j ranges over L, R and C values within $\{0, 1, \ldots, n-1\}$ respectively.

Hence $\Delta_{\mathrm{PDE}}(\rho) = \det \mathcal{A}(\rho)$ has terms

$$\rho^X P_{l\,\pi}(\omega) e^{-i\sum_{r=1}^{R+l}\omega^{\pi(r)}\rho}$$

for each $l \in \{0, 1, \dots, C\}$ and $\pi \in S_n$ where $P_{l\,\pi}$ are polynomials and X is some (fixed) integer. The terms appearing in $\eta_k(\rho)$ are

$$\text{if } L \geqslant 1 \qquad \rho^{Z_{l\,\pi}} L_{l\,\pi}(\omega) \int_0^1 e^{-i\sum_{r=1}^{R+l} \omega^{\pi(r)}\rho} e^{-i\omega^{\pi(R+l+1)}\rho x} q_T(x)\,\mathrm{d}x \qquad l \in \{0, 1, \dots, C\}$$

$$\text{if } R \geqslant 1 \qquad \rho^{\widehat{Z}_{l\,\pi}} R_{l\,\pi}(\omega) \int_0^1 e^{-i\sum_{r=1}^{R-1+l} \omega^{\pi(r)}\rho} e^{-i\omega^{\pi(R+l)}\rho x} q_T(x)\,\mathrm{d}x \qquad l \in \{0, 1, \dots, C\}$$

$$\text{if } C \geqslant 1 \qquad \rho^{\overline{Z}_{l\,\pi}} C_{l\,\pi}(\omega) \int_0^1 e^{-i\sum_{r=1}^{R+l} \omega^{\pi(r)}\rho} e^{-i\omega^{\pi(R+l+1)}\rho x} q_T(x)\,\mathrm{d}x \qquad l \in \{0, 1, \dots, C-1\}$$

for each $\pi \in S_n$ where $R_{l\,\pi}$, $L_{l\,\pi}$ and $C_{l\,\pi}$ are polynomials and $Z_{l\,\pi}$, $\widehat{Z}_{l\,\pi}$ and $\overline{Z}_{l\,\pi}$ are integers.

Remark 3.21. It should be noted that the polynomials $P_{l\,\pi}$ and $R_{l\,\pi}$ and the integer $\widehat{Z}_{l\,\pi}$ depend not upon π but upon the first $R+l$ entries in π only while the polynomials $L_{l\,\pi}$ and $C_{l\,\pi}$ and the integers $Z_{l\,\pi}$ and $\overline{Z}_{l\,\pi}$ depend upon the first $R+l+1$ entries in π only.

If $R \leqslant \nu \leqslant R + C$ then there exists a particular $l \in \{0, 1, \dots, C\}$ such that $R + l = \nu$. Also, for each $j \in \{1, 2, \dots, n\}$ there exist permutations $\pi_j \in S_n$ such that $\pi_j(r) = r - j$ for each $k \in \{1, 2, \dots, \nu\}$. Hence, for each $j \in \{1, 2, \dots, n\}$, the exponential (3.2.6) (with $\pi = \pi_j$) appears in $\Delta_{\mathrm{PDE}}(\rho)$ with some monomial coefficient $\rho^X P_{l\,\pi_j}(\omega)$. Provided we can show that each of these monomials are not identically zero this guarantees that the ratio (3.2.5) is bounded and decaying at infinity for $\rho \in D$.

3.2.1.2. The second condition

We now turn our attention to the second condition. This is a technical condition needed to ensure that the relevant coefficients are nonzero. We only state the condition here and develop the necessary notation in Section B.3 of the Appendix.

Condition 3.22. Let $(\mathcal{R}_p)_{p=1}^R$ be an ordering of the elements of $\mathcal{R}$ and $(\mathcal{L}_p)_{p=1}^L$ be an ordering of the elements of $\mathcal{L}$. Let the permutation $\tau_j \in S_n$ be defined by

$$p - j = \begin{cases} \tau_j r(\mathcal{R}_p) & \text{if } p \in \{1, 2, \dots, R\}, \\ \tau_j cc'(p - R) & \text{if } p \in \{R+1, R+2, \dots, R+C\}, \\ \tau_j l(\mathcal{L}_{p-R-C}) & \text{if } p \in \{R+C+1, R+C+2, \dots, n\}. \end{cases} \tag{3.2.7}$$

Let τ' to be the identity permutation on S_C and

$$k = \begin{cases} \nu - R & a = i, \\ \nu - 1 - R & a = -i. \end{cases}$$

Then the boundary conditions are such that for each $j \in \{1, 2, \dots, n\}$ the expression

$$\sum_{(\sigma,\sigma') \in S_{k\,\tau_j\,\tau'}} \operatorname{sgn}(\sigma) \omega^{-\sum_{m\in\mathcal{R}} \sigma r(m) m - \sum_{m\in\mathcal{C}} \sigma c(m) m - \sum_{m\in\mathcal{L}} \sigma l(m) m} \prod_{m=k+1}^{C} \widetilde{\beta}_{c'\sigma'(m)} \tag{3.2.8}$$

is nonzero if $k \geqslant 1$ or the expression

$$\sum_{\substack{\sigma\in S_n:\\ \forall\, m\in\mathcal{R}\ \exists\, p\in\mathcal{R}:\\ \tau_j r(m)=\sigma r(p)}} \operatorname{sgn}(\sigma)\omega^{-\sum_{m\in\mathcal{R}}\sigma r(m)m-\sum_{m\in\mathcal{C}}\sigma c(m)m-\sum_{m\in\mathcal{L}}\sigma l(m)m} \tag{3.2.9}$$

is nonzero if $k = 0$.

Note that in the case $\widetilde{\beta}_j = \beta$ for all $j \in \mathcal{C}$ expression (3.2.8) simplifies to

$$\sum_{\substack{\sigma\in S_n:\ \exists\ \sigma'\in S_C:\\ (\sigma,\sigma')\in S_{k\,\tau_j\,\tau'}}} \operatorname{sgn}(\sigma)\omega^{-\sum_{m\in\mathcal{R}}\sigma r(m)m-\sum_{m\in\mathcal{C}}\sigma c(m)m-\sum_{m\in\mathcal{L}}\sigma l(m)m}. \tag{3.2.10}$$

The set $S_{k\,\tau\,\tau'}$, the functions l, r and c and their domains $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{C}$ are given in Definition B.7 and Lemma B.8.

This condition is checked for particular boundary conditions in the examples of Subsubsection 3.2.1.4.

3.2.1.3. Sufficient conditions for Assumption 3.2

THEOREM 3.23. *Assume n is odd. If the boundary conditions of initial-boundary value problem* (2.1.1)–(2.1.3) *are homogeneous and non-Robin, and obey Conditions 3.19 and 3.22, then Assumption 3.2 holds.*

PROOF. If the boundary conditions obey Condition 3.19 then $0 \leqslant k \leqslant C$ in Condition 3.22 so the set $S_{k\,\tau_j\,\tau'}$ and the relevant expression (3.2.8) or (3.2.9) are all well defined.

Fix $j \in \{1, 2, \ldots, n\}$ and let $\rho \in \widetilde{D}_j$. Then the modulus of

$$e^{-i\sum_{y\in Y}\omega^y\rho} \tag{3.2.11}$$

is uniquely maximised for the index set

$$Y = \{j-1, j, \ldots, j-1+R+k-1\}.$$

By Condition 3.19 and Lemma B.8, $\Delta_{\mathrm{PDE}}(\rho)$ has a term given by that exponential multiplied by a polynomial coefficient given by the right hand side of equation (B.3.6) if $k \geqslant 1$ or equation (B.3.7) if $k = 0$, with τ replaced by τ_j. These expressions are monomial multiples of expressions (3.2.8) and (3.2.9) respectively. As $\rho \in D_j$, $\rho \neq 0$ so the coefficient is guaranteed to be nonzero by Condition 3.22.

As Y *uniquely* maximises the exponential (3.2.11) this exponential dominates all other terms in $\Delta_{\mathrm{PDE}}(\rho)$. But it also dominates all terms in $\eta_j(\rho)$, that is those of the form

$$Z(\rho)e^{-i\rho\sum_{p\in P}\omega^p}\int_0^1 e^{-i\rho\omega^{p'}x}q_T(x)\,\mathrm{d}x$$

where $P \subsetneq \{0, 1, \ldots, n-1\}$ and $p' \notin P$. Hence the ratio (3.2.5) is bounded in $\widetilde{D}_j$ for each $j \in \{1, 2, \ldots, n\}$ and decaying as $\rho \to \infty$ from within $\widetilde{D}_j$. □

3.2.1.4. Checking Assumption 3.2 for particular examples

We now give three examples of how Theorem 3.23 can be used to check that a particular set of boundary conditions specifies a problem in which Assumption 3.2 holds. The first, Example 3.24, shows the necessity of checking Condition 3.22 by describing a class of pseudoperiodic boundary conditions for which Condition 3.19 holds but Condition 3.22 does not. This is the only known 3^{rd} order example.

EXAMPLE 3.24. Let $n = 3$ and the boundary coefficient matrix be given by

$$A = \begin{pmatrix} 1 & \widetilde{\beta}_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \widetilde{\beta}_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \widetilde{\beta}_1 \end{pmatrix}, \tag{3.2.12}$$

for $\widetilde{\beta}_j \in \mathbb{R} \setminus \{0\}$ so that the problem is *pseudoperiodic*. Indeed the boundary conditions are

$$\begin{aligned} q_{xx}(0,t) + \widetilde{\beta}_3 q_{xx}(1,t) &= 0, \\ q_x(0,t) + \widetilde{\beta}_2 q_x(1,t) &= 0 \text{ and} \\ q(0,t) + \widetilde{\beta}_1 q(1,t) &= 0. \end{aligned}$$

We check for which values of $\widetilde{\beta}_j$ Assumption 3.2 holds, first if $a = i$ and then if $a = -i$.

All three boundary conditions couple the ends of the space interval so $L = R = 0$ and $C = 3$. This ensures that, for $a = \pm i$, Condition 3.19 holds.

We adopt the notation of Condition 3.22, with c' the identity permutation on $\{1, 2, 3\}$ and $c(m) = 4 - m$ on the same domain, hence $\tau_j c(m) = m - j$. We simplify expression (3.2.8) to

$$\sum_{(\sigma,\sigma') \in S_{k\,\tau_j\,\tau'}} \operatorname{sgn}(\sigma) \omega^{-\sum_{m=1}^{3} m\sigma(4-m)} \prod_{m=k+1}^{3} \widetilde{\beta}_{\sigma'(m)} \tag{3.2.13}$$

for each j.

Assume first $a = i$, hence $k = 2$ and expression (3.2.13) simplifies further to

$$\sum_{(\sigma,\sigma') \in S_{2\,\tau_j\,\tau'}} \operatorname{sgn}(\sigma) \omega^{-\sum_{m=1}^{3} m\sigma(4-m)} \widetilde{\beta}_{\sigma'(3)} \tag{3.2.14}$$

The definition (B.3.4) of $S_{2\,\tau_j\,\tau'}$ simplifies here to $(\sigma, \sigma') \in S_{2\,\tau_j\,\tau'}$ if and only if

$$\begin{aligned} & \{\sigma c c' \sigma'(p) : p \in \{1,2\}\} = \{\tau_j c c' \tau'(p) : p \in \{1,2\}\} \\ \Leftrightarrow \quad & \{\sigma(4 - \sigma'(p)) : p \in \{1,2\}\} = \{1 - j, 2 - j\} \\ \Leftrightarrow \quad & \sigma(4 - \sigma'(3)) = 3 - j \\ \Leftrightarrow \quad & \sigma'(3) = 4 - \sigma^{-1}(3 - j) \end{aligned}$$

so the $\sim_2$ equivalence class of (τ_j, τ') is shown in Table 1.

Using this characterisation of $S_{2\,\tau_j\,\tau'}$ we see that expression (3.2.14) does not evaluate to 0 provided

$$\widetilde{\beta}_1 + \widetilde{\beta}_2 + \widetilde{\beta}_3 \neq 0. \tag{3.2.15}$$

$\sigma =$	$\begin{pmatrix}1-j\\2-j\\3-j\end{pmatrix}$	$\begin{pmatrix}2-j\\3-j\\1-j\end{pmatrix}$	$\begin{pmatrix}3-j\\1-j\\2-j\end{pmatrix}$	$\begin{pmatrix}2-j\\1-j\\3-j\end{pmatrix}$	$\begin{pmatrix}1-j\\3-j\\2-j\end{pmatrix}$	$\begin{pmatrix}3-j\\2-j\\1-j\end{pmatrix}$
$\sigma' \in$	$\left\{\begin{pmatrix}3\\2\\1\end{pmatrix}, \begin{pmatrix}2\\3\\1\end{pmatrix}\right\}$	$\left\{\begin{pmatrix}1\\3\\2\end{pmatrix}, \begin{pmatrix}3\\1\\2\end{pmatrix}\right\}$	$\left\{\begin{pmatrix}2\\1\\3\end{pmatrix}, \begin{pmatrix}1\\2\\3\end{pmatrix}\right\}$	$\left\{\begin{pmatrix}3\\2\\1\end{pmatrix}, \begin{pmatrix}2\\3\\1\end{pmatrix}\right\}$	$\left\{\begin{pmatrix}1\\3\\2\end{pmatrix}, \begin{pmatrix}3\\1\\2\end{pmatrix}\right\}$	$\left\{\begin{pmatrix}2\\1\\3\end{pmatrix}, \begin{pmatrix}1\\2\\3\end{pmatrix}\right\}$

TABLE 1. The $\sim_2$ equivalence class of (τ_j, τ')

$\sigma =$	$\begin{pmatrix}1-j\\2-j\\3-j\end{pmatrix}$	$\begin{pmatrix}2-j\\3-j\\1-j\end{pmatrix}$	$\begin{pmatrix}3-j\\1-j\\2-j\end{pmatrix}$	$\begin{pmatrix}2-j\\1-j\\3-j\end{pmatrix}$	$\begin{pmatrix}1-j\\3-j\\2-j\end{pmatrix}$	$\begin{pmatrix}3-j\\2-j\\1-j\end{pmatrix}$
$\sigma' \in$	$\left\{\begin{pmatrix}3\\2\\1\end{pmatrix}, \begin{pmatrix}3\\1\\2\end{pmatrix}\right\}$	$\left\{\begin{pmatrix}1\\3\\2\end{pmatrix}, \begin{pmatrix}1\\2\\3\end{pmatrix}\right\}$	$\left\{\begin{pmatrix}2\\1\\3\end{pmatrix}, \begin{pmatrix}2\\3\\1\end{pmatrix}\right\}$	$\left\{\begin{pmatrix}2\\1\\3\end{pmatrix}, \begin{pmatrix}2\\3\\1\end{pmatrix}\right\}$	$\left\{\begin{pmatrix}3\\2\\1\end{pmatrix}, \begin{pmatrix}3\\1\\2\end{pmatrix}\right\}$	$\left\{\begin{pmatrix}1\\3\\2\end{pmatrix}, \begin{pmatrix}1\\2\\3\end{pmatrix}\right\}$

TABLE 2. The $\sim_1$ equivalence class of (τ_j, τ')

If $a = -i$ then $k = 1$ and expression (3.2.13) simplifies further to

$$\widetilde{\beta}_1\widetilde{\beta}_2\widetilde{\beta}_3 \sum_{(\sigma,\sigma')\in S_{2\,\tau_j\,\tau'}} \operatorname{sgn}(\sigma)\omega^{-\sum_{m=1}^{3} m\sigma(4-m)} \frac{1}{\widetilde{\beta}_{\sigma'(1)}} \tag{3.2.16}$$

The definition (B.3.4) of $S_{1\,\tau_j\,\tau'}$ simplifies to $(\sigma, \sigma') \in S_{2\,\tau_j\,\tau'}$ if and only if

$$\begin{aligned} & \sigma c c' \sigma'(1) = \tau_j c c' \tau'(1) \\ \Leftrightarrow \quad & \sigma(4 - \sigma'(1)) = 1 - j \\ \Leftrightarrow \quad & \sigma'(1) = 4 - \sigma^{-1}(1-j) \end{aligned}$$

so the $\sim_1$ equivalence class of (τ_j, τ') is shown in Table 1.

Using this characterisation of $S_{1\,\tau_j\,\tau'}$ we see that expression (3.2.16) does not evaluate to 0 provided

$$\frac{1}{\widetilde{\beta}_1} + \frac{1}{\widetilde{\beta}_2} + \frac{1}{\widetilde{\beta}_3} \neq 0. \tag{3.2.17}$$

REMARK 3.25. The conditions for Assumption 3.2 derived in Example 3.24 using Theorem 3.23 are necessary as well as sufficient. We consider a particular example that violates these conditions and show that it is does not satisfy Assumption 3.2, noting that the general argument is identical but notationally a little more complex. Let $a = i$, hence

$$\widetilde{D}_1 \subseteq \left\{\rho \in \mathbb{C} : 0 < \arg\rho < \frac{\pi}{3}\right\},$$

and

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix},$$

so that

$$\mathcal{A}(\rho) = \begin{pmatrix} -c_2(\rho)(e^{-i\rho}-1) & -c_1(\rho)(e^{-i\rho}-1) & -c_0(\rho)(e^{-i\rho}+2) \\ -c_2(\rho)(e^{-i\omega\rho}-1) & -\omega c_1(\rho)(e^{-i\omega\rho}-1) & -\omega^2 c_0(\rho)(e^{-i\omega\rho}+2) \\ -c_2(\rho)(e^{-i\omega^2\rho}-1) & -\omega^2 c_1(\rho)(e^{-i\omega^2\rho}-1) & -\omega c_0(\rho)(e^{-i\omega^2\rho}+2) \end{pmatrix}.$$

We calculate

$$\begin{aligned}\Delta_{\mathrm{PDE}}(\rho) = (\omega-\omega^2)c_2(\rho)c_1(\rho)c_0(\rho)\Big[9 + (2-2)(e^{i\rho}+e^{i\omega\rho}+e^{i\omega^2\rho}) \\ +(1-4)(e^{-i\rho}+e^{-i\omega\rho}+e^{-i\omega^2\rho})\Big],\end{aligned}$$

as expected, the failure of Condition 3.22 causes the coefficients of $e^{i\omega^j\rho}$ to cancel one another for each j,

$$\begin{aligned}&= 3(\omega-\omega^2)c_2(\rho)c_1(\rho)c_0(\rho)\left[3-(e^{-i\rho}+e^{-i\omega\rho}+e^{-i\omega^2\rho})\right], \\ \eta_3(\rho) &= (\omega^2-\omega)c_2(\rho)c_1(\rho)c_0(\rho)\sum_{j=0}^{2}\omega^j\hat{q}_T(\omega^j\rho)(e^{i\omega^j\rho}-e^{-i\omega^{j+1}\rho}-e^{-i\omega^{j+2}\rho}+1).\end{aligned}$$

Consider a sequence, $(\rho_j)_{j\in\mathbb{N}}$ which lies on the intersection of the ray $\arg\rho = \frac{\pi}{12}$ with $\widetilde{D}_1$ such that $\rho_j \to \infty$ as $j\to\infty$. It may be shown, using the same argument as presented in Example 3.16, that the sequence $\frac{\eta_3(\rho_j)}{\Delta_{\mathrm{PDE}}(\rho_j)} \to \infty$. This establishes the failure of Assumption 3.2.

It is a conjecture that, among third (or, more speculatively, odd) order problems with homogeneous, non-Robin boundary conditions Conditions 3.19 and 3.22 are necessary as well as sufficient for Assumption 3.2. That is, we conjecture that, at least for third order, Theorem 3.23 may be strengthened to an equivalence.

Example 3.26. We extend Example 3.24 to arbitrary odd order. Let $n = 2\nu-1$ for some integer $\nu \geqslant 2$ and the boundary coefficient matrix be given by

$$A = \begin{pmatrix} 1 & \widetilde{\beta}_n & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \widetilde{\beta}_{n-1} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & \widetilde{\beta}_1 \end{pmatrix}, \tag{3.2.18}$$

for $\widetilde{\beta}_j \in \mathbb{R}\setminus\{0\}$ so that the problem is pseudoperiodic. We check for which values of $\widetilde{\beta}_j$ Assumption 3.2 holds.

All three boundary conditions couple the ends of the space interval so $L = R = 0$ and $C = n$. This ensures that, for $a = \pm i$, Condition 3.19 holds.

We adopt the notation of Condition 3.22, with c' the identity permutation on $\{1,2,\dots,n\}$ and $c(m) = n+1-m$ on the same domain, hence $\tau_j c(m) = m-j$. We evaluate expression (3.2.8), which simplifies to

$$\sum_{(\sigma,\sigma')\in S_{k\,\tau_j\,\tau'}} \mathrm{sgn}(\sigma)\omega^{-\sum_{m=1}^{n} m\sigma(n+1-m)} \prod_{m=k+1}^{n} \widetilde{\beta}_{\sigma'(m)} \tag{3.2.19}$$

for each j, where

$$k = \begin{cases} \nu & a = i, \\ \nu - 1 & a = -i. \end{cases}$$

As $\mathcal{R}$ is empty $(\sigma, \sigma') \in S_{k\,\tau_j\,\tau'}$ if and only if

$$\begin{aligned} & \{\sigma c c' \sigma'(p) : p \in \{1, 2, \ldots, k\}\} = \{\tau_j c c' \tau'(p) : p \in \{1, 2, \ldots, k\}\} \\ \Leftrightarrow \quad & \{\sigma(n + 1 - \sigma'(p)) : p \in \{1, 2, \ldots, k\}\} = \{1 - j, 2 - j \ldots, k - j\} \end{aligned} \tag{3.2.20}$$

but if $(\sigma, \sigma') \in S_{k\,\tau_j\,\tau'}$ then $(\sigma, \sigma'') \in S_{k\,\tau_j\,\tau'}$ if and only if

$$\begin{aligned} & \forall\, q \in \{1, 2, \ldots, k\}\ \exists\, p \in \{1, 2, \ldots, k\} : \sigma''(q) = \sigma'(p) \\ \Leftrightarrow \quad & \forall\, q \in \{k + 1, k + 2, \ldots, n\}\ \exists\, p \in \{k + 1, k + 2, \ldots, n\} : \sigma''(q) = \sigma'(p). \end{aligned}$$

Hence, for any given $\sigma \in S_n$ there exists a σ' for which $(\sigma, \sigma') \in S_{k\,\tau_j\,\tau'}$ but the choice of such a σ' does not affect the product

$$\prod_{m=k+1}^{n} \widetilde{\beta}_{\sigma'(m)}$$

in expression (3.2.19) and there are $k!(n-k)! = \nu!(\nu-1)!$ choices of σ'. So any particular choice of σ' will suffice, provided we multiply by $\nu!(\nu - 1)!$. Given $\sigma \in S_n$, define $\sigma' \in S_n$ such that

$$\sigma(n + 1 - \sigma'(p)) = p - j.$$

It is clear that (σ, σ') satisfies condition (3.2.20) but as σ is a bijection we may obtain an explicit expression

$$\sigma'(p) = n + 1 - \sigma^{-1}(p - j).$$

Expression (3.2.19) may now be simplified to

$$\nu!(\nu - 1)! \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \omega^{-\sum_{m=1}^{n} m\sigma(n+1-m)} \prod_{m=k+1}^{n} \widetilde{\beta}_{n+1-\sigma^{-1}(m-j)}. \tag{3.2.21}$$

Making the substitution $\pi(m) = \sigma^{-1}(m - j)$, for which $\sigma(n + 1 - m) = \pi^{-1}(n + 1 - m) - j$ and $\operatorname{sgn}(\pi) = (-1)^{(n-1)j} \operatorname{sgn}(\sigma) = \operatorname{sgn}(\sigma)$, expression (3.2.21) may be written

$$\nu!(\nu - 1)! \sum_{\pi \in S_n} \operatorname{sgn}(\pi) \omega^{-\sum_{m=1}^{n} m(\pi^{-1}(n+1-m)-j)} \prod_{m=k+1}^{n} \widetilde{\beta}_{n+1-\pi(m)}. \tag{3.2.22}$$

Expression (3.2.22) evaluates to zero if and only if

$$\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \omega^{-\sum_{m=1}^{n} m\sigma^{-1}(n+1-m)} \prod_{m=k+1}^{n} \widetilde{\beta}_{n+1-\sigma(m)} \tag{3.2.23}$$

evaluates to zero. By Theorem 3.23, a sufficient condition for Assumption 3.2 to hold is that expression (3.2.23) is nonzero for

$$k = \begin{cases} \nu & a = i, \\ \nu - 1 & a = -i. \end{cases}$$

EXAMPLE 3.27. Let the boundary conditions be simple (hence uncoupled and non-Robin) and such that

$$R = \begin{cases} \nu, \\ \nu - 1, \end{cases} \qquad L = \begin{cases} \nu - 1, & a = i, \\ \nu, & a = -i. \end{cases} \tag{3.2.24}$$

Note these conditions on R and L are precisely those proven to be necessary and sufficient for well-posedness of the boundary value problem in [**53**].

Clearly Condition 3.19 holds. To show these boundary conditions satisfy Condition 3.22 we must show that expression (3.2.9), that is

$$\sum_{\substack{\sigma\in S_n: \\ \forall\ m\in\mathcal{R}\ \exists\ p\in\mathcal{R}: \\ \tau_j r(m)=\sigma r(p)}} \operatorname{sgn}(\sigma)\omega^{-\sum_{m\in\mathcal{R}}\sigma r(m)m-\sum_{m\in\mathcal{L}}\sigma l(m)m} \tag{3.2.25}$$

does not evaluate to zero for any j.

By definition (3.2.7) of τ_j, the requirements on the $\sigma \in S_n$ indexing the first sum in expression (3.2.25) are equivalent to

$$\sigma \in S_n : \ \forall\, m \in \{1, 2, \ldots, R\}\ \exists\, p \in \mathcal{R} : m - j = \sigma r(p).$$

Hence any such σ is τ_j with a permutation applied to $r(\mathcal{R})$ and a permutation applied to $l(\mathcal{L})$ so expression (3.2.25) may be written

$$\begin{aligned} &\pm \sum_{(\pi,\pi')\in S_R\times S_L} \operatorname{sgn}(\pi)\operatorname{sgn}(\pi')\omega^{-\left(\sum_{m=1}^{R}(\pi(m)-j)\mathcal{R}_m+\sum_{m=1}^{L}(\pi'(m)+R+C-j)\mathcal{L}_m\right)} \\ &\qquad = \pm\omega^{j\sum_{p\in\mathcal{R}}p+(j+R)\sum_{p\in\mathcal{L}}p} \sum_{(\pi,\pi')\in S_R\times S_L} \operatorname{sgn}(\pi)\operatorname{sgn}(\pi')\omega^{-\left(\sum_{m=1}^{R}\pi(m)\mathcal{R}_m+\sum_{m=1}^{L}\pi'(m)\mathcal{L}_m\right)}. \end{aligned}$$

Hence expression (3.2.25) being nonzero is equivalent to both Y and Z being nonzero, where

$$\begin{aligned} Y &= \sum_{\pi\in S_R} \operatorname{sgn}(\pi)\omega^{-\sum_{m=1}^{R}\pi(m)\mathcal{R}_m}, \\ Z &= \sum_{\pi\in S_L} \operatorname{sgn}(\pi)\omega^{-\sum_{m=1}^{L}\pi(m)\mathcal{L}_m}. \end{aligned}$$

But Y and Z are the determinants of Vandermonde matrices with leading terms $\omega^{-\mathcal{R}_m}$ for $m \in \{1, 2, \ldots, R\}$ and $\omega^{-\mathcal{L}_m}$ for $m \in \{1, 2, \ldots, L\}$ respectively. Hence Y and Z are nonzero.

REMARK 3.28. The relatively simple formulation of Condition 3.19 depends upon the boundary conditions being non-Robin. The equivalent of this condition may be formulated to allow for Robin boundary conditions but to do so requires defining four quantities to replace L, R and C.

3.2.2. Well-posedness

Theorem 3.23 gives sufficient conditions for the boundary conditions to satisfy Assumption 3.2. In this subsection we show that the holding of Assumption 3.2 is equivalent to the initial-boundary value problem being well-posed.

3.2.2.1. Assumption 3.2 implies well-posedness

Theorems 3.1 and 3.13 give an explicit representation of a unique solution to the initial-boundary value problem in terms of only known data provided Assumptions 3.2 and 3.3 both hold. It remains to be shown that Assumption 3.3 is not necesary.

Without Assumption 3.3 the expressions for I_2 and I_4 in equations (3.1.5) and (3.1.7) are not valid hence we must replace their representations in equations (3.1.9) and (3.1.11) with

$$I_2 = \left\{ \int_{-\mathbb{R}} + \left[\int_{\partial E^+} - \int_{\Gamma_0^{E+}} - \sum_{\substack{k\in \\ K^+\cup K_D^E}} \int_{\Gamma_k^E} \right] + \left[\int_{\Gamma_0^{E+}} + \sum_{\substack{k\in \\ K^+\cup K_D^E}} \int_{\Gamma_k^E} \right] \right\} P(\rho) \sum_{j\in J^+} \frac{\zeta_j(\rho)}{\Delta_{\mathrm{PDE}}(\rho)}\,\mathrm{d}\rho,$$

$$I_4 = \left\{ \int_{\mathbb{R}} + \left[\int_{\partial E^-} - \int_{\Gamma_0^{E-}} - \sum_{\substack{k\in \\ K^-\cup K_E^D}} \int_{\Gamma_k^E} \right] + \left[\int_{\Gamma_0^{E-}} + \sum_{\substack{k\in \\ K^-\cup K_E^D}} \int_{\Gamma_k^E} \right] \right\} \widehat{P}(\rho) \sum_{j\in J^-} \frac{\zeta_j(\rho)}{\Delta_{\mathrm{PDE}}(\rho)}\,\mathrm{d}\rho.$$

With this adjustment to the calculation in Section 3.1, we may derive an integral representation of the solution in terms of the known data only.

THEOREM 3.29. *Let the initial-boudary value problem* (2.1.1)–(2.1.3) *be well-posed and obey Assumption 3.2. Then its solution may be expressed as follows:*

If n is odd and $a = i$,

$$\begin{aligned} q(x,t) = \frac{i}{2} \sum_{\substack{k\in K^+ \\ \cup K^{D^+} \\ \cup K^{\mathbb{R}}\cup\{0\}}} \operatorname*{Res}_{\rho=\sigma_k} \frac{P(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^+} \zeta_j(\sigma_k) + \frac{i}{2} \sum_{\substack{k\in K^- \\ \cup K^{D^-}}} \operatorname*{Res}_{\rho=\sigma_k} \frac{\widehat{P}(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^-} \zeta_j(\sigma_k) \\ + \frac{1}{2\pi} \left\{ \int_{\partial E^+} - \int_{\Gamma_0^{E+}} - \sum_{\substack{k\in \\ K^+\cup K_D^E}} \int_{\Gamma_k^E} \right\} P(\rho) \sum_{j\in J^+} \frac{\zeta_j(\rho)}{\Delta_{\mathrm{PDE}}(\rho)}\,\mathrm{d}\rho \\ + \frac{1}{2\pi} \left\{ \int_{\partial E^-} - \int_{\Gamma_0^{E-}} - \sum_{\substack{k\in \\ K^-\cup K_E^D}} \int_{\Gamma_k^E} \right\} \widehat{P}(\rho) \sum_{j\in J^-} \frac{\zeta_j(\rho)}{\Delta_{\mathrm{PDE}}(\rho)}\,\mathrm{d}\rho \\ - \frac{1}{2\pi} \left\{ \sum_{k\in K^{\mathbb{R}}} \int_{\Gamma_k^E} + \int_{\Gamma_0^-} + \int_{\mathbb{R}} \right\} P(\rho) \left(\frac{1}{\Delta_{\mathrm{PDE}}(\rho)} - 1 \right) H(\rho)\,\mathrm{d}\rho, \quad (3.2.26) \end{aligned}$$

If n is odd and $a = -i$,

$$q(x,t) = \frac{i}{2} \sum_{\substack{k \in K^+ \\ \cup K^{D^+}}} \operatorname*{Res}_{\rho=\sigma_k} \frac{P(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j \in J^+} \zeta_j(\sigma_k) + \frac{i}{2} \sum_{\substack{k \in K^- \\ \cup K^{D^-} \\ \cup K^{\mathbb{R}} \cup \{0\}}} \operatorname*{Res}_{\rho=\sigma_k} \frac{\widehat{P}(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j \in J^-} \zeta_j(\sigma_k)$$
$$+ \frac{1}{2\pi} \left\{ \int_{\partial E^+} - \int_{\Gamma_0^{E^+}} - \sum_{\substack{k \in \\ K^+ \cup K_D^E}} \int_{\Gamma_k^E} \right\} P(\rho) \sum_{j \in J^+} \frac{\zeta_j(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \, \mathrm{d}\rho$$
$$+ \frac{1}{2\pi} \left\{ \int_{\partial E^-} - \int_{\Gamma_0^{E^-}} - \sum_{\substack{k \in \\ K^- \cup K_E^D}} \int_{\Gamma_k^E} \right\} \widehat{P}(\rho) \sum_{j \in J^-} \frac{\zeta_j(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \, \mathrm{d}\rho$$
$$+ \frac{1}{2\pi} \left\{ \sum_{k \in K^{\mathbb{R}}} \int_{\Gamma_k^E} + \int_{\Gamma_0^+} - \int_{\mathbb{R}} \right\} P(\rho) \left(\frac{1}{\Delta_{\mathrm{PDE}}(\rho)} - 1 \right) H(\rho) \, \mathrm{d}\rho, \quad (3.2.27)$$

If n is even and $a = \pm i$,

$$q(x,t) = \frac{i}{2} \sum_{\substack{k \in K^+ \\ \cup K^{D^+} \\ \cup K_E^D}} \operatorname*{Res}_{\rho=\sigma_k} \frac{P(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j \in J^+} \zeta_j(\sigma_k) + \frac{i}{2} \sum_{\substack{k \in K^- \\ \cup K^{D^-} \\ \cup K_D^E}} \operatorname*{Res}_{\rho=\sigma_k} \frac{\widehat{P}(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j \in J^-} \zeta_j(\sigma_k)$$
$$+ \frac{i}{4} \frac{1}{\Delta_{\mathrm{PDE}}{}'(0)} \, \mathrm{d}\rho \sum_{j \in J^+ \cup J^-} \zeta_j(0)$$
$$+ \frac{1}{2\pi} \left\{ \int_{\partial E^+} - \int_{\Gamma_0^{E^+}} - \sum_{\substack{k \in \\ K^+ \cup K_D^E}} \int_{\Gamma_k^E} \right\} P(\rho) \sum_{j \in J^+} \frac{\zeta_j(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \, \mathrm{d}\rho$$
$$+ \frac{1}{2\pi} \left\{ \int_{\partial E^-} - \int_{\Gamma_0^{E^-}} - \sum_{\substack{k \in \\ K^- \cup K_E^D}} \int_{\Gamma_k^E} \right\} \widehat{P}(\rho) \sum_{j \in J^-} \frac{\zeta_j(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \, \mathrm{d}\rho$$
$$+ \frac{1}{2\pi} \left\{ \sum_{k \in K_E^D} \int_{\Gamma_k^E} + \frac{1}{2} \int_{\Gamma_0^-} - \int_0^{-\infty} - \sum_{k \in K_D^E} \int_{\Gamma_k^E} - \frac{1}{2} \int_{\Gamma_0^+} + \int_0^{\infty} \right\}$$
$$P(\rho) \left(\frac{1}{\Delta_{\mathrm{PDE}}(\rho)} - 1 \right) H(\rho) \, \mathrm{d}\rho, \quad (3.2.28)$$

If n is even and $a = e^{i\theta}$,

$$q(x,t) = \frac{i}{2} \sum_{\substack{k\in K^+ \\ \cup K^{D^+} \\ \cup K^{\mathbb{R}} \cup \{0\}}} \operatorname*{Res}_{\rho=\sigma_k} \frac{P(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^+} \zeta_j(\sigma_k) + \frac{i}{2} \sum_{\substack{k\in K^- \\ \cup K^{D^-}}} \operatorname*{Res}_{\rho=\sigma_k} \frac{\widehat{P}(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^-} \zeta_j(\sigma_k)$$

$$+ \frac{1}{2\pi} \left\{ \int_{\partial E^+} - \int_{\Gamma_0^{E^+}} - \sum_{\substack{k\in \\ K^+\cup K_D^E}} \int_{\Gamma_k^E} \right\} P(\rho) \sum_{j\in J^+} \frac{\zeta_j(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \,\mathrm{d}\rho$$

$$+ \frac{1}{2\pi} \left\{ \int_{\partial E^-} - \int_{\Gamma_0^{E^-}} - \sum_{\substack{k\in \\ K^-\cup K_E^D}} \int_{\Gamma_k^E} \right\} \widehat{P}(\rho) \sum_{j\in J^-} \frac{\zeta_j(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \,\mathrm{d}\rho$$

$$- \frac{1}{2\pi} \left\{ \sum_{k\in K^{\mathbb{R}}} \int_{\Gamma_k^-} + \int_{\Gamma_0^-} + \int_{\mathbb{R}} \right\} P(\rho) \left(\frac{1}{\Delta_{\mathrm{PDE}}(\rho)} - 1 \right) H(\rho) \,\mathrm{d}\rho. \quad (3.2.29)$$

Note that if the boundary conditions are homogeneous then $H(\rho) = 0$ so the last integral evaluates to 0 in each case.

As indicated above, the proof of Theorem 3.29 for well-posed problems is a simple derivation. It remains to be shown that Assumption 3.2 implies well-posedness of the initial-boundary value problem a priori. Using the following Lemma, we appeal to the arguments presented in [**27**] and [**53**].

LEMMA 3.30. *Let $n \in \mathbb{N}$ and let $a \in \mathbb{C}$ be such that $a = \pm i$ if n is odd and $\mathrm{Re}(a) \geqslant 0$ if n is even. Let $D = \{\rho \in \mathbb{C}$ such that $\mathrm{Re}(a\rho^n) < 0\}$ and let the polynomials c_j be defined by $c_j(\rho) = -a\rho^n(i\rho)^{-(j+1)}$. Let $\alpha_{j\,k}, \beta_{j\,k} \in \mathbb{R}$ be such that the matrix*

$$A = \begin{pmatrix} \alpha_{1\,n-1} & \beta_{1\,n-1} & \alpha_{1\,n-2} & \beta_{1\,n-2} & \dots & \alpha_{1\,0} & \beta_{1\,0} \\ \alpha_{2\,n-1} & \beta_{2\,n-1} & \alpha_{2\,n-2} & \beta_{2\,n-2} & \dots & \alpha_{2\,0} & \beta_{2\,0} \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \alpha_{n\,n-1} & \beta_{n\,n-1} & \alpha_{n\,n-2} & \beta_{n\,n-2} & \dots & \alpha_{n\,0} & \beta_{n\,0} \end{pmatrix}$$

is in reduced row-echelon form. Let $q_0 \in C^\infty[0,1]$ and $h_k \in C^\infty[0,T]$ for each $k \in \{1,2,\dots,n\}$ be compatible in the sense of equation (2.1.4). Let $U : \mathbb{C} \to \mathbb{C}$ be defined by equation (2.2.17), let $\mathcal{A} : \mathbb{C} \to \mathbb{C}^{n\times n}$ be defined by equation (2.2.19) and let $\widehat{A} \in \mathbb{R}^{n\times n}$ be defined by equation (2.2.20). Let $\zeta_j, \eta_j, \Delta_{\mathrm{PDE}} : \mathbb{C} \to \mathbb{C}$ be given by Definition 2.19, where $q_T : [0,1] \to \mathbb{C}$ is some function such that Assumption 3.2 is satisfied. Let the functions $\widetilde{f}_j, \widetilde{g}_j : [0,T] \to \mathbb{C}$ be defined by equation (2.3.4). Let $f_j, g_j : [0,T] \to \mathbb{C}$ be the functions for which

$$\widetilde{f}_j(\rho) = \int_0^T e^{a\rho^n t} f_j(t) \,\mathrm{d}t, \quad \widetilde{g}_j(\rho) = \int_0^T e^{a\rho^n t} g_j(t) \,\mathrm{d}t, \quad \rho \in \mathbb{C}. \quad (3.2.30)$$

*Then $\{f_j, g_j : j \in \{0,1,\dots,n-1\}\}$ is an admissible set in the sense of Definition 1.3 of [**27**].*

The proof of Lemma 3.30 may be found in Section B.4.

COROLLARY 3.31. *Let the initial-boundary value problem specified by equations* (2.1.1)–(2.1.3) *obey Assumption 3.2. Then the problem is well-posed and its solution may be found using Theorem 3.29.*

Corollary 3.31 is a restatement of Theorems 1.1 and 1.2 of [**27**]. For this reason we refer the reader to the proof presented in Section 4 of that paper. The only difference is that we make use of the above Lemma 3.30 in place of Proposition 4.1. We have not yet shown the reverse, that well-posedness of the initial-boundary value problem implies Assumption 3.2 holds.

3.2.2.2. Assumption 3.2 holds for a well-posed problem

We now present the converse of Corollary 3.31.

THEOREM 3.32. *If the initial-boundary value problem* (2.1.1)–(2.1.3) *is well-posed, in the sense that it admits a unique solution* $q \in C^\infty([0,1]\times[0,T])$*, then Assumption 3.2 holds.*

PROOF. As the problem is well-posed, it has a unique solution q and that solution is C^∞ smooth on Ω. Hence, if we evaluate q at any fixed value of x or t we are left with a $C^\infty[0,T]$ or $C^\infty[0,1]$ function respectively. In particular,

$$\begin{aligned} q(\cdot,T) &= q_T \in C^\infty[0,1], \\ \partial_x^k q(0,\cdot) &= f_k \in C^\infty[0,T], \text{ for } k \in \{0,1,\dots,n-1\}, \\ \partial_x^k q(1,\cdot) &= g_k \in C^\infty[0,T], \text{ for } k \in \{0,1,\dots,n-1\}. \end{aligned}$$

Of course $q(\cdot,0) = q_0 \in C^\infty[0,1]$ is given in the problem. As $q_T \in C^\infty$, the Fourier transform, $\hat{q}_T$, of q_T is entire hence η_j is entire for each $j \in \{1,2,\dots,n\}$. As each boundary function is smooth their t-transforms, as defined by equations (2.1.6), are entire and decay as $\rho \to \infty$ from within D. By equation (2.3.4), each $\tilde{f}_k$ and each $\tilde{g}_k$ is given by

$$\frac{\zeta_j(\rho) - e^{a\rho^n T}\eta_j(\rho)}{\Delta_{\text{PDE}}(\rho)c_k(\rho)} \tag{3.2.31}$$

for a particular choice of j.

We define the new complex set

$$D' = \{\rho \in D \text{ such that } -\text{Re}(a\rho^n T) > 2n|\rho|\}.$$

As $D' \subset D$, the ratio (3.2.31) is analytic on D' and decays as $\rho \to \infty$ from within D'. Note also that for each $p \in \{1,2,\dots,n\}$ there exists $R > 0$ such that the set

$$D'_p = \widetilde{D}_p \cap D' \setminus \overline{B}(0,R)$$

is simply connected, open and unbounded.

By its definition (2.3.3), $\Delta_{\text{PDE}}(\rho)$ is an exponential polynomial whose terms are each

$$X(\rho)e^{-i\sum_{r\in R}\omega^r\rho}$$

where X is a polynomial of degree less than n^2 and $R \subset \{0,1,2,\dots,n-1\}$ is an index set. Hence

$$\Delta_{\text{PDE}}(\rho) = o(e^{n|\rho|}\rho^{n^2}) \text{ as } \rho \to \infty \text{ or as } \rho \to 0.$$

As ζ_j and η_j also decay and grow no faster than $o(e^{n|\rho|})$, the ratios

$$\frac{\zeta_j(\rho)}{\Delta_{\text{PDE}}(\rho)c_k(\rho)}, \quad \frac{\eta_j(\rho)}{\Delta_{\text{PDE}}(\rho)c_k(\rho)} = o(e^{2n|\rho|}\rho^{n^2+n-1}), \text{ as } \rho \to \infty.$$

Hence the ratio

$$\frac{e^{a\rho^n T}\eta_j(\rho)}{\Delta_{\text{PDE}}(\rho)c_k(\rho)} \tag{3.2.32}$$

decays as $\rho \to \infty$ from within D' and away from the zeros of Δ_{PDE}. However the ratio

$$\frac{\zeta_j(\rho)}{\Delta_{\text{PDE}}(\rho)c_k(\rho)} \tag{3.2.33}$$

is the sum of ratios (3.2.31) and (3.2.32) hence it also decays as $\rho \to \infty$ from within D' and away from the zeros of Δ_{PDE}.

The terms in each of $\zeta_j(\rho)$ and $\Delta_{\text{PDE}}(\rho)$ are exponentials, each of which either decays or grows as $\rho \to \infty$ from within one of the simply connected components $\widetilde{D}_p$ of $\widetilde{D}$. Hence as $\rho \to \infty$ from within each component $\widetilde{D}_p$ the ratio (3.2.33) either decays or grows. But as observed above, these ratios all decay as $\rho \to \infty$ from within each D'_p. Hence the ratio (3.2.33) decays as $\rho \to \infty$ from within $\widetilde{D}_p$.

Now it is a simple observation that the ratio

$$\frac{\eta_j(\rho)}{\Delta_{\text{PDE}}(\rho)c_k(\rho)} \tag{3.2.34}$$

must also decay as $\rho \to \infty$. Indeed ratio (3.2.34) is the same as ratio (3.2.33) but with q_T replacing q_0 and, by well-posedness of the problem, we know that $q_T \in C^\infty[0,1]$ also. Finally, the exponentials in η_j and Δ_{PDE} ensure that the ratio

$$\frac{\eta_j(\rho)}{\Delta_{\text{PDE}}(\rho)} \tag{3.2.35}$$

also decays as $\rho \to \infty$ from within $\widetilde{D}_p$. Indeed the transforms that multiply each term in η_j ensure that the decay of ratio (3.2.34) must come from the decay of ratio (3.2.35), not from $\frac{1}{c_k(\rho)}$. □

3.3. Series representation

In this section we investigate Assumption 3.3, giving some sufficient conditions for it to hold. We also investigate how if this assumption holds but Assumption 3.2 does not then we may specify $q_T(x)$ instead of $q_0(x)$ and get a well-posed *final*-boundary value problem. As in Section 3.2 we begin with two examples.

EXAMPLE 3.33. We consider the same initial-boundary value problems as in Example 3.14. The ζ_j are the same as the η_j calculated in the previous example but with $\hat{q}_0$ in place of $\hat{q}_T$. If $a = -i$ then $\widetilde{E}_1 \subseteq \{\rho \in \mathbb{C} \text{ such that } 0 < \arg\rho < \frac{\pi}{n}\}$ and the calculation from the previous example may be used to show that the ratio

$$\frac{\zeta_1(\rho)}{\Delta_{\text{PDE}}(\rho)}$$

is bounded within $\widetilde{E}_1$ and decaying as $\rho \to \infty$ from within $\widetilde{E}_1$. Indeed similar calculations may be performed for each ζ_k and for each $\widetilde{E}_j$ for $a = \pm i$ to show that Assumption 3.3 holds.

EXAMPLE 3.34. We present an example with the same partial differential equation and boundary conditions as Example 3.16 but we replace the initial condition with a final condition, thus specifying a *final*-boundary value problem instead of an initial-boundary value problem.

Consider the final-boundary value problem specified by equations (2.1.1) and (2.1.3) with $n = 3$, $a = i$ and the final condition

$$q(x, T) = q_T(x),$$

where $q_T : [0, 1] \to \mathbb{R}$ is a given, sufficiently smooth initial datum, and that the boundary coefficient matrix is

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The reduced global relation matrix and reduced boundary coefficient matrix are unnaffected by the fact we have specified the final data and left the initial function unknown so they are the same as in Example 3.16. Following Definition 2.19 we calculate

$$\begin{aligned}
\Delta_{\mathrm{PDE}}(\rho) &= (\omega^2 - \omega)c_2^2(\rho)c_1(\rho)\sum_{k=0}^{2}\omega^k e^{-i\omega^k\rho}, \\
\zeta_1(\rho) &= c_2(\rho)c_1(\rho)\sum_{k=0}^{2}\omega^{k+2}\left(e^{-i\omega^k\rho}\hat{q}_0(\omega^{k+1}\rho) - e^{-i\omega^{k+1}\rho}\hat{q}_0(\omega^k\rho)\right), \\
\zeta_2(\rho) &= -(\omega^2 - \omega)c_2(\rho)c_1(\rho)\sum_{k=0}^{2}\omega^k\hat{q}_0(\omega^k\rho), \\
\zeta_3(\rho) &= -c_2^2(\rho)\sum_{k=0}^{2}\left(e^{-i\omega^k\rho}\hat{q}_0(\omega^{k+1}\rho) - e^{-i\omega^{k+1}\rho}\hat{q}_0(\omega^k\rho)\right) \text{ and} \\
\zeta_4(\rho) &= \zeta_5(\rho) = \zeta_6(\rho) = 0,
\end{aligned}$$

as in Example 3.16. As $a = i$, the sets D and E are also the same as in Example 3.16 but, as we are now interested in Assumption 3.3, we are interested in the behaviour of certain meromorphic functions on the set E instead of D.

If $\rho \to \infty$ from within $\widetilde{E}_1 = \{\rho \in \mathbb{C} \text{ such that } \frac{\pi}{n} < \arg\rho < \frac{2\pi}{n}\}$ then the exponentials $e^{-i\rho}$, $e^{i\omega\rho}$ and $e^{i\omega^2\rho}$ grow while the exponentials $e^{i\rho}$, $e^{-i\omega\rho}$ and $e^{-i\omega^2\rho}$ decay. Hence the function $\hat{q}_T(\rho)$ also grows but $\hat{q}_T(\omega\rho)$ and $\hat{q}_T(\omega^2\rho)$ are decaying. Hence the dominant term in

$$\begin{aligned}
&\Delta_{\mathrm{PDE}}(\rho) \text{ is } (\omega^2 - \omega)c_2^2(\rho)c_1(\rho)e^{-i\rho} \text{ and the dominant term in} \\
&\eta_1(\rho) \qquad \text{is } c_2(\rho)c_1(\rho)\left\{\omega^2\left(e^{-i\rho}\hat{q}_0(\omega\rho) - e^{-i\omega\rho}\hat{q}_0(\rho)\right) + \omega\left(e^{-i\omega^2\rho}\hat{q}_0(\rho) - e^{-i\rho}\hat{q}_0(\omega^2\rho)\right)\right\}.
\end{aligned}$$

The ratio

$$\frac{c_2(\rho)c_1(\rho)\left\{\omega^2\left(e^{-i\rho}\hat{q}_0(\omega\rho)-e^{-i\omega\rho}\hat{q}_0(\rho)\right)+\omega\left(e^{-i\omega^2\rho}\hat{q}_0(\rho)-e^{-i\rho}\hat{q}_0(\omega^2\rho)\right)\right\}}{(\omega^2-\omega)c_2^2(\rho)c_1(\rho)e^{-i\rho}}$$
$$=\frac{1}{(\omega^2-\omega)c_2(\rho)}\left\{\omega^2\hat{q}_0(\omega\rho)-\omega\hat{q}_0(\omega^2\rho)+\left(\omega e^{-i\omega^2\rho}-\omega^2e^{-i\omega\rho}\right)\int_0^1 e^{i\rho(1-x)}\hat{q}_0(x)\,\mathrm{d}x\right\}$$

is decaying as $\rho\to\infty$ from within $\widetilde{E}_1$ because $c_2(\rho)$ is a monomial and, as noted above, the exponential $e^{i\rho(1-x)}$ is decaying for $x\in(0,1)$. Hence the ratio

$$\frac{\zeta_1(\rho)}{\Delta_{\mathrm{PDE}}(\rho)}$$

also decays as $\rho\to\infty$ from within $\widetilde{E}_1$.

It may be checked similarly that the other ratios

$$\frac{\zeta_j(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \tag{3.3.1}$$

decay as $\rho\to\infty$ from within $\widetilde{E}_k$ for each $j\in\{1,2,\dots,2n\}$ and for each $k\in\{1,2,\dots,n\}$, so that Assumption 3.3 holds. Note that Remark 3.15 also applies here; one must check all the ratios (3.3.1) decay in all the sectors $\widetilde{E}_k$ but, to be economical with space, the calculations are not presented here in full.

3.3.1. n odd, homogeneous, non-Robin

Using Lemma B.8 we construct the analogue of Theorem 3.23 to check for existence of a series representation of a solution. First we formulate a pair of conditions analogous to Conditions 3.19 and 3.22.

Condition 3.35. If $a=i$ then the $2\nu-1$ boundary conditons are such that

$$R\leqslant\nu-1\leqslant R+C$$

and if $a=-i$ then the $2\nu-1$ boundary conditons are such that

$$R\leqslant\nu\leqslant R+C$$

where R and C are defined by (3.2.3) and (3.2.4).

Condition 3.36. Let $(\mathcal{R}_p)_{p=1}^R$ be an ordering of the elements of $\mathcal{R}$ and $(\mathcal{L}_p)_{p=1}^L$ be an ordering of the elements of $\mathcal{L}$. Let the permutation $\tau_j\in S_n$ be defined by

$$p-j=\begin{cases}\tau_j r(\mathcal{R}_p) & \text{if } p\in\{1,2,\dots,R\},\\ \tau_j cc'(p-R) & \text{if } p\in\{R+1,R+2,\dots,R+C\},\\ \tau_j l(\mathcal{L}_{p-R-C}) & \text{if } p\in\{R+C+1,R+C+2,\dots,n\}.\end{cases} \tag{3.3.2}$$

Let τ' to be the identity permutation on S_C and

$$k=\begin{cases}\nu-1-R & a=i,\\ \nu-R & a=-i,\end{cases}$$

the boundary conditions are such that for each $j \in \{1, 2, \ldots, n\}$ the expression

$$\sum_{(\sigma,\sigma')\in S_{k\,\tau_j\,\tau'}} \operatorname{sgn}(\sigma)\omega^{-\sum_{m\in\mathcal{R}}\sigma r(m)m-\sum_{m\in\mathcal{C}}\sigma c(m)m-\sum_{m\in\mathcal{L}}\sigma l(m)m} \prod_{m=k+1}^{C} \widetilde{\beta}_{c'\sigma'(m)} \tag{3.3.3}$$

is nonzero if $k \geqslant 1$ or the expression

$$\sum_{\substack{\sigma\in S_n:\\ \forall\, m\in\mathcal{R}\ \exists\, p\in\mathcal{R}:\\ \tau_j r(m)=\sigma r(p)}} \operatorname{sgn}(\sigma)\omega^{-\sum_{m\in\mathcal{R}}\sigma r(m)m-\sum_{m\in\mathcal{C}}\sigma c(m)m-\sum_{m\in\mathcal{L}}\sigma l(m)m} \tag{3.3.4}$$

is nonzero if $k = 0$.

Note that in the case $\widetilde{\beta}_j = \beta$ for all $j \in \mathcal{C}$ expression (3.3.3) simplifies to

$$\sum_{\substack{\sigma\in S_n:\ \exists\ \sigma'\in S_C:\\ (\sigma,\sigma')\in S_{k\,\tau_j\,\tau'}}} \operatorname{sgn}(\sigma)\omega^{-\sum_{m\in\mathcal{R}}\sigma r(m)m-\sum_{m\in\mathcal{C}}\sigma c(m)m-\sum_{m\in\mathcal{L}}\sigma l(m)m}. \tag{3.3.5}$$

The set $S_{k\,\tau\,\tau'}$, the functions l, r and c and their domains $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{C}$ are given in Definition B.7 and Lemma B.8.

THEOREM 3.37. *Suppose that a final-boundary value problem is specified by equations* (2.1.1) *and* (2.1.3) *and the final condition*

$$q(x, T) = q_T(x) \tag{3.3.6}$$

where $q_T : [0, 1] \to \mathbb{R}$ is a given, sufficiently smooth initial datum, n is odd and that the boundary conditions are homogeneous and non-Robin and satisfy Conditions 3.35 and 3.36. Then Assumption 3.3 holds.

REMARK 3.38. Theorem 3.37 is formulated in terms of a final-boundary value problem instead of an intial-boundary value problem. This is done to emphasize the duality between Conditions 3.19 and 3.35 and between Conditions 3.22 and 3.36. It would not make sense to discuss the existence of a series representation of a solution to a problem that may not be well-posed so to state a theorem about the initial-boundary value problem we would have to require that the boundary conditions also obey Assumption 3.2, destroying the duality with Theorem 3.23. We will of course go on to do such a thing in Theorem 3.50.

Note also that Assumption 3.3 refers to the initial-boundary value problem so it should properly be restated to refer to this new final-boundary value problem but the meaning is clear. Also note that the ζ_j are now defined in terms of the *unknown* initial function as there is no initial datum, while the η_j are given in terms of the *known* final datum.

PROOF OF THEOREM 3.37. The proof mirrors that of Theorem 3.23.

If the boundary conditions obey Condition 3.35 then $0 \leqslant k \leqslant C$ in Condition 3.36 so the set $S_{k\,\tau_j\,\tau'}$ and the relevant expression (3.3.3) or (3.3.4) are all well defined.

Fix $j \in \{1, 2, \ldots, n\}$ and let $\rho \in \widetilde{E}_j$. Then the modulus of

$$e^{-i\sum_{y\in Y}\omega^y\rho} \tag{3.3.7}$$

is uniquely maximised for the index set

$$Y = \{j - 1, j, \ldots, j - 1 + R + k - 1\}.$$

By Condition 3.35 and Lemma B.8, $\Delta_{\mathrm{PDE}}(\rho)$ has a term given by that exponential multiplied by a polynomial coefficient given by the right hand side of equation (B.3.6) if $k \geqslant 1$ or equation (B.3.7) if $k = 0$, with τ replaced by τ_j. As $\rho \in E_j$, $\rho \neq 0$ so the coefficient is guaranteed to be nonzero by Condition 3.36.

As Y *uniquely* maximises the exponential (3.3.7) this exponential dominates all other terms in $\Delta_{\mathrm{PDE}}(\rho)$. But it also dominates all terms in $\zeta_j(\rho)$, that is those of the form

$$Z(\rho)e^{-i\rho\sum_{p\in P}\omega^p}\int_0^1 e^{-i\rho\omega^{p'}x}q_0(x)\,\mathrm{d}x$$

where $P \subsetneq \{0, 1, \ldots, n-1\}$ and $p' \notin P$. Hence the ratio

$$\frac{\zeta_j(\rho)}{\Delta_{\mathrm{PDE}}(\rho)}$$

is bounded in $\widetilde{E}_j$ for each $j \in \{1, 2, \ldots, n\}$ and decaying as $\rho \to \infty$ from within $\widetilde{E}_j$. □

We now present some examples illustrating the use of Theorem 3.37.

Example 3.39. Consider the odd-order final-boundary value problem with pseudoperiodic boundary coefficient matrix (3.2.18) as discussed in Example 3.26. Clearly Condition 3.35 is satisfied for $a = \pm i$ as $C = n$. In Condition 3.36 the value of k is given by

$$k = \begin{cases} \nu - 1 & a = i, \\ \nu & a = -i. \end{cases}$$

The same argument as presented in Example 3.26 now shows that the final-boundary value problem is well-posed provided expression (3.2.23) is non-zero for the the relevant value of k.

Consider the special case $n = 3$. Following the argument presented in Example 3.24, expression (3.2.23) simplifies to expression (3.2.17) for $a = i$ and expression (3.2.15) for $a = -i$.

Example 3.40. Let the boundary conditions be simple and such that

$$R = \begin{cases} \nu - 1, \\ \nu, \end{cases} \qquad L = \begin{cases} \nu, & a = i, \\ \nu - 1, & a = -i. \end{cases} \tag{3.3.8}$$

Hence, compared to condition (3.26), there are the opposite number of boundary conditions specified at the right and left ends of the interval. In particular, Condition 3.35 holds. The same argument as presented in Example 3.27 may now be used to show that Condition 3.36 holds.

3.3.2. Well-posed FBVP

We now state the analogue of Theorem 3.29

Theorem 3.41. *Given a final-boudary value problem specified by partial differential equation* (2.1.1)*, boundary conditions* (2.1.3) *and the final condition* (3.3.6) *that obeys Assumption 3.3 we may write its solution as follows:*

If n is odd and $a = i$,

$$q(x,t) = \frac{i}{2} \sum_{\substack{k\in K^+ \\ \cup K^{D^+} \\ \cup K^{\mathbb{R}} \cup \{0\}}} \operatorname*{Res}_{\rho=\sigma_k} \frac{P(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^+} e^{i\sigma_k^n T} \eta_j(\sigma_k) + \frac{i}{2} \sum_{\substack{k\in K^- \\ \cup K^{D^-}}} \operatorname*{Res}_{\rho=\sigma_k} \frac{\widehat{P}(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^-} e^{i\sigma_k^n T} \eta_j(\sigma_k)$$
$$+ \frac{1}{2\pi} \left\{ \int_{\partial E^+} - \int_{\Gamma_0^{E^+}} - \sum_{\substack{k\in \\ K^+ \cup K_D^E}} \int_{\Gamma_k^E} \right\} P(\rho) \sum_{j\in J^+} \frac{e^{i\rho^n T} \eta_j(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \,\mathrm{d}\rho$$
$$+ \frac{1}{2\pi} \left\{ \int_{\partial E^-} - \int_{\Gamma_0^{E^-}} - \sum_{\substack{k\in \\ K^- \cup K_E^D}} \int_{\Gamma_k^E} \right\} \widehat{P}(\rho) \sum_{j\in J^-} \frac{e^{i\rho^n T} \eta_j(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \,\mathrm{d}\rho$$
$$- \frac{1}{2\pi} \left\{ \sum_{k\in K^{\mathbb{R}}} \int_{\Gamma_k^E} + \int_{\Gamma_0^-} + \int_{\mathbb{R}} \right\} P(\rho) \left(\frac{1}{\Delta_{\mathrm{PDE}}(\rho)} - 1 \right) H(\rho) \,\mathrm{d}\rho, \quad (3.3.9)$$

If n is odd and $a = -i$,

$$q(x,t) = \frac{i}{2} \sum_{\substack{k\in K^+ \\ \cup K^{D^+}}} \operatorname*{Res}_{\rho=\sigma_k} \frac{P(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^+} e^{-i\sigma_k^n T} \eta_j(\sigma_k) + \frac{i}{2} \sum_{\substack{k\in K^- \\ \cup K^{D^-} \\ \cup K^{\mathbb{R}} \cup \{0\}}} \operatorname*{Res}_{\rho=\sigma_k} \frac{\widehat{P}(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^-} e^{-i\sigma_k^n T} \eta_j(\sigma_k)$$
$$+ \frac{1}{2\pi} \left\{ \int_{\partial E^+} - \int_{\Gamma_0^{E^+}} - \sum_{\substack{k\in \\ K^+ \cup K_D^E}} \int_{\Gamma_k^E} \right\} P(\rho) \sum_{j\in J^+} \frac{e^{-i\rho^n T} \eta_j(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \,\mathrm{d}\rho$$
$$+ \frac{1}{2\pi} \left\{ \int_{\partial E^-} - \int_{\Gamma_0^{E^-}} - \sum_{\substack{k\in \\ K^- \cup K_E^D}} \int_{\Gamma_k^E} \right\} \widehat{P}(\rho) \sum_{j\in J^-} \frac{e^{-i\rho^n T} \eta_j(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \,\mathrm{d}\rho$$
$$+ \frac{1}{2\pi} \left\{ \sum_{k\in K^{\mathbb{R}}} \int_{\Gamma_k^E} + \int_{\Gamma_0^+} - \int_{\mathbb{R}} \right\} P(\rho) \left(\frac{1}{\Delta_{\mathrm{PDE}}(\rho)} - 1 \right) H(\rho) \,\mathrm{d}\rho, \quad (3.3.10)$$

If n is even and $a = \pm i$,

$$q(x,t) = \frac{i}{2} \sum_{\substack{k\in K^+ \\ \cup K^{D^+} \\ \cup K_E^D}} \operatorname*{Res}_{\rho=\sigma_k} \frac{P(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^+} e^{\pm i\sigma_k^n T}\eta_j(\sigma_k) + \frac{i}{2} \sum_{\substack{k\in K^- \\ \cup K^{D^-} \\ \cup K_D^E}} \operatorname*{Res}_{\rho=\sigma_k} \frac{\widehat{P}(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^-} e^{\pm i\sigma_k^n T}\eta_j(\sigma_k)$$

$$+ \frac{i}{4}\frac{1}{\Delta_{\mathrm{PDE}}{}'(0)}\,\mathrm{d}\rho \sum_{j\in J^+\cup J^-} \eta_j(0)$$

$$+ \frac{1}{2\pi}\left\{ \int_{\partial E^+} - \int_{\Gamma_0^{E^+}} - \sum_{\substack{k\in \\ K^+\cup K_D^E}} \int_{\Gamma_k^E} \right\} P(\rho) \sum_{j\in J^+} \frac{e^{\pm i\rho^n T}\eta_j(\rho)}{\Delta_{\mathrm{PDE}}(\rho)}\,\mathrm{d}\rho$$

$$+ \frac{1}{2\pi}\left\{ \int_{\partial E^-} - \int_{\Gamma_0^{E^-}} - \sum_{\substack{k\in \\ K^-\cup K_E^D}} \int_{\Gamma_k^E} \right\} \widehat{P}(\rho) \sum_{j\in J^-} \frac{e^{\pm i\rho^n T}\eta_j(\rho)}{\Delta_{\mathrm{PDE}}(\rho)}\,\mathrm{d}\rho$$

$$+ \frac{1}{2\pi}\left\{ \sum_{k\in K_E^D} \int_{\Gamma_k^E} + \frac{1}{2}\int_{\Gamma_0^-} - \int_0^{-\infty} - \sum_{k\in K_D^E} \int_{\Gamma_k^E} - \frac{1}{2}\int_{\Gamma_0^+} + \int_0^{\infty} \right\}$$

$$P(\rho)\left(\frac{1}{\Delta_{\mathrm{PDE}}(\rho)} - 1\right) H(\rho)\,\mathrm{d}\rho, \quad (3.3.11)$$

If n is even and $a = e^{i\theta}$,

$$q(x,t) = \frac{i}{2} \sum_{\substack{k\in K^+ \\ \cup K^{D^+} \\ \cup K^{\mathbb{R}}\cup\{0\}}} \operatorname*{Res}_{\rho=\sigma_k} \frac{P(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^+} e^{a\sigma_k^n T}\eta_j(\sigma_k) + \frac{i}{2} \sum_{\substack{k\in K^- \\ \cup K^{D^-}}} \operatorname*{Res}_{\rho=\sigma_k} \frac{\widehat{P}(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^-} e^{a\sigma_k^n T}\eta_j(\sigma_k)$$

$$+ \frac{1}{2\pi}\left\{ \int_{\partial E^+} - \int_{\Gamma_0^{E^+}} - \sum_{\substack{k\in \\ K^+\cup K_D^E}} \int_{\Gamma_k^E} \right\} P(\rho) \sum_{j\in J^+} \frac{e^{a\rho^n T}\eta_j(\rho)}{\Delta_{\mathrm{PDE}}(\rho)}\,\mathrm{d}\rho$$

$$+ \frac{1}{2\pi}\left\{ \int_{\partial E^-} - \int_{\Gamma_0^{E^-}} - \sum_{\substack{k\in \\ K^-\cup K_E^D}} \int_{\Gamma_k^E} \right\} \widehat{P}(\rho) \sum_{j\in J^-} \frac{e^{a\rho^n T}\eta_j(\rho)}{\Delta_{\mathrm{PDE}}(\rho)}\,\mathrm{d}\rho$$

$$- \frac{1}{2\pi}\left\{ \sum_{k\in K^{\mathbb{R}}} \int_{\Gamma_k^-} + \int_{\Gamma_0^-} + \int_{\mathbb{R}} \right\} P(\rho)\left(\frac{1}{\Delta_{\mathrm{PDE}}(\rho)} - 1\right) H(\rho)\,\mathrm{d}\rho. \quad (3.3.12)$$

Note that if the boundary conditions are homogeneous then $H(\rho) = 0$ so the last integral evaluates to 0 in each case.

The proof is identical to the proof of Theorem 3.29 except that in each step we replace occurrences of $\zeta_j(\rho)$ with $e^{a\rho^n T}\eta_j(\rho)$ instead of the other way around.

The equivalents of Lemma 3.30, Corollary 3.31 and Theorem 3.32 are given below. Their proofs are analogous to the previous proofs.

LEMMA 3.42. *Let $n \in \mathbb{N}$ and let $a \in \mathbb{C}$ be such that $a = \pm i$ if n is odd and $\mathrm{Re}(a) \geqslant 0$ if n is even. Let $D = \{\rho \in \mathbb{C}$ such that $\mathrm{Re}(a\rho^n) < 0\}$ and let the polynomials c_j be defined by $c_j(\rho) = -a\rho^n(i\rho)^{-(j+1)}$. Let $\alpha_{j\,k}, \beta_{j\,k} \in \mathbb{R}$ be such that the matrix*

$$A = \begin{pmatrix} \alpha_{1\,n-1} & \beta_{1\,n-1} & \alpha_{1\,n-2} & \beta_{1\,n-2} & \dots & \alpha_{1\,0} & \beta_{1\,0} \\ \alpha_{2\,n-1} & \beta_{2\,n-1} & \alpha_{2\,n-2} & \beta_{2\,n-2} & \dots & \alpha_{2\,0} & \beta_{2\,0} \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \alpha_{n\,n-1} & \beta_{n\,n-1} & \alpha_{n\,n-2} & \beta_{n\,n-2} & \dots & \alpha_{n\,0} & \beta_{n\,0} \end{pmatrix}$$

is in reduced row-echelon form. Let $q_T \in C^\infty[0,1]$ and $h_k \in C^\infty[0,T]$ for each $k \in \{1,2,\dots,n\}$ be compatible in the sense that

$$\sum_{j=0}^{n-1} \alpha_{k\,j} \partial_x^j q_T(0) + \sum_{j=0}^{n-1} \beta_{k\,j} \partial_x^j q_T(1) = h_k(T)$$

holds. Let $\mathcal{A} : \mathbb{C} \to \mathbb{C}^{n\times n}$ be defined by equation (2.2.19) *and let $\widehat{A} \in \mathbb{R}^{n\times n}$ be defined by equation* (2.2.20). *Let $\zeta_j, \eta_j, \Delta_{\mathrm{PDE}} : \mathbb{C} \to \mathbb{C}$ be given by Definition 2.19, where $U : \mathbb{C} \to \mathbb{C}$ be defined by equation* (2.2.17) *in terms of some function $q_0 : [0,1] \to \mathbb{C}$ such that Assumption 3.3 is satisfied. Let the functions $\widetilde{f}_j, \widetilde{g}_j : [0,T] \to \mathbb{C}$ be defined by equation* (2.3.4). *Let $f_j, g_j : [0,T] \to \mathbb{C}$ be the functions for which*

$$\widetilde{f}_j(\rho) = \int_0^T e^{a\rho^n t} f_j(t)\,\mathrm{d}t, \quad \widetilde{g}_j(\rho) = \int_0^T e^{a\rho^n t} g_j(t)\,\mathrm{d}t, \quad \rho \in \mathbb{C}.$$

Then $\{f_j, g_j : j \in \{0,1,\dots,n-1\}\}$ is an admissible set in the sense of Definition 1.3 of [**27**].

COROLLARY 3.43. *Let the final-boundary value problem obey Assumption 3.3. Then the problem is well-posed and its solution may be found using Theorem 3.41.*

THEOREM 3.44. *If the final-boundary value problem* (2.1.1), (2.1.2) *and* (3.3.6) *is well-posed, in the sense that it admits a unique solution $q \in C^\infty([0,1]\times[0,T])$, then Assumption 3.3 holds.*

3.3.3. Existence of a series representation

We now combine the results of the present section with those of Section 3.2 to give necessary and sufficient conditions for an initial- (or final-) boundary value problem to be well-posed and for its solution to admit a series representation.

THEOREM 3.45. *Let $n \geqslant 2$, let $a = \pm i$ if n is odd and $\mathrm{Re}(a) \geqslant 0$ if n is even, and let $A \in \mathbb{R}^{n\times 2n}$ be a rank n matrix in reduced row-echelon form. Let $X \in C^\infty[0,1]$ and $H_k \in C^\infty[0,T]$*

be such that the compatability condition

$$A\begin{pmatrix} X^{(n-1)}(0) \\ X^{(n-1)}(1) \\ X^{(n-2)}(0) \\ X^{(n-2)}(1) \\ \vdots \\ X(0) \\ X(1) \end{pmatrix} = \begin{pmatrix} H_1(0) \\ H_2(0) \\ \vdots \\ H_n(0) \end{pmatrix}$$

holds.

Let Π be the following initial-boundary value problem:

Find $q \in C^\infty([0,1] \times [0,T])$ such that the partial differential equation

$$\partial_t q(x,t) + a(-i\partial_x)^n q(x,t) = 0$$

holds on $[0,1] \times [0,T]$ with boundary conditions

$$A\begin{pmatrix} \partial_x^{n-1} q(0,t) \\ \partial_x^{n-1} q(1,t) \\ \partial_x^{n-2} q(0,t) \\ \partial_x^{n-2} q(1,t) \\ \vdots \\ q(0,t) \\ q(1,t) \end{pmatrix} = \begin{pmatrix} H_1(t) \\ H_2(t) \\ \vdots \\ H_n(t) \end{pmatrix}$$

and initial condition

$$q(x,0) = X(x).$$

Let Π' be the following final-boundary value problem:

Find $q \in C^\infty([0,1] \times [0,T])$ such that the partial differential equation

$$\partial_t q(x,t) + a(-i\partial_x)^n q(x,t) = 0$$

holds on $[0,1] \times [0,T]$ with boundary conditions

$$A\begin{pmatrix} \partial_x^{n-1} q(0,t) \\ \partial_x^{n-1} q(1,t) \\ \partial_x^{n-2} q(0,t) \\ \partial_x^{n-2} q(1,t) \\ \vdots \\ q(0,t) \\ q(1,t) \end{pmatrix} = \begin{pmatrix} H_1(T-t) \\ H_2(T-t) \\ \vdots \\ H_n(T-t) \end{pmatrix}$$

and final condition

$$q(x,T) = X(x).$$

If $a = \pm i$ then let Π'' be the following final-boundary value problem:

Find $q \in C^\infty([0,1] \times [0,T])$ *such that the partial differential equation*

$$\partial_t q(x,t) - a(-i\partial_x)^n q(x,t) = 0$$

holds on $[0,1] \times [0,T]$ *with boundary conditions*

$$A \begin{pmatrix} \partial_x^{n-1} q(0,t) \\ \partial_x^{n-1} q(1,t) \\ \partial_x^{n-2} q(0,t) \\ \partial_x^{n-2} q(1,t) \\ \vdots \\ q(0,t) \\ q(1,t) \end{pmatrix} = \begin{pmatrix} H_1(t) \\ H_2(t) \\ \vdots \\ H_n(t) \end{pmatrix}$$

and initial condition

$$q(x,0) = X(x).$$

The following are equivalent:

(1) *The problems* Π *and* Π' *are all well-posed in the sense that they have unique solutions.*
(2) *The problem* Π *is well-posed and its solution admits a series representation with an integral of the boundary data.*
(3) *The problem* Π' *is well-posed and its solution admits a series representation with an integral of the boundary data.*
(4) *Assumption 3.2 and Assumption 3.3 both hold.*

If $a = \pm i$ *then the following are equivalent to one another and to (1):*

(5) *The problems* Π *and* Π'' *are all well-posed in the sense that they have unique solutions.*
(6) *The problem* Π'' *is well-posed and its solution admits a series representation with an integral of the boundary data.*

If n *is even then the following are equivalent to one another and to (1):*

(7) *The problem* Π *is well-posed.*
(8) *The problem* Π' *is well-posed.*
(9) *Assumption 3.2 holds.*
(10) *Assumption 3.3 holds.*

Proof. Corollaries 3.31 and 3.43 and Theorems 3.32 and 3.44 show that (1) is equivalent to (4).

If (4) holds then Π is well-posed so Theorem 3.13 implies (2). If Assumption 3.2 is false then, by Theorem 3.32, (2) is false. If Assumption 3.3 is not true then it is not possible to close the contours of integration in I_2 and I_4, defined in equations (3.1.1), hence there exists no series representation of the solution to Π. Hence (2) implies (4). In the same way, (3) is equivalent to (4).

If $a = \pm i$ then Lemma 3.47 states that Π' and Π'' are equivalent problems. Hence (1) and (5) are equivalent and (3) and (6) are equivalent.

Corollary 3.31 and Theorem 3.32 show that (7) is equivalent to (9). Corollary 3.43 and Theorem 3.44 show that (8) is equivalent to (10).[2] As n is even the theory of differential operators yields that (7) is equivalent to (2), hence (3) and finally (8). □

Example 3.46. Let us consider once again the third-order problems with pseudo-periodic boundary conditions, that is boundary coefficient matrix (3.2.12), as studied in Example 3.2.12. For concreteness let us assume $a = i$, noting that analagous results hold for $a = -i$. We have already noted in Remark 3.25 that such an initial-boundary value problem is ill-posed if and only if

$$\sum_{j=1}^{3} \widetilde{\beta}_j = 0. \tag{3.3.13}$$

The same argument may be used to show that the final-boundary value problem is ill-posed if and only if

$$\sum_{j=1}^{3} \frac{1}{\widetilde{\beta}_j} = 0. \tag{3.3.14}$$

Now, by Theorem 3.45, we conclude that a third-order pseudo-periodic initial-boundary value problem is well-posed and its solution admits a series representation if and only if both equations 3.3.13 and 3.3.14 are false. In particular, we have the interesting example of an initial-boundary value problem specified by the highly coupled boundary coefficient matrix

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \frac{1}{2} \end{pmatrix}$$

which is well-posed but whose solution may not be represented as a series.

Theorem 3.45 is the main result of this Chapter. Part of the proof depends upon Lemma 3.47. This lemma provides a link between initial- and final-boundary value problems by switching the direction of the time variable.

Lemma 3.47. *Let $n \geqslant 2$, $a = \pm i$ and $A \in \mathbb{R}^{n \times 2n}$ be a rank n matrix in reduced row-echelon form. Let $X \in C^\infty[0,1]$ and $H_k \in C^\infty[0,1]$ be such that the compatability condition*

$$A \begin{pmatrix} X^{(n-1)}(0) \\ X^{(n-1)}(1) \\ X^{(n-2)}(0) \\ X^{(n-2)}(1) \\ \vdots \\ X(0) \\ X(1) \end{pmatrix} = \begin{pmatrix} H_1(0) \\ H_2(0) \\ \vdots \\ H_n(0) \end{pmatrix}$$

holds.

Let Π be the following initial-boundary value problem:

[2] Of course these hold for odd n also, it is the following statement that is unique to n even.

Find $q \in C^\infty([0,1]\times[0,T])$ *such that the partial differential equation*

$$\partial_t q(x,t) + a(-i\partial_x)^n q(x,t) = 0 \tag{3.3.15}$$

holds on $[0,1]\times[0,T]$ *with boundary conditions*

$$A\begin{pmatrix} \partial_x^{n-1}q(0,t) \\ \partial_x^{n-1}q(1,t) \\ \partial_x^{n-2}q(0,t) \\ \partial_x^{n-2}q(1,t) \\ \vdots \\ q(0,t) \\ q(1,t) \end{pmatrix} = \begin{pmatrix} H_1(t) \\ H_2(t) \\ \vdots \\ H_n(t) \end{pmatrix} \tag{3.3.16}$$

and initial condition

$$q(x,0) = X(x). \tag{3.3.17}$$

Let Π' be the following final-boundary value problem:

Find $q \in C^\infty([0,1]\times[0,T])$ *such that the partial differential equation*

$$\partial_t q(x,t) - a(-i\partial_x)^n q(x,t) = 0 \tag{3.3.18}$$

holds on $[0,1]\times[0,T]$ *with boundary conditions*

$$A\begin{pmatrix} \partial_x^{n-1}q(0,t) \\ \partial_x^{n-1}q(1,t) \\ \partial_x^{n-2}q(0,t) \\ \partial_x^{n-2}q(1,t) \\ \vdots \\ q(0,t) \\ q(1,t) \end{pmatrix} = \begin{pmatrix} H_1(T-t) \\ H_2(T-t) \\ \vdots \\ H_n(T-t) \end{pmatrix} \tag{3.3.19}$$

and final condition

$$q(x,T) = X(x). \tag{3.3.20}$$

Then Π *and* Π' *are equivalent problems in the sense that* Π *is well-posed if and only if* Π' *is well-posed and if* q *is a solution of* Π *then* Q *is a solution of* Π' *where* $Q(x,T-t) = q(x,t)$.

Remark 3.48. Comparing the boundary conditions (3.3.16) and (3.3.19) it is clear that the boundary data are different; the direction of time has been reversed. If the boundary conditions are homogeneous, as in the examples discussed in Remark 3.49, then this effect is hidden but homogeneity is not necessary. Indeed, provided the boundary data each satisfy $H_k(\frac{T}{2}-t) = H_k(t-\frac{T}{2})$ for all $t \in [0,T]$ the problems Π and Π' have the same boundary conditions.

Remark 3.49. Lemma 3.47 gives an alternative argument that may be used to deduce the results of Examples 3.39 and 3.40 from Examples 3.26 and 3.27.

PROOF OF LEMMA 3.47. Assume Π is well-posed, in the sense that it has a unique C^∞ smooth solution q. We apply the map $t \mapsto T-t$ to the problem Π'. Then $\partial_t q(x,t) \mapsto -\partial_t q(x,T-t)$ hence partial differential equation (3.3.18) becomes

$$\partial_t q(x,T-t) + a(-i\partial_x)^n q(x,T-t) = 0. \tag{3.3.21}$$

Clearly the function Q defined by $Q(x,T-t) = q(x,t)$ satisfies equation (3.3.21) if and only if q satisfies equation (3.3.15). Similarly, Q satisfies boundary conditions (3.3.19) and final condition (3.3.20) if and only if q satisfies boundary conditions (3.3.16) and initial condition (3.3.17). Finally $Q \in C^\infty([0,1]\times[0,T])$ because $q \in C^\infty([0,1]\times[0,T])$. This establishes that Q is a solution of Π'.

Now assume that $\overline{Q}$ is a solution of Π'. Then $\overline{q}$ defined by $\overline{q}(x,t) = \overline{Q}(x,T-t)$ is a solution of Π. Hence, by the well-posedness of Π, $\overline{q} = q$ and $\overline{Q} = Q$. This justifies the well-posedness of Π'.

The equivalence is justified by repeating the above argument in the opposite direction, initially assuming Π' is well-posed with solution Q and defining q as the function such that $q(x,t) = Q(x,T-t)$. □

To conclude this chapter we give Theorem 3.50 that gives sufficient and easily checked conditions for an initial or final-boundary value problem to be well-posed and for its solution to admit a series representation.

THEOREM 3.50. *Suppose n is odd and the initial-boundary value problem Π from Theorem 3.45 has homogeneous, non-Robin boundary conditions that satisfy Conditions 3.19 and 3.22 and Conditions 3.35 and 3.36. Then the problem is well-posed and its solution admits a series representation.*

The corresponding final-boundary value problem Π' and initial-boundary value problem Π'' are also well-posed and their solutions also admit series representations.

PROOF. We consider the initial-boundary value problem Π. Theorem 3.23 guarantees that Assumption 3.2 holds. By Corollary 3.31 this is enough to ensure well-posedness. Although Theorem 3.37 is stated in terms of a final-boundary value problem, the difference is purely symbolic. Indeed, its proof may be used to justify that Assumption 3.3 holds for the initial-boundary value problem, treating q_T (hence each η_j) as an unknown function. Now Theorem 3.1 allows us to write the solution to the problem as:

If n is odd and $a = i$,

$$q(x,t) = \frac{i}{2} \sum_{\substack{k\in K^+ \\ \cup K^{D^+}\cup K^{E^+} \\ \cup K^{\mathbb{R}}\cup\{0\}}} \operatorname*{Res}_{\rho=\sigma_k} \frac{P(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^+} \zeta_j(\sigma_k) + \frac{i}{2} \sum_{\substack{k\in K^- \\ \cup K^{D^-}\cup K^{E^-}}} \operatorname*{Res}_{\rho=\sigma_k} \frac{\widehat{P}(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^-} \zeta_j(\sigma_k).$$

If n is odd and $a = -i$,

$$q(x,t) = \frac{i}{2} \sum_{\substack{k\in K^+ \\ \cup K^{D^+}\cup K^{E^+}}} \operatorname*{Res}_{\rho=\sigma_k} \frac{P(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^+} \zeta_j(\sigma_k) + \frac{i}{2} \sum_{\substack{k\in K^- \\ \cup K^{D^-}\cup K^{E^-} \\ \cup K^{\mathbb{R}}\cup\{0\}}} \operatorname*{Res}_{\rho=\sigma_k} \frac{\widehat{P}(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^-} \zeta_j(\sigma_k).$$

In either case, the representation is a discrete series. By the same argument, the final-boundary value problem is also well-posed and its solution admits a series representation. □

CHAPTER 4

Spectral theory

The main aims of this chapter are to describe a problem in the spectral theory of ordinary differential operators and show how it is related to the issues of well-posedness and the existence of a series representation of a solution that were considered in Chapter 3. We give sets of sufficient conditions that allow each of these problems to be investigated using results pertaining to the other.

In Section 4.1 we give the definition of a differential operator T in terms of a formal differential operator and a particular domain, characterised by a set of boundary conditions. We give some properties of the differential operator, as presented in [**47**], including sufficient conditions for the eigenfunctions of the operator to form a complete system. Even in the later monograph [**48**] the completeness of the eigenfunctions of a large class of differential operators is undecided. It is these, "degenerate irregular", operators that we aim to investigate.

The operator T may be considered as the spatial part of the partial differential equation (2.1.1) and the boundary conditions of T may be seen as homogeneous boundary conditions specifying an initial-boundary value problem. Sections 4.2 and 4.3 investigate the deep link between the operator and the initial-boundary value problem.

In Section 4.2 we discuss the relationship between Δ_{PDE} and Δ, Birkhoff's characteristic determinant of T. We give sufficient conditions, in terms of the boundary conditions of the operator, for these functions to have the same zeros. This allows us to infer properties of the initial-boundary value problem associated with T from the study of T itself.

In Section 4.3 we show a link in the opposite direction. Specifically, we show that if Assumptions 3.2 and 3.3 hold for an initial-boundary value problem and the zeros of Δ_{PDE} are each of order 1 then the eigenfunctions of the operator T associated with that problem form a complete system.

Some standard results on biorthogonal systems and bases in Banach spaces are presented in Section 4.4. The aim is to describe a method for showing that a system, which may be both complete and biorthogonal, is not a basis. This method is used for a particular example in Chapter 5.

4.1. The problem in operator theory

In this section we give the definition of a differential operator and summarize the results of [**47**]. The principal results are Theorems 4.8 and 4.11 which give results concerning the eigenvalues and eigenfunctions of that operator. We do not reproduce Locker's proofs, or even sketch them, as they are long and technical. The results themselves are of use in investigating the the initial-boundary value problems and the differential operators in parallel as the extended, worked examples of Chapter 5 illustrate.

4.1.1. The linear differential operator T

We present some of the definitions and results of [**47**] and [**48**]. Locker is interested in general two-point linear differential operators but we restrict ourselves to the case where the differential

operator is equal to its principal part. This means we have no need to define operators of the form

$$\sum_{j=0}^{n} a_j(t) \left(\frac{\mathrm{d}}{\mathrm{d}t} \right)^j .$$

Locker studies the principal part of this operator to yield results about the full operator. He uses perturbation methods to show that the properties of the full operator may be inferred from the properties of the principal part but such deductions about Locker's more general operator are beyond the scope of this work. The partial differential equations studied in Chapters 2 and 3 have only a single spatial derivative term so the "principal part" is the central object of interest. With this in mind we make the following:

DEFINITION 4.1. *For $n \in \mathbb{N}$, define the space*

$$H^n[0,1] = \{u \in C^{n-1}[0,1] : u^{(n-1)} \text{ absolutely continuous on } [0,1], u^{(n)} \in L^2[0,1]\}. \tag{4.1.1}$$

Define the linear, two-point, single-order differential operator T on the domain

$$\mathcal{D}(T) = \left\{ u \in H^n[0,1] : \sum_{j=0}^{n-1} \alpha_{k\,j} u^{(j)}(0) + \sum_{j=0}^{n-1} \beta_{k\,j} u^{(j)}(1) = 0,\ \forall\ k = 1, 2, \ldots, n \right\}$$

by

$$T = \tau,$$

where $\tau : H^n[0,1] \to L^2[0,1]$ is the linear, two-point, n^{th}-order formal differential operator

$$\tau = \left(-i \frac{\mathrm{d}}{\mathrm{d}t} \right)^n$$

and the constants $\alpha_{k\,j}$, $\beta_{k\,j} \in \mathbb{R}$, known as the boundary coefficients*, are such that the* boundary coefficient matrix

$$A = \begin{pmatrix} \alpha_{1\,n-1} & \beta_{1\,n-1} & \alpha_{1\,n-2} & \beta_{1\,n-2} & \cdots & \alpha_{1\,0} & \beta_{1\,0} \\ \alpha_{2\,n-1} & \beta_{2\,n-1} & \alpha_{2\,n-2} & \beta_{2\,n-2} & \cdots & \alpha_{2\,0} & \beta_{2\,0} \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \alpha_{n\,n-1} & \beta_{n\,n-1} & \alpha_{n\,n-2} & \beta_{n\,n-2} & \cdots & \alpha_{n\,0} & \beta_{n\,0} \end{pmatrix}. \tag{4.1.2}$$

is of rank n and in reduced row-echelon form.

Henceforth we use the terms differential operator *and* formal differential operator *to refer to operators of the form T and τ respectively. Some properties of T are given in Theorem 4.4.*

In equation (2.1.32) we stated the boundary conditions for the initial-boundary value problems in terms of a single boundary coefficient matrix. This is not the usual form, as presented in [**10**] but the form of [**47**]; it was chosen to aid comparison with the analysis presented in the present chapter. This allows us to extend many of the terms given in Definition 2.7 from the setting of initial-boundary value problems posed on partial differential equations to the new setting of ordinary differential operators.

DEFINITION 4.2 (Classification of boundary conditions). *The boundary conditions* (2.1.3) *of the differential operator T may be written in the matrix form*

$$A\begin{pmatrix} u^{(n-1)}(0) \\ u^{(n-1)}(1) \\ u^{(n-2)}(0) \\ u^{(n-2)}(1) \\ \vdots \\ u(0) \\ u(1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

where A is the boundary coefficient matrix.

- *If each boundary condition only involves derivatives of the same order then we call the boundary conditions* non-Robin. *Otherwise we say that a boundary condition is* of Robin type.
- *Boundary conditions with the property*

  Every non-zero entry in the boundary coefficient matrix is a pivot.

  are called simple.
- *A set of boundary conditions is* uncoupled *(or* does not couple the ends of the interval*) if*

  $$\text{If } \alpha_{kj} \text{ is a pivot in } A \text{ then } \beta_{kr} = 0 \,\forall\, r \text{ and}$$
  $$\text{If } \beta_{kj} \text{ is a pivot in } A \text{ then } \alpha_{kr} = 0 \,\forall\, r.$$

  Otherwise we say that the boundary conditions are coupled *(or that they* couple the ends of the interval*).*

Note that we require homogeneous boundary conditions for the differential operator as any inhomogeneity precludes the closure under addition and scalar multiplication of the domain $\mathcal{D}(T)$.

NOTATION 4.3. We also extend the domain of Notation 2.12 so that the sets $\widehat{J}^+$ and $\widehat{J}^-$ and their dependents $\widetilde{J}^+$, $\widetilde{J}^-$, J and J' may be defined directly in terms of the boundary coefficients of a differential operator T.

4.1.1.1. Properties of the differential operator

THEOREM 4.4. *The differential operator T is formally self-adjoint but (in general) non-self-adjoint. The domain $\mathcal{D}(T)$ is dense in $L^2[0,1]$ and T is a closed linear operator on $L^2[0,1]$.*

PROOF. *Formally self-adjoint*: Let $u \in \mathcal{D}(T)$, $v \in H^n[0,1]$. Then the inner product may be evaluated

$$\begin{aligned}\langle Tu, v\rangle &= \int_0^1 (-i)^n u(n)(x)\bar{v}(x)\,\mathrm{d}x \\ &= (-i)^n \sum_{j=1}^{n} (-i)^{j-1}\left[u^{(n-j)}(1)\bar{v}^{(j-1)}(1) - u^{(n-j)}(0)\bar{v}^{(j-1)}(0)\right] \\ &\qquad + (-i)^n(-1)^n \int_0^1 u(x)\bar{v}^{(n)}(x)\,\mathrm{d}x \\ &= (-i)^n \sum_{j=1}^{n} (-i)^{j-1}\left[u^{(n-j)}(1)\bar{v}^{(j-1)}(1) - u^{(n-j)}(0)\bar{v}^{(j-1)}(0)\right] + \langle u, \tau v\rangle. \qquad (4.1.3)\end{aligned}$$

As $u \in \mathcal{D}(T)$ and the boundary coefficient matrix is rank n we have n linear equations in $u^{(j)}$. Hence we may construct another n linear equations in $v^{(j)}$ to ensure that the sum in equation (4.1.3) evaluates to 0, so that an adjoint boundary coefficient matrix may be constructed. Indeed Chapter 3 of [**10**] gives a method for finding the adjoint boundary coefficients in terms of the boundary coefficients of T using Green's functions. Then one may define the operator $T^\star$ as in Definition 4.1 but using the adjoint boundary coefficient matrix so that

$$\langle Tu, v\rangle = \langle u, T^\star v\rangle.$$

However, T and $T^\star$ are, on their respective domains, given by the same formal differential operator τ hence they are formally self-adjoint.

Non-self-adjoint: Let T be the differential operator defined by $n = 2$ and the boundary coefficient matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Then

$$\langle Tu, v\rangle = -u'(1)\bar{v}(1) + u(1)\bar{v}'(1) + \langle u, \tau v\rangle$$

hence the adjoint boundary coefficient matrix is

$$A^\star = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \neq A.$$

This proves that T is not in general self-adjoint. Indeed for this example $\sigma(T) = \emptyset$, as is shown in Example 10.1 of [**48**].

The remaining properties are given in Example 2.21 of [**47**]. □

The counterexample in the above proof is an example of a degenerate operator, defined below. However, it is not necessary to find an operator with empty spectrum to construct an example of a non-self-adjoint operator T, or even to find a degenerate T. The third order operator with boundary coefficient matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

is non-self adjoint and degenerate but has non-empty spectrum. The third order operator with boundary coefficient matrix

$$A = \begin{pmatrix} 0 & 0 & 1 & \beta & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

is non-self-adjoint, has non-empty spectrum and is not degenerate.

To discuss the properties of T further, we require another definition:

DEFINITION 4.5. *For $\lambda \in \mathbb{C}$, the* eigenspace *is the nullity $\mathcal{N}(\lambda I - T)$ and its dimension is known as the* geometric multiplicity *of λ. If $\lambda \in \rho(T)$ then geometric multiplicity of λ is zero and if λ is an eigenvalue of T then the geometric multiplicity of the eigenspace is the number of linearly independent eigenfunctions associated with T.*

It may occur that $\mathcal{N}(\lambda I - T) \subsetneq \mathcal{N}((\lambda I - T)^2)$, in which case we require the notions of the generalised eigenspace*,*

$$\bigcup_{k \in \mathbb{N}} \mathcal{N}((\lambda I - T)^k),$$

and its dimension, the algebraic multiplicity *of λ, denoted $\nu(\lambda)$. If $\lambda \in \sigma(T)$ and $\nu(\lambda) \geqslant 2$ then it makes sense to define the* generalised eigenfunctions *associated with λ to be the functions $u \in \mathcal{D}$ such that for some $k \in \mathbb{N}$*

$$(\lambda I - T)^k u = 0$$

Clearly, the generalised eigenfunctions include all of the usual eigenfunctions for $k = 1$.

A Fredholm operator *is a densely defined, closed linear operator between Hilbert spaces with closed range $\mathcal{R}(T)$ for which the nullspace, $\mathcal{N}(T)$, and the orthogonal complement of the range, $\mathcal{R}(T)^\perp$, are finite-dimensional. The* index *is the difference between the dimensions of these spaces, hence a Fredholm operator of index 0 is an operator for which these spaces have the same finite dimension.*

In the first three chapters of [**47**], Locker gives an introduction to the concepts of Definition 4.5 with a discussion of their relevance to the differential operator T. The only exception is the generalised eigenfunction which is not defined. The definition of generalised eigenfunctions can be found in [**49**] but the concept goes back at least to [**64**], which was first published in Russian in 1917.

The operators T are Fredholm operators of index 0, with range $\mathcal{R}(T)$ a closed, linear subspace of $L^2[0,1]$. Hence, for all $\lambda \in \mathbb{C}$, the operator $(\lambda I - T)$ is also Fredholm of index 0. Such operators fall into two classes:

Class 1: The resolvent set, $\rho(T)$, is nonempty. Then the spectrum, $\sigma(T)$, is a countable complex set with no finite limit points. Further, each spectral point of T is an eigenvalue with finite algebraic multiplicity.

Class 2: $\rho(T) = \emptyset$. Then $\sigma(T) = \mathbb{C}$ and each point has infinite algebraic multiplicity. This can occur, as is shown in Example 10.2 of [**48**].

Most of the operators we study fall into Class 1 but the operator associated to the third-order pseudo-periodic initial-boundary value problems studied in Example 3.46 for which either of equations (3.3.13) or (3.3.14) hold is an (apparently new) example that falls into Class 2.

4.1.2. Characteristic determinant and regularity

We give the definitions of three matrices that depend on the boundary coefficent matrix and list Locker's conditions of regularity.

NOTATION 4.6. Let $\alpha_{k\,j}$, $\beta_{k\,j}$ be the boundary coefficients of a differential operator T. Then we define the integer m_k to be the greatest nonnegative integer j such that at least one of $\alpha_{k\,j}$, $\beta_{k\,j}$ is nonzero. As we require the boundary coefficients to be such that the boundary coefficient matrix, A, is in reduced row echelon form, this means that either $\alpha_{k\,m_k}$ or $\beta_{k\,m_k}$ is a pivot in A. For any $k \in \{1, 2, \ldots, n\}$, the integer m_k is called the *order* of the k^{th} boundary condition.

We define two families of polynomials for $\rho \in \mathbb{C}$,

$$P_k(\rho) = \sum_{j=0}^{m_k} \alpha_{k\,j}\rho^j, \qquad Q_k(\rho) = \sum_{j=0}^{m_k} \beta_{k\,j}\rho^j \qquad \text{for } k = 1, 2, \ldots, n. \tag{4.1.4}$$

We also define the constants

$$\omega = e^{\frac{2\pi i}{n}}$$

$$\nu = \begin{cases} \frac{n}{2} & n \text{ even}, \\ \frac{n+1}{2} & n \text{ odd}, \end{cases}$$

where n is the order of the differential operator T.

DEFINITION 4.7. *The* characteristic matrix *M of the differential operator T is defined entrywise as follows. For $j \in \{1, 2, \ldots, n\}$,*

$$M_{j\,k}(\rho) = \begin{cases} P_j(i\omega^{k-1}\rho) + Q_j(i\omega^{k-1}\rho)e^{i\omega^{k-1}\rho}, & k \in \{1, 2, \ldots, \nu - 1\}, \\ P_j(i\omega^{k-1}\rho)e^{-i\omega^{k-1}\rho} + Q_j(i\omega^{k-1}\rho), & k \in \{\nu, \nu + 1, \ldots, n\}. \end{cases} \tag{4.1.5}$$

The characteristic determinant*, Δ, of the differential operator T is then*

$$\Delta(\rho) = \det M(\rho). \tag{4.1.6}$$

The above explicit definition of the characteristic determinant follows Locker's definition [**47**]. The general definition goes back to Birkhoff [**4**].

The similarity in notation between the characteristic determinant, Δ, of a differential operator and the function Δ_{PDE} given in Definition 2.19 associated with an initial-boundary value problem is expected. Indeed in Section 4.2 we compare the associated matrices M and $\mathcal{A}$ to discuss the relationship between Δ and Δ_{PDE}.

We state, without proof, a theorem of Birkhoff [**4**]. Locker gives this as Theorem 2.1 in Chapter 4 of [**47**].

THEOREM 4.8. *Assume that Δ is not identically zero on $\mathbb{C}$. Then $\sigma \neq 0$ is a zero of Δ if and only if σ^n is an eigenvalue of T.*

In order to state Locker's classification of boundary conditions we must develop some further notation. The following polynomials essentially split the characteristic determinant into two parts.

NOTATION 4.9. For a differential operator T, we define the polynomials π_1 and π_0 as follows.

$$\pi_1(\rho) = \det \begin{pmatrix} & 2 \leqslant k \leqslant \nu & \nu+1 \leqslant k \leqslant n \\ Q_1(i\rho) & P_1(i\rho\omega^{k-1}) & Q_1(i\rho\omega^{k-1}) \\ \vdots & \vdots & \vdots \\ Q_n(i\rho) & P_n(i\rho\omega^{k-1}) & Q_n(i\rho\omega^{k-1}) \end{pmatrix}, \tag{4.1.7}$$

$$\pi_0(\rho) = \det \begin{pmatrix} P_1(i\rho) & P_1(i\rho\omega^{k-1}) & Q_1(i\rho\omega^{k-1}) \\ \vdots & \vdots & \vdots \\ P_n(i\rho) & P_n(i\rho\omega^{k-1}) & Q_n(i\rho\omega^{k-1}) \end{pmatrix}. \tag{4.1.8}$$

As each of the polynomials P_k, Q_k are of degree no greater than m_k, the maximum degree of the polynomials π_1, π_0 is $\sum_{k=1}^{n} m_k$. For this reason we also define the integer

$$p_0 = \sum_{k=1}^{n} m_k.$$

We are ready to give Locker's classification of boundary conditions. We choose the terms used in the second monograph, [**48**].

DEFINITION 4.10. *Let n be even. Let π_0 be the polynomial and let p_0 be the integer, defined in Notation 4.9, associated with the differential operator T. Then the boundary conditions are, hence the differential operator is, said to be*

- regular *if* $\deg \pi_0 = p_0$.
- simply irregular *if* $0 \leqslant \deg \pi_0 < p_0$.
- degenerate irregular *if π_0 is identically zero.*

Let n be odd. Let π_1, π_0 be the polynomials and let p_0 be the integer, defined in Notation 4.9, associated with the differential operator. Then the boundary conditions are, hence the differential operator is, said to be

- regular *if* $\deg \pi_1 = \deg \pi_0 = p_0$.
- simply irregular *if neither π_1 nor π_0 is identically zero and at least one of* $\deg \pi_1 < p_0$, $\deg \pi_0 < p_0$ *holds.*
- degenerate irregular *if π_1 is identically zero or π_0 is identically zero.*

This work concentrates upon odd order, differential operators by linking their study to the study of their associated initial- and final-boundary value problems. We use the same link and the results presented in [**47**] to investigate the initial- and final-boundary value problems associated with regular differential operators. We collate those essential results, Theorems 3.1–3.3 in Chapter 5 of [**47**], into the single Theorem 4.11.

THEOREM 4.11. *Let T be a regular differential operator, as characterised in Definitions 4.1 and 4.10. Then its generalised eigenfunctions form a complete system in $L^2[0,1]$.*

We do not attempt to present Locker's proof in this work for two reasons:

- The proof is both technical and long. As the main result, it occupies the entirety of Chapters 4 and 5 of [**47**], over 100 pages.

- As stated above, we focus upon degenerate irregular differential operators. It is not clear that Locker's proof of Theorem 4.11 illuminates the study of these differential operators.

4.2. Eigenvalues of T

In this section we show that, under certain conditions, the functions Δ and Δ_{PDE} have the same zeros. When Theorem 4.8 is applied to this fact we have a way of determining the eigenvalues of a differential operator T directly from the initial-boundary value problem associated to that operator.

This theorem is important as it permits the study of the initial-boundary value problem through the study of the associated ordinary differential operator. It is a useful result that this connection can be made, although the theorem has only been proved for non-Robin boundary conditions that have a certain symmetry. It is known that this theorem is not sharp and it is a conjecture that it holds for all third order problems. This is a particularly interesting topic for further study.

First, we formally define the notion of an initial-boundary value problem associated with a given differential operator.

DEFINITION 4.12. *Let T be the differential operator of order n with boundary coefficient matrix A given in Definition 4.1. Let $a \in \mathbb{C}$ be specified such that* $\text{Re}(a) \geqslant 0$ *if n is even and $a = \pm i$ if n is odd. We define the* initial-boundary value problem associated with (T, a) *as the following problem:*

Find $q \in C^\infty([0,1] \times [0,T])$ that satisfies the partial differential equation

$$\partial_t q(x,t) + a(-i\partial_x)^n q(x,t) = 0 \tag{4.2.1}$$

on $[0,1] \times [0,T]$, subject to the initial condition

$$q(x,0) = q_0(x) \quad \text{for } x \in [0,1] \tag{4.2.2}$$

and the boundary conditions

$$A \begin{pmatrix} f_{n-1} \\ g_{n-1} \\ f_{n-2} \\ g_{n-2} \\ \vdots \\ f_0 \\ g_0 \end{pmatrix} = \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{pmatrix}, \tag{4.2.3}$$

where $f_j(t) = \partial_x^j q(0,t)$, $g_j(t) = \partial_x^j q(1,t)$. We assume $q_0 \in C^\infty[0,1]$ and $h_k \in C^\infty[0,T]$ are known data.

We also define the final-boundary value problem associated with (T, a) *as the problem:*

Find $q \in C^\infty([0,1]\times[0,T])$ that satisfies the partial differential equation (4.2.1) *on* $[0,1]\times[0,T]$*, subject to the final condition*

$$q(x,T) = q_T(x) \quad \textit{for } x \in [0,1] \tag{4.2.4}$$

and the boundary conditions (4.2.3) *where* $f_j(t) = \partial_x^j q(0,t)$*,* $g_j(t) = \partial_x^j q(1,t)$*. We assume* $q_T \in C^\infty[0,1]$ *and* $h_k \in C^\infty[0,T]$ *are known data.*

Further, we define the homogeneous initial-boundary value problem associated with (T,a) *as the initial-boundary value problem associated with* (T,a) *for which the boundary data are all identically zero. We define the* homogeneous final-boundary value problem associated with (T,a) *as the final-boundary value problem associated with* (T,a) *for which the boundary data are all identically zero.*

Finally we refer to the initial- and final-boundary value problems defined above, for any permissible a, as the boundary-value problems associated with T*.*

The following lemma is useful in generalising the results of this section. Its use reduces the proof of a statement 'X holds for all boundary-value problems associated with T,' to the proof of the statement 'X holds for the homogeneous initial-boundary value problem associated with (T,i).'

LEMMA 4.13. *Let T be the differential operator given in Definition 4.1 and let* $\mathrm{Re}(a) \geqslant 0$ *if n is even and $a = \pm i$ if n is odd. Then the reduced global relation matrices $\mathcal{A}$ associated with each of the boundary-value problems associated with T differ only by a constant multiple.*

PROOF. From equation (2.2.19) each entry depends upon a only through $c_j(\rho)$. As each c_j depends upon a in the same way, by equation (2.1.6) we could rewrite equation (2.2.19) with

$$\mathcal{A}_{k\,j}(\rho) = \begin{cases} a\bar{c}_{(J_j-1)/2}(\rho)\left(\omega^{(n-1-[J_j-1]/2)(k-1)}\right. & \\ \quad -\sum_{r\in\widehat{J}^+}\alpha_{\widehat{J}_r^+\,(J_j-1)/2}\omega^{(n-1-r)(k-1)}(i\rho)^{(J_j-1)/2-r} & J_j \text{ odd},\\ \quad \left.+e^{-i\omega^{k-1}\rho}\sum_{r\in\widehat{J}^-}\alpha_{\widehat{J}_r^-\,(J_j-1)/2}\omega^{(n-1-r)(k-1)}(i\rho)^{(J_j-1)/2-r}\right) & \\ a\bar{c}_{J_j/2}(\rho)\left(-\omega^{(n-1-J_j/2)(k-1)}e^{-i\omega^{k-1}\rho}\right. & \\ \quad -\sum_{r\in\widehat{J}^+}\beta_{\widehat{J}_r^+\,J_j/2}\omega^{(n-1-r)(k-1)}(i\rho)^{J_j/2-r} & J_j \text{ even},\\ \quad \left.+e^{-i\omega^{k-1}\rho}\sum_{r\in\widehat{J}^-}\beta_{\widehat{J}_r^-\,J_j/2}\omega^{(n-1-r)(k-1)}(i\rho)^{J_j/2-r}\right) & \end{cases}$$

where $\bar{c}_j(\rho) = -\rho^n(i\rho)^{-(j+1)}$, which does not depend upon a. Hence the map $a \mapsto a'$ induces the map $\mathcal{A} \mapsto \mathcal{A}'$ defined by

$$\mathcal{A}' = \frac{a'}{a}\mathcal{A}.$$

The boundary data do not affect $\mathcal{A}$ hence the inhomogeneous / homogeneous boundary value problems associated with a have the same boundary coefficient matrix. Finally, note that Lemma 2.17 holds for final-boundary value problems as well as initial-boundary value problems. □

4.2.1. Non-Robin with a symmetry condition

In this section we assume that the boundary conditions of the differential operator are non-Robin and obey Condition 4.14 below.

CONDITION 4.14. Let the boundary coefficients of a differential operator, an initial-boundary value problem or a final-boundary value problem be such that

$$\begin{aligned} r \in \widehat{J}^+ &\Leftrightarrow n-1-r \in \widetilde{J}^- \text{ and} \\ r \in \widehat{J}^- &\Leftrightarrow n-1-r \in \widetilde{J}^+. \end{aligned} \tag{4.2.5}$$

Also, for all $j \in \widetilde{J}^- \cap \widehat{J}^+$,

$$\beta_{\widehat{J}^+_j\, j} = \beta_{\widehat{J}^+_{n-1-j}\, n-1-j}.$$

We note that the index sets $\widehat{J}^\pm$, $\widetilde{J}^\pm$ are defined in Notation 2.12 and Notation 4.3 in terms of the boundary coefficients (and implicitly of n) only, not in terms of a or the data of the initial- or final- boundary value problems. Hence, by Lemma 4.13, Condition 4.14 holds for a particular differential operator T if and only if it holds for any particular boundary value problem associated with T.

Although Condition 4.14 imposes a strong symmetry on the boundary conditions it turns out to be quite a natural condition. Almost all of our earlier examples of initial- and final-boundary value problems have boundary coefficients that obey this condition, as do boundary coefficient matrices such as

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \beta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \beta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

However the boundary coefficient matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \beta_{2\,2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \beta_{3\,1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

does not obey Condition 4.14. The condition requires that if a certain boundary function, f_r or g_r, is prescribed then its mirror image, the boundary function at the other end with the opposite derivative, g_{n-1-r} or f_{n-1-r}, cannot be prescribed. Further the number of different coupling constants is reduced.

THEOREM 4.15. *Let T be a differential operator of the form described in Definition 4.1 with non-Robin boundary conditions obeying Condition 4.14. Then the determinant function Δ has the same zeros as the determinant functions Δ_{PDE} from each of the associated boundary value problems.*

It is known that Condition 4.14 is not sharp. It is an open and interesting question whether it may be discarded entirely.

PROOF. Fixing a particular, permissible value of a, and showing that the nonzero zeros of Δ and of Δ_{PDE} from the homogeneous initial-boundary value problem associated with (T, a) are equal is sufficient proof as we may extend this to the full result using Lemma 4.13. We choose $a = i$ as it is one of the two values permissible for both odd and even n.

The function Δ_{PDE} is defined by equation (2.3.3) as the determinant of the reduced global relation matrix $\mathcal{A}$, which is defined by equation (2.2.5) for homogeneous, non-Robin boundary conditions. Indeed, for each $j \in \widetilde{J}^-$, there is a column of $\mathcal{A}$ given by

$$\omega^{(n-1-j)(k-1)} c_j(\rho), \quad k \in \{1, 2, \ldots, n\} \tag{4.2.6}$$

and for each $j \in \widetilde{J}^+$, there is a column of $\mathcal{A}$ given by

$$-\omega^{(n-1-j)(k-1)} c_j(\rho) \left(e^{-i\omega^{k-1}\rho} + \beta_{\widehat{J}_j^+ j} \right), \quad k \in \{1, 2, \ldots, n\}. \tag{4.2.7}$$

The monomials are defined by

$$c_j(\rho) = -i\rho^n (i\rho)^{-(j+1)}.$$

Substituting this into equations (4.2.6) and (4.2.7) we draw table (4.2.8) to summarise the columns of $\mathcal{A}(\rho)$. For each j in the intersection of the sets indicated, there is a column of $\mathcal{A}(\rho)$ whose entries are given by the formulae shown, indexed by $k = 1, 2, \ldots, n$.

$\mathcal{A}(\rho)$	$\widehat{J}^+$	$\widetilde{J}^+$
$\widehat{J}^-$	No Columns	$-i^{-(n-1)}\omega^{(n-1-j)(k-1)}(i\rho)^{n-1-j}$
$\widetilde{J}^-$	$i^{-(n-1)}\omega^{(n-1-j)(k-1)}(i\rho)^{n-1-j} \times \left(e^{-i\omega^{k-1}\rho} + \beta_{\widehat{J}_j^+ j}\right)$	$i^{-(n-1)}\omega^{(n-1-j)(k-1)}(i\rho)^{n-1-j}e^{-i\omega^{k-1}\rho}$, $-i^{-(n-1)}\omega^{(n-1-j)(k-1)}(i\rho)^{n-1-j}$

(4.2.8)

But Condition 4.14 implies that table (4.2.8) is equivalent to the following table (4.2.9).

$\mathcal{A}(\rho)$	$\widehat{J}^+$	$\widetilde{J}^+$
$\widehat{J}^-$	$i^{-(n-1)}\omega^{j(k-1)}(i\rho)^j e^{-i\omega^{k-1}\rho}$, $-i^{-(n-1)}\omega^{j(k-1)}(i\rho)^j$	$-i^{-(n-1)}\omega^{j(k-1)}(i\rho)^j$
$\widetilde{J}^-$	$i^{-(n-1)}\omega^{j(k-1)}(i\rho)^j \left(e^{-i\omega^{k-1}\rho} + \beta_{\widehat{J}_j^+ j}\right)$	No Columns

(4.2.9)

The characteristic determinant Δ is the determinant of the matrix-valued function M. This means we may define a new matrix $M'(\rho)$ by multiplying $\nu - 1$ of the rows of $M(\rho)$ by exponential functions and taking the transpose as follows,

$$M'_{k\,j}(\rho) = P_j(i\omega^{k-1}\rho)e^{-i\omega^{k-1}\rho} + Q_j(i\omega^{k-1}\rho), \tag{4.2.10}$$

so that

$$\Delta(\rho) = e^{i\sum_{k=1}^{\nu-1}\omega^{k-1}\rho} \det M'(\rho).$$

The exponential $e^{i\sum_{k=1}^{\nu-1}\omega^{k-1}\rho}$ is entire and nonzero on $\mathbb{C}$ so the zeros of Δ are the same as the zeros of $\det M'$.

We now study the columns of $M'(\rho)$. Note first that, as the boundary conditions are non-Robin, the polynomials P_j and Q_j are each monomials of order m_j, hence

$$M'_{j\,k}(\rho) = \omega^{m_j(k-1)}(i\rho)^{m_j}\left(\alpha_{j\,m_j}e^{-i\omega^{k-1}\rho} + \beta_{j\,m_j}\right).$$

By the definition of m_j, for each $j \in \{1,2,\ldots,n\}$, m_j lies in at least one of $\widehat{J}^+$ and $\widehat{J}^-$ and $m_j \in \widehat{J}^+ \cap \widehat{J}^-$ if and only if $m_j = m_{j'}$ for some $j' \neq j$. If $j < j'$ are such that $m_j = m_{j'} \in \widehat{J}^+ \cap \widehat{J}^-$ then M' has columns

$$\omega^{m_j(k-1)}(i\rho)^{m_j}, \qquad \omega^{m_j(k-1)}(i\rho)^{m_j}e^{-i\omega^{k-1}\rho}, \qquad k \in \{1,2,\ldots,n\}.$$

The former column corresponds to the boundary condition in which $\beta_{j\,m_j}$ is a pivot of A, in which $\alpha_{j\,m_j} = 0$ because $\beta_{j\,m_j}$ is the first nonzero entry in its row. The latter corresponds to the boundary condition in which $\alpha_{j'\,m_j}$ is a pivot of A, in which $\beta_{j'\,m_j} = 0$ because $\beta_{j\,m_j}$ is a pivot in the next row of A. If $m_j \in \widetilde{J}^+ \cap \widehat{J}^-$ then $M'(\rho)$ has a column

$$\omega^{m_j(k-1)}(i\rho)^{m_j},$$

as $\beta_{j\,m_j}$ is a pivot, hence the previous entry, $\alpha_{j\,m_j}$, in that row of A must be zero. If $m_j \in \widehat{J}^+ \cap \widetilde{J}^-$ then $M'(\rho)$ has a column

$$\omega^{m_j(k-1)}(i\rho)^{m_j}\left(e^{-i\omega^{k-1}\rho} + \beta_{j\,m_j}\right),$$

as $\alpha_{j\,m_j}$ is a pivot. We have defined all n columns of M' hence there are no columns for which $m_j \in \widetilde{J}^+ \cap \widetilde{J}^-$. With the change of variable $m_j \mapsto r$, hence $j \mapsto \widehat{J}^+_r$ for $m_j \in \widehat{J}^+$, we present these columns in table form as before:

$M'(\rho)$	$\widehat{J}^+$	$\widetilde{J}^+$
$\widehat{J}^-$	$\omega^{r(k-1)}(i\rho)^r e^{-i\omega^{k-1}\rho}$, $\omega^{r(k-1)}(i\rho)^r$	$\omega^{r(k-1)}(i\rho)^r$
$\widetilde{J}^-$	$\omega^{r(k-1)}(i\rho)^r\left(e^{-i\omega^{k-1}\rho} + \beta_{\widehat{J}^+_r\,r}\right)$	No Columns

(4.2.11)

Comparing tables (4.2.9) and (4.2.11), we see that there is a one-to-one correspondence between the columns of M' and the columns of $\mathcal{A}$, the difference being multiplication by a constant $\pm i^{-(n-1)}$. Hence

$$\Delta_{\mathrm{PDE}}(\rho) = (-1)^{|\widehat{J}^+|}i^{-n(n-1)}\det M'(\rho) = \pm e^{-i\sum_{k=1}^{\nu-1}\omega^{k-1}\rho}\Delta(\rho). \tag{4.2.12}$$

□

In equation (4.2.12) we have actually proven a stronger result than is required for Theorem 4.15. We might try to take advantage of this and construct $\mathcal{A}$ directly from M', at least under Condition 4.14, but any such method still requires use of the index sets of Notation 2.12 so it is no easier than constructing $\mathcal{A}$ from its definition in Lemma 2.14.

To find the solution of an initial-boundary value problem it is necessary to know $\mathcal{A}$, as the functions ζ_j, given in Definition 2.19, depend upon this matrix and appear in Theorems 3.1, 3.13 and 3.29. However, much can be learned about the behaviour of such a solution from the columns

of $\mathcal{A}$, without knowledge of their arrangement. Indeed the arguments in Sections 3.2 and 3.3 of Chapter 3 depend only upon the columns of $\mathcal{A}$.

4.2.2. General boundary conditions

Condition 4.14 is not necessary for Theorem 4.15 to hold. We consider two examples in which the condition does not hold and show that the theorem still holds.

EXAMPLE 4.16. Consider the differential operator T with boundary coefficient matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \beta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

and the initial-boundary value problem associated with (T, i). We calculate, using the notation of the proof of Theorem 4.15,

$$\begin{aligned} \det M'(\rho) &= (i\rho)^3 \det \begin{pmatrix} e^{-i\rho} & (e^{-i\rho}+\beta) & 1 \\ \omega^2 e^{-i\omega\rho} & \omega(e^{-i\omega\rho}+\beta) & 1 \\ \omega e^{-i\omega^2\rho} & \omega^2(e^{-i\omega^2\rho}+\beta) & 1 \end{pmatrix} \\ &= (i\rho)^3 \sum_{j=0}^{2} \omega^j (e^{-i\omega^j\rho}+\beta)(\omega^{1-j}e^{-i\omega^{j+2}\rho} - \omega^{2-j}e^{-i\omega^{j+1}\rho}) \\ &= (i\rho)^3(\omega-\omega^2)\sum_{j=0}^{2}\left(\beta e^{-i\omega^j\rho} + e^{i\omega^j\rho}\right), \end{aligned}$$

and

$$\begin{aligned} \det \mathcal{A}(\rho) &= -(i\rho)^3 \det \begin{pmatrix} e^{-i\rho} & (e^{-i\rho}+\beta) & 1 \\ e^{-i\omega\rho} & \omega(e^{-i\omega\rho}+\beta) & \omega^2 \\ e^{-i\omega^2\rho} & \omega^2(e^{-i\omega^2\rho}+\beta) & \omega \end{pmatrix} \\ &= -(i\rho)^3 \sum_{j=0}^{2} \omega^j (e^{-i\omega^j\rho}+\beta)(\omega^{2-j}e^{-i\omega^{j+2}\rho} - \omega^{1-j}e^{-i\omega^{j+1}\rho}) \\ &= \det M'(\rho). \end{aligned}$$

It may be checked case-by-case that Theorem 4.15 holds for all 3rd order non-Robin boundary conditions with a single coupling. Further, it may checked case-by-case that the result always holds for simple 3rd order boundary conditions.

EXAMPLE 4.17. Let T be the differential operator of order 3 with pseudoperiodic boundary conditions, that is

$$A = \begin{pmatrix} 1 & \beta_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \beta_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \beta_3 \end{pmatrix}.$$

Then

$$\begin{aligned}\Delta(\rho)e^{i\sum_{k=1}^{\nu}\omega^{k-1}\rho} &= \det M'(\rho)\\ &= (i\rho)^3 \det\begin{pmatrix}(e^{-i\rho}+\beta_1) & (e^{-i\rho}+\beta_2) & (e^{-i\rho}+\beta_3)\\ \omega^2(e^{-i\omega\rho}+\beta_1) & \omega(e^{-i\omega\rho}+\beta_2) & (e^{-i\omega\rho}+\beta_3)\\ \omega(e^{-i\omega^2\rho}+\beta_1) & \omega^2(e^{-i\omega^2\rho}+\beta_2) & (e^{-i\omega^2\rho}+\beta_3)\end{pmatrix}\\ &= (i\rho)^3 \det\begin{pmatrix}(e^{-i\rho}+\beta_1) & (e^{-i\rho}+\beta_2) & (e^{-i\rho}+\beta_3)\\ (e^{-i\omega\rho}+\beta_1) & \omega^2(e^{-i\omega\rho}+\beta_2) & \omega(e^{-i\omega\rho}+\beta_3)\\ (e^{-\omega^2\rho}+\beta_1) & \omega(e^{-i\omega^2\rho}+\beta_2) & \omega^2(e^{-i\omega^2\rho}+\beta_3)\end{pmatrix}\end{aligned}$$

and

$$\begin{aligned}\Delta_{\mathrm{PDE}}(\rho) &= \det \mathcal{A}(\rho)\\ &= -c_0(\rho)c_1(\rho)c_2(\rho)\det\begin{pmatrix}(e^{-i\rho}+\beta_1) & (e^{-i\rho}+\beta_2) & (e^{-i\rho}+\beta_3)\\ (e^{-i\omega\rho}+\beta_1) & \omega(e^{-i\omega\rho}+\beta_2) & \omega^2(e^{-i\omega\rho}+\beta_3)\\ (e^{-\omega^2\rho}+\beta_1) & \omega^2(e^{-i\omega^2\rho}+\beta_2) & \omega(e^{-i\omega^2\rho}+\beta_3)\end{pmatrix}.\end{aligned}$$

By evaluating the determinants it may be shown that they differ only by a constant even if the β_j are all different. Similarly, it may be checked case-by-case that Theorem 4.15 holds for all sets of third order non-Robin boundary conditions with two couplings.

The other condition, that the boundary conditions be non-Robin, at first appears to be more fundamental for Theorem 4.15. Indeed, we compare the sums in the expressions for $\mathcal{A}$ and M as defined in equations (2.2.19) and (4.1.5). The coefficients of the sums in the reduced global relation matrix are the boundary coefficients lying in a particular *column* of the boundary coefficient matrix whereas the coefficients in the sums in the charactistic matrix are the boundary coefficients lying in a particular *row* of the boundary coefficient matrix. Nevertheless, there exists an example of third order uncoupled non-Robin boundary conditions, specified by the boundary coefficient matrix

$$A = \begin{pmatrix}1 & 0 & \alpha & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\end{pmatrix},$$

for which Δ_{PDE} and Δ have the same zeros. An interesting direction for future study would be to check if Theorem 4.15 holds for all third order Robin-type boundary conditions and, if so, to try to extend this result to arbitrary order.

4.3. Eigenfunctions of T

In this section we prove directly that the expression for q in Theorem 3.1 must be an expansion in the eigenfunctions of T, under the condition that the zeros of Δ_{PDE} are all simple. This gives an alternative to using Locker's method for showing that the generalised eigenfunctions of a differential operator form a complete system, using the study of the associated initial-boundary value problem. The result is contained in the following Theorem 4.18. Its full proof requires the

notion of minimality of a system of functions. We leave the definition of this concept and the associated concept of completeness to Section 4.4.

THEOREM 4.18. *Consider a homogeneous initial-boundary value problem for which Assumptions 3.2 and 3.3 are satisfied and which is associated to the differential operator T. Assume further that every zero of Δ_{PDE} is simple. Then the solution $q(x,t)$ as expressed in Theorem 3.1 is an expansion in the eigenfunctions of T. Further, for every σ in the PDE discrete spectrum, σ^n is an eigenvalue of T.*

PROOF. By Theorem 3.1 a series representation for q exists. From equations (3.0.6)–(3.0.9) we may define two index sets K_1, K_2 such that

$$q(x,t) = \frac{i}{2}\sum_{k\in K_1}\operatorname*{Res}_{\rho=\sigma_k}\frac{P(\rho)}{\Delta_{\mathrm{PDE}}(\rho)}\sum_{j\in J^+}\zeta_j(\sigma_k) + \frac{i}{2}\sum_{k\in K_2}\operatorname*{Res}_{\rho=\sigma_k}\frac{\widehat{P}(\rho)}{\Delta_{\mathrm{PDE}}(\rho)}\sum_{j\in J^-}\zeta_j(\sigma_k). \tag{4.3.1}$$

By Definitions 2.19 and 3.4, Δ_{PDE} and ζ_j are independent of x and t; only P and $\widehat{P}$ have x and t as parameters. The functions P and $\widehat{P}$ are separable in x and t; they may be written in the forms

$$P(\rho,x,t) = P_1(\rho,x)P_2(\rho,t),$$
$$\widehat{P}(\rho,x,t) = \widehat{P}_1(\rho,x)P_2(\rho,t),$$

for P_1, $\widehat{P}_1$ independent of t and P_2 independent of x. Hence we may express equation (4.3.1) in the form

$$q(x,t) = \sum_{k\in\mathbb{N}}\phi_k(x)\psi_k(t) \text{ where} \tag{4.3.2}$$

$$\phi_k(x)\psi_k(t) = \begin{cases} \frac{i}{2}\operatorname{Res}_{\rho=\sigma_k}\frac{P(\rho)}{\Delta_{\mathrm{PDE}}(\rho)}\sum_{j\in J^+}\zeta_j(\sigma_k) & k\in K_1, \\ \frac{i}{2}\operatorname{Res}_{\rho=\sigma_k}\frac{\widehat{P}(\rho)}{\Delta_{\mathrm{PDE}}(\rho)}\sum_{j\in J^-}\zeta_j(\sigma_k) & k\in K_2. \end{cases} \tag{4.3.3}$$

By expanding P and $\widehat{P}$ in equation (4.3.3) and applying the limit formula for the residue at a simple pole, we obtain

$$\psi_k(t) = e^{a\sigma_k^n t}, \tag{4.3.4}$$

where $(\sigma_k)_{k\in\mathbb{N}}$ is a sequence containing each nonzero zero of Δ_{PDE} for which $0 \leqslant \arg(\sigma) < \frac{2\pi}{n}$ precisely once and obeying $|\sigma_k| \leqslant |\sigma_{k+1}|$. From Theorem B.4 we know that $\sigma_k = Ak + O(1)$ as $k\to\infty$ for some nonzero complex constant A. Hence $\sigma_k^n = A^n k^n + O(k^{n-1})$ as $k\to\infty$. This is sufficient to ensure that the series

$$\sum_{k=1}^{\infty}\frac{1}{|\frac{a}{i}\sigma_k^n|}$$

converges. This is the criterion for Theorem 3.3.3 of [**56**] to guarantee that, for any $\lambda > 0$, there exists a continuous function, h, compactly supported on $[-\lambda,\lambda]$ such that its inverse Fourier transform,

$$H(\rho) = \int_{-\lambda}^{\lambda} e^{i\rho t}h(t)\,\mathrm{d}t,$$

is entire and has zeros at each $\frac{a}{i}\sigma_k^n$. We choose $\lambda = \frac{T}{2}$, where T here denotes the final time of the initial-boundary value problem. Note that, as it is continuous on a compact interval,

h is bounded hence $h \in L^2[-\frac{T}{2}, \frac{T}{2}]$ and $H \in \mathscr{F}L^2_{1,\frac{T}{2}}$, a set of inverse Fourier transforms of L^2 functions defined in equation (4.1.5) of [**56**]. Though it does not appear explicitly in the statement of the theorem, it is noted in the proof of Theorem 3.3.3 of [**56**] that H is a function of exponential type at most $\frac{T}{2}$. Now define the function $F : \mathbb{C} \to \mathbb{C}$ by

$$F(\rho) = H(\rho)\left(\rho - \frac{a}{i}\sigma_1^n\right).$$

It is clear that F is entire, is of exponential type at most $\frac{T}{2}+1$ and has zeros at each $\frac{a}{i}\sigma_k^n$. Hence, by Theorem 4.1.1 of [**56**][1], the system of exponential functions $(\psi_k)_{k\in\mathbb{N}}$ is *minimal* in $L_2[-\frac{T}{2}, \frac{T}{2}]$, in the sense that each function lies outside the closure of the linear span of the others. By the linear reparameterization $t \mapsto t + \frac{T}{2}$, the system $(\psi_k)_{k\in\mathbb{N}}$ is also minimal in $L_2[0, T]$.

We use the separated form (4.3.2) to express the time and space partial derivatives of q in terms of ordinary derivatives of ϕ_n and ψ_n:

$$\partial_t q(x,t) = \sum_{k\in\mathbb{N}} \phi_k(x)\psi_k'(t), \tag{4.3.5}$$

$$(\partial_x)^j q(x,t) = \sum_{k\in\mathbb{N}} \phi_k^{(j)}(x)\psi_k(t). \tag{4.3.6}$$

From equation (4.3.6) we may deduce that

$$(-i\partial_x)^n q(x,t) = \sum_{k\in\mathbb{N}} \tau(\phi_k)(x)\psi_k(t), \tag{4.3.7}$$

where τ is the formal differential operator associated with T in Definition 4.1, and

$$A\begin{pmatrix} f_{n-1}(t) \\ g_{n-1}(t) \\ \vdots \\ f_0(t) \\ g_0(t) \end{pmatrix} = A\begin{pmatrix} \sum_{k\in\mathbb{N}} \phi_k^{(n-1)}(0)\psi_k(t) \\ \sum_{k\in\mathbb{N}} \phi_k^{(n-1)}(1)\psi_k(t) \\ \vdots \\ \sum_{k\in\mathbb{N}} \phi_k(0)\psi_k(t) \\ \sum_{k\in\mathbb{N}} \phi_k(1)\psi_k(t) \end{pmatrix}, \tag{4.3.8}$$

where A is the boundary coefficient matrix common to the initial-boundary value problem and the differential operator. By equation (2.1.32) and the homogeneity of the boundary conditions, the left hand side of equation (4.3.8) is equal to zero. Hence, as each line of equation (4.3.8) holds for every $t \in [0, T]$ and the ψ_k are minimal in $L_2[0, T]$, we may conclude that $\phi_k \in \mathcal{D}(T)$ for each $k \in \mathbb{N}$. Hence we may rewrite equation (4.3.7) as

$$(-i\partial_x)^n q(x,t) = \sum_{k\in\mathbb{N}} T(\phi_k)(x)\psi_k(t). \tag{4.3.9}$$

Differentiating equation (4.3.4) with respect to t we obtain

$$\psi_k'(t) = -a\sigma_k^n\psi_k(t). \tag{4.3.10}$$

Using equation (4.3.10) we rewrite equation (4.3.5) as

$$\partial_t q(x,t) = -a\sum_{k\in\mathbb{N}} \sigma_k^n\phi_k(x)\psi_k(t). \tag{4.3.11}$$

[1] See also [**51**]

From the partial differential equation, we obtain a relation between the left hand sides of equations (4.3.9) and (4.3.11), indeed

$$-a\sum_{k\in\mathbb{N}}\sigma_k^n\phi_k(x)\psi_k(t)+a\sum_{k\in\mathbb{N}}T(\phi_k)(x)\psi_k(t)=0.$$

Hence

$$\sum_{k\in\mathbb{N}}(T-\sigma_k^n)\phi_k(x)\psi_k(t)=0,$$

hence, by the minimality of the ψ_k, each ϕ_k is an eigenfunction of T with eigenvalue σ_k^n. □

THEOREM 4.19. *If the boundary conditions of an initial-boundary value problem are such that the problem is well-posed, its solution has a series representation and all zeros of Δ_{PDE} are simple then the eigenfunctions of the associated ordinary differential operator T form a complete system in $L_2[0,1]$.*

PROOF. Choose some $q_0\in C^\infty[0,1]$, to specify a particular initial-boundary value problem. Solving that problem and expressing its solution as a discrete series, we know from Theorem 4.18 that the series expansion (3.0.6)–(3.0.9) is in terms of the eigenfunctions of T. Evaluating both sides of this equation at $t=0$ we obtain an expansion of the initial datum in terms of the eigenfunctions. Hence the eigenfunctions form a complete system in $C^\infty[0,1]$. As $C^\infty[0,1]$ is dense in $L^2[0,1]$, the result is proven. □

Another immediate corollary to Theorem 4.18 is

COROLLARY 4.20. *The PDE discrete spectrum of a well-posed initial-boundary value problem that admits a series representation and for which all zeros of Δ_{PDE} are simple is a subset of the discrete spectrum of the ordinary differential operator with which it is associated.*

4.4. The failure of the system of eigenfunctions to be a basis

In this section we develop some of the theory of biorthogonal sequences as presented in Section 3.3 of [**15**], giving expanded versions of the proofs Davies presents. These definitions are also given in the survey [**56**]. Sedletskii's survey and its references also give an extensive treatment of the exponential systems we investigate. Biorthogonal sequences are essential to the study of our differential operators as they are non-self-adjoint. This means that their eigenfunctions, together with the eigenfunctions of the adjoint operator, form a biorthogonal pair of sequences. We use the following notational convention:

NOTATION 4.21. Let $\mathcal{B}$ be a Banach space with dual space $\mathcal{B}^\star$, the space of linear functionals defined on $\mathcal{B}$. Let $f\in\mathcal{B}$ and let $\phi\in\mathcal{B}^\star$. We define the use of angled brackets,

$$\langle f,\phi\rangle=\phi(f),$$

to mean the functional ϕ acting upon the element f of the Banach space.

This notation is intentionally similar to inner product notation on Hilbert spaces. Indeed if $\mathcal{B}$ is a Hilbert space then the Fréchet-Riesz theorem guarantees a one-one correspondence between functionals $\phi \in \mathcal{B}^\star$ and $g \in \mathcal{B}$ such that $\phi(f) = \langle f, g\rangle$ for all $f \in \mathcal{B}$ and $\|\phi\| = \|g\|$, see Theorem IV.4.5 of [**19**]. So, at least for Hilbert spaces, this notation could be seen as an abuse of inner product notation by identifying each ϕ with its corresponding g. We, like Davies, use the notation in the more general setting of Banach spaces without inner products to emphasize that orthonormal sequences in Hilbert spaces are the prototype for the more general idea of bases in Banach spaces.

4.4.1. Biorthogonal sequences

DEFINITION 4.22. *Let $\mathcal{B}$ be a Banach space and let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{B}$. Then $(f_n)_{n\in\mathbb{N}}$ is a* complete *sequence if*

$$\operatorname{span}\{f_n : n \in \mathbb{N}\} \text{ is dense in } \mathcal{B}.$$

A sequence $(f_n)_{n\in\mathbb{N}}$ is said to be minimal complete *if it is complete and for all $k \in \mathbb{N}$ the sequence $(f_n)_{n\in\mathbb{N}\setminus\{k\}}$ is not complete.*[2] *The sequence $(f_n)_{n\in\mathbb{N}}$ is a* basis *for $\mathcal{B}$ if every $f \in \mathcal{B}$ has a unique expansion*

$$f = \lim_{n\to\infty} \left(\sum_{r=1}^{n} \alpha_r f_r \right),$$

where the scalars α_r are known as the Fourier coefficients *of f with respect to the basis $(f_n)_{n\in\mathbb{N}}$.*

Clearly a complete sequence that is not minimal complete is not a basis but it should also be noted that minimal completeness does *not* imply that the sequence is a basis. Indeed Lemma 4.26 gives the extra condition needed for a minimal complete sequence to be a basis.

DEFINITION 4.23. *Let $\mathcal{B}$ be a Banach space with dual space $\mathcal{B}^\star$ and let $(f_n)_{n\in\mathbb{N}}$ and $(\phi_n)_{n\in\mathbb{N}}$ be sequences in $\mathcal{B}$ and $\mathcal{B}^\star$, respectively. The sequences $(f_n)_{n\in\mathbb{N}}$ and $(\phi_n)_{n\in\mathbb{N}}$ are* biorthogonal *or $((f_n)_{n\in\mathbb{N}}, (\phi_n)_{n\in\mathbb{N}})$ is a* biorthogonal pair *for $\mathcal{B}$ if*

$$\langle f_n, \phi_m\rangle = \delta_{m,n} \qquad \forall\, m, n \in \mathbb{N}.$$

Let $((f_n)_{n\in\mathbb{N}}, (\phi_n)_{n\in\mathbb{N}})$ be a biorthogonal pair for $\mathcal{B}$. For each $n \in \mathbb{N}$, the operators $P_n : \mathcal{B} \to \mathcal{B}$ are defined by

$$P_n f = \sum_{r=1}^{n} \langle f, \phi_r\rangle f_r$$

and the operators $Q_n : \mathcal{B} \to \mathcal{B}$ are defined by

$$Q_1 = P_1, \qquad Q_n = P_n - P_{n-1} \text{ for } n \geqslant 2.$$

The following lemma gives sufficient (but not necessary) conditions for a biorthogonal pair to exist. We include it as it is useful in the proof of Lemma 4.26.

[2] It is possible to define minimality without completeness, by requiring that each f_n is disjoint from the closure of the linear span of the others, see [**56**].

LEMMA 4.24. *Let $\mathcal{B}$ be a Banach space with basis $(f_n)_{n\in\mathbb{N}}$. Then there exists a sequence $(\phi_n)_{n\in\mathbb{N}}$ in $\mathcal{B}^\star$ such that the Fourier coefficients of f with respect to $(f_n)_{n\in\mathbb{N}}$ are given by $\alpha_n = \langle f, \phi_n\rangle$. Furthermore, $((f_n)_{n\in\mathbb{N}}, (\phi_n)_{n\in\mathbb{N}})$ is a biorthogonal pair.*

PROOF. Equip $\mathbb{N}$ with the discrete topology and let $K = \mathbb{N} \cup \{\infty\}$ be the Alexandrov one-point compactification of $\mathbb{N}$. Let

$$\mathcal{C} = \{s : K \to \mathcal{B} \text{ such that } s \text{ is continuous, } s(1) \in \mathbb{C}f_1 \text{ and } \forall\, n \geqslant 2, (s(n) - s(n-1)) \in \mathbb{C}f_n\}.$$

Define a norm on $\mathcal{C}$ by

$$\|s\|_{\mathcal{C}} = \sup_{n\in K} \|s(n)\|_{\mathcal{B}} = \max_{n\in K} \|s(n)\|_{\mathcal{B}},$$

the latter equality is justified by the compactness of K and the continuity of s. Then $(\mathcal{C}, \|\cdot\|_{\mathcal{C}})$ is a Banach space. Let the operator $X : \mathcal{C} \to \mathcal{B}$ be defined by $Xs = s(\infty)$. Then X is a bounded, linear operator with norm 1. We show in the next two paragraphs that X is a bijection.

As $\mathcal{B}$ has a basis, for any $g \in \mathcal{B}$ there exist Fourier coefficients β_r such that $g = \sum_{r=1}^{\infty} \beta_r f_r$. Let $s_g : K \to \mathcal{B}$ denote the function defined by

$$s_g(n) = \sum_{r=1}^{n} \beta_r f_r.$$

Certainly $s_g(1) \in \mathbb{C}f_1$ and $(s_g(n) - s_g(n-1)) \in \mathbb{C}f_n$. Any open ball in $\mathcal{B}$ contains either no $s_g(n)$, finitely many $s_g(n)$ or finitely many plus all $s_g(n)$ for n greater than some N. Each of these are open sets in the topology on K so s_g is continuous. This establishes $s_g \in \mathcal{C}$, the domain of X. But $Xs_g = g$ and g may be any point in $\mathcal{B}$, so X is onto.

Let $s, t \in \mathcal{C}$ be such that $Xs = Xt$, that is $s(\infty) = t(\infty)$. By the definition of $\mathcal{C}$, there exist sequences $(\gamma_n)_{n\in\mathbb{N}}$ and $(\delta_n)_{n\in\mathbb{N}}$ of complex numbers such that $s(\infty) = \sum_{n=1}^{\infty} \gamma_n f_n$ and $t(\infty) = \sum_{n=1}^{\infty} \delta_n f_n$ hence

$$\sum_{n=1}^{\infty} \gamma_n f_n = \sum_{n=1}^{\infty} \delta_n f_n.$$

Now, by the uniqueness of the expansion in a basis, we have that $\gamma_n = \delta_n$ for all $n \in \mathbb{N}$ so $s(n) = t(n)$ for all $n \in K$ and $s = t$. This establishes that X is one-one.

The inverse mapping theorem now provides that X^{-1}, the inverse of X, exists and is linear and bounded. Now

$$(X^{-1}f)(n) - (X^{-1}f)(n-1) = \alpha_n f_n$$

implies that α_n depends continuously on f. That is, there exists a bounded linear functional $\phi_n : \mathcal{B}^\star \to \mathbb{C}$ such that $\langle f, \phi_n\rangle = \alpha_n$.

As $f_n \in \mathcal{B}$ it has a unique expansion

$$f_n = \lim_{k\to\infty} \sum_{r=1}^{k} \varepsilon_{n,r} f_r,$$

which must be given by $\varepsilon_{n,r} = \delta_{n,r}$. Hence

$$\langle f_n, \phi_m\rangle = \varepsilon_{n,m} = \delta_{n,m}.$$

□

In Definition 4.23 it is not required that the sequence $(f_n)_{n\in\mathbb{N}}$ be complete for the pair of sequences $((f_n)_{n\in\mathbb{N}},(\phi_n)_{n\in\mathbb{N}})$ to be biorthogonal. Lemma 4.25 concerns the existence of a biorthogonal pair in the case that one sequence is known to be complete.

LEMMA 4.25. *Let $(f_n)_{n\in\mathbb{N}}$ be a complete sequence in a Banach space $\mathcal{B}$. There exists a sequence $(\phi_n)_{n\in\mathbb{N}}$ in $\mathcal{B}^\star$ biorthogonal to $(f_n)_{n\in\mathbb{N}}$ if and only if $(f_n)_{n\in\mathbb{N}}$ is minimal complete.*

PROOF. Assume $k\in\mathbb{N}$ is such that $(f_n)_{n\in\mathbb{N}\setminus\{k\}}$ is complete, that is $(f_n)_{n\in\mathbb{N}}$ is not minimal complete. Then there exist scalars $(\alpha_r)_{r\in\mathbb{N}\setminus\{k\}}$ such that

$$f_k=\lim_{n\to\infty}\sum_{r=1}^{n}\alpha_{r\,n}f_r.$$

Now any sequence $(\phi_n)_{n\in\mathbb{N}}$ biorthogonal to $(f_n)_{n\in\mathbb{N}}$ has the properties

$$\begin{aligned}\langle f_r,\phi_m\rangle&=\delta_{r,m} &&\forall\, r,m\in\mathbb{N}\setminus\{k\},\\ \langle f_k,\phi_m\rangle&=0 &&\forall\, m\in\mathbb{N}\setminus\{k\}.\end{aligned}$$

Hence

$$1=\langle f_k,\phi_k\rangle=\lim_{n\to\infty}\sum_{\substack{r=1\\ r\neq k}}^{n}\alpha_{r\,n}\langle f_r,\phi_k\rangle=0,$$

as each $\langle f_r,\phi_k\rangle=0$.

Conversely, assume there does not exist a sequence biorthogonal to $(f_n)_{n\in\mathbb{N}}$. Then there exists some $k\in\mathbb{N}$ such that

$$\begin{aligned}\exists\,(\phi_n)_{n=1}^{k-1}\in\mathcal{B}^\star \text{ such that } \forall\, m\in\mathbb{N}, n\in\{1,2,\ldots,k-1\},\quad \langle f_m,\phi_n\rangle=\delta_{m,n},\\ \forall\,\phi\in\mathcal{B}^\star\ \exists\, m\in\mathbb{N}\text{ such that }\langle f_m,\phi\rangle\neq\delta_{k,m}.\end{aligned}\tag{4.4.1}$$

It cannot happen that

$$\exists\,\phi\in\mathcal{B}^\star \text{ such that } \forall\, m\in\mathbb{N}\setminus\{k\}\quad \langle\phi,f_m\rangle=0 \text{ and } \langle\phi,f_k\rangle\in\mathbb{C}\setminus\{0\},$$

as then

$$\phi_k=\frac{\phi}{\langle\phi,f_k\rangle}$$

would contradict statement (4.4.1). Hence

$$\forall\,\phi\in\mathcal{B}^\star \text{ such that } \forall\, m\in\mathbb{N}\setminus\{k\},\ \langle\phi,f_m\rangle=0,\quad \langle\phi_k,f_k\rangle=0.$$

But then $(f_n)_{n\in\mathbb{N}\setminus\{k\}}$ is complete and $(f_n)_{n\in\mathbb{N}}$ is not minimal complete. □

4.4.2. A test for a basis

In this subsection we derive a property of biorthogonal sequences which specify a basis. This provides us with a test for a basis that is used in Chapter 5.

LEMMA 4.26. *Let $((f_n)_{n\in\mathbb{N}},(\phi_n)_{n\in\mathbb{N}})$ be a biorthogonal pair in a Banach space $\mathcal{B}$. Then the operators P_n and Q_n given in Definition 4.23 are finite rank bounded projections. If $(f_n)_{n\in\mathbb{N}}$ is a basis then P_n are uniformly bounded in norm and converge strongly to the identity operator,*

I, as $n \to \infty$. If P_n are uniformly bounded in norm and $(f_n)_{n\in\mathbb{N}}$ is complete then $(f_n)_{n\in\mathbb{N}}$ is a basis.

REMARK 4.27. Before giving a formal proof of the above Lemma we give a heuristic idea of the reason that these projection operators must be uniformly bounded in norm.

Let $((a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}})$ and $((c_n)_{n\in\mathbb{N}}, (d_n)_{n\in\mathbb{N}})$ be pairs of sequences in a Banach space $\mathcal{B}$ such that $\langle a_j, b_k\rangle = 0$ and $\langle c_j, d_k\rangle = 0$ for all $j \neq k$ and $\langle a_k, b_k\rangle \neq 0$, $\langle c_k, d_k\rangle \neq 0$ for all k. Then each pair can be normalised into a biorthogonal pair in the following way.

Define new sequences $(A_n)_{n\in\mathbb{N}}$, $(B_n)_{n\in\mathbb{N}}$, $(C_n)_{n\in\mathbb{N}}$, $(D_n)_{n\in\mathbb{N}}$ by

$$A_n = \frac{a_n}{\sqrt{\langle a_n, b_n\rangle}}, \qquad B_n = \frac{b_n}{\sqrt{\langle a_n, b_n\rangle}},$$
$$C_n = \frac{c_n}{\sqrt{\langle c_n, d_n\rangle}}, \qquad D_n = \frac{d_n}{\sqrt{\langle c_n, d_n\rangle}}.$$

Our pairs of systems, $((A_n)_{n\in\mathbb{N}}, (B_n)_{n\in\mathbb{N}})$ and $((C_n)_{n\in\mathbb{N}}, (D_n)_{n\in\mathbb{N}})$, are both biorthogonal systems; we have performed a biorthonormalisation on the original pairs. But this does not mean that they are necessarily normalised. Indeed it could be that one pair is normalised,

$$\|A_n\| = O(1) = \|B_n\|,$$

but the other is not,

$$\|C_n\| = O(e^n) = \|D_n\|.$$

If $(C_n)_{n\in\mathbb{N}}$ is a basis then when a function, u, is expanded in that basis the Fourier coefficients are given by Lemma 4.25[3] as $\langle u, D_n\rangle$. Because $\|C_n\| = O(e^n)$, we require

$$\langle u, D_n\rangle = o\left(\frac{e^k}{k}\right).$$

As $\|D_n\| = O(e^k)$ also, this puts quite a tight restriction on u.

If $(A_n)_{n\in\mathbb{N}}$ were a basis and a function u were expanded in that basis then we require only

$$\langle u, B_n\rangle = o\left(\frac{1}{k}\right).$$

As $\|B_n\| = O(1)$, u does not have such a restriction placed upon it.

The result of Lemma 4.26 is essentially that the basis vectors are not just biorthonormalised but that they and the sequence biorthogonal to them may be simultaneously normalised and mutually biorthonormalised.

PROOF OF LEMMA 4.26. The inner product is linear and continuous so the operator P_n is a bounded linear operator on $\mathcal{B}$. Further the rank of P_n is the dimension of the range of P_n,

$$\dim(\operatorname{span}\{f_r : r \in \{1, 2, \ldots, n\}\}) \leqslant n$$

[3]As biorthogonal sequences need not be unique it might in fact be some other biorthogonal sequence $(E_n)_{n\in\mathbb{N}}$ instead of $(D_n)_{n\in\mathbb{N}}$. This is why the projection operators are essential in the statement of the Lemma.

hence P_n is of finite rank. This also shows Q_n is finite rank. We show that P_n satisfies the definition of a projection operator:

$$\begin{aligned} P_n^2 f &= \sum_{r=1}^{n} \left\langle \sum_{k=1}^{n} \langle f, \phi_k \rangle f_k, \phi_r \right\rangle f_r \\ &= \sum_{r,k=1}^{n} \langle f, \phi_k \rangle \langle f_k, \phi_r \rangle f_r \\ &= \sum_{r=1}^{n} \langle f, \phi_r \rangle f_r, \end{aligned}$$

by the biorthogonality of $(f_k)_{k\in\mathbb{N}}$ and $(\phi_r)_{r\in\mathbb{N}}$,

$$= P_n f.$$

As $Q_n f = \langle \phi_n, f \rangle f_n$, Q_n is trivially a projection operator.

Assume $(f_n)_{n\in\mathbb{N}}$ is a basis. Then Lemma 4.24 states that there exists a sequence $(\psi_n)_{n\in\mathbb{N}}$ in $\mathcal{B}$ such that the Fourier coefficients f may be expressed $\alpha_n = \langle f, \psi_n \rangle$ and $((f_n)_{n\in\mathbb{N}}, (\psi_n)_{n\in\mathbb{N}})$ is a biorthogonal pair. Let $f \in \mathbb{C}f_1$, then

$$\begin{aligned} \langle f, \psi_1 \rangle f_1 = \sum_{r=1}^{\infty} \langle f, \psi_r \rangle f_r = f = P_1 f = \langle f, \phi_1 \rangle f_1 \\ \Leftrightarrow \langle f, \psi_1 \rangle = \langle f, \phi_1 \rangle. \end{aligned}$$

Note that $\langle g, \psi_r \rangle = \langle g, \phi_r \rangle = 0$ for all $g \in \operatorname{span}\{f_n : n \in \mathbb{N} \setminus \{r\}\}$. Assume that for $f \in \operatorname{span}\{f_k : k \in \{1, 2, \dots, n\}\}$ we have $\langle f, \psi_k \rangle = \langle f, \phi_k \rangle$ for each $k = 1, 2, \dots, n$. Now let $f \in \operatorname{span}\{f_k : k \in \{1, 2, \dots, n+1\}\}$. Then

$$\begin{aligned} \sum_{r=1}^{n+1} \langle f, \psi_r \rangle f_r = \sum_{r=1}^{\infty} \langle f, \psi_r \rangle f_r = f = P_{n+1} f = \sum_{r=1}^{n+1} \langle f, \phi_r \rangle f_r \\ \Leftrightarrow \langle f, \psi_r \rangle = \langle f, \phi_r \rangle. \end{aligned}$$

Hence, by induction, $\langle f, \psi_r \rangle = \langle f, \phi_r \rangle$ for all $r \in \mathbb{N}$, $f \in \mathcal{B}$, so $\psi_r = \phi_r$. Hence, by comparing the limit of $P_n f$, that is the expansion of f, with If we see that $P_n \xrightarrow{\text{S}} I$. That is

$$\lim_{n\to\infty} \|(P_n - I)f\| = 0$$

hence $\sup_{n\in\mathbb{N}} \|(P_n - I)f\| \leqslant \infty$ and, by the uniform boundedness theorem we have that

$$\sup_{n\in\mathbb{N}} \|P_n - I\| < \infty,$$

so P_n is also uniformly bounded.

Now assume the projection operators P_n are uniformly bounded in norm and $(f_n)_{n\in\mathbb{N}}$ is complete. Let $\mathcal{L} = \operatorname{span}\{f_n : n \in \mathbb{N}\}$. If $f \in \mathcal{L}$ then there exists a natural number N such that $P_n f = f$ for all $n > N$. We have that $\overline{\mathcal{L}} = \mathcal{B}$ and $\sup_{n\in\mathbb{N}} \|P_n\| < \infty$, hence $\sup_{n\in\mathbb{N}} \|P_n - I\| < \infty$. Now for $f \in \mathcal{B}$ we may choose a sequence $(g_n)_{n\in\mathbb{N}}$ in $\mathcal{L}$ such that $g_n \to f$ as $n \to \infty$ and there

exists a sequence $(N_n)_{n\in\mathbb{N}}$ such that $P_r g_n = g_n$ for all $r > N_n$. Then

$$\begin{aligned}\lim_{k\to\infty} \|(P_k - I)f\| &= \lim_{k\to\infty} \|(P_k - I) \lim_{n\to\infty} g_n\| \\ &= \lim_{k,n\to\infty} \|(P_k - I)g_n\| \\ &= \lim_{k,n\to\infty} \|g_k - g_n\| = 0,\end{aligned}$$

as $(g_n)_{n\in\mathbb{N}}$ is Cauchy. Hence $(f_n)_{n\in\mathbb{N}}$ is a basis and the Fourier coefficients are given by

$$\alpha_n = P_n f.$$

□

LEMMA 4.28. *Let $((f_n)_{n\in\mathbb{N}}, (\phi_n)_{n\in\mathbb{N}})$ be a biorthogonal pair in a Banach space $\mathcal{B}$. Then*

$$\|Q_n\| = \|\phi_n\|\|f_n\| \geqslant 1.$$

PROOF. We calculate the ratio

$$\frac{\|Q_n(\phi_n)\|}{\|\phi_n\|} = \frac{\|\langle \phi_n, \phi_n\rangle f_n\|}{\|\phi_n\|} = \|\phi_n\|\|f_n\|.$$

But if $f \in \mathcal{B}$ with $\|f\| = 1$ then

$$\|Q_n(f)\| = \|\langle f, \phi_n\rangle f_n\| \leqslant \|\phi_n\|\|f_n\|.$$

This justifies the equality $\|Q_n\| = \|\phi_n\|\|f_n\|$. By the definition of a biorthogonal sequence, $\langle \phi_n, f_n\rangle = 1$ hence

$$\|\phi_n\|\|f_n\| \geqslant 1.$$

□

DEFINITION 4.29. *A biorthogonal pair is said to be* tame *and have a* polynomial growth bound *if there exist $c, \alpha \geqslant 0$ such that $\|Q_n\| \leqslant cn^\alpha$ for all $n \in \mathbb{N}$. If no such bound exists, the biorthogonal pair is* wild.

THEOREM 4.30. *Let $((f_n)_{n\in\mathbb{N}}, (\phi_n)_{n\in\mathbb{N}})$ be a biorthogonal pair and $(f_n)_{n\in\mathbb{N}}$ be a basis. Then $((f_n)_{n\in\mathbb{N}}, (\phi_n)_{n\in\mathbb{N}})$ has a biorthogonal growth bound with $\alpha = 0$.*

PROOF. By Lemma 4.26, the projection operators P_n are uniformly bounded in norm. That is, there exists $M > 0$ such that $\sup_{n\in\mathbb{N}} \|P_n\| = M$. Hence $\sup_{n\geqslant 2} \|Q_n\| = \sup_{n\geqslant 2} \|P_n - P_{n-1}\| \leqslant 2M$. Also $\|Q_1\| = \|P_1\| \leqslant M$, so $\|Q_n\| \leqslant 2M$ for all $n \in \mathbb{N}$. So the polynomial growth bound exists with $\alpha = 0$, $c = 2M$. □

Theorem 4.30 may be used to show that a particular system is not a basis. Indeed, for a given biorthogonal system, $((f_n)_{n\in\mathbb{N}}, (\phi_n)_{n\in\mathbb{N}})$, the norms of the projections Q_n may be calculated using Lemma 4.28. If this shows that the Q_n are not uniformly bounded in norm then Theorem 4.30 implies that $(f_n)_{n\in\mathbb{N}}$ is not a basis. This method is applied in [**14**] and [**16**]. It is also followed in the present work in Subsection 5.2.2, specifically Theorem 5.1 and its requisite Lemmata 5.5 and 5.6.

CHAPTER 5

Two interesting examples

In this chapter we present the detailed analysis of two examples, both for $q_t = q_{xxx}$ with boundary conditions

$$q_x(0,t) + \beta q_x(1,t) = 0 \text{ or } q_x(0,t) = 0,$$
$$q(0,t) = 0,$$
$$q(1,t) = 0.$$

The second of these may be considered as the limit of the first as the coupling constant β approaches 0. For each example we investigate both the homogeneous initial-boundary value problem and the associated the differential operator.

For each example we break the analysis into major themes by section. In Section 5.1 we adapt the standard notation used throughout the thesis to include a superscript β or 0 to distinguish between the two examples. In Sections 5.2 and 5.3 each example has its own subsection so no additional notational identification is necessary. At the end of each section we present a third subsection comparing and contrasting the two cases. In the final subsection we also indicate how the arguments presented in that section may (or may not) be generalised to higher order and to other kinds of boundary conditions.

We conclude the chapter by comparing and contrasting all four of the calculations presented, discussing their relative usefulness and complexity. In the next chapter we discuss some directions for further work, informed by the results of this chapter.

5.1. The problems and regularity

In this section we set up the boundary conditions to be investigated in this chapter. We define the differential operators and the initial-boundary value problems we wish to discuss and calculate some of the simple quantities associated with each. One set of boundary conditions is *coupled* and the other *uncoupled*; we use these words to distinguish between the two problems in this chapter but this does not imply our conclusions are true for all coupled or uncoupled boundary conditions.

5.1.1. The differential operator

Let T^β, respectively T^0, be the differential operator of Definition 4.1 specified by $n = 3$ and the boundary coefficient matrix

$$A^\beta = \begin{pmatrix} 0 & 0 & 1 & \beta & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \text{ respectively } A^0 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{5.1.1}$$

where $\beta \in \mathbb{R} \setminus \{-1, 0, 1\}$.

The corresponding values of ν, ω defined by Notation 4.6 are

$$\nu = 2, \tag{5.1.2}$$

$$\omega = e^{\frac{2\pi i}{3}} = -\frac{1}{2} + \frac{\sqrt{3}}{2}. \tag{5.1.3}$$

We also calculate

$$\Delta^\beta(\rho) = i\rho \sum_{j=0}^{2} \omega^j (e^{-i\omega^j \rho} + \beta)(e^{-i\omega^{j+1}\rho} - e^{-i\omega^{j+2}\rho}),$$

$$\text{respectively } \Delta^0(\rho) = i\rho e^{i\rho}(\omega - \omega^2) \sum_{r=0}^{2} \omega^r e^{i\omega^r \rho}. \tag{5.1.4}$$

Note also that Condition 4.14 holds so, by Theorems 4.8 and 4.15 the PDE discrete spectrum is equal to the discrete spectrum of the operator.

5.1.2. The initial-boundary value problem

We also study the homogeneous initial-boundary value problem associated with (T^β, i), respectively (T^0, i). That is the problem of finding a function $q \in C^\infty([0,1] \times [0,T])$ satisfying the partial differential equation

$$q_t(x,t) - q_{xxx}(x,t) = 0$$

subject to the initial condition

$$q(x,0) = q_0(x),$$

and the boundary conditions

$$A^\beta(f_2(t), g_2(t), f_1(t), g_1(t), f_0(t), g_0(t))^{\mathrm{T}} = (0,0,0)^{\mathrm{T}},$$
$$\text{respectively } A^0(f_2(t), g_2(t), f_1(t), g_1(t), f_0(t), g_0(t))^{\mathrm{T}} = (0,0,0)^{\mathrm{T}}.$$

As in the previous chapters, $q_0 \in C^\infty[0,1]$ is a known function and the boundary functions are defined by

$$f_j(t) = \partial_x^j q(0,t), \qquad g_j(t) = \partial_x^j q(1,t).$$

5.1.3. Regularity

The polynomials P_k^β and Q_k^β, respectively P_k^0 and Q_k^0, defined in Notation 4.6 are given by

$$P_1^\beta(\rho) = \rho, \qquad P_2^\beta(\rho) = 1, \qquad P_3^\beta(\rho) = 0,$$
$$Q_1^\beta(\rho) = \beta\rho, \qquad Q_2^\beta(\rho) = 0, \qquad Q_3^\beta(\rho) = 1,$$
$$P_1^0(\rho) = \rho, \qquad P_2^0(\rho) = 1, \qquad P_3^0(\rho) = 0,$$
$$Q_1^0(\rho) = 0, \qquad Q_2^0(\rho) = 0, \qquad Q_3^0(\rho) = 1,$$

and, using Notation 4.9, $p_0 = 1$.

The polynomials π_1^β, π_0^β associated with T^β, respectively π_1^0, π_0^0 associated with T^0, from Notation 4.9, are given by

$$\pi_1^\beta(\rho) = \det\begin{pmatrix} \beta i\rho & i\rho\omega & \beta i\rho\omega^2 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \qquad \text{respectively} \qquad \pi_1^0(\rho) = \det\begin{pmatrix} 0 & i\rho\omega & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$
$$= \beta i\rho(1-\omega^2), \qquad = 0,$$
$$\pi_0^\beta(\rho) = \det\begin{pmatrix} i\rho & i\rho\omega & \beta i\rho\omega^2 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{respectively} \qquad \pi_0^0(\rho) = \det\begin{pmatrix} i\rho & i\rho\omega & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$= i\rho(1-\omega). \qquad = i\rho(1-\omega).$$

Since $\beta \neq 0$, we find

$$\deg \pi_1^\beta = \deg \pi_0^\beta = p_0,$$

hence the differential operator T^β satisfies the regularity condition of Definition 4.10 but, although $\deg \pi_0^0 = p_0$, the differential operator T^0 is degenerate irregular.

The only difference between the coupled and uncoupled operators is the first boundary condition, the top row of the boundary coefficient matrix. However it is clear that the operators have very different behaviour; the first is regular while the second is degenerate irregular. This difference is reflected in the spectral behaviour of the two differential operators, as is shown in Section 5.2. The initial-boundary value problems also have very different properties. These are discussed in Section 5.3.

5.2. The spectral theory

In this section we use Theorems 4.11 and 4.30 to decide whether the eigenfunctions of each differential operator form a basis. We use only the methods of Chapter 4, the spectral theory of differential operators, without appealing to arguments from the theory of initial-boundary value problems.

5.2.1. Coupled

In Subsection 5.1.3 we showed that this differential operator is regular according the classification of Definition 4.10. This means that we may apply Locker's Theorem 4.11 to see that the generalised eigenfunctions form a complete system in $L_2[0,1]$.

5.2.2. Uncoupled

This differential operator is degenerate irregular so Locker's Theorem 4.11 does not apply. Indeed, we show the opposite:

THEOREM 5.1. *Let T be the differential operator of Definition 4.1 specified by $n = 3$ and the boundary coefficient matrix A^0. Then the eigenfunctions of T do not form a basis in $L^2[0,1]$.*[1]

The remainder of this subsection is devoted to a direct calculation to prove Theorem 5.1. We break the proof into a sequence of lemmata:

LEMMA 5.2. *Let T be the differential operator of Definition 4.1 specified by $n = 3$ and the boundary coefficient matrix A^0. Then the eigenvalues of T are the cubes of the nonzero zeros of*

$$e^{i\rho} + \omega e^{i\omega\rho} + \omega^2 e^{i\omega^2\rho}. \tag{5.2.1}$$

The nonzero zeros of expression (5.2.1) *may be expressed as complex numbers σ_k, $\omega\sigma_k$, $\omega^2\sigma_k$ for each $k \in \mathbb{N}$, where $\mathrm{Re}(\sigma_k) = 0$ and $\mathrm{Im}(\sigma_k) > 0$. Then σ_k is given asymptotically by*

$$-i\sigma_k = \frac{2\pi}{\sqrt{3}}\left(k + \frac{1}{6}\right) + O\left(e^{-\sqrt{3}\pi k}\right) \text{ as } k \to \infty. \tag{5.2.2}$$

LEMMA 5.3. *Let T be the differential operator of Definition 4.1 specified by $n = 3$ and the boundary coefficient matrix A^0 and let $(\sigma_k)_{k\in\mathbb{N}}$ be the increasing sequence of positive imaginary zeros of expression* (5.2.1). *Let*

$$\phi_k(x) = \sum_{r=0}^{2} e^{i\omega^r\sigma_k x}\left(e^{i\omega^{r+2}\sigma_k} - e^{i\omega^{r+1}\sigma_k}\right), \quad k \in \mathbb{N}. \tag{5.2.3}$$

Then, for each $k \in \mathbb{N}$, ϕ_k is an eigenfunction of T with eigenvalue σ_k^3.

LEMMA 5.4. *Let T and $(\sigma_k)_{k\in\mathbb{N}}$ be the differential operator and sequence of imaginary numbers of Lemma 5.3. Then:*

- *The adjoint operator $T^\star$ is the differential operator of Definition 4.1 specified by $n = 3$ and the boundary coefficient matrix*

$$A^\star = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{5.2.4}$$

- *The set of eigenvalues of $T^\star$ is $\{-\sigma_k^3 : k \in \mathbb{N}\}$.*
- *Let*

$$\psi_k(x) = \sum_{r=0}^{2} e^{-i\omega^r\sigma_k x}\left(e^{-i\omega^{r+2}\sigma_k} - e^{-i\omega^{r+1}\sigma_k}\right), \quad k \in \mathbb{N}. \tag{5.2.5}$$

  Then, for each $k \in \mathbb{N}$, ψ_k is an eigenfunction of $T^\star$ with eigenvalue $-\sigma_k^3$ and there are at most finitely many eigenfunctions of $T^\star$ that are not in the set $\{\psi_k : k \in \mathbb{N}\}$.

LEMMA 5.5. *Let σ_k, ϕ_k and ψ_k be the eigenvalues and eigenfunctions from Lemma 5.4. Let*

$$\Psi_k(x) = \frac{\psi_k(x)}{\langle \psi_k, \phi_k \rangle}. \tag{5.2.6}$$

[1] See also [**52**].

Then there exists a minimal $Y \in \mathbb{N}$ *such that* $((\phi_k)_{k=Y}^{\infty}, (\Psi_k)_{k=Y}^{\infty})$ *is a biorthogonal sequence in* $L^2[0,1]$. *Moreover*

$$\langle \psi_k, \phi_k \rangle = 2\cos\left(\frac{\sqrt{3}}{2}(-i\sigma_k)\right)\left[\cosh\left(\frac{3}{2}(-i\sigma_k)\right) + 2\cos\left(\frac{\sqrt{3}}{2}(-i\sigma_k)\right)\right] - 8 \tag{5.2.7}$$

$$= (-1)^k \frac{\sqrt{3}}{2} e^{\sqrt{3}\pi\left(k+\frac{1}{6}\right)} + O(1) \text{ as } k \to \infty. \tag{5.2.8}$$

LEMMA 5.6. *Let* σ_k, ϕ_k *and* ψ_k *be the eigenvalues and eigenfunctions from Lemma 5.4. Then*

$$\|\psi_k\|^2 = \|\phi_k\|^2 \tag{5.2.9}$$

$$= \frac{1}{-i\sigma_k}\Big(\sinh(-i\sigma_k)[\cos(\sqrt{3}(-i\sigma_k)) - 6] + \sqrt{3}\cosh(-i\sigma_k)\sin(\sqrt{3}(-i\sigma_k))$$
$$+ 3\sinh(2(-i\sigma_k)) - 3e^{\frac{1}{2}(-i\sigma_k)}\left[\cos\left(\tfrac{\sqrt{3}}{2}(-i\sigma_k)\right) + \sqrt{3}\sin\left(\tfrac{\sqrt{3}}{2}(-i\sigma_k)\right)\right] \tag{5.2.10}$$
$$+3e^{\frac{-1}{2}(-i\sigma_k)}\left[\cos\left(\tfrac{\sqrt{3}}{2}(-i\sigma_k)\right) - \sqrt{3}\sin\left(\tfrac{\sqrt{3}}{2}(-i\sigma_k)\right)\right]\Big)$$

$$= \frac{3\sqrt{3}e^{\frac{4\pi}{\sqrt{3}}\left(k+\frac{1}{6}\right)}}{4\pi\left(k+\frac{1}{6}\right)} + O\left(\frac{e^{\frac{2\pi}{\sqrt{3}}k}}{k}\right) \text{ as } k \to \infty. \tag{5.2.11}$$

PROOF OF LEMMA 5.2. Equation (5.1.4) gives an expression for $\Delta(\rho)$ in this example. Theorem 4.8 shows that the nonzero zeros of Δ are the n^{th} roots of the nonzero eigenvalues of T. The right hand side of equation (5.1.4) is expression (5.2.1) multiplied by

$$i\rho e^{i\rho}(\omega^2 - \omega),$$

so the nonzero solutions of equation (5.1.4) are the nonzero zeros of expression (5.2.1). If 0 is an eigenvalue of T then any eigenfunction ϕ associated with this eigenvalue must have the properties

$$\phi'''(x) = 0 \; \forall \; x \in [0,1],$$
$$\phi(0) = \phi(1) = \phi'(0).$$

It is trivial that no function, except the zero function, has these properties. Hence 0 is not an eigenvalue of T.

It is shown in the appendix of [**54**] that the zeros of expression (5.2.1) lie on three rays emanating from the origin, the rays $z = i\omega^r x$, $x > 0$ for $r = 0, 1, 2$. Further, by the rotational symmetry of equation (5.2.1), σ_k is a solution if and only if $\omega^r\sigma_k$ is a solution for each $r \in \mathbb{Z}$. Hence we define the sequence $(\sigma_k)_{k\in\mathbb{N}}$ to be an increasing sequence of imaginary numbers, in the sense that $\text{Re}(\sigma_k) = 0$ and $\text{Im}(\sigma_{k+1}) > \text{Im}(\sigma_k) > 0$, such that the set

$$\{0, \sigma_k, \omega\sigma_k, \omega^2\sigma_k : k \in \mathbb{N}\} \tag{5.2.12}$$

is precisely the set of zeros of expression (5.2.1).

Under the condition $-i\sigma_k \in \mathbb{R}^+$, we now asymptotically solve

$$\begin{aligned} 0 &= e^{i\sigma_k} + \omega e^{i\omega\sigma_k} + \omega^2 e^{i\omega^2\sigma_k} \\ &= e^{i\sigma_k} + \frac{1}{2}\left(-1+\sqrt{3}i\right) e^{-\frac{1}{2}i\sigma_k} e^{\frac{\sqrt{3}i}{2}i\sigma_k} + \frac{1}{2}\left(-1-\sqrt{3}i\right) e^{-\frac{1}{2}i\sigma_k} e^{-\frac{\sqrt{3}i}{2}i\sigma_k} \\ &= e^{i\sigma_k} - 2e^{-\frac{1}{2}i\sigma_k} \sin\left(\tfrac{\sqrt{3}}{2}i\sigma_k + \tfrac{\pi}{6}\right) \\ \Rightarrow \qquad -i\sigma_k &= \frac{2\pi}{\sqrt{3}}\left(k+\tfrac{1}{6}\right) + O\left(e^{-\sqrt{3}\pi k}\right) \text{ as } k \to \infty. \end{aligned}$$

□

Proof of Lemma 5.3. To show that ϕ_k is an eigenfunction with eigenvalue σ_k^3 we need to show that ϕ_k is in the domain of T and that $\tau\phi_k = \sigma_k^3\phi_k$. The latter is trivial; for $k \in \mathbb{N}$,

$$\begin{aligned} \tau\phi_k(x) &= (-i)^3 \sum_{r=0}^{2} \left(e^{i\omega^{r+2}\sigma_k} - e^{i\omega^{r+1}\sigma_k}\right) \frac{\mathrm{d}^3}{\mathrm{d}x^3} e^{i\omega^r\sigma_k x} \\ &= (-i)^3 (i\sigma_k)^3 \sum_{r=0}^{2} \omega^{3r} \left(e^{i\omega^{r+2}\sigma_k} - e^{i\omega^{r+1}\sigma_k}\right) e^{i\omega^r\sigma_k x} \\ &= \sigma_k^3 \phi_k(x). \end{aligned}$$

It is immediate that $\phi_k(0) = 0$ and $\phi_k(1) = 0$ so the second and third boundary conditions are satisfied. We now evaluate

$$\begin{aligned} \phi_k'(0) &= i\sigma_k \sum_{r=0}^{2} \omega^r \left(e^{i\omega^{r+2}\sigma_k} - e^{i\omega^{r+1}\sigma_k}\right) \\ &= i\sigma_k(\omega - \omega^2) \sum_{r=0}^{2} \omega^r e^{i\omega^r\sigma_k} \\ &= e^{-i\sigma_k}\Delta(\sigma_k) = 0, \end{aligned}$$

the latter equality being justified by the definition of σ_k as a zero of Δ. This establishes that, for $k \in \mathbb{N}$, ϕ_k is an eigenfunction of T with eigenvalue σ_k^3.

Lemma 5.2 establishes that the only eigenvalues of T are the cubes of σ_k. We have found one eigenfunction for each eigenvalue of T. It remains to be shown that there are no eigenvalues of algebraic multiplicity greater than or equal to 2. Theorem 2.1 in Chapter 4 of [**47**] states that the algebraic multiplicity of σ_k^3 as an eigenvalue of T is the order of σ_k as a zero of the characteristic determinant.[2] As Δ is entire, the order of σ_k as a zero of Δ is greater than or equal to 2 if and only if

$$\frac{\mathrm{d}}{\mathrm{d}\rho}\Delta(\sigma_k) = 0.$$

We calculate

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}\rho}\Delta(\rho) \\ \Leftrightarrow 0 &= e^{i\rho} + \omega^2 e^{i\omega\rho} + \omega e^{i\omega^2\rho}. \end{aligned}$$

[2]This theorem requires σ_k is nonzero but we established in Lemma 5.2 that 0 is not an eigenvalue of T.

But if $\Delta(\rho) = 0$ also then

$$\begin{aligned} 0 &= e^{i\rho} + \omega e^{i\omega\rho} + \omega^2 e^{i\omega^2\rho}, \\ \Rightarrow 0 &= (\omega - \omega^2)\left(e^{i\omega\rho} - e^{i\omega^2\rho}\right) \\ \Rightarrow \rho &= 2k\pi i \; \exists \; k \in \mathbb{Z}. \end{aligned}$$

But

$$\Delta(2k\pi i) = \begin{cases} \sqrt{3}e^{-\sqrt{3}k\pi} & k \text{ even}, \\ 2 + \sqrt{3}e^{-\sqrt{3}k\pi} & k \text{ odd}, \end{cases}$$

which is strictly positive, hence $2k\pi i$ is not a zero of Δ for any $k \in \mathbb{Z}$ and every zero of Δ is simple. □

Proof of Lemma 5.4. The adjoint boundary coefficient matrix $A^\star$ may be constructed using the method presented in Section 3 of Chapter 11 of [**10**], particularly Theorem 3.1, but in this case a direct calculation easily shows that $T^\star$ is adjoint to T.

The matrix $A^\star$ is in reduced row echelon form so we may calculate $\Delta^\star$ using Definition 4.7:

$$\begin{aligned} \Delta^\star(\rho) &= i\rho e^{i\rho}(\omega^2 - \omega)\sum_{r=0}^{2} \omega^r e^{-i\omega^r\rho} \\ &= e^{2i\rho}\Delta(-\rho), \end{aligned} \tag{5.2.13}$$

where Δ is the characteristic determinant of T. The argument in the proof of Lemma 5.2 may be applied to establish that the set of eigenvalues of $T^\star$ is $\{-\sigma_k^3 : k \in \mathbb{N}\}$.

The final statement may be proved using the argument in the proof of Lemma 5.3. □

Proof of Lemma 5.5. For any $j, k \in \mathbb{N}$,

$$\sigma_k^3\langle\phi_k, \psi_j\rangle = \langle T\phi_k, \psi_j\rangle = \langle\phi_k, T^\star\psi_j\rangle = \sigma_j^3\langle\phi_k, \psi_j\rangle,$$

hence if $j \neq k$ then $\langle\phi_k, \psi_j\rangle = 0$. Hence, provided there does not exist $k \in \mathbb{N}$ such that $\langle\phi_k, \psi_k\rangle = 0$, the eigenfunctions of T and $T^\star$ form a biorthogonal sequence. Indeed, by the following asymptotic calculation there must exist some $Y \geqslant 1$ such that $\langle\phi_k, \psi_k\rangle \neq 0$ for all $k \geqslant Y$.

As $-i\sigma_k \in \mathbb{R}^+$,

$$\begin{aligned} \overline{\phi_k} &= -\phi_k \\ \overline{\psi_k} &= -\psi_k. \end{aligned} \tag{5.2.14}$$

Using equations (5.2.14), we calculate

$$\begin{aligned}
\langle \phi_k, \psi_k \rangle &= -\int_0^1 \phi_k(x)\psi_k(x)\,\mathrm{d}x \\
&= -\int_0^1 \sum_{r,l=0}^{2} e^{i(\omega^r-\omega^l)\sigma_k x}\Big(e^{i(\omega^{r+1}-\omega^{l+1})\sigma_k} + e^{i(\omega^{r+2}-\omega^{l+2})\sigma_k} \\
&\qquad\qquad -e^{i(\omega^{r+1}\omega^{l+2})\sigma_k} - e^{i(\omega^{r+2}-\omega^{l+1})\sigma_k}\Big)\,\mathrm{d}x \\
&= -\sum_{r=0}^{2}(2 - e^{i\omega^r(1-\omega)\sigma_k} - e^{-i\omega^r(1-\omega)\sigma_k}) - \\
&\quad \sum_{r=0}^{2}\int_0^1 e^{i\omega^r(1-\omega)\sigma_k x}\,\mathrm{d}x\left(e^{i\omega^{r+1}(1-\omega)\sigma_k} + e^{i\omega^{r+2}(1-\omega)\sigma_k} - e^{i\omega^{r+1}(1-\omega^2)\sigma_k} - 1\right) \\
&\quad + \int_0^1 e^{i\omega^r(1-\omega^2)\sigma_k x}\,\mathrm{d}x\left(e^{i\omega^{r+1}(1-\omega^2)\sigma_k} + e^{i\omega^{r+2}(1-\omega^2)\sigma_k} - e^{i\omega^{r+2}(1-\omega)\sigma_k} - 1\right).
\end{aligned} \tag{5.2.15}$$

The first sum in equation (5.2.15) evaluates to the right hand side of equation (5.2.7). The second sum evaluates to 0.

The asymptotic expression (5.2.8) follows from

$$\cos\left(\frac{\sqrt{3}}{2}(-i\sigma_k)\right) = (-1)^k\frac{\sqrt{3}}{2} + O(ke^{-\sqrt{3}\pi k}).$$

□

Proof of Lemma 5.6. The proof of equations (5.2.9) and (5.2.10) is a simple but lengthy calculation and is omitted here in the interest of brevity. Equation (5.2.11) is justified by applying the asymptotic approximation (5.2.2) to equation (5.2.10). □

Proof of Theorem 5.1. By Lemma 5.5, the pair $((\phi_k)_{k=Y}^{\infty}, (\psi_k)_{k=Y}^{\infty})$ is a biorthogonal system. If $Y > 1$ then the sequence $(\phi_k)_{k=L}^{\infty}$ is not complete as $\psi_1 \notin \overline{\text{span}}\{\phi_k : k \geqslant L\}$. Certainly this implies $(\phi_k)_{k=L}^{\infty}$ is not a basis.

Now assume $Y = 1$. This means that the projections Q_k are well defined in Definition 4.23. Using Lemmata 4.28, 5.5 and 5.6,

$$\begin{aligned}
\|Q_k\| &= \|\phi_k\|\|\Psi_k\| \\
&= \frac{\|\phi_k\|^2}{|\langle \psi_k, \phi_k\rangle|} \\
&= \frac{3e^{\frac{\pi}{\sqrt{3}}\left(k+\frac{1}{6}\right)}}{2\pi\left(k+\frac{1}{6}\right)} + O\left(\frac{e^{-\frac{\pi}{\sqrt{3}}k}}{k}\right) \text{ as } k\to\infty.
\end{aligned} \tag{5.2.16}$$

Hence the biorthogonal sequence is wild. Now Theorem 4.30 shows that $(\phi_k)_{k\in\mathbb{N}}$ is not a basis in $L_2[0,1]$. □

Numerical evidence suggests that $Y = 1$.

5.2.3. The limit $\beta \to 0$

We may wish to consider the calculations in Subsection 5.2.2 as the limit $\beta \to 0$ of such calculations for the coupled operator. For concreteness, we specify $\beta \in (-1, 0)$ and consider the one-sided limit $\beta \to 0^-$. Indeed, we may show that if σ_k is a zero of Δ_{PDE} then $\omega^r \sigma_k$ is also a zero of Δ_{PDE} and $-\sigma_k$ is a zero of $\Delta^\star_{\text{PDE}}$. We may express the zeros of Δ_{PDE} as the complex numbers σ_k, $\omega\sigma_k$ and $\omega^2\sigma_k$ for $k \in \mathbb{N}$ where the σ_k are given asymptotically by the expression

$$\sigma_k = \begin{cases} \left(k - \frac{1}{3}\right)\pi + i\log(-\beta) + O\left(e^{\frac{-\sqrt{3}k\pi}{2}}\right) & k \text{ even}, \\ \left(-k - \frac{2}{3}\right)\pi + i\log(-\beta) + O\left(e^{\frac{-\sqrt{3}k\pi}{2}}\right) & k \text{ odd}. \end{cases} \tag{5.2.17}$$

With this definition of σ_k the eigenfunctions of T and $T^\star$ are given by equations (5.2.3) and (5.2.5) respectively. After a suitable scaling, the eigenfunctions of the operator and its adjoint form a biorthogonal sequence. As above, the norms $\|\phi_k\|$ and $\|\psi_k\|$ are equal. The difference is that, for $\beta \neq 0$, the fastest-growing terms in $\|\phi_k\|$ cancel out so that, for large k,

$$\|\phi_k\|^2 = O\left(e^{\frac{\sqrt{3}k\pi}{2}} k^{-1}\right) = \langle \phi_k, \psi_k \rangle.$$

This suggests that the biorthonormalised eigenfunctions,

$$\frac{\phi_k}{\sqrt{|\langle \phi_k, \psi_k \rangle|}},$$

are in fact normalised in the coupled case whereas they are not normalised in the uncoupled case, where their norms grow exponentially with k.

In the notation of Remark 4.27, the eigenfunctions of the coupled operator and its adjoint are the sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ whereas the eigenfunctions of the uncoupled operator and its adjoint are the sequences $(c_n)_{n\in\mathbb{N}}$ and $(d_n)_{n\in\mathbb{N}}$. For the coupled operator it is possible to divide by sequences of scalars so that the eigenfunctions and adjoint eigenfunctions may be simultaneously normalised and mutually biorthonormalised. Such an undertaking is not possible for the uncoupled operator.

We may compare the positioning of the eigenvalues, the cube of each σ_k, between the coupled and uncoupled cases. The big-O terms in equation (5.2.17) depend upon β but for any particular $\beta \in (-1, 0)$ there exists a sufficiently large R that we may be sure each annulus

$$\{\rho \in \mathbb{C} : R + k\pi < |\rho| < R + (k+1)\pi\}, \quad j \in \mathbb{N}$$

contains precisely six zeros of Δ_{PDE}. However if $\beta = 0$ then there exists a sufficiently large R that we may be sure each annulus

$$\left\{\rho \in \mathbb{C} : R + k\frac{4\pi}{\sqrt{3}} < |\rho| < R + (k+1)\frac{4\pi}{\sqrt{3}}\right\}, \quad j \in \mathbb{N}$$

contains precisely six zeros of Δ_{PDE}.

Numerical work suggests that for any particular $\beta \in (-1, 0)$ the zeros of Δ_{PDE} are distributed approximately at the crosses in Figure 5.1. The red rays and line segments represent the assymptotic locations of the zeros. The grey lines are ∂D, the contours of integration in the associated initial-boundary value problem. As $\beta \to 0^-$, hence $\log(-\beta) \to -\infty$, the red rays

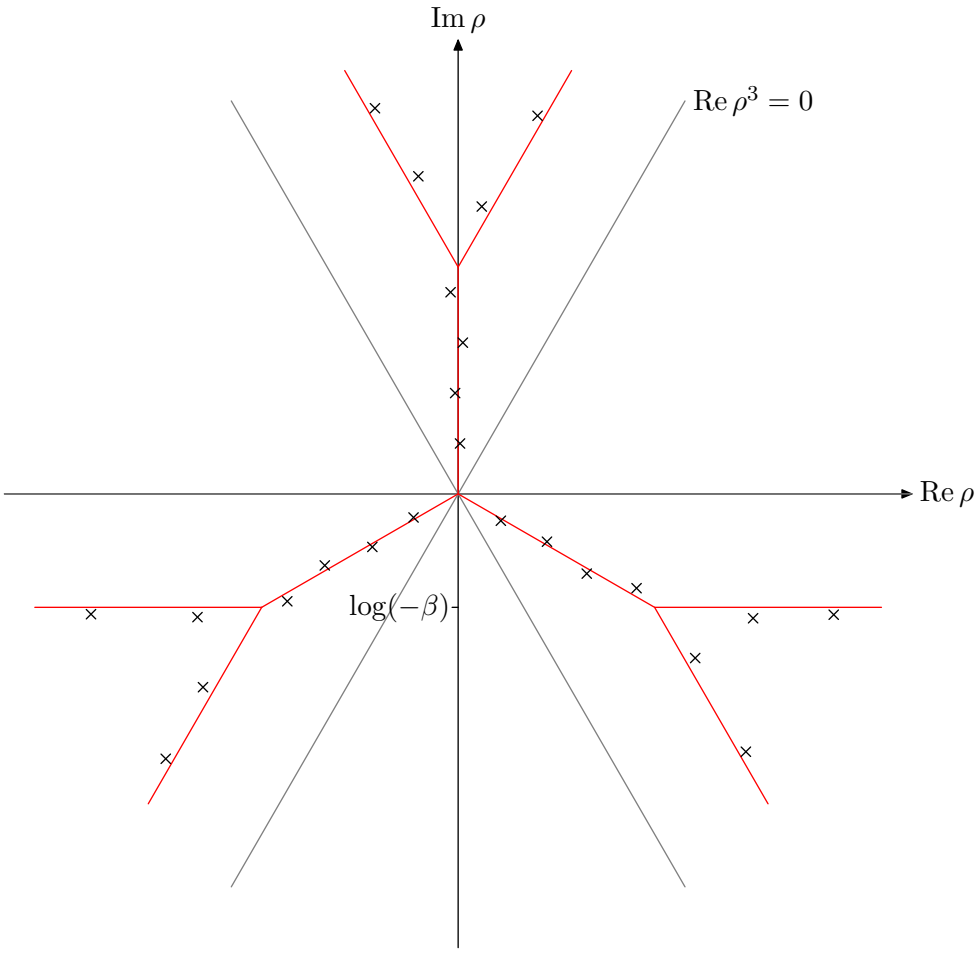

FIGURE 5.1. The asymptotic position of σ_k for $\beta \in (-1, 0)$.

move further from the origin, leaving the complex plane entirely in the limit, so that the red line segments emanating from the origin extend to infinity.

5.2.4. Comparison

It has been shown that the generalised eigenfunctions of the differential operator with coupled boundary conditions form a complete system in $L^2[0, 1]$, but when the boundary conditions are uncoupled the eigenfunctions do not form a basis. It was very easy to use regularity to conclude that the generalised eigenfunctions form a complete system but to show that the eigenfunctions of the degenerate irregular differential operator do not form a basis involved a lengthy asymptotic calculation. The deep symmetry of the boundary conditions,[3] and the specific identities used, make the argument of Subsection 5.2.2 an unattractive method for generalisation.

[3]Indeed, in the limit $\beta \to -1$ the operator becomes self-adjoint.

We consider one particular generalisation. Let $n \geqslant 3$ be an odd number with $n = 2\nu - 1$ and let the boundary coefficient matrices be

$$A = \begin{cases} \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ & & & & 0 & 0 & & & & \\ & & & & 0 & 0 & & & & \\ & 0_{n-1} & & & \vdots & \vdots & & I_{n-1} & & \\ & & & & 0 & 0 & & & & \end{pmatrix} & \nu \text{ even}, \\ (0_n | I_n) & \nu \text{ odd}. \end{cases} \tag{5.2.18}$$

It is not hard to show that $A^\star$ takes the opposite form to A in equation (5.2.18) but certain steps become significantly more difficult. The eigenfunctions have a more complex form and the proofs of Lemma 5.3 and the final statement of Lemma 5.4 become much more difficult, requiring combinatorial arguments. The argument in the appendix of [**54**] does not generalise to higher order so it can only be shown that $\mathrm{Re}(\sigma_k) \to 0$ which is insufficient for equations (5.2.14) to hold and makes the calculations of Lemmata 5.5 and 5.6 significantly more complex.

Even the above generalisation of uncoupled boundary conditions still has a great deal of symmetry. If general uncoupled (even non-Robin) boundary conditions are considered then this symmetry is destroyed. This introduces further complication to proving equivalent forms of Theorem 5.1.

5.3. The IBVP theory

In this section we use Theorem 3.50, Theorem 3.23 and a direct calculation to show that both initial-boundary value problems are well-posed and show that only the solution of the coupled case has a series representation. We use only the methods of Chapter 3, the theory of initial-boundary value problems, without appealing to arguments from spectral theory.

5.3.1. Coupled

THEOREM 5.7. *The initial-boundary value problem associated with (T^β, i) is well-posed and its solution admits a series representation.*

PROOF OF THEOREM 5.7. Using Notation 3.18, $L = R = C = 1$. Hence Conditions 3.19 and 3.35 hold. To check Conditions 3.22 and 3.36 we write

$$\mathcal{R} = \{3\}, \qquad \mathcal{L} = \{3\}, \qquad \mathcal{C} = \{2\},$$
$$r : 3 \mapsto 2, \qquad l : 3 \mapsto 1, \qquad c : 2 \mapsto 3,$$
$$c' : 1 \mapsto 2, \qquad \tau_j = \begin{pmatrix} 3-j \\ 1-j \\ 2-j \end{pmatrix}.$$

We also define I_1 to be the identity transformation on $\{1\}$, the only element of S_1. For Condition 3.22, $k = 1$ so we must check that

$$\sum_{\substack{\sigma\in S_3:\\ (\sigma,1)\in S_{1\,\tau_j\,I_1}}} \operatorname{sgn}(\sigma)\omega^{-2\sigma(3)} \tag{5.3.1}$$

is nonzero. However $(\sigma, 1) \in S_{1\,\tau_j\,I_1}$ if and only if $\sigma(2) \in \{1 - j, 2 - j\}$ so expression 5.3.1 evaluates to

$$\omega^{2j}(\omega^{-6} - \omega^{-6} - \omega^{-1} + \omega^{-2}) = \omega^{2j}(\omega - \omega^2) \neq 0.$$

For Condition 3.36, $k = 0$ so we check

$$\sum_{\substack{\sigma\in S_3:\\ \sigma(2)=\tau_j(2)}} \operatorname{sgn}(\sigma)\omega^{-2\sigma(3)} = \omega^{2j}(1 - \omega^2) \neq 0.$$

As n is odd, the boundary conditions are homogeneous and non-Robin, Conditions 3.19 and 3.22 hold and Conditions 3.35 and 3.36 hold, Theorem 3.50 guarantees the initial-boundary value problem is well-posed and that its solution admits a series representation. □

5.3.2. Uncoupled

THEOREM 5.8. *The initial-boundary value problem associated with (T^0, i) is well-posed but its solution does not admit a series representation.*

PROOF. Using Notation 3.18, $L = 1$, $R = 2$ and $C = 0$. Hence Condition 3.19 holds. To check Condition 3.22 we write

$$\begin{aligned} &\mathcal{R} = \{3, 2\}, && \mathcal{L} = \{3\}, \\ &r : 3 \mapsto 2, && l : 3 \mapsto 1, \\ &r : 2 \mapsto 1, && \tau_j = \begin{pmatrix} 3 - j \\ 1 - j \\ 2 - j \end{pmatrix}. \end{aligned}$$

We check

$$\sum_{\substack{\sigma\in S_3:\\ \sigma(1)=\tau_j(1)}} \operatorname{sgn}(\sigma)\omega^{-2\sigma(3)} = \omega^{2j}(\omega^2 - \omega) \neq 0.$$

As n is odd, the boundary conditions are homogeneous and non-Robin and Conditions 3.19 and 3.22 hold, Theorem 3.23 guarantees Assumption 3.2 holds which, by Corollary 3.31, guarantees the initial-boundary value problem is well-posed.

Because $R > \nu - 1$, Condition 3.35 does not hold so Theorem 3.37 does not apply. Indeed, in this example we show that Assumption 3.3 does not hold. From its definition in Lemma 2.14, we calculate the reduced global relation matrix,

$$\mathcal{A}(\rho) = \begin{pmatrix} c_2(\rho) & c_2(\rho)e^{-i\rho} & c_1(\rho)e^{-i\rho} \\ c_2(\rho) & c_2(\rho)e^{-i\omega\rho} & c_1(\rho)\omega e^{-i\omega\rho} \\ c_2(\rho) & c_2(\rho)e^{-i\omega^2\rho} & c_1(\rho)\omega^2 e^{-i\omega^2\rho} \end{pmatrix},$$

hence its determinant,

$$\Delta_{\mathrm{PDE}}(\rho) = i\rho(\omega^2-\omega)\sum_{r=0}^{2}\omega^r e^{i\omega^r\rho},$$

and the functions

$$\begin{aligned}
\zeta_1(\rho) &= i\rho(\omega^2-\omega)\sum_{r=0}^{2}\omega^r\hat{q}_0(\omega^r\rho)e^{i\omega^r\rho},\\
\zeta_2(\rho) &= i\rho\sum_{r=0}^{2}\hat{q}_0(\omega^r\rho)\left(\omega^{r+1}e^{-i\omega^{r+1}\rho}-\omega^{r+2}e^{-i\omega^{r+2}\rho}\right),\\
\zeta_3(\rho) &= i\rho\sum_{r=0}^{2}\hat{q}_0(\omega^r\rho)\left(e^{-i\omega^{r+2}\rho}-e^{-i\omega^{r+1}\rho}\right),\\
\zeta_4(\rho) &= \zeta_5(\rho) = \zeta_6(\rho) = 0.
\end{aligned}$$

As $a = i$, the regions of interest in Assumption 3.3 are

$$\widetilde{E}_j \subseteq E_j = \left\{\rho\in\mathbb{C} : \frac{(2j-1)\pi}{3} < \arg(\rho) < \frac{2j\pi}{3}\right\}.$$

We consider the particular ratio

$$\frac{\zeta_3(\rho)}{\Delta_{\mathrm{PDE}}(\rho)}, \quad \rho\in\widetilde{E}_2. \tag{5.3.2}$$

For $\rho\in\widetilde{E}_2$, $\mathrm{Re}(i\omega^r\rho) < 0$ if and only if $r = 2$ so we may approximate ratio (5.3.2) by its dominant terms as $\rho\to\infty$ from within $\widetilde{E}_2$,

$$\frac{(\hat{q}_0(\rho)-\hat{q}_0(\omega\rho))e^{-i\omega^2\rho}+\hat{q}_0(\omega^2\rho)(e^{-i\omega\rho}-e^{-i\rho})+o(1)}{(\omega^2-\omega)e^{i\rho}+(1-\omega^2)e^{i\omega\rho}+o(1)}.$$

We expand the integrals from the $\hat{q}_0$ in the numerator and multiply the numerator and denominator by $e^{-i\omega\rho}$ to obtain

$$\frac{i\int_0^1\left(e^{i\rho(1-x)}-e^{i\rho(1-\omega x)}-e^{i\rho\omega^2(1-x)}+e^{-i\rho(2\omega-\omega^2x)}\right)\hat{q}_0(x)\,\mathrm{d}x+o\left(e^{\mathrm{Im}(\omega\rho)}\right)}{\sqrt{3}(e^{i\rho(1-\omega)}+\omega)+o\left(e^{\mathrm{Im}(\omega\rho)}\right)}. \tag{5.3.3}$$

Let $(R_j)_{j\in\mathbb{N}}$ be a strictly increasing sequence of positive real numbers such that $\rho_j = R_j e^{i\frac{7\pi}{6}}\in\widetilde{E}_2$, R_j is bounded away from $\{\frac{2\pi}{\sqrt{3}}(k+\frac{1}{6}) : k\in\mathbb{N}\}$[4] and $R_j\to\infty$ as $j\to\infty$. Then $\rho_j\to\infty$ from within $\widetilde{E}_2$. We evaluate ratio (5.3.3) at $\rho = \rho_j$,

$$\frac{i\int_0^1\left(2ie^{\frac{R_j}{2}(1-x)-\frac{\sqrt{3}R_j}{2}i}\sin\left(\frac{\sqrt{3}R_jx}{2}\right)-e^{-R_j(1-x)}\left(1-e^{-\sqrt{3}R_ji}\right)\right)\hat{q}_0(x)\,\mathrm{d}x+o\left(e^{-\frac{R_j}{2}}\right)}{\sqrt{3}(e^{-\sqrt{3}R_ji}+\omega)+o\left(e^{-\frac{R_j}{2}}\right)}. \tag{5.3.4}$$

The denominator of ratio (5.3.4) is bounded away from 0 by the definition of R_j and the numerator tends to ∞ for any nonzero initial datum. This establishes that Assumption 3.3 does not hold which imples that there is no series representation of the solution. □

[4]Of course this is guaranteed by $\rho_j\in\widetilde{E_2}$ and the asymptotic expression (5.2.2) for σ_k but by adding this condition explicitly we avoid having to resort to Lemma 5.2

REMARK 5.9. In the proof of Theorem 5.8 we use the example of the ratio $\frac{\zeta_3(\rho)}{\Delta_{\mathrm{PDE}}(\rho)}$ being unbounded as $\rho \to \infty$ from within $\widetilde{E}_2$. It may be shown using the same argument that $\frac{\zeta_2(\rho)}{\Delta_{\mathrm{PDE}}(\rho)}$ is unbounded in the same region and that both these ratios are unbounded for $\rho \in \widetilde{E}_3$ using $\rho_j = R_j e^{i\frac{11\pi}{6}}$ for appropriate choice of $(R_j)_{j\in\mathbb{N}}$. However the ratio

$$\frac{\zeta_1(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} = \frac{\sum_{j\in J^+}\zeta_j(\rho)}{\Delta_{\mathrm{PDE}}(\rho)}$$

is bounded in $\widetilde{E}_1 = \widetilde{E}^+$ hence it is possible to deform the contours of integration in the upper half-plane. This permits a partial series representation of the solution to the initial-boundary value problem.

5.3.3. Comparison

We recall that Condition 3.19 requires that appropriate numbers of boundary conditions be specified at each end of the interval and that Condition 3.22 requires that n particular quantities be nonzero. The pair of Conditions 3.19 and 3.22 is easier to check than the direct calculation presented in Example 3.14. This is because they focus only upon the exponentials appearing in Δ_{PDE}, meaning that only $n+1$ checks must be performed (one to ensure R and C take appropriate values and one for ρ in each $\widetilde{E}_j$) whereas the direct calculation of Example 3.14 requires n^2 checks (one for each ζ_k in each $\widetilde{E}_j$) although only one is presented in that example.

The disadvantage of using the pair of Conditions 3.19 and 3.22, or the pair of Conditions 3.35 and 3.36, is that they are not known to be necessary. So when they fail, as in our uncoupled example, a direct computation is still required. To show the failure of Assumption 3.2 or 3.3 only a single counterexample is required so the direct computation is more suited to proving that a problem is ill-posed or that the solution of a well-posed problem does not a have a series representation than proving the positive results.

The proofs of Theorems 5.7 and 5.8 generalise quite easily to other examples, at least on a case-by-case basis. They may even be proven for the example of boundary coefficient matrix (5.2.18) for arbitrary odd n but more care must be taken with the asymptotic argument in the second part of the proof of Theorem 5.8.

CHAPTER 6

Conclusion and further work

6.1. Conclusion

In this work, we have discussed the mutual interaction of two conceptually separate approaches to the study of linear differential operators and linear partial differential equations. The derivation of characterising conditions and results in one context can be transferred to conditions and results in the other. In some cases, one approach emerges as preferable, because it is simpler or more exhaustive.

Although the use of Conditions 3.19, 3.22, 3.35 and 3.36 is easier than a direct verification of Assumptions 3.2 and 3.3, derived in the present work, it is still significantly easier to check Locker's regularity conditions, Definition 4.10. Hence, if a boundary value problem is such that Theorem 4.15 holds then Locker's regularity conditions are useful as they suggest that the solution has a series representation without having to construct $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{C}$. Thus the study of an ordinary differential operator informs the study of its associated boundary value problems on partial differential equations.

However the calculation in the proof of Theorem 5.8 is far simpler than that in the proof of Theorem 5.1 and, as discussed in Subsections 5.2.4 and 5.3.3, it generalises more easily. This suggests that the study of an initial-boundary value problem on a partial differential equation informs the study of the ordinary differential operator with which it is associated.

6.2. Open problems

Below, we detail a few problems left open by the preceding chapters. This is not intended to be a complete survey of such problems but a sample of those that the author considers to be of greatest interest.

6.2.1. Eigenvalues

Theorem 4.15 gives sufficient conditions for the PDE discrete spectrum to be precisely the zeros of the characteristic determinant, that is the n^{th} roots of the discrete spectrum, of the associated ordinary differential operator. In Subsection 4.2.2 we discuss attempts to extend this result to arbitrary boundary conditions, observing that there are no known counterexamples.

The proof of Theorem 4.15 actually gives a stronger result than the theorem states; it is proven that $\mathcal{A}$ and M have the same columns as one another. An immediate corollary of this is that under the conditions of Theorem 4.15

$$\Delta_{\text{PDE}}(\rho) = X\rho^{Y}e^{Z\rho}\Delta(\rho) \quad \exists\, X \in \mathbb{C}\setminus\{0\}, \quad \exists\, Y \in \mathbb{Z}, \quad \exists\, Z \in \mathbb{C}. \tag{6.2.1}$$

This means that the nonzero zeros of Δ_{PDE} are precisely the nonzero zeros of Δ and each zero is of the same order in both determinant functions. By Theorem 2.1 in Chapter 4 of [**47**] this gives not only the eigenvalues of T but also their algebraic multiplicities directly from the initial-boundary value problem. Therefore, it is reasonable to call Δ_{PDE} the *PDE chracteristic determinant* for initial-boundary value problems such that equation (6.2.1) holds. We make the following

CONJECTURE 6.1. *Equation* (6.2.1) *holds for arbitrary boundary conditions of any order.*

In the third order this conjecture has been established for all non-Robin boundary conditions but it was necessary to investigate many sets of boundary conditions individually. A general argument that does not require symmetry conditions has thus-far been elusive.

It has been shown that equation (6.2.1) holds for general Robin boundary conditions but the symmetry condition has not been removed. For Robin boundary conditions, the symmetry condition becomes very complex to express. Completely removing the symmetry condition is a topic of current research. The zeros of the two characteristic determinants are of great interest in their respective problems and to show that the zeros are the same and of the same orders in general would be of great utility.

6.2.2. Regularity conditions for well-posedness

In Example 3.24 we derive necessary and sufficient conditions for well-posedness of the initial-boundary value problems associated with (T, i) and $(T, -i)$ where T is the third order operator with pseudo-periodic boundary conditions. In the second row of Table 1 on page 134 we note that these conditions correspond to the polynomials π_1 and π_0 from Notation 4.9 not being identically zero. The same result holds for simple boundary conditions, as is shown in Table 1. It would be interesting to know if this correspondence extends to arbitrary boundary conditions:

CONJECTURE 6.2. *For any differential operator T, $\pi_1 = 0$ only if the initial-boundary value problem associated with (T, i) is ill-posed and $\pi_0 = 0$ only if the initial-boundary value problem associated with $(T, -i)$ is ill-posed.*

If this conjecture is true then it gives even simpler conditions for well-posedness of an initial-boundary value problem. Even if it holds only for odd order initial-boundary value problems with non-Robin boundary conditions, problems whose well-posedness may already be checked by Conditions 3.19 and 3.22, it is still gives a much easier check for well-posedness than the existing conditions. Indeed, a related conjecture is:

CONJECTURE 6.3. *Let T be an odd order differential operator with non-Robin boundary conditions. Then*

- $\deg(\pi_1) = p_0$ *if and only if Conditions 3.19 and 3.22 hold for the initial-boundary value problem associated with (T, i).*
- $\deg(\pi_0) = p_0$ *if and only if Conditions 3.19 and 3.22 hold for the initial-boundary value problem associated with $(T, -i)$.*

6.2.3. Rates of blow-up

There is a further suggestion of a link in the comparison of the two calculations for the uncoupled case in Chapter 5. In the proof of Theorem 5.8 we require that the R_j are bounded away from the set $\{\frac{2\pi}{\sqrt{3}}(k + \frac{1}{6}) : k \in \mathbb{N}\}$. Consider what happens in the limit where this bound

approaches 0; let each $R_j = \frac{2\pi}{\sqrt{3}}(j+\frac{1}{6}) + X$ and consider the limit $X \to 0$. Then the ratio of the dominant term in expression (5.3.4) to R_j is given by

$$\frac{(-1)^j e^{i\frac{\pi}{6}} e^{\frac{\pi}{\sqrt{3}}(j+\frac{1}{6})}}{\sqrt{3}\pi(j+\frac{1}{6})}, \tag{6.2.2}$$

which is a scalar multiple of the dominant term in equation (5.2.16) hence these two quantities, one from operator theory and the other from the study of the IBVP, have the same rate of blowup. This suggests that the (greatest) rate of blowup of the integrands

$$\frac{\eta_j(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \to 0 \text{ as } \rho \to \infty \text{ within } \widetilde{D}^{\pm} \text{ and}$$
$$\frac{\zeta_j(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \to 0 \text{ as } \rho \to \infty \text{ within } \widetilde{E}^{\pm},$$

of Assumptions 3.2 and 3.3 is equal to the rate of blowup of $\|Q_k\|$.

In the following, we work towards a formal statement of the above conjecture. In order to do so, we make several other conjectures. We assume that every zero of Δ_{PDE} is of order one. This assumption may not always hold but it is expected that any result obtained under it may be extended to the general case. The function of this assumption is to simplify residue calculations and to ensure that each eigenvalue has a unique (up to scalar multiples) eigenfunction. We define the

NOTATION 6.4. Let $(\sigma_k)_{k\in\mathbb{N}}$ be a sequence containing all the nonzero roots of $\Delta_{\mathrm{PDE}}(\rho) = 0$ within the sector $0 \leqslant \arg\rho < \frac{2\pi}{n}$ precisely once such that $\sigma_k \leqslant \sigma_{k+1}$ for all $k \in \mathbb{N}$.

Then we have

THEOREM 6.5. *If a series representation may be obtained by the methods of Chapter 3 (the decay assumptions both hold) then, for each $k \in \mathbb{N}$, the function*

$$\phi_k(x) = \sum_{r=0}^{\nu-1} \omega^r \lim_{\rho\to\sigma_k} \left(\frac{\rho-\sigma_k}{\Delta_{\mathrm{PDE}}(\rho)}\right) e^{i\omega^r\sigma_k x} \sum_{j\in J^+} \zeta_j(\omega^r\sigma_k) + \sum_{r=\nu}^{n-1} \omega^r \lim_{\rho\to\sigma_k} \left(\frac{\rho-\sigma_k}{\Delta_{\mathrm{PDE}}(\rho)}\right) e^{i\omega^r\sigma_k(x-1)} \sum_{j\in J^-} \zeta_j(\omega^r\sigma_k) \tag{6.2.3}$$

is an eigenfunction of the ordinary differential operator T.

This theorem is not stated in Section 4.3 but it is implicit in the proof Theorem 4.18. Indeed, its proof requires only an evaluation of the function ϕ_k in the original proof, observing that $\Delta_{\mathrm{PDE}}(\omega^r\rho) = \Delta_{\mathrm{PDE}}(\rho)$. The difficulty is in choosing the correct sector in which to specify the sequence $(\sigma_k)_{k\in\mathbb{N}}$.

On its own the above theorem is not particularly helpful as there may be any number of eigenvalues and eigenfunctions of T that do not appear in the series expansion. We conjecture that we may extend Theorem 6.5 to an equivalence:

CONJECTURE 6.6. *Under the conditions of Theorem 6.5, the functions* (6.2.3) *are the* only *eigenfunctions of T.*

The essential missing link in the proof of this Conjecture 6.6 is Conjecture 6.1. We assume this is the case and proceed with a proof.

PROOF OF CONJECTURE 6.6 UNDER AN ASSUMPTION. Under Conjecture 6.1, the nonzero zeros of Δ are precisely the nonzero zeros of Δ_{PDE}. Further, if $\sigma \neq 0$ is a zero of Δ_{PDE}, then σ has the same order in both Δ and Δ_{PDE} hence, as we assume all the zeros of Δ_{PDE} are of order one, σ is of order one in Δ. Then, by Theorem 2.1 in Chapter 4 of [**47**], σ^n is an eigenvalue of T with algebraic multiplicity one. Now, as there are no eigenvalues that are not nonzero zeros of Δ, there can be no eigenfunctions other than the ϕ_k. □

So far, we only have results when a series representation of the solution exists but we are primarily interested in the case where one of the initial- or final- boundary value problems are ill-posed, that is where a series representation does not exist.

CONJECTURE 6.7. *Theorem 6.5 and Conjecture 6.6 both hold even when the initial- or final-boundary value problems are ill-posed.*

This conjecture is certainly true for the only example investigated in the thesis—the uncoupled example in Chapter 5.

Let $T^\star$ be the adjoint operator to T. Then the boundary value problems associated to $T^\star$ have PDE characteristic determinant $\Delta^\star_{\text{PDE}}$ whose zeros are given by the sequence $(\sigma_k^\star)_{k\in\mathbb{N}}$, defined as in Notation 6.4. Then, by conjecture 6.7, the eigenfunctions of $T^\star$ are

$$\psi_k(x) = \sum_{r=0}^{\nu-1} \omega^r \lim_{\rho\to\sigma_k^\star} \left(\frac{\rho-\sigma_k^\star}{\Delta^\star_{\text{PDE}}(\rho)}\right) e^{i\omega^r\sigma_k^\star x} \sum_{j\in J^+} \zeta_j^\star(\omega^r\sigma_k^\star) + \sum_{r=\nu}^{n-1} \omega^r \lim_{\rho\to\sigma_k^\star} \left(\frac{\rho-\sigma_k^\star}{\Delta^\star_{\text{PDE}}(\rho)}\right) e^{i\omega^r\sigma_k^\star(x-1)} \sum_{j\in J^-} \zeta_j^\star(\omega^r\sigma_k^\star). \quad (6.2.4)$$

We require one further conjecture, regarding the asymptotic position of σ_k.

CONJECTURE 6.8. *Let T be such that at least one of the initial-boundary value problem associated with (T, i) and the initial-boundary value problem associated with $(T, -i)$ is well-posed. Then either*

Case 1: *There exist $X, \varepsilon > 0$, $Y \in \mathbb{R}$ and $\theta \in [0, \frac{2\pi}{n})$ such that*

$$\sigma_k = (Xk + Y)e^{i\theta} + z + O(e^{-\varepsilon k}) \quad \text{as } k \to \infty, \quad (6.2.5)$$

or

Case 2: *There exist $X, \varepsilon > 0$, $Y_1, Y_2 \in \mathbb{R}$, $\theta_1 \in [0, \frac{\pi}{n})$ and $\theta_2 \in [\frac{\pi}{n}, \frac{2\pi}{n})$ such that*

$$\sigma_{2k} = (Xk + Y_1)e^{i\theta_1} + z + O(e^{-\varepsilon k}) \quad \text{as } k \to \infty, \quad (6.2.6)$$

$$\sigma_{2k+1} = (Xk + Y_2)e^{i\theta_2} + z + O(e^{-\varepsilon k}) \quad \text{as } k \to \infty. \quad (6.2.7)$$

The above conjecture is not a great strengthening of the results that follow immediately from the distribution theory of zeros of exponential polynomials [**42**]. Certainly the σ_k fall within one or two semi-strips and are asymptotically regularly spaced, we are only strengthening this from semi-strips to rays. It is a further conjecture that Case 2 holds if and only if the series

representation exists and Case 1 holds if and only if precisely one of the initial-boundary value problems associated with (T, i) and $(T, -i)$ is well-posed.

If both the initial-boundary value problems are ill-posed, for example all boundary conditions at the same end, then it is possible that neither of these cases hold but we are not really interested in that situation. Indeed, Examples 10.1 and 10.2 of [**48**] show that the discrete spectra of such problems may be empty or the entirity of $\mathbb{C}$.

Finally, we formally state our original conjecture.

CONJECTURE 6.9. *The rate of blow-up depends upon the location of the eigenvalues.*

Case 1: *Let*

$$\rho_k(\delta, \theta') = (Xk + Y + \delta)e^{i\theta'} + z. \tag{6.2.8}$$

Then there exists some $W \in \mathbb{C} \setminus \{0\}$ *such that*

$$\lim_{\delta \to 0} \left[\frac{\left(\frac{\|\phi_k\| \|\psi_k\|}{\langle \phi_k, \psi_k \rangle} \right)}{\sup\limits_{\substack{j \in \{1,2,\ldots,2n\} \\ \theta' \in V}} \left(\frac{\xi_j[\rho_k(\delta, \theta')]}{\Delta_{\mathrm{PDE}}[\rho_k(\delta, \theta')] \rho_k(\delta, \theta')} \right)} \right] = W + o(1) \quad \text{as } \rho \to \infty, \tag{6.2.9}$$

where V *is the interval* $[0, \frac{\pi}{n})$ *or* $[\frac{\pi}{n}, \frac{2\pi}{n})$ *which contains* θ' *and* $\xi_j = \zeta_j$ *or* $\xi_j = \eta_j$*, whichever is appropriate for the given* V *(this will depend upon* a*).*

Case 2: *Then*

$$\left(\frac{\|\phi_k\| \|\psi_k\|}{\langle \phi_k, \psi_k \rangle} \right) = O(1) \quad \text{as } k \to \infty \tag{6.2.10}$$

but the ratios $\frac{\eta_j(\rho)}{\Delta_{\mathrm{PDE}}(\rho)}$ *and* $\frac{\zeta_j(\rho)}{\Delta_{\mathrm{PDE}}(\rho)}$ *decay as* $\rho \to \infty$ *from within* $\widetilde{D}$ *and* $\widetilde{E}$ *respectively.*

Case 1 is the interesting part. In the denominator we ensure $\rho_k \to \infty$ at the same rate as σ_k but we allow ρ_k to lie on a different ray to σ_k. The reason δ appears in the expression is to ensure that $\rho_k \in \widetilde{E}$. If the optimal $\theta' \neq \theta$ then it is unnecessary; we may take $\delta = 0$ before evaluating the ratio. So in the denominator we essentially have the ratio

$$\frac{\zeta_j(\sigma_k')}{\Delta_{\mathrm{PDE}}(\sigma_k')},$$

where σ_k' is some rotation of σ_k about z. In the uncoupled example of Chapter 5 this is precisely what happens as $\theta = \frac{\pi}{6}$ but an optimal value of θ' is $\frac{\pi}{12}$.

Towards proving Conjecture 6.9, we hope to show that:

- θ is always the mid-point of the interval V.
- The optimal values of θ' are always

$$\frac{\min(V) + \theta}{2} \quad \text{and} \quad \frac{\max(V) + \theta}{2}.$$

- $\sigma_k^\star = -\sigma_k$ but some rotation is necessary to ensure σ_k lies in the correct sector.
- $\Delta_{\mathrm{PDE}}^\star(\rho) = \Delta_{\mathrm{PDE}}(-\rho)$.
- $\|\psi_k\| = \|\phi_k\|$.

Certainly ϕ_k and ψ_k are each essentially linear combinations of terms of the form

$$\frac{\zeta_j(\sigma_k)}{\Delta_{\mathrm{PDE}}(\sigma_k)},$$

hence the product of their norms is

$$O\left[\left(\frac{\zeta_j(\sigma_k)}{\Delta_{\mathrm{PDE}}(\sigma_k)}\right)^2\right].$$

The inner product is also

$$O\left[\left(\frac{\zeta_j(\sigma_k)}{\Delta_{\mathrm{PDE}}(\sigma_k)}\right)^2\right].$$

Any argument we attempt along the lines of cancelling these two is not only unrigorous but incorrect. Indeed, for the uncoupled example of Chapter 5 we find that in fact the inner product is

$$O\left[\frac{\zeta_j(\sigma_k)}{\Delta_{\mathrm{PDE}}(\sigma_k)}\right].$$

Hence it is necessary to evaluate the eigenfunction norms and inner product directly.

APPENDIX A

Results

Boundary Conditions Name	Boundary Coefficient Matrix	Boundary Conditions For BVP	Locker's Regularity	Well-posed IBVP on $q_t - q_{xxx} = 0$	Well-posed IBVP on $q_t + q_{xxx} = 0$	Can Be Solved Using Separation of Variables
Quasiperiodic Periodic $\Leftrightarrow \beta = -1$	$\begin{pmatrix} 1 & \beta & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \beta & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \beta \end{pmatrix}$ $\beta \in \mathbb{R} \setminus \{0\}$	$\partial_x^j q(0,t) + \beta \partial_x^j q(1,t) = 0$ $j \in \{0,1,2\}$	Regular	✓	✓	Iff $\beta = -1$
Pseudoperiodic	$\begin{pmatrix} 1 & \beta_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \beta_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \beta_3 \end{pmatrix}$ $\beta_j \in \mathbb{R} \setminus \{0\}$	$\partial_x^j q(0,t) + \beta_j \partial_x^j q(1,t) = 0$ $j \in \{0,1,2\}$	Regular $\Leftrightarrow$ $\sum_{j=1}^3 \beta_j \neq 0$ and $\sum_{j=1}^3 \beta_j^{-1} \neq 0$ otherwise degenerate	if $\sum_{j=1}^3 \beta_j \neq 0$	if $\sum_{j=1}^3 \beta_j^{-1} \neq 0$	×
Coupled in First Order	$\begin{pmatrix} 0 & 0 & 1 & \beta & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ $\beta \in \mathbb{R} \setminus \{0\}$	$q(0,t) = q(1,t) = 0$ $q_x(0,t) = q_x(1,t)$	Regular	✓	✓	×
Simplest for Well-posed IBVP if a=i	$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$q(0,t) = q(1,t) = 0$ $q_x(0,t) = 0$	Degenerate as $\pi_1 = 0$	✓	×	×
Simplest for Well-posed FBVP if a=i	$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$q(0,t) = q(1,t) = 0$ $q_x(1,t) = 0$	Degenerate as $\pi_0 = 0$	×	✓	×
Simple	Any uncoupled non-Robin A		Degenerate as $\pi_0 = 0$ or $\pi_1 = 0$	Well-Posed $\Leftrightarrow$ 2 at left 1 at right	Well-Posed $\Leftrightarrow$ 1 at left 2 at right	×
General Uncoupled	Any uncoupled including Robin A		Degenerate as $\pi_0 = 0$ or $\pi_1 = 0$	Unknown. Conjecture: As above.	Unknown. Conjecture: As above.	×
Coupled, no Series	$\begin{pmatrix} 1 & \beta & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$ $\beta \in \mathbb{R} \setminus \{0\}$	$q(0,t) = q_x(0,t) = 0$ $q_{xx}(0,t) = -\beta q_{xx}(1,t)$	Degenerate as $\pi_1 = 0$	✓	×	×

TABLE 1. Results: Third order

Boundary Conditions Name	Boundary Conditions For BVP	Locker's Regularity	Well-posed IBVP on $\partial_t q + i(-i\partial_x)^n q = 0$	Well-posed IBVP on $\partial_t q - i(-i\partial_x)^n q = 0$	Can Be Solved Using Separation of Variables
Quasiperiodic Periodic $\Leftrightarrow \beta = -1$	$\partial_x^j q(0,t) + \beta \partial_x^j q(1,t) = 0$ $j \in \{0, 1, \ldots, n-1\}$	Regular	✓	✓	Iff $\beta = -1$
Pseudoperiodic	$\partial_x^j q(0,t) + \beta_j \partial_x^j q(1,t) = 0$ $j \in \{0, 1, \ldots, n-1\}$	Regular $\Leftrightarrow$ two identities on β_j do not hold. Otherwise degenerate.	See Example 3.26	See Example 3.26	×
Coupled in order $\frac{n+1}{2}$	$\partial_x^j q(0,t) = \partial_x^j q(1,t) = 0$ $j \in \{0, 1, \ldots, \frac{n-3}{2}\}$ $\partial_x^{\frac{n-1}{2}} q(0,t) - \beta \partial_x^{\frac{n-1}{2}} q(1,t) = 0$	Regular	See Example B.6 Conjecture: true for $n \geqslant 7$	Conjecture: true for $n \geqslant 7$	×
Simplest for Well-posed IBVP if a=i	$\partial_x^j q(0,t) = \partial_x^j q(1,t) = 0$ $j \in \{0, 1, \ldots, \frac{n-3}{2}\}$ $\partial_x^{\frac{n-1}{2}} q(0,t) = 0$	Degenerate as $\pi_0 = 1$	✓	×	×
Simplest for Well-posed FBVP if a=-i	$\partial_x^j q(0,t) = \partial_x^j q(1,t) = 0$ $j \in \{0, 1, \ldots, \frac{n-3}{2}\}$ $\partial_x^{\frac{n-1}{2}} q(1,t) = 0$	Degenerate as $\pi_1 = 1$	×	✓	×
Simplest for Well-posed FBVP	$q(0,t) = q(1,t) = 0$ $q_x(1,t) = 0$	Degenerate as $\pi_0 = 0$	×	✓	×
Simple		Degenerate as $\pi_0 = 0$ or $\pi_1 = 0$	See Example 3.27 and [**53**]	See Example 3.27 and [**53**]	×
General Uncoupled		Degenerate as $\pi_0 = 0$ or $\pi_1 = 0$	Unknown. Conjecture: As above.	Unknown. Conjecture: As above.	×

TABLE 2. Results: Odd order $n \geqslant 3$

APPENDIX B

Some proofs

B.1. Standard theorems

Here we list some standard mathematical results. The first three are well-known and the fourth is not obscure. They have in common that they do not fit into the areas of mathematics covered by this thesis but are necessary fundamentals for those topics. We list them, without proof, to remind the reader of the results.

THEOREM B.1 (Green). *Let Ω be a simply connected open set in $\mathbb{R}^2$ whose boundary, $\partial\Omega$, is a positively oriented piecewise smooth, simple closed curve. Let $F, G : \overline{\Omega} \to \mathbb{C}$ be continuous functions with continuous partial derivatives. Then*

$$\int_{\partial\Omega} (F\,\mathrm{d}x + G\,\mathrm{d}y) = \iint_{\Omega} \left(\frac{\partial G}{\partial x} - \frac{\partial F}{\partial y} \right) \mathrm{d}x\,\mathrm{d}y.$$

This theorem originally appears appears in [**34**]. It is used in Section 2.1 to obtain the 'global relation', an essential step in Fokas' unified transform method

THEOREM B.2 (Cramer's rule). *If the square matrix A is full rank then the equation*

$$Ax = y$$

has solution

$$x_j = \frac{\det A_j}{\det A}$$

where A_j is the matrix A with the j^{th} column replaced by the vector y.

A proof is given in [**50**] but it may be found either as a result or an exercise in any first textbook on linear algebra. This theorem is used in Section 2.3 to solve a full rank system of linear equations which is obtained in Section 2.2.

LEMMA B.3 (Jordan). *If the only singularities of $f : \mathbb{C} \to \mathbb{C}$ are poles then, for any $a > 0$,*

$$\lim_{R\to\infty} \int_0^{\pi} \exp(iaRe^{i\theta}) f(Re^{i\theta})\,\mathrm{d}\theta = 0.$$

A trivial corollary is the following:

If the only singularities of $g : \mathbb{C} \to \mathbb{C}$ are poles then

$$\lim_{R\to\infty} \int_0^{\frac{\pi}{n}} \exp(iaR^n e^{ni\theta}) g(Re^{i\theta})\,\mathrm{d}\theta = 0.$$

It is the above corollary to Jordan's Lemma that we make use of in Section 3.1 of this thesis. A proof may be found in any introductory text on complex analysis.

THEOREM B.4 (Langer). *Consider the exponential sum*

$$f(z) = \sum_{j=1}^{n} P_j e^{\lambda_j z},$$

where the coefficients $P_j \in \mathbb{C} \setminus \{0\}$ are constant and the $\lambda_j \in \mathbb{C}$ are all different. Let Λ be the indicator diagram *of f, the closed convex hull of the complex conjugates of the λ_j. Assume the λ_j are not collinear, then Λ is a convex polygon in $\mathbb{C}$ with $N \leqslant n$ vertices. For each $k = 1, 2, \ldots, N$, define L_k to be a unique edge of Λ and let l_k be the length of L_k. Then there exists some $r > 0$,*

$s_k > 0$ such that all zeros of f outside $B(0,r)$ lie in N semi-strips, S_k, perpendicular to L_k and of width s_k. Further, as $R \to \infty$, the number of zeros of f within $B(0,R) \cap S_k$ is asymptotically given by

$$\frac{2\pi R}{l_k}.$$

This result appears as Theorem 8 of [**42**] and is proved in the preceeding Section 9 using a simple geometric argument. The same result is obtained by analytic means and given as the final Theorem of [**41**]. The result also appears on page 298 of [**45**] as a corollary to Theorem 8 of Chapter 6. The distribution of zeros of exponential polynomials is of particular importance in understanding the discrete spectra of differential operators; the result is used in Chapter 5.

B.2. Proof of the discrete series representation

In this section we establish the following:

- Evaluate I_k for $k \in \{2,3,4,5\}$ via a residue calculation.
- Verify that $I_1 -$ (remaining integrals) $= 0$.
- Show that q is independent of the final function, q_T.

The following fact will be used repeatedly to replace $\hat{q}_T$ with $\hat{q}_0$. For all $k \in \mathbb{N}$ and for $j = 1, 2, \ldots, n$, the function

$$\frac{\zeta_j(\rho) - e^{a\rho^n T}\eta_j(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \text{ is entire.} \tag{B.2.1}$$

Indeed, the t-transforms of the boundary functions are entire, as are the monomials c_j, hence the product of a t-transform and a monomial c_j is also entire. Equation (B.2.1) is then justified by equation (2.3.4).

1. $\mathbf{k \in K^-}$

We concentrate first on the integrals in the lower half-plane. For $k \in K^-$ we note that, by adding the entire function (B.2.1),

$$\begin{aligned}\int_{\Gamma_k^D} \frac{\widehat{P}(\rho)e^{a\rho^n T}}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^-} \eta_j(\rho)\,\mathrm{d}\rho &= \int_{\Gamma_k^D} \frac{\widehat{P}(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^-} \left[e^{a\rho^n T}\eta_j(\rho) - e^{a\sigma_k^n T}\eta_j(\rho) + \zeta_j(\rho)\right] \mathrm{d}\rho \\ &= \int_{\Gamma_k^D} \frac{\widehat{P}(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^-} \zeta_j(\rho)\,\mathrm{d}\rho.\end{aligned}$$

We observe that

$$\int_{\Gamma_k^D \cup \Gamma_k^E} \frac{\widehat{P}(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^-} \zeta_j(\rho)\,\mathrm{d}\rho = \int_{\Gamma_k} \frac{\widehat{P}(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^-} \zeta_j(\rho)\,\mathrm{d}\rho,$$

as we integrate along the diameter that divides D^- from E^- once in each direction so the contribution from this section of the contours Γ_k^D and Γ_k^E cancels out. We conclude for $k \in K^-$

$$\int_{\Gamma_k^E} \frac{\widehat{P}(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^-} \zeta_j(\rho)\,\mathrm{d}\rho + \int_{\Gamma_k^D} \frac{\widehat{P}(\rho)e^{a\rho^n T}}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^-} \eta_j(\rho)\,\mathrm{d}\rho = \int_{\Gamma_k} \frac{\widehat{P}(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^-} \zeta_j(\rho)\,\mathrm{d}\rho. \tag{B.2.2}$$

2. $\mathbf{k \in K^+}$

A similar calculation may be performed in the upper half-plane; for $k \in K^+$

$$\int_{\Gamma_k^E} \frac{P(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^+} \zeta_j(\rho)\,\mathrm{d}\rho + \int_{\Gamma_k^D} \frac{P(\rho)e^{a\rho^n T}}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^+} \eta_j(\rho)\,\mathrm{d}\rho = \int_{\Gamma_k} \frac{P(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^+} \zeta_j(\rho)\,\mathrm{d}\rho. \tag{B.2.3}$$

3. $\mathbf{k \in K_E^D}$

We now turn our attention to the integrals whose contours touch the real axis but do not pass through 0. Using fact (B.2.1) once again,

$$\int_{\Gamma_k^D} \frac{P(\rho)e^{a\rho^n T}}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^+} \eta_j(\rho)\,\mathrm{d}\rho = \int_{\Gamma_k^D} \frac{P(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^+} \zeta_j(\rho)\,\mathrm{d}\rho,$$

but for the contours in the lower half-plane we must be more careful. We rewrite

$$\begin{aligned}\int_{\Gamma_k^E} \frac{\widehat{P}(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^-} \zeta_j(\rho)\,\mathrm{d}\rho &= \int_{\Gamma_k^E} \frac{P(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^+} \zeta_j(\rho)\,\mathrm{d}\rho \\ &\quad + \int_{\Gamma_k^E} \frac{P(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \left[e^{-i\rho} \sum_{j\in J^-} \zeta_j(\rho) - \sum_{j\in J^+} \zeta_j(\rho) \right] \mathrm{d}\rho.\end{aligned}$$

Hence for $k \in K_E^D$

$$\begin{aligned}&\int_{\Gamma_k^E} \frac{\widehat{P}(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^-} \zeta_j(\rho)\,\mathrm{d}\rho + \int_{\Gamma_k^D} \frac{P(\rho)e^{a\rho^n T}}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^+} \eta_j(\rho)\,\mathrm{d}\rho \\ &\quad = \int_{\Gamma_k} \frac{P(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^+} \zeta_j(\rho)\,\mathrm{d}\rho + \int_{\Gamma_k^E} \frac{P(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \left[e^{-i\rho} \sum_{j\in J^-} \zeta_j(\rho) - \sum_{j\in J^+} \zeta_j(\rho) \right] \mathrm{d}\rho, \end{aligned} \tag{B.2.4}$$

cancelling the integrals in each direction along the real interval $(\sigma_k - \varepsilon_k, \sigma_k + \varepsilon_k)$.

4. $\mathbf{k \in K_D^E}$

For the contours in the lower half-plane

$$\int_{\Gamma_k^D} \frac{\widehat{P}(\rho)e^{a\rho^n T}}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^-} \eta_j(\rho)\,\mathrm{d}\rho = \int_{\Gamma_k^D} \frac{\widehat{P}(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^-} \zeta_j(\rho)\,\mathrm{d}\rho$$

and for the contours in the upper half-plane

$$\int_{\Gamma_k^E} \frac{P(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^+} \zeta_j(\rho)\,\mathrm{d}\rho = \int_{\Gamma_k^E} \frac{\widehat{P}(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^-} \zeta_j(\rho)\,\mathrm{d}\rho + \int_{\Gamma_k^E} \frac{P(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \left[\sum_{j\in J^+} \zeta_j(\rho) - e^{-i\rho} \sum_{j\in J^-} \zeta_j(\rho)\right] \mathrm{d}\rho.$$

Hence for $k \in K_D^E$

$$\int_{\Gamma_k^E} \frac{P(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^+} \zeta_j(\rho)\,\mathrm{d}\rho + \int_{\Gamma_k^D} \frac{\widehat{P}(\rho)e^{a\rho^n T}}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^-} \eta_j(\rho)\,\mathrm{d}\rho = \int_{\Gamma_k} \frac{\widehat{P}(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^-} \zeta_j(\rho)\,\mathrm{d}\rho + \int_{\Gamma_k^E} \frac{P(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \left[\sum_{j\in J^+} \zeta_j(\rho) - e^{-i\rho} \sum_{j\in J^-} \zeta_j(\rho)\right] \mathrm{d}\rho. \quad \text{(B.2.5)}$$

5. $\mathbf{k \in K_E^E}$

Similarly to the above,

$$\int_{\Gamma_k^+} \frac{P(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^+} \zeta_j(\rho)\,\mathrm{d}\rho + \int_{\Gamma_k^-} \frac{\widehat{P}(\rho)e^{a\rho^n T}}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^-} \eta_j(\rho)\,\mathrm{d}\rho = \int_{\Gamma_k} \frac{P(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^+} \zeta_j(\rho)\,\mathrm{d}\rho + \int_{\Gamma_k^-} \frac{P(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \left[e^{-i\rho} \sum_{j\in J^-} \zeta_j(\rho) - \sum_{j\in J^+} \zeta_j(\rho)\right] \mathrm{d}\rho. \quad \text{(B.2.6)}$$

6. $\mathbf{k = 0}$

We now investigate the integrals whose contours pass through 0. We use fact (B.2.1) to obtain following identities:

$$\int_{\Gamma_0^{D+}} \frac{P(\rho)e^{a\rho^n T}}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^+} \eta_j(\rho)\,\mathrm{d}\rho = \int_{\Gamma_0^{D+}} \frac{P(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^+} \zeta_j(\rho)\,\mathrm{d}\rho,$$
$$\int_{\Gamma_0^{D-}} \frac{\widehat{P}(\rho)e^{a\rho^n T}}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^-} \eta_j(\rho)\,\mathrm{d}\rho = \int_{\Gamma_0^{D-}} \frac{\widehat{P}(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^-} \zeta_j(\rho)\,\mathrm{d}\rho.$$

Putting these equations together we obtain

$$\int_{\Gamma_0^{E+}} \frac{P(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^+} \zeta_j(\rho)\,\mathrm{d}\rho + \int_{\Gamma_0^{D+}} \frac{P(\rho)e^{a\rho^n T}}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^+} \eta_j(\rho)\,\mathrm{d}\rho = \int_{\Gamma_0^+} \frac{P(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^+} \zeta_j(\rho)\,\mathrm{d}\rho, \quad \text{(B.2.7)}$$

$$\int_{\Gamma_0^{E-}} \frac{\widehat{P}(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^-} \zeta_j(\rho)\,\mathrm{d}\rho + \int_{\Gamma_0^{D-}} \frac{\widehat{P}(\rho)e^{a\rho^n T}}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^-} \eta_j(\rho)\,\mathrm{d}\rho = \int_{\Gamma_0^-} \frac{\widehat{P}(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^-} \zeta_j(\rho)\,\mathrm{d}\rho. \quad \text{(B.2.8)}$$

We combine equations (B.2.7) and (B.2.8) in different ways depending upon the values of n and a, as detailed in the following paragraphs.

1. n odd, a = i. Then

$$\int_{\Gamma_0^{E^-}} \frac{\widehat{P}(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^-} \zeta_j(\rho)\,\mathrm{d}\rho + \int_{\Gamma_0^{D^-}} \frac{\widehat{P}(\rho)e^{a\rho^n T}}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^-} \eta_j(\rho)\,\mathrm{d}\rho$$
$$= \int_{\Gamma_0^6} \frac{P(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^+} \zeta_j(\rho)\,\mathrm{d}\rho + \int_{\Gamma_0^6} \frac{P(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \left[e^{-i\rho} \sum_{j\in J^-} \zeta_j(\rho) - \sum_{j\in J^+} \zeta_j(\rho)\right] \mathrm{d}\rho. \tag{B.2.9}$$

From equations (B.2.7) and (B.2.9) we obtain

$$\int_{\Gamma_0^{E^+}} \frac{P(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^+} \zeta_j(\rho)\,\mathrm{d}\rho + \int_{\Gamma_0^{D^+}} \frac{P(\rho)e^{i\rho^n T}}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^+} \eta_j(\rho)\,\mathrm{d}\rho$$
$$+ \int_{\Gamma_0^{E^-}} \frac{\widehat{P}(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^-} \zeta_j(\rho)\,\mathrm{d}\rho + \int_{\Gamma_0^{D^-}} \frac{\widehat{P}(\rho)e^{i\rho^n T}}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^-} \eta_j(\rho)\,\mathrm{d}\rho$$
$$= \int_{\Gamma_0} \frac{P(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^+} \zeta_j(\rho)\,\mathrm{d}\rho + \int_{\Gamma_0^-} \frac{P(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \left[e^{-i\rho} \sum_{j\in J^-} \zeta_j(\rho) - \sum_{j\in J^+} \zeta_j(\rho)\right] \mathrm{d}\rho. \tag{B.2.10}$$

We now use equations (B.2.2), (B.2.3), (B.2.4) and (B.2.10) together with equations (3.1.9)–(3.1.12) to write

$$\sum_{l=2}^{5} I_l = \sum_{\substack{k\in K^+ \\ \cup K^{D^+}\cup K^{E^+} \\ \cup K^{\mathbb{R}}\cup\{0\}}} \int_{\Gamma_k} \frac{P(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^+} \zeta_j(\rho)\,\mathrm{d}\rho + \sum_{\substack{k\in K^- \\ \cup K^{D^-}\cup K^{E^-}}} \int_{\Gamma_k} \frac{\widehat{P}(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^-} \zeta_j(\rho)\,\mathrm{d}\rho$$
$$+ \left\{\sum_{k\in K^{\mathbb{R}}} \int_{\Gamma_k^E} + \int_{\Gamma_0^-}\right\} \frac{P(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \left[e^{-i\rho} \sum_{j\in J^-} \zeta_j(\rho) - \sum_{j\in J^+} \zeta_j(\rho)\right] \mathrm{d}\rho$$
$$+ \int_{-\mathbb{R}} \frac{P(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^+} \zeta_j(\rho)\,\mathrm{d}\rho + \int_{\mathbb{R}} \frac{\widehat{P}(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^-} \zeta_j(\rho)\,\mathrm{d}\rho. \tag{B.2.11}$$

We rewrite the last term as

$$\int_{\mathbb{R}} \frac{P(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^+} \zeta_j(\rho)\,\mathrm{d}\rho + \int_{\mathbb{R}} \frac{P(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \left[e^{-i\rho} \sum_{j\in J^-} \zeta_j(\rho) - \sum_{j\in J^+} \zeta_j(\rho)\right] \mathrm{d}\rho,$$

hence equation (B.2.11) as

$$\sum_{l=2}^{5} I_l = \sum_{\substack{k\in K^+ \\ \cup K^{D^+}\cup K^{E^+} \\ \cup K^{\mathbb{R}}\cup\{0\}}} \int_{\Gamma_k} \frac{P(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^+} \zeta_j(\rho)\,\mathrm{d}\rho + \sum_{\substack{k\in K^- \\ \cup K^{D^-}\cup K^{E^-}}} \int_{\Gamma_k} \frac{\widehat{P}(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^-} \zeta_j(\rho)\,\mathrm{d}\rho$$
$$+ \left\{\sum_{k\in K^{\mathbb{R}}} \int_{\Gamma_k^E} + \int_{\Gamma_0^-} + \int_{\mathbb{R}}\right\} \frac{P(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \left[e^{-i\rho} \sum_{j\in J^-} \zeta_j(\rho) - \sum_{j\in J^+} \zeta_j(\rho)\right] \mathrm{d}\rho. \tag{B.2.12}$$

2. n odd, a = −i. Then

$$\begin{aligned}
&\int_{\Gamma_0^{E+}} \frac{P(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^+} \zeta_j(\rho)\,\mathrm{d}\rho + \int_{\Gamma_0^{D+}} \frac{P(\rho)e^{-i\rho^n T}}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^+} \eta_j(\rho)\,\mathrm{d}\rho \\
&\qquad + \int_{\Gamma_0^{E-}} \frac{\widehat{P}(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^-} \zeta_j(\rho)\,\mathrm{d}\rho + \int_{\Gamma_0^{D-}} \frac{\widehat{P}(\rho)e^{-i\rho^n T}}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^-} \eta_j(\rho)\,\mathrm{d}\rho \\
&= \int_{\Gamma_0} \frac{\widehat{P}(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^-} \zeta_j(\rho)\,\mathrm{d}\rho + \int_{\Gamma_0^+} \frac{P(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \left[\sum_{j\in J^+} \zeta_j(\rho) - e^{-i\rho} \sum_{j\in J^-} \zeta_j(\rho)\right] \mathrm{d}\rho. \quad \text{(B.2.13)}
\end{aligned}$$

Using a similar argument to that above we write

$$\begin{aligned}
\sum_{l=2}^{5} I_l = &\sum_{\substack{k\in K^+ \\ \cup K^{D^+}\cup K^{E^+}}} \int_{\Gamma_k} \frac{P(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^+} \zeta_j(\rho)\,\mathrm{d}\rho + \sum_{\substack{k\in K^- \\ \cup K^{D^-}\cup K^{E^-} \\ \cup K^{\mathbb{R}}\cup\{0\}}} \int_{\Gamma_k} \frac{\widehat{P}(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^-} \zeta_j(\rho)\,\mathrm{d}\rho \\
&+ \left\{ \sum_{k\in K^{\mathbb{R}}} \int_{\Gamma_k^E} + \int_{\Gamma_0^+} - \int_{\mathbb{R}} \right\} \frac{P(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \left[\sum_{j\in J^+} \zeta_j(\rho) - e^{-i\rho} \sum_{j\in J^-} \zeta_j(\rho)\right] \mathrm{d}\rho. \quad \text{(B.2.14)}
\end{aligned}$$

3. n even, a = ±i. Then

$$\begin{aligned}
&\int_{\Gamma_0^{E+}} \frac{P(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^+} \zeta_j(\rho)\,\mathrm{d}\rho + \int_{\Gamma_0^{D+}} \frac{P(\rho)e^{a\rho^n T}}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^+} \eta_j(\rho)\,\mathrm{d}\rho \\
&\qquad + \int_{\Gamma_0^{E-}} \frac{\widehat{P}(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^-} \zeta_j(\rho)\,\mathrm{d}\rho + \int_{\Gamma_0^{D-}} \frac{\widehat{P}(\rho)e^{a\rho^n T}}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^-} \eta_j(\rho)\,\mathrm{d}\rho \\
&\qquad = \frac{1}{2}\int_{\Gamma_0} \frac{P(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^+} \zeta_j(\rho)\,\mathrm{d}\rho + \frac{1}{2}\int_{\Gamma_0} \frac{\widehat{P}(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^-} \zeta_j(\rho)\,\mathrm{d}\rho \\
&\qquad\qquad + \frac{1}{2}\int_{\Gamma_0^-} \frac{P(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \left[e^{-i\rho} \sum_{j\in J^-} \zeta_j(\rho) - \sum_{j\in J^+} \zeta_j(\rho)\right] \mathrm{d}\rho \\
&\qquad\qquad\qquad + \frac{1}{2}\int_{\Gamma_0^+} \frac{P(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \left[\sum_{j\in J^+} \zeta_j(\rho) - e^{-i\rho} \sum_{j\in J^-} \zeta_j(\rho)\right] \mathrm{d}\rho. \quad \text{(B.2.15)}
\end{aligned}$$

We may now write

$$\sum_{l=2}^{5} I_l = \left\{ \sum_{\substack{k\in K^+ \\ \cup K^{D^+}\cup K^{E^+} \\ \cup K_E^D}} \int_{\Gamma_k} + \frac{1}{2}\int_{\Gamma_0} \right\} \frac{P(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^+} \zeta_j(\rho)\,\mathrm{d}\rho$$
$$+ \left\{ \sum_{\substack{k\in K^- \\ \cup K^{D^-}\cup K^{E^-} \\ \cup K_D^E}} \int_{\Gamma_k} + \frac{1}{2}\int_{\Gamma_0} \right\} \frac{\widehat{P}(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^-} \zeta_j(\rho)\,\mathrm{d}\rho$$
$$+ \left\{ \sum_{k\in K_E^D} \int_{\Gamma_k^E} + \frac{1}{2}\int_{\Gamma_0^-} - \int_0^{-\infty} - \sum_{k\in K_D^E} \int_{\Gamma_k^E} - \frac{1}{2}\int_{\Gamma_0^+} + \int_0^{\infty} \right\}$$
$$\frac{P(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \left[e^{-i\rho} \sum_{j\in J^-} \zeta_j(\rho) - \sum_{j\in J^+} \zeta_j(\rho) \right] \mathrm{d}\rho. \quad \text{(B.2.16)}$$

4. n even, $\mathbf{a} = \mathbf{e}^{\mathbf{i}\theta}$, $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Then

$$\sum_{l=2}^{5} I_l = \sum_{\substack{k\in K^+ \\ \cup K^{D^+}\cup K^{E^+} \\ \cup K^{\mathbb{R}}\cup\{0\}}} \int_{\Gamma_k} \frac{P(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^+} \zeta_j(\rho)\,\mathrm{d}\rho + \sum_{\substack{k\in K^- \\ \cup K^{D^-}\cup K^{E^-}}} \int_{\Gamma_k} \frac{\widehat{P}(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \sum_{j\in J^-} \zeta_j(\rho)\,\mathrm{d}\rho$$
$$+ \left\{ \sum_{k\in K^{\mathbb{R}}} \int_{\Gamma_k^-} + \int_{\Gamma_0^-} + \int_{\mathbb{R}} \right\} \frac{P(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \left[e^{-i\rho} \sum_{j\in J^-} \zeta_j(\rho) - \sum_{j\in J^+} \zeta_j(\rho) \right] \mathrm{d}\rho. \quad \text{(B.2.17)}$$

Cancelling the remaining integrals

The following lemma deals with the last term of equations (B.2.12), (B.2.14), (B.2.16) and (B.2.17).

Lemma B.5. *For any choice of boundary conditions*

$$\sum_{j\in J^+} \zeta_j(\rho) - e^{-i\rho} \sum_{j\in J^-} \zeta_j(\rho) = \Delta_{\mathrm{PDE}}(\rho) \left[\hat{q}_0(\rho) + \left(\frac{1}{\Delta_{\mathrm{PDE}}(\rho)} - 1 \right) H(\rho) \right], \quad \text{(B.2.18)}$$

where

$$H(\rho) = \sum_{j\in \widehat{J}^+} c_j(\rho) \tilde{h}_{\widehat{J}_j^+}(\rho) - e^{-i\rho} \sum_{j\in \widehat{J}^-} c_j(\rho) \tilde{h}_{\widehat{J}_j^-}(\rho),$$

where $\widehat{J}^+$, $\widehat{J}^-$, $\widehat{J}_j^+$ and $\widehat{J}_j^-$ are from Notation 2.12 and h_j are the boundary data from equation (2.1.3).

Proof. Using Definition 2.19, we expand the left hand side of equation (B.2.18) in terms of $u(\rho, l)$ (the linear combination of transforms of the initial and boundary data defined in

equation (2.2.18)) and rearrange the result. To this end we define the matrix-valued function $X^{l\,j} : \mathbb{C} \to \mathbb{C}^{(n-1)\times(n-1)}$ entrywise by

$$(X^{l\,j})_{s\,r}(\rho) = \mathcal{A}_{(s+l)\,(r+j)}(\rho),$$

where $(s+l)$ is taken to be the least positive integer equal to $s+l$ modulo n and $(r+j)$ is taken to be the least positive integer equal to $r+j$ modulo n. So the matrix $X^{l\,j}$ is the $(n-1)\times(n-1)$ submatrix of

$$\begin{pmatrix} \mathcal{A} & \mathcal{A} \\ \mathcal{A} & \mathcal{A} \end{pmatrix}$$

whose $(1,1)$ entry is the $(l+1, r+j)$ entry. Then

$$\widehat{\zeta}_j(\rho) = \sum_{l=1}^{n} u(\rho, l) \det X^{l\,j}(\rho). \tag{B.2.19}$$

Using equations (2.3.1) of Definition 2.19,

$$\begin{aligned}
&\sum_{j\in J^+} \zeta_j(\rho) - e^{-i\rho} \sum_{j\in J^-} \zeta_j(\rho) \\
&= \left(\sum_{j:J_j \text{ odd}} c_{(J_j-1)/2}(\rho)\widehat{\zeta}_j(\rho) + \sum_{j:J'_j \text{ odd}} c_{(J'_j-1)/2}(\rho)\widehat{\zeta}_{j+n}(\rho) \right) \\
&\qquad - e^{-i\rho} \left(\sum_{j:J_j \text{ even}} c_{J_j/2}(\rho)\widehat{\zeta}_j(\rho) + \sum_{j:J'_j \text{ even}} c_{J'_j/2}(\rho)\widehat{\zeta}_{j+n}(\rho) \right) \\
&= \left(\sum_{j:J_j \text{ odd}} c_{(J_j-1)/2}(\rho)\widehat{\zeta}_j(\rho) + \sum_{j:J'_j \text{ odd}} c_{(J'_j-1)/2}(\rho) \left[\tilde{h}_j(\rho) - \sum_{k=1}^{n} \widehat{A}_{j\,k}\widehat{\zeta}_k(\rho) \right] \right) \\
&\qquad - e^{-i\rho} \left(\sum_{j:J_j \text{ even}} c_{J_j/2}(\rho)\widehat{\zeta}_j(\rho) + \sum_{j:J'_j \text{ even}} c_{J'_j/2}(\rho) \left[\tilde{h}_j(\rho) - \sum_{k=1}^{n} \widehat{A}_{j\,k}\widehat{\zeta}_k(\rho) \right] \right),
\end{aligned} \tag{B.2.20}$$

and, by equation (B.2.19), the right hand side of equation (B.2.20) equals

$$\begin{aligned}
\sum_{l=1}^{n} u(\rho, l) \Bigg[&\left(\sum_{j:J_j \text{ odd}} c_{(J_j-1)/2}(\rho) \det X^{l\,j} - \sum_{j:J'_j \text{ odd}} c_{(J'_j-1)/2}(\rho) \sum_{k=1}^{n} \widehat{A}_{j\,k} \det X^{l\,j} \right) \\
&-e^{-i\rho} \left(\sum_{j:J_j \text{ even}} c_{J_j/2}(\rho) \det X^{l\,j} - \sum_{j:J'_j \text{ even}} c_{J'_j/2}(\rho) \sum_{k=1}^{n} \widehat{A}_{j\,k} \det X^{l\,j} \right) \Bigg] \\
&+ \sum_{j:J'_j \text{ odd}} c_{(J'_j-1)/2}(\rho)\tilde{h}_j(\rho) - e^{-i\rho} \sum_{j:J'_j \text{ even}} c_{J'_j/2}(\rho)\tilde{h}_j(\rho).
\end{aligned}$$

If j is such that J'_j is odd then $j = \widehat{J}^+_r$ and $r = (J'_j - 1)/2$ for some $r \in \widehat{J}^+$ and if j is such that J'_j is even then $j = \widehat{J}^-_r$ and $r = J'_j/2$ for some $r \in \widehat{J}^-$, so the right hand side of equation (B.2.20)

equals

$$= \sum_{l=1}^{n} u(\rho, l) \left[\left(\sum_{j:J_j \text{ odd}} c_{(J_j-1)/2}(\rho) \det X^{l\,j} - \sum_{j:J'_j \text{ odd}} c_{(J'_j-1)/2}(\rho) \sum_{\substack{k:J_k \text{ odd} \\ k:J_k \text{ even}}} \widehat{A}_{j\,k} \det X^{l\,j} \right) \right.$$
$$\left. - e^{-i\rho} \left(\sum_{j:J_j \text{ even}} c_{J_j/2}(\rho) \det X^{l\,j} - \sum_{j:J'_j \text{ even}} c_{J'_j/2}(\rho) \sum_{\substack{k:J_k \text{ odd} \\ k:J_k \text{ even}}} \widehat{A}_{j\,k} \det X^{l\,j} \right) \right] + H(\rho).$$

In the line above we have split the index set $\{1, 2, \ldots, n\}$ for the sums over k into two sets. We now seperate the sums for these sets, in the process interchanging the dummy variables j and k in the final two sums on each line, rewriting the square bracket as

$$\left[\sum_{j:J_j \text{ odd}} \left(c_{(J_j-1)/2}(\rho) - \sum_{k:J'_k \text{ odd}} c_{(J'_k-1)/2}(\rho) \widehat{A}_{k\,j} + e^{-i\rho} \sum_{k:J'_k \text{ even}} c_{J'_k/2}(\rho) \widehat{A}_{k\,j} \right) \det X^{l\,j} \right.$$
$$\left. + \sum_{j:J_j \text{ even}} \left(-c_{J_j/2}(\rho) e^{-i\rho} - \sum_{k:J'_k \text{ odd}} c_{(J'_k-1)/2}(\rho) \widehat{A}_{k\,j} + e^{-i\rho} \sum_{k:J'_k \text{ even}} c_{J'_k/2}(\rho) \widehat{A}_{k\,j} \right) \det X^{l\,j} \right]. \tag{B.2.21}$$

We now change the dummy variable k in the sums of expression (B.2.21). For k such that J'_k is odd we may define $r \in \widehat{J}^+$ such that $r = (J'_k - 1)/2$ and $k = \widehat{J}^+_r$. Then, by the definition of the reduced boundary coefficient matrix (2.2.20),

$$\widehat{A}_{k\,j} = \begin{cases} \alpha_{\widehat{J}^+_r\,(J_j-1)/2} & J_j \text{ odd}, \\ \beta_{\widehat{J}^+_r\,J_j/2} & J_j \text{ even}. \end{cases}$$

Similarly, for k such that J'_k is even we may define $r \in \widehat{J}^-$ such that $r = J'_k/2$ and $k = \widehat{J}^-_r$. Then

$$\widehat{A}_{k\,j} = \begin{cases} \alpha_{\widehat{J}^-_r\,(J_j-1)/2} & J_j \text{ odd}, \\ \beta_{\widehat{J}^-_r\,J_j/2} & J_j \text{ even}. \end{cases}$$

This and the definition of the reduced global relation matrix (2.2.19) allow us to write

$$c_{(J_j-1)/2}(\rho) - \sum_{k:J'_k \text{ odd}} c_{(J'_k-1)/2}(\rho) \widehat{A}_{k\,j} + e^{-i\rho} \sum_{k:J'_k \text{ even}} c_{J'_k/2}(\rho) \widehat{A}_{k\,j}$$
$$= c_{(J_j-1)/2}(\rho) \left(1 - \sum_{r \in \widehat{J}^+} (i\rho)^{(J_j-1)/2-r} \alpha_{\widehat{J}^+_r\,(J_j-1)/2} + e^{-i\rho} \sum_{r \in \widehat{J}^-} (i\rho)^{(J_j-1)/2-r} \alpha_{\widehat{J}^-_r\,(J_j-1)/2} \right)$$
$$= \mathcal{A}_{1\,j}(\rho) \tag{B.2.22}$$

for j such that J_j is odd and

$$
\begin{aligned}
&- c_{J_j/2}(\rho)e^{-i\rho} - \sum_{k:J'_k \text{ odd}} c_{(J'_k-1)/2}(\rho)\widehat{A}_{k\,j} + e^{-i\rho}\sum_{k:J'_k \text{ even}} c_{J'_k/2}(\rho)\widehat{A}_{k\,j} \\
&\qquad = c_{J_j/2}(\rho)\left(-e^{-i\rho} - \sum_{r\in\widehat{J}^+}(i\rho)^{J_j/2-r}\beta_{\widehat{J}^+_r\,J_j/2} + e^{-i\rho}\sum_{r\in\widehat{J}^-}(i\rho)^{J_j/2-r}\beta_{\widehat{J}^-_r\,J_j/2}\right) \\
&\qquad\qquad = \mathcal{A}_{1\,j}(\rho) \quad \text{(B.2.23)}
\end{aligned}
$$

for j such that J_j is even.

Substituting equations (B.2.22) and (B.2.23) into expression (B.2.21) we obtain

$$
\sum_{j\in J^+}\zeta_j(\rho) - e^{-i\rho}\sum_{j\in J^-}\zeta_j(\rho) = \sum_{l=1}^{n} u(\rho,l)\sum_{j=1}^{n}\mathcal{A}_{1\,j}(\rho)\det X^{l\,j}(\rho) + H(\rho),
$$

but $\sum_{j=1}^{n}\mathcal{A}_{1\,j}(\rho)\det X^{l\,j}(\rho)$ is the determinant of the matrix obtained by replacing the l^{th} row of $\mathcal{A}$ with the first row of $\mathcal{A}$ hence

$$
\sum_{j=1}^{n}\mathcal{A}_{1\,j}(\rho)\det X^{l\,j}(\rho) = \delta_{l\,1}\Delta_{\text{PDE}}(\rho).
$$

Finally note that, from the defintion of $u(\rho,l)$ in equation (2.2.18) we may write $u(\rho,1) = \hat{q}_0(\rho) - H(\rho)$. Thus

$$
\sum_{j\in J^+}\zeta_j(\rho) - e^{-i\rho}\sum_{j\in J^-}\zeta_j(\rho) = (\hat{q}_0(\rho) - H(\rho))\Delta_{\text{PDE}}(\rho) + H(\rho). \qquad \square
$$

n odd, a = i. Lemma B.5 ensures that for homogeneous boundary conditions, that is boundary conditions for which $H(\rho) = 0$, the integrand in the final term of equation (B.2.12) is given by $-P(\rho)\hat{q}_0(\rho)$. This is an entire function so it is certainly analytic on $\overline{B}(\sigma_k,\varepsilon_k)$ for $k \in K^{\mathbb{R}} \cup \{0\}$ so the integrals around Γ_k^E and Γ_0^- will evaluate to zero. We are left with the integral

$$
-\int_{\mathbb{R}} P(\rho)\hat{q}_0(\rho)\,\mathrm{d}\rho = -I_1.
$$

Hence, for homogeneous boundary conditions, equation (3.1.2) becomes

$$
2\pi q(x,t) = \sum_{\substack{k\in K^+\\ \cup K^{D^+}\cup K^{E^+}\\ \cup K^{\mathbb{R}}\cup\{0\}}}\int_{\Gamma_k}\frac{P(\rho)}{\Delta_{\text{PDE}}(\rho)}\sum_{j\in J^+}\zeta_j(\rho)\,\mathrm{d}\rho + \sum_{\substack{k\in K^-\\ \cup K^{D^-}\cup K^{E^-}}}\int_{\Gamma_k}\frac{\widehat{P}(\rho)}{\Delta_{\text{PDE}}(\rho)}\sum_{j\in J^-}\zeta_j(\rho)\,\mathrm{d}\rho. \tag{B.2.24}
$$

The same argument works for other values of n and a; the last integral terms of equations (B.2.14), (B.2.16) and (B.2.17) cancel with I_1.

Each of the integrals in equation (B.2.24) is along a closed, circular contour containing at most one singularity which is either a pole or a removable singularity. This completes the proofs of Theorems 3.1 and 3.13.

B.3. The coefficients in $\mathcal{A}$

The following example gives a sketch of the general method, indicating how it is possible to check that the relevant coefficients are nonzero. Below, we give formal definitions of the sets used in Condition 3.22. Finally, we state and prove the technical Lemma B.8 which gives the values of the coefficients we must check.

EXAMPLE B.6 (5^{th} order with a single coupling). Let $a = i$ and the boundary coefficient matrix be

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & \beta_{1\,2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

This specifies a single coupled boundary condition and two at each end of the interval hence $C = 1$, $L = R = 2$. Following Notation 2.12 and Lemma 2.14 we see that

$$\begin{aligned} \widehat{J}^{+} &= \{0,1,2\} & \widetilde{J}^{+} &= \{3,4\}, \\ \widehat{J}^{-} &= \{0,1\} & \widetilde{J}^{-} &= \{2,3,4\}, \\ J &= \{4,6,7,8,9\}, & J' &= \{0,1,2,3,5\} \\ (J_j)_{j=1}^{3} &= (9,8,7,6,4) & (J'_j)_{j=1}^{3} &= (5,3,2,1,0) \end{aligned}$$

and

$$V(\rho) = \begin{pmatrix} \tilde{f}_4(\rho) \\ \tilde{g}_4(\rho) \\ \tilde{f}_3(\rho) \\ \tilde{g}_3(\rho) \\ \tilde{g}_2(\rho) \end{pmatrix}.$$

Hence, by applying a cyclic permutation to the rows of the reduced global relation, we obtain

$$\mathcal{A}(\rho)V(\rho) = \begin{pmatrix} \hat{q}_0(\omega\rho) \\ \hat{q}_0(\omega^2\rho) \\ \hat{q}_0(\omega^3\rho) \\ \hat{q}_0(\omega^4\rho) \\ \hat{q}_0(\rho) \end{pmatrix} - e^{i\rho^5 T} \begin{pmatrix} \hat{q}_T(\omega\rho) \\ \hat{q}_T(\omega^2\rho) \\ \hat{q}_T(\omega^3\rho) \\ \hat{q}_T(\omega^4\rho) \\ \hat{q}_T(\rho) \end{pmatrix}$$

where

$$\mathcal{A}(\rho) = \begin{pmatrix} c_4(\rho) & -e^{-i\omega\rho}c_4(\rho) & \omega c_3(\rho) & -\omega e^{-i\omega\rho}c_3(\rho) & -\omega^2(e^{-i\omega\rho}+\beta_{1\,2})c_2(\rho) \\ c_4(\rho) & -e^{-i\omega^2\rho}c_4(\rho) & \omega^2 c_3(\rho) & -\omega^2 e^{-i\omega^2\rho}c_3(\rho) & -\omega^4(e^{-i\omega^2\rho}+\beta_{1\,2})c_2(\rho) \\ c_4(\rho) & -e^{-i\omega^3\rho}c_4(\rho) & \omega^3 c_3(\rho) & -\omega^3 e^{-i\omega^3\rho}c_3(\rho) & -\omega(e^{-i\omega^3\rho}+\beta_{1\,2})c_2(\rho) \\ c_4(\rho) & -e^{-i\omega^4\rho}c_4(\rho) & \omega^4 c_3(\rho) & -\omega^4 e^{-i\omega^4\rho}c_3(\rho) & -\omega^3(e^{-i\omega^4\rho}+\beta_{1\,2})c_2(\rho) \\ c_4(\rho) & -e^{-i\rho}c_4(\rho) & c_3(\rho) & -e^{-i\rho}c_3(\rho) & -(e^{-i\rho}+\beta_{1\,2})c_2(\rho) \end{pmatrix}.$$

We define the functions l, r and c that map the indices of each boundary datum to the position of that boundary datum within V as follows:

$$\begin{aligned} l &: 5 \mapsto 1, & l &: 4 \mapsto 3, \\ r &: 5 \mapsto 2, & r &: 4 \mapsto 4, \\ c &: 3 \mapsto 5. \end{aligned}$$

These functions are defined such that

$$\begin{aligned} \text{Domain}\,(l) &= \{j+1 : \tilde{f}_j(\rho) \text{ is an entry of } V\} \\ \text{Domain}\,(r) &= \{j+1 : \tilde{g}_j(\rho) \text{ is an entry of } V \text{ which corresponds to a BC} \\ &\qquad \text{which does NOT couple the ends of the interval}\} \\ \text{Domain}\,(c) &= \{j+1 : \tilde{g}_j(\rho) \text{ is an entry of } V \text{ which corresponds to a} \\ &\qquad \text{BC which couples the ends of the interval}\}. \end{aligned}$$

They are also injective, and their ranges are all $\{1,2,3,4,5\}$ but their codomains are disjoint. This is guaranteed by defining the functions so that for j in the relevant domains

$$\begin{aligned} \mathcal{A}_{k\,l(j)} &= \omega^{k(n-j)} c_{j-1}(\rho) \\ \mathcal{A}_{k\,r(j)} &= -\omega^{k(n-j)} e^{-i\omega^k \rho} c_{j-1}(\rho) \\ \mathcal{A}_{k\,c(j)} &= -\omega^{k(n-j)} \left(e^{-i\omega^k \rho} + \beta_{p\,j-1}\right) c_{j-1}(\rho) \end{aligned}$$

where p is the index of the unique boundary condition that couples f_{j-1} and g_{j-1}. We may now write

$$\begin{aligned} \Delta_{\text{PDE}}(\rho) = c_4^2(\rho) c_3^2(\rho) c_2(\rho) \sum_{\sigma \in S_5} &\text{sgn}(\sigma) \omega^{\sum_{j\in\{4,5\}} \sigma l(j)(n-j)} \\ &\times (-1)^2 \omega^{\sum_{j\in\{4,5\}} \sigma r(j)(n-j)} e^{-i\sum_{j\in\{4,5\}} \omega^{\sigma r(j)}\rho} \\ &\times (-1) \omega^{\sum_{j\in\{3\}} \sigma c(j)(n-j)} \prod_{j\in\{3\}} \left(e^{-i\omega^{\sigma c(j)}\rho} + \beta_{1\,2}\right) c_4^2(\rho) c_3^2(\rho) c_2(\rho) \\ = \rho^4 \beta_{1\,2} \sum_{\sigma\in S_5} &\text{sgn}(\sigma) \omega^{-(4[\sigma(3)+\sigma(4)]+3\sigma(5))} e^{-i\left(\omega^{\sigma(2)}+\omega^{\sigma(4)}\right)\rho} \\ &+ \rho^4 \sum_{\sigma\in S_5} \text{sgn}(\sigma) \omega^{-(4[\sigma(3)+\sigma(4)]+3\sigma(5))} e^{-i\left(\omega^{\sigma(2)}+\omega^{\sigma(4)}+\omega^{\sigma(5)}\right)\rho}. \quad \text{(B.3.1)} \end{aligned}$$

For any given $\pi \in S_5$ we can now calculate coefficients of $e^{-i\sum_{j=1}^2 \omega^{\pi(j)}\rho}$ and $e^{-i\sum_{j=1}^3 \omega^{\pi(j)}\rho}$ in $\Delta_{\text{PDE}}(\rho)$; the first coming from the first sum in the right hand side of equation (B.3.1) and the second from the second sum.

We look first at $e^{-i\sum_{j=1}^2 \omega^{\pi(j)}\rho}$. If we choose some $\tau \in S_5$ such that

$$\begin{aligned} \sum_{j\in\{2,4\}} \omega^{\tau(j)} &= \sum_{j=1}^{2} \omega^{\pi(j)} \\ \Leftrightarrow \quad \{\tau(j) : j \in \{2,4\}\} &= \{\pi(j) : j \in \{1,2\}\} \end{aligned}$$

then the coefficient of $e^{-i\sum_{j=1}^{2}\omega^{\pi(j)}\rho}$ in Δ_{PDE} is given by

$$\rho^4\beta_{1\,2}\sum_{\substack{\sigma\in S_5:\\ \{\sigma(2),\sigma(4)\}\\ =\{\tau(2),\tau(4)\}}}\operatorname{sgn}(\sigma)\omega^{-(4[\sigma(3)+\sigma(4)]+3\sigma(5))}. \tag{B.3.2}$$

Similarly, we may choose some $\tau\in S_5$ such that

$$\begin{aligned}&\sum_{j\in\{4,5\}}\omega^{\tau r(j)}+\omega^{\sigma(5)}=\sum_{j=1}^{3}\omega^{\pi(j)}\\ \Leftrightarrow\quad&\{\tau(j):j\in\{2,4,5\}\}=\{\pi(j):j\in\{1,2,3\}\}\end{aligned}$$

then the coefficient of $e^{-i\sum_{j=1}^{3}\omega^{\pi(j)}\rho}$ in Δ_{PDE} is given by

$$\rho^4\sum_{\substack{\sigma\in S_5:\\ \{\sigma(2),\sigma(4),\sigma(5)\}\\ =\{\tau(2),\tau(4),\tau(5)\}}}\operatorname{sgn}(\sigma)\omega^{-(4[\sigma(3)+\sigma(4)]+3\sigma(5))}. \tag{B.3.3}$$

To ensure well-posedness we must check that the coefficients of $e^{-i\sum_{j=k}^{k+2}\omega^{\pi(j)}\rho}$ are nonzero for each $k=1,2,3,4,5$. Hence we must check the values of

$$\rho^4\sum_{\substack{\sigma\in S_5:\\ \{\sigma(2),\sigma(4),\sigma(5)\}\\ =\{k,k+1,k+2\}}}\operatorname{sgn}(\sigma)\omega^{-(4[\sigma(3)+\sigma(4)]+3\sigma(5))}$$

for each k. The σ that index the sum are

$$\begin{pmatrix}k+3\\k\\k+4\\k+1\\k+2\end{pmatrix},\begin{pmatrix}k+3\\k+2\\k+4\\k\\k+1\end{pmatrix},\begin{pmatrix}k+3\\k+1\\k+4\\k+2\\k\end{pmatrix},\begin{pmatrix}k+4\\k\\k+3\\k+2\\k+1\end{pmatrix},\begin{pmatrix}k+4\\k+1\\k+3\\k\\k+2\end{pmatrix},\begin{pmatrix}k+4\\k+2\\k+3\\k+1\\k\end{pmatrix},$$
$$\begin{pmatrix}k+3\\k\\k+4\\k+2\\k+1\end{pmatrix},\begin{pmatrix}k+3\\k+1\\k+4\\k\\k+2\end{pmatrix},\begin{pmatrix}k+3\\k+2\\k+4\\k+1\\k\end{pmatrix},\begin{pmatrix}k+4\\k\\k+3\\k+1\\k+2\end{pmatrix},\begin{pmatrix}k+4\\k+2\\k+3\\k\\k+1\end{pmatrix},\begin{pmatrix}k+4\\k+1\\k+3\\k+2\\k\end{pmatrix}.$$

Denoting the first of these $\hat{\sigma}$, the first six have $\operatorname{sgn}(\sigma)=\operatorname{sgn}(\hat{\sigma})$ and the last six have $\operatorname{sgn}(\sigma)=-\operatorname{sgn}(\hat{\sigma})$. Evaluating expression (B.3.3) at each of these σ we see that the coefficients we require are given by

$$\rho^4\operatorname{sgn}(\hat{\sigma})\left[2\omega^{-(k+1)}+2\omega^{-(k+3)}+2\omega^{-(k+4)}-3\omega^{-k}-3\omega^{-(k+2)}\right]\neq 0,$$

guaranteeing the ratio (3.2.5) is bounded and decaying at infinity for $\rho\in D$.

To construct the argument of Example B.6 in general we require the additional notation of Definition B.7. With this it is possible to state Lemma B.8, the tool we will use to find the relevant coefficients. This completes the formulation of the notation used in Condition 3.22.

DEFINITION B.7. *Let us define the index sets $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{C}$ and, as the boundary conditions are non-Robin, simplify the notation for the coupling coefficients so that only a single index is used:*

- $\mathcal{L} = \{j+1 : \alpha_{r\,j} = 0 \,\forall\, r\}$ *so that* $L{=}|\mathcal{L}|$
- $\mathcal{R} = \{j+1 : \beta_{r\,j} = 0 \,\forall\, r\}$ *so that* $R{=}|\mathcal{R}|$
- $\mathcal{C} = \{j+1 : \exists\, r : \beta_{r\,j}, \alpha_{r\,j} \neq 0\}$ *so that* $C{=}|\mathcal{C}|$
- $\widetilde{\beta}_j = \beta_{p\,j-1}$, *where p is the index of the (unique, as the boundary conditions are non-Robin) boundary condition that couples f_{j-1} and g_{j-1}.*

LEMMA B.8. *Let $c' : \{1, 2, \ldots, C\} \to \mathcal{C}$ be a bijection and let the three injective functions*

$$l : \mathcal{L} \to \{1, 2, \ldots, n\}$$
$$r : \mathcal{R} \to \{1, 2, \ldots, n\}$$
$$c : \mathcal{C} \to \{1, 2, \ldots, n\}$$

be chosen such that their codomains are disjoint. For any $k \in \{0, 1, \ldots, C\}$, $\tau \in S_n$ and $\tau' \in S_C$ let

$$S_{k\,\tau\,\tau'} = \left\{ (\sigma, \sigma') \in S_n \times S_C : \sum_{j\in\mathcal{R}} \omega^{\sigma r(j)} + \sum_{j=1}^{k} \omega^{\sigma c c' \sigma'(j)} = \sum_{j\in\mathcal{R}} \omega^{\tau r(j)} + \sum_{j=1}^{k} \omega^{\tau c c' \tau'(j)} \right\}. \quad \text{(B.3.4)}$$

If the pair $(\tau, \tau') \in S_n \times S_C$ is chosen such that

$$\{\pi(j) : j \in \{1, 2, \ldots, R+k\}\} = \{\tau r(p) : p \in \mathcal{R}\} \cup \{\tau c c' \tau'(p) : p \in \{1, 2, \ldots, k\}\} \quad \text{(B.3.5)}$$

then the coefficient of $e^{-i\sum_{r=1}^{R+k} \omega^{\pi(r)}\rho}$ in Δ_{PDE} is given by

$$\pm\, P_{k\,\pi} = (-1)^{R+C}(-a)^n (i\rho)^{-(\sum_{j\in\mathcal{R}} j + \sum_{j\in\mathcal{C}} j + \sum_{j\in\mathcal{L}} j)} \sum_{(\sigma,\sigma')\in S_{k\,\tau\,\tau'}} \mathrm{sgn}(\sigma)\omega^{-\sum_{j\in\mathcal{R}} \sigma r(j) j - \sum_{j\in\mathcal{C}} \sigma c(j) j - \sum_{j\in\mathcal{L}} \sigma l(j) j} \prod_{j=k+1}^{C} \widetilde{\beta}_{c'\sigma'(j)} \quad \text{(B.3.6)}$$

for $k > 0$ and

$$\pm\, P_{0\,\pi} = (-1)^{R+C}(-a)^n (i\rho)^{-(\sum_{j\in\mathcal{R}} j + \sum_{j\in\mathcal{C}} j + \sum_{j\in\mathcal{L}} j)} C! \prod_{j=1}^{C} \widetilde{\beta}_j \sum_{\substack{\sigma\in S_n: \\ \forall\, j\in\mathcal{R}\ \exists\, q\in\mathcal{R}: \\ \tau r(j) = \sigma r(q)}} \mathrm{sgn}(\sigma)\omega^{-\sum_{j\in\mathcal{R}} \sigma r(j) j - \sum_{j\in\mathcal{C}} \sigma c(j) j - \sum_{j\in\mathcal{L}} \sigma l(j) j} \quad \text{(B.3.7)}$$

if $k = 0$.

In the special case $\widetilde{\beta}_j = \beta$ for all $j \in \mathcal{C}$ equation (B.3.6) *simplifies to*

$$\pm\, P_{k\,\pi} = (-1)^{R+C}(-a)^n (i\rho)^{-(\sum_{j\in\mathcal{R}} j + \sum_{j\in\mathcal{C}} j + \sum_{j\in\mathcal{L}} j)} \beta^{C-k} k!(C-k)! \sum_{\substack{\sigma\in S_n:\ \exists\ \sigma'\in S_C: \\ (\sigma,\sigma')\in S_{k\,\tau\,\tau'}}} \mathrm{sgn}(\sigma)\omega^{-\sum_{j\in\mathcal{R}} \sigma r(j) j - \sum_{j\in\mathcal{C}} \sigma c(j) j - \sum_{j\in\mathcal{L}} \sigma l(j) j}. \quad \text{(B.3.8)}$$

PROOF. Note that for any valid choice of l, r and c their codomains have disjoint union $\{1, 2, \ldots, n\}$. The uncertainty in the sign in equations (B.3.6), (B.3.7) and (B.3.8) comes from different choices of the functions l, r and c. For concreteness, we require that l, r and c satisfy the conditions

$$\begin{aligned}
\mathcal{A}_{p\,l(j)}(\rho) &= \omega^{(p-1)(n-j)} c_{j-1}(\rho) && j \in \mathcal{L}, \\
\mathcal{A}_{p\,r(j)}(\rho) &= -\omega^{(p-1)(n-j)} e^{-i\omega^{p-1}\rho} c_{j-1}(\rho) && j \in \mathcal{R} \text{ and} \\
\mathcal{A}_{p\,c(j)}(\rho) &= -\omega^{(p-1)(n-j)} \left(e^{-i\omega^{p-1}\rho} + \widetilde{\beta}_j \right) c_{j-1}(\rho) && j \in \mathcal{C}.
\end{aligned}$$

Then the columns of $\mathcal{A}(\rho)$ are

$$\underset{j \in \mathcal{R}}{\begin{pmatrix} -e^{-i\rho} c_{j-1}(\rho) \\ -\omega^{(n-j)} e^{-i\omega\rho} c_{j-1}(\rho) \\ -\omega^{2(n-j)} e^{-i\omega^2\rho} c_{j-1}(\rho) \\ \vdots \\ -\omega^{(n-1)(n-j)} e^{-i\omega^{n-1}\rho} c_{j-1}(\rho) \end{pmatrix}} \quad \underset{j \in \mathcal{C}}{\begin{pmatrix} -(e^{-i\rho} + \widetilde{\beta}_j) c_{j-1}(\rho) \\ -\omega^{(n-j)} (e^{-i\omega\rho} + \widetilde{\beta}_j) c_{j-1}(\rho) \\ -\omega^{2(n-j)} (e^{-i\omega^2\rho} + \widetilde{\beta}_j) c_{j-1}(\rho) \\ \vdots \\ -\omega^{(n-1)(n-j)} (e^{-i\omega^{n-1}\rho} + \widetilde{\beta}_j) c_{j-1}(\rho) \end{pmatrix}} \quad \underset{j \in \mathcal{L}}{\begin{pmatrix} c_{j-1}(\rho) \\ \omega^{(n-j)} c_{j-1}(\rho) \\ \omega^{2(n-j)} c_{j-1}(\rho) \\ \vdots \\ \omega^{(n-1)(n-j)} c_{j-1}(\rho) \end{pmatrix}}$$

and

$$\begin{aligned}
\Delta_{\mathrm{PDE}}(\rho) = \sum_{\sigma \in S_n} & \operatorname{sgn}(\sigma) (-1)^R \omega^{\sum_{j \in \mathcal{R}} \sigma r(j)(n-j)} e^{-i \sum_{j \in \mathcal{R}} \omega^{\sigma r(j)} \rho} (-1)^C \omega^{\sum_{j \in \mathcal{C}} \sigma c(j)(n-j)} \\
& \times \prod_{j \in \mathcal{C}} \left(e^{-i\omega^{\sigma c(j)}\rho} + \widetilde{\beta}_j \right) \omega^{\sum_{j \in \mathcal{L}} \sigma l(j)(n-j)} \\
& \times \prod_{j \in \mathcal{R}} c_{j-1}(\rho) \prod_{j \in \mathcal{C}} c_{j-1}(\rho) \prod_{j \in \mathcal{L}} c_{j-1}(\rho) \\
= (-1)^{R+C} & (-a)^n \rho^{n^2} (i\rho)^{-(\sum_{j \in \mathcal{R}} j + \sum_{j \in \mathcal{C}} j + \sum_{j \in \mathcal{L}} j)} \\
& \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \omega^{-\sum_{j \in \mathcal{R}} \sigma r(j) j - \sum_{j \in \mathcal{C}} \sigma c(j) j - \sum_{j \in \mathcal{L}} \sigma l(j) j} \\
& \times e^{-i \sum_{j \in \mathcal{R}} \omega^{\sigma r(j)} \rho} \sum_{k=0}^{C} \left(\sum_{\sigma' \in S_C} e^{-i \sum_{j=1}^{k} \omega^{\sigma c c' \sigma'(j)} \rho} \prod_{j=k+1}^{C} \widetilde{\beta}_{c'\sigma'(j)} \right). \qquad \text{(B.3.9)}
\end{aligned}$$

The latter equality is justified by a binomial expansion.

Having established representation (B.3.9) for Δ_{PDE} we investigate the structure of $S_{k\,\tau\,\tau'}$. Initially we work under the assumption $1 \leqslant k \leqslant C$. In the following two paragraphs we find necessary conditions for $\sigma \in S_n$ to be such that there exists some $\sigma' \in S_C$ such that $(\sigma, \sigma') \in S_{k\,\tau\,\tau'}$ and the third shows that these conditions are also sufficient by constructing a σ' to complete the pair. The fourth paragraph completes the characterisation of $S_{k\,\tau\,\tau'}$ by giving necessary and sufficient conditions for σ' to complete such a pair.

First consider $\sigma \in S_n$ such that there is some $j \in \{r(p) : p \in \mathcal{R}\} \cup \{cc'\tau'(p) : p \in \{1, 2, \ldots, k\}\}$ and some $q \in \{l(p) : p \in \mathcal{L}\}$ such that $\tau(j) = \sigma(q)$. As the domains of $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{C}$ are disjoint, there is no $s \in \{r(p) : p \in \mathcal{R}\}$ for which $\tau(j) = \sigma(s)$ and there is no $s \in \{c(p) : p \in \mathcal{C}\}$ with $\tau(j) = \sigma(s)$ hence, as c' and τ' are bijections, there is no $s \in \{1, 2, \ldots, C\}$ for which

$\tau(j) = \sigma c c' \sigma'(s)$. This establishes that there is no $\sigma' \in S_C$ such that $(\sigma, \sigma') \in S_{k\,\tau\,\tau'}$. Hence every $\sigma \in S_n$ for which there exists some $\sigma' \in S_C$ such that $(\sigma, \sigma') \in S_{k\,\tau\,\tau'}$ has the property

$$\forall\, j \in \{r(p) : p \in \mathcal{R}\} \cup \{cc'\tau'(p) : p \in \{1, 2, \ldots, k\}\}$$
$$\exists\, q \in \{r(p) : p \in \mathcal{R}\} \cup \{cc'\sigma'(p) : p \in \{1, 2, \ldots, k\}\} \text{ such that } \tau(j) = \sigma(q). \quad \text{(B.3.10)}$$

We define

$$X = |\{j \in \mathcal{R} \text{ such that } \forall\, p \in \mathcal{R}\ \tau r(j) \neq \sigma r(p)\}| \text{ and}$$
$$Y = |\{j \in \{1, 2, \ldots, k\} \text{ such that } \forall\, p \in \mathcal{R}\ \tau cc'\tau'(j) \neq \sigma r(p)\}|.$$

It is immediate that, for any $\sigma \in S_n$, $X + Y \leqslant R + k$ but, as $\{\tau r(j) : j \in \mathcal{R}\}$ and $\{\tau cc'\tau'(j) : j \in \{1, 2, \ldots, k\}\}$ are disjoint, τ, c' and τ' are bijections and r and c are injections with disjoint codomains, $X + Y \geqslant k$. Definition (B.3.4) implies that every $\sigma \in S_n$ for which there exists some $\sigma' \in S_C$ such that $(\sigma, \sigma') \in S_{k\,\tau\,\tau'}$ has the property $X + Y \leqslant k$ hence any such σ has the property

$$X + Y = k. \quad \text{(B.3.11)}$$

For $\sigma \in S_n$ obeying conditions (B.3.10) and (B.3.11) we now show that there exists $\sigma' \in S_C$ such that $(\sigma, \sigma') \in S_{k\,\tau\,\tau'}$. Condition (B.3.11) ensures $Y = k - X$ so we may choose some sequence $(j_p)_{p=1}^k$ such that, for $1 \leqslant p \leqslant X$, $j_p \in \mathcal{R}$ and for every $q \in \mathcal{R}$ $\tau r(j_p) \neq \sigma r(q)$ and, for $X + 1 \leqslant p \leqslant k$, $j_p \in \{1, 2, \ldots, k\}$ and for every $q \in \mathcal{R}$ $\tau cc'\tau'(j_p) \neq \sigma r(q)$. Now by condition (B.3.11), for $1 \leqslant p \leqslant X$, there exists some $q \in \mathcal{C}$ such that $\tau r(j_p) = \sigma c(q)$ so we may define $c'\sigma'(p) = q$ hence

$$\sigma'(p) = c'^{-1} c^{-1} \sigma^{-1} \tau r(j_p)$$

and, for $X + 1 \leqslant p \leqslant k$, there exists some $q \in \mathcal{C}$ such that $\tau cc'\tau'(j_p) = \sigma c(q)$ so we may define $c'\sigma'(p) = q$ hence

$$\sigma'(p) = c'^{-1} c^{-1} \sigma^{-1} \tau cc'\tau'(j_p).$$

This is a unique (up to choice of $(j_p)_{p=1}^k$) definition of the first k entries of σ'. Note that, as $\{\tau r(j) : j \in \mathcal{R}\}$ and $\{\tau cc'\tau'(j) : j \in \{1, 2, \ldots, k\}\}$ are disjoint, τ, c'^{-1} and σ^{-1} are bijections and r and c are injections with disjoint codomains, for any $1 \leqslant j, q \leqslant k$, $\sigma'(j) \neq \sigma'(q)$. The remaining $C - k$ entries of σ' may be chosen in any other way such that $\sigma' \in S_C$.

Now assume $(\sigma, \sigma') \in S_{k\,\tau\,\tau'}$ and let $\sigma'' \in S_C$. Then

$$\begin{aligned}
& (\sigma, \sigma'') \in S_{k\,\tau\,\tau'} \\
\Leftrightarrow\quad & \sum_{j \in \mathcal{R}} \omega^{\sigma r(j)} + \sum_{j=1}^{k} \omega^{\sigma cc'\sigma''(j)} = \sum_{j \in \mathcal{R}} \omega^{\tau r(j)} + \sum_{j=1}^{k} \omega^{\tau cc'\tau'(j)} \\
\Leftrightarrow\quad & \sum_{j \in \mathcal{R}} \omega^{\sigma r(j)} + \sum_{j=1}^{k} \omega^{\sigma cc'\sigma''(j)} = \sum_{j \in \mathcal{R}} \omega^{\sigma r(j)} + \sum_{j=1}^{k} \omega^{\sigma cc'\sigma'(j)} \\
\Leftrightarrow\quad & \sum_{j=1}^{k} \omega^{\sigma cc'\sigma''(j)} = \sum_{j=1}^{k} \omega^{\tau cc'\sigma'(j)} \\
\Leftrightarrow\quad & \forall\, j \in \{1, 2, \ldots, k\}\ \exists\, p \in \{1, 2, \ldots, k\} \text{ such that } \sigma''(j) = \sigma'(p) \qquad \text{(B.3.12)}
\end{aligned}$$

This yields a complete characterisation of $S_{k\,\tau\,\tau'}$ for $k \geqslant 1$.

If $k = 0$ the definition (B.3.4) of $S_{k\,\tau\,\tau'}$ simplifies to

$$\begin{aligned} S_{0\,\tau\,\tau'} &= \left\{ (\sigma, \sigma') \in S_n \times S_C : \sum_{j\in\mathcal{R}} \omega^{\sigma r(j)} = \sum_{j\in\mathcal{R}} \omega^{\tau r(j)} \right\} \\ &= \{\sigma \in S_n : \ \forall\, j \in \mathcal{R}\ \exists\, p \in \mathcal{R} : \tau r(j) = \sigma r(p)\} \times S_C. \end{aligned} \tag{B.3.13}$$

In this paragraph we argue that each $S_{k\,\tau\,\tau'}$ gives rise to a single unique exponential. The relation $\sim_k$ defined by

$$(\sigma, \sigma') \sim_k (\tau, \tau') \Leftrightarrow (\sigma, \sigma') \in S_{k\,\tau\,\tau'}$$

is an equivalence. From the definition of $S_{k\,\tau\,\tau'}$ it is clear that the $\sim_k$-classes are of equal size. Let $(\sigma, \sigma') \in S_{k\,\tau\,\tau'}$. Then, by the definition of $S_{k\,\tau\,\tau'}$,

$$e^{-i\sum_{j\in\mathcal{R}}\omega^{\sigma r(j)}\rho} e^{-i\sum_{j=1}^{k}\omega^{\sigma c c'\sigma'(j)}\rho} = e^{-i\sum_{j\in\mathcal{R}}\omega^{\tau r(j)}\rho} e^{-i\sum_{j=1}^{k}\omega^{\tau c c'\tau'(j)}\rho}.$$

Hence the equivalence classes of $\sim_k$ each identify a unique sum of powers of ω in the exponent and for each possible sum of $R + k$ powers of ω there exists an equivalence class of $\sim_k$. Hence for a given exponential $e^{-i\sum_{r=1}^{R+k}\omega^{\pi(r)}\rho}$ we may find its coefficient in equation (B.3.9) by choosing some pair $(\tau, \tau') \in S_n \times S_C$ such that equation (B.3.5) holds and evaluating expression (B.3.6) if $k \geqslant 1$ or expression (B.3.7) if $k = 0$.

If $k \geqslant 1$ and $\widetilde{\beta}_j = \beta$ for all $j \in \mathcal{C}$ then expression (B.3.6) simplifies to

$$P_{k\,\pi} = (-1)^{R+C}(-a)^n (i\rho)^{-(\sum_{j\in\mathcal{R}} j + \sum_{j\in\mathcal{C}} j + \sum_{j\in\mathcal{L}} j)} \sum_{(\sigma,\sigma')\in S_{k\,\tau\,\tau'}} \operatorname{sgn}(\sigma)\omega^{-\sum_{j\in\mathcal{R}}\sigma r(j) j - \sum_{j\in\mathcal{C}}\sigma c(j) j - \sum_{j\in\mathcal{L}}\sigma l(j) j} \beta^{C-k}.$$

If $\sigma \in S_n$ is such that there exists $\sigma' \in S_C$ such that $(\sigma, \sigma') \in S_{k\,\tau\,\tau'}$ then, as above, for $\sigma'' \in S_C$, $(\sigma, \sigma'') \in S_{k\,\tau\,\tau'}$ if and only if

$$\forall\, j \in \{1, 2, \ldots, k\}\ \exists\, p \in \{1, 2, \ldots, k\} \text{ such that } \sigma''(j) = \sigma'(p).$$

Hence, for a given σ, provided there exists some $\sigma' \in S_C$ such that $(\sigma, \sigma') \in S_{k\,\tau\,\tau'}$ there exist $k!(C - k)!$ such σ'. Hence expression (B.3.6) simplifies to expression (B.3.8). □

B.4. Admissible functions

This section gives the proof of Lemma 3.30. For convenience we reproduce Definition 1.3 of [**27**], adjusting to the notation of this work.

Definition B.9 (Admissible Functions). *Let $q_0 \in \mathbb{C}^\infty[0, 1]$, and let*

$$\{f_j, g_j : j \in \{0, 1, \ldots, n - 1\}\}$$

be a set of $2n$ C^∞ *functions on* $[0,T]$ *such that* $\partial_x^j q_0(0) = f_j(0)$ *and* $\partial_x^j q_0(1) = g_j(0)$ *for each* $j \in \{0,1,\dots,n-1\}$. *Let*

$$\widetilde{F}(\rho) = \sum_{j=1}^{n-1} c_j(\rho)\widetilde{f}_j(\rho), \tag{B.4.1}$$

$$\widetilde{G}(\rho) = \sum_{j=1}^{n-1} c_j(\rho)\widetilde{g}_j(\rho), \tag{B.4.2}$$

where $\widetilde{f}_j, \widetilde{g}_j$ *are defined in Lemma 3.30.*

The set of functions $\{f_j, g_j : j \in \{0,1,\dots,n-1\}\}$ *is called* admissible with respect to q_0 *if and only if* $q_T \in C^\infty[0,1]$ *and the functions* $\widetilde{F}, \widetilde{G}$ *satisfy the following relation:*

$$\widetilde{F}(\rho) - e^{-i\rho}\widetilde{G}(\rho) = -\hat{q}_0(\rho) + e^{a\rho^n T}\hat{q}_T(\rho). \tag{B.4.3}$$

Proof of Lemma 3.30. By the definition of $\widetilde{f}_j, \widetilde{g}_j$ in the statement of Lemma 3.30 and the definition of the index sets $J^\pm$ in Definition 2.19 we may write equations (B.4.1) and (B.4.2) as

$$\widetilde{F}(\rho) = \sum_{j\in J^+} \frac{\zeta_j(\rho) - e^{a\rho^n T}\eta_j(\rho)}{\Delta_{\mathrm{PDE}}(\rho)}, \tag{B.4.4}$$

$$\widetilde{G}(\rho) = \sum_{j\in J^-} \frac{\zeta_j(\rho) - e^{a\rho^n T}\eta_j(\rho)}{\Delta_{\mathrm{PDE}}(\rho)}. \tag{B.4.5}$$

Now by Cramer's rule and the calculations in the proof of Lemma 2.17 equation (B.4.3) is satisfied. The following remains to be shown:

(1) $q_T \in C^\infty[0,1]$.
(2) $f_j, g_j \in C^\infty[0,T]$ for each $j \in \{0,1,\dots,n\}$.
(3) $\partial_x^j q_0(0) = f_j(0)$ and $\partial_x^j q_0(1) = g_j(0)$ for each $j \in \{0,1,\dots,n-1\}$.

(1) By Assumption 3.2, η_j is entire. Hence $\hat{q}_T$ is entire so, by the standard results on the inverse Fourier transform, $q_T : [0,1] \to \mathbb{C}$, defined by

$$q_T(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\rho x}\hat{q}_T(\rho)\,\mathrm{d}\rho,$$

is a C^∞ smooth function.

(2) We know ζ_j is entire by construction and η_j is entire by Assumption 3.2 hence $\widetilde{F}$ and $\widetilde{G}$ are meromorphic on $\mathbb{C}$ and analytic on $\widetilde{D}$. Assumption 3.2 also guarantees that $\frac{\eta_j(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \to 0$ as $\rho \to \infty$ from within $\widetilde{D}$ hence, by the definition of D,

$$\frac{e^{a\rho^n T}\eta_j(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \to 0 \text{ as } \rho \to \infty \text{ from within } \widetilde{D}.$$

We know $\hat{q}_T$ is entire hence, as $\hat{q}_0$ is entire and the definitions of ζ_j and η_j differ only by which of these Fourier transforms appears, the ratio $\frac{\zeta_k(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \to 0$ as $\rho \to \infty$ from within $\widetilde{D}$ also. This establishes that

$$\frac{\zeta_j(\rho) - e^{a\rho^n T}\eta_j(\rho)}{\Delta_{\mathrm{PDE}}(\rho)} \to 0 \text{ as } \rho \to \infty \text{ from within } \widetilde{D}.$$

Hence, by equations (B.4.4) and (B.4.5), $\widetilde{F}(\rho), \widetilde{G}(\rho) \to 0$ as $\rho \to \infty$ within $\widetilde{D}$. By Lemma B.10 we have the direct definitions (B.4.6) and (B.4.7) of f_j and g_j in terms of $\widetilde{F}$ and $\widetilde{G}$ and, because $\widetilde{F}(\rho), \widetilde{G}(\rho) \to 0$ as $\rho \to \infty$ within $\widetilde{D}$, these definitions guarantee that f_j and g_j are C^∞ smooth.

(3) Equation (2.1.4) guarantees equations (B.4.4) and (B.4.6), imply that the compatibility condition $\partial_x^j q_0(0) = f_j(0)$ is satisfied by construction. Equations (B.4.5) and (B.4.7), imply that the compatibility condition $\partial_x^j q_0(1) = g_j(0)$ is satisfied. □

LEMMA B.10. *Let $\widetilde{F}$ and $\widetilde{G}$ be defined by equations* (B.4.1) *and* (B.4.2) *where $\widetilde{f}_j, \widetilde{g}_j$ are analytic on $\widetilde{D}$, given by Definition 3.9. Then the functions $f_j : [0,T] \to \mathbb{C}$ defined by*

$$f_j(t) = -\frac{i^j}{2\pi} \int_{\partial D} \lambda^j e^{-a\lambda^n t} \widetilde{F}(\lambda)\, \mathrm{d}\lambda, \tag{B.4.6}$$

$$g_j(t) = -\frac{i^j}{2\pi} \int_{\partial D} \lambda^j e^{-a\lambda^n t} \widetilde{G}(\lambda)\, \mathrm{d}\lambda, \tag{B.4.7}$$

satisfy equation (3.2.30).

PROOF. We use a scalar, inhomogeneous Riemann-Hilbert problem to derive the inverse to the t-transform (2.1.30). This mirrors the derivation of the Fourier transform presented in Example 7.4.6, particularly Section 7.4.2, of [**1**]. We perform this derivation for the pair $((\widetilde{f}_j)_{j=0}^{n-1}, (f_j)_{j=0}^{n-1})$, noting that the derivation is identical for the pair $((\widetilde{g}_j)_{j=0}^{n-1}, (g_j)_{j=0}^{n-1})$.

Consider the partial differential equation

$$\mu_t(t,\rho) + a\rho^n \mu(t,\rho) = \sum_{j=0}^{n-1} c_j(\rho) f_j(t), \tag{B.4.8}$$

where $f_j \in C^1[0,T]$, $t \in [0,T]$ and $\rho \in \mathbb{C}$. By direct integration we obtain solutions

$$\mu_+(t,\rho) = \int_0^t e^{-a\rho^n(t-s)} \sum_{j=0}^{n-1} c_j(\rho) f_j(s)\, \mathrm{d}s, \tag{B.4.9}$$

$$\mu_-(t,\rho) = -\int_t^T e^{-a\rho^n(t-s)} \sum_{j=0}^{n-1} c_j(\rho) f_j(s)\, \mathrm{d}s, \tag{B.4.10}$$

where μ_+ is analytic on $\mathbb{C} \setminus \overline{D}$, μ_- is analytic on D and

$$\max_{t\in[0,T]} \mu_+(t,\rho) \to 0 \text{ as } \rho \to \infty \text{ from within } \mathbb{C} \setminus \overline{D},$$

$$\max_{t\in[0,T]} \mu_-(t,\rho) \to 0 \text{ as } \rho \to \infty \text{ from within } D.$$

Hence on ∂D we may calculate the difference or *jump* function

$$(\mu_+ - \mu_-)(t,\rho) = e^{-a\rho^n t} \widetilde{F}.$$

This specifies a Riemann-Hilbert problem which, by Section 7.3 of [**1**], has the sectionally analytic function

$$\mu(t,\rho) = \frac{-1}{2\pi i} \int_{\partial D} \frac{e^{-a\lambda^n t} \widetilde{F}(\lambda)}{\lambda - \rho}\, \mathrm{d}\lambda \tag{B.4.11}$$

as its solution. Taking the partial derivative of equation (B.4.11) with respect to t we obtain

$$\mu_t(t,\rho) = \frac{a}{2\pi i} \int_{\partial D} \frac{\lambda^n e^{-a\lambda^n t} \widetilde{F}(\lambda)}{\lambda - \rho}\, \mathrm{d}\lambda. \tag{B.4.12}$$

Combining equations (B.4.8), (B.4.11) and (B.4.12) and equating coefficients of ρ^j we obtain equations (B.4.6).

We have shown that the transforms (B.4.6) are the inverse of the transforms (3.2.30), hence the pair $((\widetilde{f}_j)_{j=0}^{n-1}, (f_j)_{j=0}^{n-1})$ satisfies (B.4.6) if and only if it satisfies (3.2.30). □

List of theorems, definitions etc.

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