

Determinants of Multiplicative Toeplitz Matrices¹

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Abstract

In this paper we study matrices $A = (a_{ij})$ whose (i, j) th-entry is a function of i/j ; that is, $a_{ij} = f(i/j)$ for some $f : \mathbb{Q}^+ \rightarrow \mathbb{C}$. We obtain a formula for the truncated determinants in the case where f is multiplicative, linking them to determinants of truncated Toeplitz matrices. We apply our formula to obtain several determinants of number-theoretic matrices.

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Introduction

In this article, we consider matrices and their determinants of the form

$$\begin{pmatrix} c(1) & c(\frac{1}{2}) & c(\frac{1}{3}) & c(\frac{1}{4}) & \cdots \\ c(2) & c(1) & c(\frac{2}{3}) & c(\frac{1}{2}) & \cdots \\ c(3) & c(\frac{3}{2}) & c(1) & c(\frac{3}{4}) & \cdots \\ c(4) & c(2) & c(\frac{4}{3}) & c(1) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad (0.1)$$

where the (i, j) th-entry is a function of i/j . They are characterised by being constant along lines where i/j is a given positive rational. In this sense they resemble Toeplitz matrices, which are constant on lines parallel to the diagonal; i.e. where $i - j$ is a given integer. For this reason we shall call them *multiplicative Toeplitz matrices*. These matrices represent linear operators between various spaces. To identify these spaces, and to determine when they are bounded are interesting questions in themselves, but for the purposes of this paper, these will not concern us. We shall be mainly concerned with determinants of truncated matrices.

Toeplitz matrices are most usefully studied by associating them with a function (or ‘symbol’) whose Fourier coefficients make up the matrix. Indeed, they generate bounded operators on ℓ^2 if and only if the matrix entries are the Fourier coefficients of an essentially bounded function on \mathbb{T} . For matrices of the form (0.1), we associate, by analogy, the series

$$\sum_{q \in \mathbb{Q}^+} c(q)q^{it}, \quad (0.2)$$

where q ranges over the positive rationals. We shall first make sense of such series in §1.

In §2, we concern ourselves with the case when $c(\cdot)$ is a multiplicative function on the positive rationals. Then the matrix (0.1) and the series (0.2) are shown to factorise as ‘Euler products’.

In §3, we consider the determinants of truncated multiplicative Toeplitz matrices for which $c(\cdot)$ is multiplicative. The ‘Euler product’ formula ceases to hold when we truncate the matrices, but we show in Theorem 3.2 that it is miraculously recovered on taking determinants. This reduces the problem of evaluating such determinants to those whose non-zero entries lie on ‘lines’ $i/j = p^k$ for p prime and $k \in \mathbb{Z}$. It is shown in Theorem 3.1 that these can be evaluated in terms of determinants of Toeplitz matrices. The proofs are done separately in §4. Using this formula, and

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known formulae for Toeplitz determinants with rational symbol, we evaluate several determinants of number-theoretic interest in §5. For example, for f and g completely multiplicative

$$\det \left(f \left(\frac{i}{(i,j)} \right) g \left(\frac{j}{(i,j)} \right) \right)_{i,j \leq n} = \prod_{p \leq n} \left(1 - f(p)g(p) \right)^{\left[\frac{n}{p} \right]}. \quad (0.3)$$

Determinants associated with arithmetical functions have been studied and evaluated by many authors. A number of generalizations have been made of Smith's determinant [7] of the matrix with coefficients (i, j) (see for example [6]). The matrices considered are however of a different form; e.g. in [6], the first column is constant. Generalizations in a different direction were made in [2], [5], and also [3], but again the matrices considered are of a different form from (0.1).

§1. \mathbb{Q}^+ -coefficients and series

For a function $c : \mathbb{Q}^+ \rightarrow \mathbb{C}$ defined on the positive rationals, we define

$$\sum_{q \in \mathbb{Q}^+} c(q) = \lim_{N \rightarrow \infty} \sum_{\substack{m, n \leq N \\ (m, n) = 1}} c \left(\frac{m}{n} \right), \quad \text{whenever this limit exists.}$$

We shall sometimes abbreviate the left-hand sum by $\sum_q c(q)$. For $q = \frac{m}{n} \in \mathbb{Q}^+$, with $(m, n) = 1$, we write $|q| = \max\{m, n\}$. Thus the above definition becomes

$$\sum_q c(q) = \lim_{N \rightarrow \infty} \sum_{|q| \leq N} c(q).$$

Note that if $f(t) = \sum_{q \in \mathbb{Q}^+} c(q)q^{it}$ and the sum converges absolutely, then

$$\begin{aligned} \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T f(t)x^{-it} dt &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \sum_{q \in \mathbb{Q}^+} c(q)q^{it}x^{-it} dt \\ &= \sum_{q \in \mathbb{Q}^+} c(q) \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T (q/x)^{it} dt \\ &= \begin{cases} c(x) & \text{if } x \in \mathbb{Q}^+ \\ 0 & \text{if } x \notin \mathbb{Q}^+. \end{cases} \end{aligned}$$

More generally, given a measurable function $f : \mathbb{R} \rightarrow \mathbb{C}$, we can ask if the limit

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T f(t)x^{-it} dt \quad (1.1)$$

exists for every $x > 0$. If it exists, we shall denote it by $c(x)$ or, if we want to emphasize the dependence on f , by $c_f(x)$. This limit is also called the 'Hadamard coefficient' (see [8], [4]). Of particular number-theoretic interest are those f for which $c(x)$ is supported on the positive rationals; that is,

$$c(x) = 0 \quad \text{for } x \notin \mathbb{Q}^+. \quad (1.2)$$

Definition 1.1

- (a) Let V denote the space of measurable functions f for which (1.1) and (1.2) hold. We call $c(q)$ (for $q \in \mathbb{Q}^+$), the \mathbb{Q}^+ -coefficients of f and write

$$f(t) \sim \sum_{q \in \mathbb{Q}^+} c(q)q^{it},$$

which we call the \mathbb{Q}^+ -series, of f . (The series formed as such is a formal series and need not converge.)

(b) Let U denote the subspace of V consisting of all absolutely convergent \mathbb{Q}^+ -series; that is,

$$U = \left\{ \sum_{q \in \mathbb{Q}^+} c(q)q^{it} : \sum_{q \in \mathbb{Q}^+} |c(q)| < \infty \right\}.$$

(c) For $f \in V$, with \mathbb{Q}^+ -coefficients $c(q)$, let $A(f)$ denote the infinite matrix

$$A(f) = \left(c\left(\frac{i}{j}\right) \right)_{i,j \geq 1}.$$

Proposition 1.2

Let $f, g \in U$. Then $fg \in U$ and

$$c_{fg}(q) = \sum_{r \in \mathbb{Q}^+} c_f(r)c_g(q/r).$$

Proof. If $f(t) = \sum_{r \in \mathbb{Q}^+} c_f(r)r^{it}$ and $g(t) = \sum_{s \in \mathbb{Q}^+} c_g(s)s^{it}$ we have, by the absolute convergence of the series,

$$f(t)g(t) = \sum_{r,s \in \mathbb{Q}^+} c_f(r)c_g(s)(rs)^{it} = \sum_{q \in \mathbb{Q}^+} q^{it} \left(\sum_{r \in \mathbb{Q}^+} c_f(r)c_g(q/r) \right)$$

from which the result follows. □

Examples 1.3

(a) Let $g(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ and $h(s) = \sum_{n=1}^{\infty} \frac{b_n}{n^s}$ be two Dirichlet series which converge absolutely for $\Re s > \sigma_0$. Let $\alpha, \beta > \sigma_0$ and put $f(t) = g(\alpha - it)h(\beta + it)$. Then $f \in U$ with

$$c(q) = \frac{1}{m^\alpha n^\beta} \sum_{d=1}^{\infty} \frac{a_{md}b_{nd}}{d^{\alpha+\beta}} \quad \text{for } q = \frac{m}{n} \text{ with } (m, n) = 1.$$

We can prove this by multiplying out the series for $g(\alpha - it)$ and $h(\beta + it)$. We have

$$\begin{aligned} f(t) &= \sum_{m=1}^{\infty} \frac{a_m}{m^{\alpha-it}} \sum_{n=1}^{\infty} \frac{b_n}{n^{\beta+it}} = \sum_{m,n \geq 1} \frac{a_m b_n}{m^\alpha n^\beta} \left(\frac{m}{n}\right)^{it} = \sum_{d=1}^{\infty} \sum_{\substack{m,n \geq 1 \\ (m,n)=d}} \frac{a_m b_n}{m^\alpha n^\beta} \left(\frac{m}{n}\right)^{it} \\ &= \sum_{d=1}^{\infty} \frac{1}{d^{\alpha+\beta}} \sum_{\substack{m,n \geq 1 \\ (m,n)=1}} \frac{a_{md}b_{nd}}{m^\alpha n^\beta} \left(\frac{m}{n}\right)^{it} = \sum_{\substack{m,n \geq 1 \\ (m,n)=1}} \frac{1}{m^\alpha n^\beta} \left(\sum_{d=1}^{\infty} \frac{a_{md}b_{nd}}{d^{\alpha+\beta}} \right) \left(\frac{m}{n}\right)^{it}. \end{aligned}$$

(b) As a special case of the above, take $a_n = b_n = 1$ for all n . Then, for $\alpha, \beta > 1$, we have

$$\frac{\zeta(\alpha - it)\zeta(\beta + it)}{\zeta(\alpha + \beta)} = \sum_{q \in \mathbb{Q}^+} c(q)q^{it} = \sum_{\substack{m,n \geq 1 \\ (m,n)=1}} \frac{1}{m^\alpha n^\beta} \left(\frac{m}{n}\right)^{it},$$

where $\zeta(\cdot)$ is the Riemann zeta function. Note that this gives for arbitrary $m, n \in \mathbb{N}$

$$c\left(\frac{m}{n}\right) = c\left(\frac{m/(m,n)}{n/(m,n)}\right) = \frac{(m,n)^{\alpha+\beta}}{m^\alpha n^\beta}.$$

§2. Multiplicative coefficients and Euler products

We start by extending the notion of a multiplicative function to functions defined on the positive rationals. Recall that a function $f : \mathbb{N} \rightarrow \mathbb{C}$ is multiplicative if it is not identically zero and

$f(mn) = f(m)f(n)$ whenever $(m, n) = 1$. This is equivalent to saying that f satisfies: $f(1) = 1$ and

$$f(p_1^{a_1} \dots p_k^{a_k}) = f(p_1^{a_1}) \dots f(p_k^{a_k}),$$

for all distinct primes p_i and all $a_i \in \mathbb{N}$.

Definition 2.1 A function $a : \mathbb{Q}^+ \rightarrow \mathbb{C}$ is *multiplicative* if $a(1) = 1$ and

$$a(p_1^{a_1} \dots p_k^{a_k}) = a(p_1^{a_1}) \dots a(p_k^{a_k}),$$

for all distinct primes p_i and all $a_i \in \mathbb{Z} \setminus \{0\}$.

We say a is *completely multiplicative* if, in addition to the above, we also have, for all primes p and $k \in \mathbb{N}$,

$$a(p^k) = a(p)^k, \quad \text{and} \quad a(p^{-k}) = a(p^{-1})^k.$$

The following proposition follows immediately from the definition:

Proposition 2.2

Let $a : \mathbb{Q}^+ \rightarrow \mathbb{C}$. Then a is multiplicative if and only if there exist multiplicative functions $f, g : \mathbb{N} \rightarrow \mathbb{C}$ such that $a(\frac{m}{n}) = f(m)g(n)$ whenever $(m, n) = 1$. Furthermore, a is completely multiplicative if and only if both f and g are.

Next we show that coefficients derived from Dirichlet series with multiplicative coefficients as in example (a) are also multiplicative.

Theorem 2.3

Let $a, b : \mathbb{N} \rightarrow \mathbb{C}$ be multiplicative functions such that $\sum_{d=1}^{\infty} a(md)b(nd)$ converges absolutely for every $m, n \geq 1$ and its sum is non-zero if $m = n = 1$. Then the function $c : \mathbb{Q}^+ \rightarrow \mathbb{C}$ is multiplicative, where $c(\cdot)$ is defined by

$$c\left(\frac{m}{n}\right) = \frac{\sum_{d=1}^{\infty} a(md)b(nd)}{\sum_{d=1}^{\infty} a(d)b(d)} \quad \text{for } (m, n) = 1.$$

Proof. First $c(1) = 1$. Now consider $(m, n) = 1$. Write $m = \prod_p p^\alpha$ and $n = \prod_p p^\beta$. Note that $\alpha = 0$ for $p|n$, while $\beta = 0$ for $p|m$. Since a and b are multiplicative we have

$$\begin{aligned} c\left(\frac{m}{n}\right) \sum_{d=1}^{\infty} a(d)b(d) &= \prod_p \sum_{k=0}^{\infty} a(p^{k+\alpha})b(p^{k+\beta}) \\ &= \prod_{p|mn} \sum_{k=0}^{\infty} a(p^k)b(p^k) \prod_{p|mn} \sum_{k=0}^{\infty} a(p^{k+\alpha})b(p^{k+\beta}) \\ &= \prod_p \sum_{k=0}^{\infty} a(p^k)b(p^k) \prod_{p|mn} \frac{\sum_{k=0}^{\infty} a(p^{k+\alpha})b(p^{k+\beta})}{\sum_{k=0}^{\infty} a(p^k)b(p^k)} \\ &= \sum_{d=1}^{\infty} a(d)b(d) \prod_{p|mn} \frac{\sum_{k=0}^{\infty} a(p^{k+\alpha})b(p^{k+\beta})}{\sum_{k=0}^{\infty} a(p^k)b(p^k)}. \end{aligned}$$

(Note that here we used the fact that $\sum_{k=0}^{\infty} a(p^k)b(p^k) \neq 0$ which follows from $\sum_{n=1}^{\infty} a(n)b(n) \neq 0$.) Hence

$$\begin{aligned} c\left(\frac{m}{n}\right) &= \prod_{p|m} \frac{\sum_{k=0}^{\infty} a(p^{k+\alpha})b(p^k)}{\sum_{k=0}^{\infty} a(p^k)b(p^k)} \prod_{p|n} \frac{\sum_{k=0}^{\infty} a(p^k)b(p^{k+\beta})}{\sum_{k=0}^{\infty} a(p^k)b(p^k)} \\ &= f(m)g(n), \end{aligned} \tag{2.1}$$

where

$$f(m) = \prod_{p|m} \frac{\sum_{k=0}^{\infty} a(p^{k+\alpha})b(p^k)}{\sum_{k=0}^{\infty} a(p^k)b(p^k)} \quad \text{and} \quad g(n) = \prod_{p|n} \frac{\sum_{k=0}^{\infty} a(p^k)b(p^{k+\beta})}{\sum_{k=0}^{\infty} a(p^k)b(p^k)}.$$

After Proposition 2.2, it remains to show that f and g are multiplicative.

Consider f . Again, let $(m, n) = 1$ and write $m = \prod_p p^\alpha$ and $n = \prod_p p^\beta$. Then

$$\begin{aligned} f(mn) &= \prod_{p|mn} \frac{\sum_{k=0}^{\infty} a(p^{k+\alpha+\beta})b(p^k)}{\sum_{k=0}^{\infty} a(p^k)b(p^k)} \\ &= \prod_{p|m} \frac{\sum_{k=0}^{\infty} a(p^{k+\alpha})b(p^k)}{\sum_{k=0}^{\infty} a(p^k)b(p^k)} \prod_{p|n} \frac{\sum_{k=0}^{\infty} a(p^{k+\beta})b(p^k)}{\sum_{k=0}^{\infty} a(p^k)b(p^k)} \\ &= f(m)f(n), \end{aligned}$$

showing f is multiplicative. Similarly, g is multiplicative. □

Euler products

Dirichlet series with multiplicative coefficients can be factorised as Euler products. We show below that this factorisation extends naturally to \mathbb{Q}^+ -series with multiplicative coefficients and, furthermore, to the corresponding multiplicative Toeplitz matrix.

Suppose $f \in U$ and f has multiplicative \mathbb{Q}^+ -coefficients $c(q)$. For a prime p and t real, we define $f_p(t)$ by

$$f_p(t) = \sum_{k \in \mathbb{Z}} c(p^k) p^{ikt}.$$

Let n be a squarefree number greater than 1. We define $f_n(t)$ by

$$f_n(t) = \prod_{p|n} f_p(t).$$

Let S_n denote the set of all rationals formed from the prime divisors of n ; i.e. if $n = p_1 \dots p_k$ then

$$S_n = \{p_1^{a_1} \dots p_k^{a_k} : a_1, \dots, a_k \in \mathbb{Z}\}.$$

By multiplicativity of $c(\cdot)$ and the Fundamental Theorem of Arithmetic, it follows that

$$f_n(t) = \sum_{q \in S_n} c(q) q^{it}.$$

Theorem 2.4

Let $f \in U$ whose \mathbb{Q}^+ -coefficients $c(q)$ are multiplicative. Then we have the Euler products

$$\begin{aligned} (i) \quad & f(t) = \prod_p f_p(t), \quad \text{and} \\ (ii) \quad & A(f) = \prod_p A(f_p). \end{aligned}$$

Proof. (i) From above we have

$$\prod_{p \leq P} f_p(t) = \sum_{q \in S_Q} c(q) q^{it},$$

where $Q = \prod_{p \leq P} p$. Note that every q with $|q| \leq P$ is in S_Q . Hence

$$\begin{aligned} \left| f(t) - \prod_{p \leq P} f_p(t) \right| &= \left| \sum_{q \in \mathbb{Q}^+} c(q) q^{it} - \sum_{q \in S_Q} c(q) q^{it} \right| \\ &\leq \sum_{|q| > P} |c(q)| \rightarrow 0, \quad \text{as } P \rightarrow \infty \end{aligned}$$

from which the result follows.

(ii) First we prove that for m, n coprime,

$$A(f_m f_n) = A(f_m)A(f_n). \quad (2.2)$$

By Proposition 1.2, $f_m, f_n \in U$ implies $f_m f_n \in U$, with

$$c_{f_m f_n}(q) = \sum_{r \in \mathbb{Q}^+} c_{f_m}(r) c_{f_n}(q/r).$$

Replacing r with i/q gives

$$A(f_m f_n)_{ij} = c_{f_m f_n}\left(\frac{i}{j}\right) = \sum_{q \in \mathbb{Q}^+} c_{f_m}(i/q) c_{f_n}(q/j). \quad (2.3)$$

On the other hand,

$$\left(A(f_m)A(f_n)\right)_{ij} = \sum_{r=1}^{\infty} c_{f_m}(i/r) c_{f_n}(r/j).$$

Hence we must show that there is no contribution in (2.3) if $q \notin \mathbb{N}$.

We can write $i = s_1 t_1 u$ and $j = s_2 t_2 v$, where $s_1, s_2 \in S_m$, $t_1, t_2 \in S_n$ and u, v are coprime to mn . Since $i, j \in \mathbb{N}$, this forces $s_1, s_2, t_1, t_2, u, v \in \mathbb{N}$. For a contribution to (2.3), we need $i/q \in S_m$ and $q/j \in S_n$. Hence

$$q = is = jt \quad \text{for some } s \in S_m \text{ and } t \in S_n.$$

But then $s_1 s t_1 u = s_2 t_2 t v$, which forces $s_1 s = s_2$, $t_1 = t_2 t$, and $u = v$. As a result, $q = \frac{is_2}{s_1} = s_2 t_1 u \in \mathbb{N}$.

Applying (2.2) repeatedly we have, for every P ,

$$A\left(\prod_{p \leq P} f_p\right) = \prod_{p \leq P} A(f_p).$$

Fix i, j , and let $P \geq \max\{i, j\}$. Then the matrices $A(f_p)$ for $p > P$ do not affect the (i, j) th-entry on the right. Similarly, on the left, the (i, j) th-entry equals that of $A(f)$. Thus

$$A(f) = \prod_p A(f_p).$$

□

§3. Determinants

In this section we obtain a formula for the determinants of truncated multiplicative Toeplitz matrices in the case where the \mathbb{Q}^+ -coefficients are multiplicative. Let $f \in U$ with multiplicative \mathbb{Q}^+ -coefficients $c(q)$. We denote by $A_N(f)$ the truncated $N \times N$ -matrix for $A(f)$; that is,

$$A_N(f) = \begin{pmatrix} c(1) & c(\frac{1}{2}) & c(\frac{1}{3}) & c(\frac{1}{4}) & \cdots & c(\frac{1}{N}) \\ c(2) & c(1) & c(\frac{2}{3}) & c(\frac{2}{4}) & \cdots & c(\frac{2}{N}) \\ c(3) & c(\frac{3}{2}) & c(1) & c(\frac{3}{4}) & \cdots & c(\frac{3}{N}) \\ c(4) & c(2) & c(\frac{4}{3}) & c(1) & \cdots & c(\frac{4}{N}) \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ c(N) & c(\frac{N}{2}) & c(\frac{N}{3}) & c(\frac{N}{4}) & \cdots & c(1) \end{pmatrix}.$$

From Theorem 2.4 we had the Euler product formula

$$A(f) = \prod_p A(f_p).$$

When we truncate the matrices, we find that

$$A_N(f) \neq \prod_p A_N(f_p).$$

However, even though this formula breaks down, the formula is miraculously recovered by taking determinants. Thus

$$\det A_N(f) = \prod_p \det A_N(f_p).$$

This reduces the problem to the evaluation of $\det A_N(f_p)$ for a given prime p . Define $\hat{f}_p : \mathbb{T} \rightarrow \mathbb{T}$ (where \mathbb{T} is the unit circle) by

$$\hat{f}_p(e^{i\theta}) = f_p\left(\frac{\theta}{\log p}\right) \quad (0 \leq \theta < 2\pi)$$

Write c_k for $c(p^k)$. The matrix $A(f_p)$ contains only entries (i, j) where i/j is a power of p ; the (i, j) -entry is c_k if $i/j = p^k$. For instance for $p = 2$ it looks like

$$A(f_2) = \begin{pmatrix} c_0 & c_{-1} & 0 & c_{-2} & 0 & 0 & 0 & c_{-3} & \cdots \\ c_1 & c_0 & 0 & c_{-1} & 0 & 0 & 0 & c_{-2} & \cdots \\ 0 & 0 & c_0 & 0 & 0 & c_{-1} & 0 & 0 & \cdots \\ c_2 & c_1 & 0 & c_0 & 0 & 0 & 0 & c_{-1} & \cdots \\ 0 & 0 & 0 & 0 & c_0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & c_1 & 0 & 0 & c_0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & c_0 & 0 & \cdots \\ c_3 & c_2 & 0 & c_1 & 0 & 0 & 0 & c_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

For $a \in L^\infty(\mathbb{T})$, we denote (as is usual) by $T(a)$, the Toeplitz matrix with symbol a , and by $T_N(a)$ its $N \times N$ -truncation. Thus

$$T(\hat{f}_p) = \begin{pmatrix} c_0 & c_{-1} & c_{-2} & c_{-3} & \cdots \\ c_1 & c_0 & c_{-1} & c_{-2} & \cdots \\ c_2 & c_1 & c_0 & c_{-1} & \cdots \\ c_3 & c_2 & c_1 & c_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

For example, writing c_k and d_k for the coefficients of f_2 and f_3 respectively, we have

$$\begin{aligned} A_4(f_2)A_4(f_3) &= \begin{pmatrix} c_0 & c_{-1} & 0 & c_{-2} \\ c_1 & c_0 & 0 & c_{-1} \\ 0 & 0 & c_0 & 0 \\ c_2 & c_1 & 0 & c_0 \end{pmatrix} \begin{pmatrix} d_0 & 0 & d_{-1} & 0 \\ 0 & d_0 & 0 & 0 \\ d_1 & 0 & d_0 & 0 \\ 0 & 0 & 0 & d_0 \end{pmatrix} \\ &= \begin{pmatrix} c_0d_0 & c_{-1}d_0 & c_0d_{-1} & c_{-2}d_0 \\ c_1d_0 & c_0d_0 & c_1d_{-1} & c_{-1}d_0 \\ c_0d_1 & 0 & c_0d_0 & 0 \\ c_2d_0 & c_1d_0 & c_2d_{-1} & c_0d_0 \end{pmatrix}, \end{aligned}$$

while

$$A_4(f_2f_3) = \begin{pmatrix} c_0d_0 & c_{-1}d_0 & c_0d_{-1} & c_{-2}d_0 \\ c_1d_0 & c_0d_0 & c_1d_{-1} & c_{-1}d_0 \\ c_0d_1 & c_{-1}d_1 & c_0d_0 & c_{-2}d_1 \\ c_2d_0 & c_1d_0 & c_2d_{-1} & c_0d_0 \end{pmatrix}.$$

So $A_4(f_2)A_4(f_3) \neq A_4(f_2f_3)$, but both the above matrices have determinants

$$c_0 \begin{vmatrix} c_0 & c_{-1} & c_{-2} \\ c_1 & c_0 & c_{-1} \\ c_2 & c_1 & c_0 \end{vmatrix} d_0^2 \begin{vmatrix} d_0 & d_{-1} \\ d_1 & d_0 \end{vmatrix}.$$

Our main results are the following:

Theorem 3.1

Let p be a prime number. Then

$$\det A_N(f_p) = \prod_{r \geq 1} (\det T_r(\hat{f}_p))^{\lfloor \frac{N}{p^{r-1}} \rfloor - 2 \lfloor \frac{N}{p^r} \rfloor + \lfloor \frac{N}{p^{r+1}} \rfloor}.$$

(The product is of course finite with $r \leq \frac{\log N}{\log p} + 1$.)

Theorem 3.2

Let $f \in U$ and suppose f has multiplicative \mathbb{Q}^+ -coefficients. Then

$$\det A_N(f) = \prod_p \det A_N(f_p) = \prod_p \prod_{r \geq 1} (\det T_r(\hat{f}_p))^{\lfloor \frac{N}{p^{r-1}} \rfloor - 2 \lfloor \frac{N}{p^r} \rfloor + \lfloor \frac{N}{p^{r+1}} \rfloor}.$$

Remarks 3.3 The restriction of f to U is clearly unnecessary (in both Theorem 3.1 and 3.2) since the results only concern the values $c(q)$ with $|q| \leq N$. The only necessary assumption is that $c(\cdot)$ is multiplicative on $\{q \in \mathbb{Q}^+ : |q| \leq N\}$. We retain the notation $A(f)$ since the form of f (and hence of \hat{f}_p) is useful in evaluating $\det T_r(\hat{f}_p)$.

On noticing that the exponent of $\det T_r(\hat{f}_p)$ is positive for $r \leq \lfloor \frac{\log N}{\log p} \rfloor - 1$, we have the following corollary:

Corollary 3.4

- (i) The matrix $A_N(f)$ is invertible if and only if $A_N(f_p)$ is invertible for all p .
- (ii) The matrix $A_N(f_p)$ is invertible for all sufficiently large N if and only if $T_r(\hat{f}_p)$ is invertible for all $r \geq 1$, in which case

$$\det A_N(f_p) = \prod_{r \geq 1} \left(\frac{|T_{r-1}(\hat{f}_p)| |T_{r+1}(\hat{f}_p)|}{|T_r(\hat{f}_p)|^2} \right)^{\lfloor \frac{N}{p^r} \rfloor},$$

where $|T_r(\hat{f}_p)| = \det T_r(\hat{f}_p)$ and $|T_0(\hat{f}_p)| = 1$. Hence, if $T_r(\hat{f}_p)$ is invertible for all $r \geq 1$ and all primes p , we have

$$\det A_N(f) = \prod_p \prod_{r \geq 1} \left(\frac{|T_{r-1}(\hat{f}_p)| |T_{r+1}(\hat{f}_p)|}{|T_r(\hat{f}_p)|^2} \right)^{\lfloor \frac{N}{p^r} \rfloor}.$$

§4. Proofs of Theorems 3.1 and 3.2

We start with some preliminaries.

Definition 4.1 Let $n > 1$ be squarefree, say $n = p_1 \dots p_k$. Define sets S_n and S_n^+ by²

$$\begin{aligned} S_n &= \{p_1^{a_1} \dots p_k^{a_k} : a_1, \dots, a_k \in \mathbb{Z}\} \quad \text{and} \\ S_n^+ &= \{p_1^{a_1} \dots p_k^{a_k} : a_1, \dots, a_k \geq 0\} = S_n \cap \mathbb{N}. \end{aligned}$$

It is straightforward to prove the following basic properties:

- (i) S_n is a group under multiplication;
- (ii) if $d|n$, then $S_d^+ \subset S_n^+$;
- (iii) if $k \in S_n^+$ and $d|k$, then $d \in S_n^+$.

Definition 4.2 Define a relation \sim_n on a subset T of \mathbb{N} as follows: for $i, j \in T$

$$i \sim_n j \iff i/j \in S_n.$$

² S_n was defined earlier in §2, but we include it here for convenience.

That \sim_n is an equivalence relation follows immediately from the fact that S_n is a group under multiplication.

Lemma 4.3

Let T be a subset of \mathbb{N} which satisfies the condition: if $m \in T$, then $d \in T$ for every divisor d of m . Then the equivalence classes of the relation \sim_n on T are given by

$$[k] = kS_n^+ \cap T = \{ks \in T : s \in S_n^+\} \text{ for each } k \in T \text{ coprime to } n.$$

Proof. For $k \in T$ such that $(k, n) = 1$, define $[k] = kS_n^+ \cap T$. Then $a, b \in [k]$ implies $a = ks$ and $b = kt$ for some $s, t \in S_n^+$. Hence $a/b = s/t \in S_n$, and so $a \sim_n b$. Next, every $j \in T$ is in some $[k]$. For we can write $j = \prod_p p^{\alpha_p}$ for some α_p , so that

$$j = \prod_{p|n} p^{\alpha_p} \prod_{p \nmid n} p^{\alpha_p} = sk,$$

where $s = \prod_{p|n} p^{\alpha_p} \in S_n^+$ and k is coprime to n , and $k \in T$ since $k|j$. Hence $j \in [k]$. □

In particular, we can take $T = T_N \stackrel{\text{def}}{=} \{1, \dots, N\}$. Further, by property (iii) above, we can (and shall) take $T = S_m^+ \cap T_N$ for some m . We shall sometimes write $[k]_n$ or $[k]_{n,T}$ to denote the equivalence classes, to emphasize the dependence on n and the set T .

Define the counting function

$$S_n(x) = \sum_{\substack{s \leq x \\ s \in S_n^+}} 1.$$

Thus $|[k]_{n,T_N}| = S_n(N/k)$, and since \sim_n partitions T_N , we have

$$\sum_{\substack{1 \leq k \leq N \\ (k, n) = 1}} S_n\left(\frac{N}{k}\right) = |T_N| = N.$$

Slightly more generally, using the fact that $[x/k] = [[x]/k]$, we have for $x \geq 0$

$$\sum_{\substack{k \leq x \\ (k, n) = 1}} S_n\left(\frac{x}{k}\right) = [x]. \tag{4.1}$$

We shall write

$$S_n^+ = \{s_1^{(n)}, s_2^{(n)}, \dots\} \text{ where } 1 = s_1^{(n)} < s_2^{(n)} < \dots$$

Proof of Theorem 3.1

Since p is prime, the equivalence relation \sim_p on $T_N = \{1, 2, \dots, N\}$ is now: for $i, j \in T_N$,

$$i \sim_p j \iff \frac{i}{j} = p^m \text{ for some integer } m.$$

The equivalence classes are given by

$$[k] = \{kp^m : m \geq 0\} \cap T_N, \quad \text{where } k \in T_N \text{ and } p \nmid k.$$

The (i, j) th-entry of the matrix $A_N(f_p)$ is $c_{f_p}(\frac{i}{j})$. This entry is zero if $i \not\sim_p j$, while if $i \sim_p j$, say $i/j = p^m$, then it is $c(p^m) = c_m$.

Consider the first equivalence class $[1] = \{1, p, p^2, \dots, p^{r-1}\}$ for some r . Swap the rows labelled $1, p, \dots, p^{r-1}$ with the first r rows, then swap the columns labelled $1, p, \dots, p^{r-1}$ with the first r

columns. The determinant remains the same³, but now the top left $(r \times r)$ -matrix has entries $a_{ij} = c_{f_p}(p^{i-1}/p^{j-1}) = c_{i-j}$; i.e. it is the Toeplitz matrix $T_r(\hat{f}_p)$. Furthermore, all remaining entries in the first r rows and columns are zero. Now apply the same procedure for the next equivalence class, say $[k_1] = \{k_1, k_1p, \dots, k_1p^{s-1}\}$ for some s . Swap rows and columns labelled⁴ $k_1, k_1p, \dots, k_1p^{s-1}$ with the s rows and columns directly after the first r . Repeating this for each equivalence class and rearranging the matrix accordingly results in a matrix with blocks down the diagonal and zeros elsewhere; that is,

$$\begin{aligned} \det A_N(f_p) &= \begin{vmatrix} T_{|[1]|}(\hat{f}_p) & & & 0 \\ & T_{|[k_1]|}(\hat{f}_p) & & \\ & & T_{|[k_2]|}(\hat{f}_p) & \\ 0 & & & \ddots \end{vmatrix} \\ &= \prod_{\substack{1 \leq k \leq N \\ p \nmid k}} \det T_{|[k]|}(\hat{f}_p). \end{aligned}$$

The size of a class $[k]$ is

$$|[k]| = \sum_{m \geq 0, kp^m \leq N} 1 = \sum_{0 \leq m \leq \frac{\log N/k}{\log p}} 1 = \left\lfloor \frac{\log N/k}{\log p} \right\rfloor + 1.$$

We require the number of equivalence classes of a given size r ; that is, for how many $k \in T_N$ with $p \nmid k$ is

$$r = \left\lfloor \frac{\log N/k}{\log p} \right\rfloor + 1?$$

This holds if and only if $\frac{N}{p^r} < k \leq \frac{N}{p^{r-1}}$, and so the required number is

$$\begin{aligned} \sum_{\substack{\frac{N}{p^r} < k \leq \frac{N}{p^{r-1}} \\ p \nmid k}} 1 &= \sum_{\frac{N}{p^r} < k \leq \frac{N}{p^{r-1}}} 1 - \sum_{\frac{N}{p^{r+1}} < k \leq \frac{N}{p^r}} 1 \\ &= \left\lfloor \frac{N}{p^{r-1}} \right\rfloor - 2 \left\lfloor \frac{N}{p^r} \right\rfloor + \left\lfloor \frac{N}{p^{r+1}} \right\rfloor. \end{aligned}$$

Hence

$$\det A_N(f_p) = \prod_{1 \leq r \leq \left\lfloor \frac{\log N}{\log p} \right\rfloor + 1} (\det T_r(\hat{f}_p))^{\left\lfloor \frac{N}{p^{r-1}} \right\rfloor - 2 \left\lfloor \frac{N}{p^r} \right\rfloor + \left\lfloor \frac{N}{p^{r+1}} \right\rfloor}.$$

□

For the proof of Theorem 3.2, we first need a preliminary result, regarding the determinants of a certain type of block matrix. For a matrix $A = (a_{ij})$, denote by $A_{m,n}$ the $(m \times n)$ -truncation of A ; that is, $A_{m,n} = (a_{ij})_{i \leq m, j \leq n}$ ($m, n \in \mathbb{N}$). Further, write A_n for $A_{n,n}$.

Proposition 4.4

Let l_1, \dots, l_n be positive integers such that $l_1 \geq \dots \geq l_n \geq 1$. Let $B = (b_{ij})$ and $C = (c_{ij})$ be

³The determinant of a matrix remains unchanged after swapping rows k and l and columns k and l .

⁴Using the labelling of the original matrix.

arbitrary matrices. Let A be the matrix formed by the blocks $(b_{rs}C_{l_r, l_s})_{r,s \leq n}$; that is,

$$A = \begin{pmatrix} b_{11} \begin{bmatrix} c_{11} & \cdots & c_{1l_1} \\ \vdots & \ddots & \vdots \\ c_{l_1 1} & \cdots & c_{l_1 l_1} \end{bmatrix} & b_{12} \begin{bmatrix} c_{11} & \cdots & c_{1l_2} \\ \vdots & \ddots & \vdots \\ c_{l_1 1} & \cdots & c_{l_1 l_2} \end{bmatrix} & \cdots & b_{1n} \begin{bmatrix} c_{11} \cdots c_{1l_n} \\ \vdots \cdots \vdots \\ c_{l_1 1} \cdots c_{l_1 l_n} \end{bmatrix} \\ b_{21} \begin{bmatrix} c_{11} & \cdots & c_{1l_1} \\ \vdots & \ddots & \vdots \\ c_{l_2 1} & \cdots & c_{l_2 l_1} \end{bmatrix} & b_{22} \begin{bmatrix} c_{11} & \cdots & c_{1l_2} \\ \vdots & \ddots & \vdots \\ c_{l_2 1} & \cdots & c_{l_2 l_2} \end{bmatrix} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} \begin{bmatrix} c_{11} & \cdots & c_{1l_1} \\ \vdots & \ddots & \vdots \\ c_{l_n 1} & \cdots & c_{l_n l_1} \end{bmatrix} & \vdots & \cdots & b_{nn} \begin{bmatrix} c_{11} \cdots c_{1l_n} \\ \vdots \cdots \vdots \\ c_{l_n 1} \cdots c_{l_n l_n} \end{bmatrix} \end{pmatrix}.$$

Then

$$\det A = \prod_{m=1}^n (\det B_m)^{l_m - l_{m+1}} (\det C_{l_m}),$$

where $l_{n+1} = 0$.

Proof. We can describe $A = (a_{ij})$ as follows: put $L_0 = 0$ and $L_r = l_1 + \cdots + l_r$, for $r = 1, \dots, n$. Then for $L_{r-1} < i \leq L_r$ and $L_{s-1} < j \leq L_s$

$$a_{ij} = b_{rs} c_{i-L_{r-1}, j-L_{s-1}}.$$

We call the submatrix where $L_{r-1} < i \leq L_r$ and $L_{s-1} < j \leq L_s$ the *block* (r, s) .

We prove the result by induction on n . If $n = 1$, we have $|b_{11}C_{l_1}| = b_{11}^{l_1}|C_{l_1}| = |B_1|^{l_1}|C_{l_1}|$, as required. Now assume the result holds for all such matrices with $k \leq n - 1$ blocks down the diagonal. Since determinants vary continuously with respect to the entries, it suffices to prove the result when $b_{11} \neq 0$, which we now assume.

Apply the first step in Gaussian elimination: subtract suitable multiples of the first number of rows from each of the blocks in the first column to obtain zeros below the leading block. i.e. for $L_{r-1} < i \leq L_r$ and $r = 2, \dots, n$, we apply the rule

$$r_i \mapsto r_i - \frac{b_{r1}}{b_{11}} r_{i-L_{r-1}}.$$

Then the (r, s) -block $L_{r-1} < i \leq L_r$, $L_{s-1} < j \leq L_s$ transforms as

$$a_{ij} \rightarrow a_{ij} - \frac{b_{r1}}{b_{11}} a_{i-L_{r-1}, j} = \left(b_{rs} - \frac{b_{r1} b_{1s}}{b_{11}} \right) c_{i-L_{r-1}, j-L_{s-1}} = b'_{rs} c_{i-L_{r-1}, j-L_{s-1}},$$

say, where $b'_{rs} = b_{rs} - \frac{b_{r1} b_{1s}}{b_{11}}$. Here we used the fact that $L_{r-1} < i \leq L_r$ implies $0 < i - L_{r-1} \leq l_r \leq l_1$, so that $a_{i-L_{r-1}, j}$ lies in block $(1, s)$. Note that $b'_{r1} = 0$ so the first column of blocks (apart from the top one) are now all zero. Hence

$$\begin{aligned} |A| &= |(b_{rs}C_{l_r, l_s})_{r,s \leq n}| = \begin{vmatrix} b_{11}C_{l_1} & & * \\ & 0 & (b'_{rs}C_{l_r, l_s})_{2 \leq r, s \leq n} \end{vmatrix} \\ &= b_{11}^{l_1} |C_{l_1}| |(b'_{rs}C_{l_r, l_s})_{2 \leq r, s \leq n}|. \end{aligned}$$

On the right we have again a determinant of a 'block' matrix, but this time with $n - 1$ blocks along the diagonal. By inductive hypothesis, we have

$$|A| = |B_1|^{l_1} |C_{l_1}| \cdot |B'_1|^{l_2 - l_3} \cdots |B'_{n-1}|^{l_n - l_{n+1}} |C_{l_2}| \cdots |C_{l_n}|, \quad (4.2)$$

where $B' = (b'_{ij})_{i,j \geq 2}$. But applying the same first step of Gaussian elimination to B_r itself, we get

$$|B_r| = \begin{vmatrix} b_{11} & * \\ 0 & B'_{r-1} \end{vmatrix} = b_{11}|B'_{r-1}|.$$

Hence $|B'_{r-1}| = |B_r|/|B_1|$, and (4.2) becomes

$$\begin{aligned} |A| &= |B_1|^{l_1} \frac{|B_2|^{l_2-l_3} \dots |B_n|^{l_n-l_{n+1}}}{|B_1|^{l_2-l_3} \dots |B_1|^{l_n-l_{n+1}}} |C_{l_1}| \dots |C_{l_n}| \\ &= |B_1|^{l_1-l_2} |B_2|^{l_2-l_3} \dots |B_n|^{l_n-l_{n+1}} |C_{l_1}| \dots |C_{l_n}|. \end{aligned}$$

□

Proof of Theorem 3.2. Note that $\det A_N(f) = \det A_N(f_n)$ where $n = \prod_{p|N} p$, since the coefficients $c(i/j)$ with i or j greater than N do not affect the matrix $A_N(f)$. Indeed, we shall evaluate $\det A_N(f_n)$ for any n squarefree and any N by proving

$$\det A_N(f_n) = \prod_{p|n} \det A_N(f_p).$$

If n is prime, then there is nothing more to prove. Suppose n has at least two prime factors. We can write $n = mp$ where p is prime and $p \nmid m$. Then the above follows if we can prove that

$$\det A_N(f_n) = \det A_N(f_p) \det A_N(f_m). \quad (4.3)$$

We begin as in the proof of Theorem 3.1, this time swapping the rows and columns of $A_N(f_n)$ corresponding to each of the equivalence classes of \sim_n on the set T_N in turn. They are

$$[k] = [k]_{n, T_N} = \{ks_1^{(n)}, ks_2^{(n)}, \dots\} \quad \text{where } (k, n) = 1 \text{ and } k \leq N.$$

Swap rows and columns labelled by the elements of $[1]$ with the first $|[1]|$ rows and columns. The top left $|[1]| \times |[1]|$ -matrix now has entries $b_{ij} = c(\frac{s_i^{(n)}}{s_j^{(n)}})$. Then swap the rows and columns labelled by the elements of the next equivalence class, say $[k_1]$, with the next $|[k_1]|$ rows and columns. Repeat this procedure for all the equivalence classes. As in the case $n = p$ for Theorem 3.1, we obtain a matrix with blocks down the diagonal and zeros elsewhere:

$$\begin{aligned} |A_N(f_n)| &= \begin{vmatrix} B_{|[1]|}(f_n) & & & 0 \\ & B_{|[k_1]|}(f_n) & & \\ & & B_{|[k_2]|}(f_n) & \\ & & & \ddots \\ 0 & & & & \ddots \end{vmatrix} \\ &= \prod_{\substack{1 \leq k \leq N \\ (k, n) = 1}} |B_{|[k]|}(f_n)|, \end{aligned} \quad (4.4)$$

where $B(f_n) = (b_{ij})$ with $b_{ij} = c(\frac{s_i^{(n)}}{s_j^{(n)}})$, and $B_r(f_n)$ is the $r \times r$ -truncation of $B(f_n)$.

Let $k \in T_N$ such that $(k, n) = 1$. Define the set $T^{(k)}$ by

$$T^{(k)} \stackrel{\text{def}}{=} [1]_{n, T_M} = S_n^+ \cap T_M, \quad \text{where } M = \left[\frac{N}{k}\right],$$

and let t_k denote its cardinality; i.e.

$$t_k = |T^{(k)}| = S_n\left(\frac{N}{k}\right).$$

Note that $|[k]_{n,T_N}| = |[1]_{n,T_M}| = t_k$, so that $B_{|[k]_{n,T_N}|}(f_n) = B_{t_k}(f_n)$, and (4.4) becomes

$$|A_N(f_n)| = \prod_{\substack{1 \leq k \leq N \\ (k, n) = 1}} |B_{t_k}(f_n)|. \quad (4.5)$$

Consider the equivalence relation \sim_m on $T^{(k)}$. By Lemma 4.3, the equivalence classes are

$$[j]_{m,T} = jS_m^+ \cap T^{(k)} = jS_m^+ \cap S_n^+ \cap T_M = jS_m^+ \cap T_M = [j]_{m,T_M}, \quad (\text{since } jS_m^+ \subset S_n^+)$$

for $j \in T$ such that $(j, m) = 1$; i.e. $j \in S_n^+$ and $j \leq M$ and $(j, m) = 1$. But $j \in S_n^+$ implies $j = p^r j'$ for some $r \geq 0$ and $j' \in S_m^+$. Since j is coprime to m , we must have $j' = 1$, hence j is a power of p only. Thus the equivalence classes are

$$[p^r]_{m,T_M} = \{p^r s_1^{(m)}, p^r s_2^{(m)}, \dots, p^r s_l^{(m)}\} \quad (\text{for some } l)$$

for $r = 0, 1, 2, \dots$ such that $p^r \leq M$. Note that $|[p^r]_{m,T_M}| = S_m\left(\frac{N}{kp^r}\right)$.

Label the rows and columns of each matrix $B_{t_k}(f_n)$ by the elements of $T^{(k)}$. Now swap the rows and columns of each $B_{t_k}(f_n)$ according to the equivalence classes of \sim_m on $T^{(k)}$. That is, swap the rows and columns labelled by the elements of $[1]_{m,T_M}$ with the first $|[1]_{m,T_M}|$ rows and columns. Then swap those labelled by $[p]_{m,T_M}$ with the next $|[p]_{m,T_M}|$ rows and columns. Repeat this for each of the equivalence classes. The resulting determinant now has its rows and columns ordered as

$$[1]_m, [p]_m, [p^2]_m, \dots$$

For this new determinant, the (i, j) th-entry in the ‘block’ corresponding to the equivalence classes $[p^{r-1}]_m$ and $[p^{s-1}]_m$ is

$$\begin{aligned} c\left(\frac{p^r s_i^{(m)}}{p^s s_j^{(m)}}\right) &= c(p^{r-s})c\left(\frac{s_i^{(m)}}{s_j^{(m)}}\right) && (\text{by multiplicativity of } c(\cdot)) \\ &= c_{r-s}B(f_m)_{ij}. \end{aligned}$$

The resulting matrix is therefore in the block form as in Proposition 4.4, with “ B ” = $T(\hat{f}_p)$, “ C ” = $B(f_m)$, and

$$“l_r” = l_{r,k} \stackrel{\text{def}}{=} |[p^{r-1}]_{m,T_M}| = S_m\left(\frac{N}{kp^{r-1}}\right).$$

By Proposition 4.4, we have

$$|B_{t_k}(f_n)| = \prod_{r \geq 1} |T_r(\hat{f}_p)|^{l_{r,k} - l_{r+1,k}} |B_{l_{r,k}}(f_m)|.$$

(This is a finite product since the $l_{r,k}$ are eventually zero. Indeed, $l_{r,k} = 0$ when $kp^{r-1} > N$.) Substituting into (4.5) gives

$$\begin{aligned} |A_N(f_n)| &= \prod_{\substack{1 \leq k \leq N \\ (k, n) = 1}} \prod_{r \geq 1} |T_r(\hat{f}_p)|^{l_{r,k} - l_{r+1,k}} |B_{l_{r,k}}(f_m)| \\ &= \left(\prod_{r \geq 1} |T_r(\hat{f}_p)|^{e_r} \right) \left(\prod_{r \geq 1} \prod_{\substack{1 \leq k \leq N \\ (k, n) = 1}} |B_{l_{r,k}}(f_m)| \right), \end{aligned} \quad (4.6)$$

where

$$e_r = \sum_{\substack{1 \leq k \leq N \\ (k, n) = 1}} (l_{r,k} - l_{r+1,k}) = \sum_{\substack{1 \leq k \leq N \\ (k, n) = 1}} \left(S_m\left(\frac{N}{kp^{r-1}}\right) - S_m\left(\frac{N}{kp^r}\right) \right).$$

Now, $(k, n) = 1$ if and only if $(k, m) = 1$ and $p \nmid k$. Hence for $x \leq N$,

$$\begin{aligned} \sum_{\substack{1 \leq k \leq N \\ (k, n) = 1}} S_m\left(\frac{x}{k}\right) &= \sum_{\substack{k \leq x \\ (k, m) = 1, p \nmid k}} S_m\left(\frac{x}{k}\right) = \sum_{\substack{k \leq x \\ (k, m) = 1}} S_m\left(\frac{x}{k}\right) - \sum_{\substack{pk \leq x \\ (k, m) = 1}} S_m\left(\frac{x}{pk}\right) \\ &= [x] - [x/p] \quad \text{from (4.1)}. \end{aligned}$$

It follows that

$$e_r = \left[\frac{N}{p^{r-1}} \right] - \left[\frac{N}{p^r} \right] - \left(\left[\frac{N}{p^r} \right] - \left[\frac{N}{p^{r+1}} \right] \right) = \left[\frac{N}{p^{r-1}} \right] - 2 \left[\frac{N}{p^r} \right] + \left[\frac{N}{p^{r+1}} \right].$$

Hence (4.6) implies (after Theorem 3.1),

$$|A_N(f_n)| = |A_N(f_p)| \prod_{r \geq 1} \prod_{\substack{1 \leq k \leq N \\ (k, n) = 1}} |B_{l_r, k}(f_m)|. \quad (4.7)$$

Furthermore, since $l_{r, k} = 0$ when $kp^{r-1} > N$, we have

$$\prod_{\substack{1 \leq k \leq N \\ (k, n) = 1}} |B_{l_r, k}(f_m)| = \frac{\prod_{\substack{k \leq N/p^{r-1} \\ (k, m) = 1}} |B_{l_r, k}(f_m)|}{\prod_{\substack{pk \leq N/p^{r-1} \\ (k, m) = 1}} |B_{l_r, pk}(f_m)|}.$$

But $l_{r, pk} = l_{r+1, k}$, so

$$\begin{aligned} \prod_{r \geq 1} \prod_{\substack{1 \leq k \leq N \\ (k, n) = 1}} |B_{l_r, k}(f_m)| &= \frac{\prod_{r \geq 1} \prod_{\substack{k \leq N/p^{r-1} \\ (k, m) = 1}} |B_{l_r, k}(f_m)|}{\prod_{r \geq 2} \prod_{\substack{k \leq N/p^{r-1} \\ (k, m) = 1}} |B_{l_r, k}(f_m)|} \\ &= \prod_{\substack{k \leq N \\ (k, m) = 1}} |B_{l_1, k}(f_m)| \quad (r = 1 \text{ term only}) \\ &= |A_N(f_m)| \quad \text{by (4.5),} \end{aligned}$$

since, $l_{1, k} = t_k$. Thus (4.7) gives

$$|A_N(f_n)| = |A_N(f_p)| |A_N(f_m)|,$$

and the result follows. \square

§5. Applications

We can use Theorem 3.2 to evaluate some interesting determinants. First suppose f has completely multiplicative coefficients $c(q)$, such that $|c(p)|, |c(1/p)| < 1$. Then, for $t \in \mathbb{T}$,

$$\begin{aligned} \hat{f}_p(t) &= \sum_{-\infty}^{\infty} c(p^k) t^k = \sum_{k=1}^{\infty} c(p)^k t^k + \sum_{k=0}^{\infty} c(1/p)^k t^{-k} \\ &= \frac{c(p)t}{1 - c(p)t} + \frac{t}{t - c(1/p)} \\ &= \frac{(1 - c(p)c(1/p))t}{(1 - c(p)t)(t - c(1/p))}. \end{aligned}$$

This is a rational function and explicit formulas for determinants of truncated Toeplitz matrices with rational symbols were found by Day (see, for example [1], p. 163). Applying Day's formula, one finds that

$$\det T_n(\hat{f}_p) = \left(1 - c(p)c(1/p)\right)^{n-1} \quad \text{for } n \geq 1.$$

Hence

$$\frac{|T_{m-1}(\hat{f}_p)||T_{m+1}(\hat{f}_p)|}{|T_m(\hat{f}_p)|^2} = \begin{cases} 1 & \text{if } m \geq 2 \\ 1 - c(p)c(1/p) & \text{if } m = 1. \end{cases}$$

Corollary 3.4 gives

$$\det A_n(f) = \prod_{p \leq n} \left(1 - c(p)c(1/p)\right)^{\lfloor \frac{n}{p} \rfloor}.$$

After Proposition 2.2, we can rewrite this more conveniently as follows: put $c(m/n) = f(m)g(n)$ whenever $(m, n) = 1$ and where f and g are completely multiplicative functions on \mathbb{N} . Then $c\left(\frac{i}{j}\right) = c\left(\frac{i/(i,j)}{j/(i,j)}\right) = f\left(\frac{i}{(i,j)}\right)g\left(\frac{j}{(i,j)}\right)$ and we obtain:

Corollary 5.1

Let $f, g : \mathbb{N} \rightarrow \mathbb{C}$ be completely multiplicative. Then

$$\det \left(f\left(\frac{i}{(i,j)}\right)g\left(\frac{j}{(i,j)}\right) \right)_{i,j \leq n} = \prod_{p \leq n} \left(1 - f(p)g(p)\right)^{\lfloor \frac{n}{p} \rfloor} = \prod_{r=1}^n \prod_{p|r} \left(1 - f(p)g(p)\right).$$

The last expression is a standard manipulation the RHS.

For example, with $f(n) = \frac{1}{n^\alpha}$ and $g(n) = \frac{1}{n^\beta}$ ($\alpha, \beta \in \mathbb{C}$), we obtain

$$\det \left(\left(\frac{(i,j)^{\alpha+\beta}}{i^\alpha j^\beta} \right)_{i,j \leq n} \right) = \prod_{r=1}^n \prod_{p|r} \left(1 - \frac{1}{p^{\alpha+\beta}}\right). \quad (5.1)$$

In particular, putting $\alpha = \beta = \frac{1}{2}$, we have

$$\det \left(\left(\frac{(i,j)}{\sqrt{ij}} \right)_{i,j \leq n} \right) = \prod_{r=1}^n \frac{\phi(r)}{r}.$$

More generally, taking $f(n) = \frac{h(n)}{\sqrt{n}}$ and $g(n) = \frac{1}{\sqrt{nh(n)}}$, with h completely multiplicative and non-zero, Corollary 5.1 gives

$$\det \left(\left(\frac{h(i)(i,j)}{h(j)\sqrt{ij}} \right)_{i,j \leq n} \right) = \prod_{r=1}^n \frac{\phi(r)}{r}.$$

As a further example, let $a(n) = \frac{\mu(n)}{n^\sigma}$ and $b(n)$ completely multiplicative. Define $c(\cdot)$ as in Theorem 2.3. Then c is multiplicative (but not completely). Moreover, by (2.1) (with $m = \prod_p p^\alpha$, $n = \prod_p p^\beta$ and $(m, n) = 1$) we have

$$\begin{aligned} c\left(\frac{m}{n}\right) &= \prod_{p|m} \frac{a(p^\alpha)}{1 + a(p)b(p)} \prod_{p|n} \frac{b(p^\beta) + a(p)b(p^{\beta+1})}{1 + a(p)b(p)} \\ &= \frac{a(m)b(n)}{\prod_{p|m} (1 + a(p)b(p))} \quad (\text{since } b \text{ is completely multiplicative}) \\ &= \frac{\mu(m)b(n)}{m^\sigma} \prod_{p|m} \frac{1}{1 - b(p)p^{-\sigma}}. \end{aligned} \quad (5.2)$$

For $m \geq 1$ and p prime, it follows that

$$c(p^m) = \begin{cases} -\frac{1}{p^\sigma - b(p)} & \text{if } m = 1 \\ 0 & \text{if } m \geq 2 \end{cases}, \quad \text{and} \quad c\left(\frac{1}{p^m}\right) = b(p)^m.$$

Hence

$$\hat{f}_p(t) = \sum_{-\infty}^{\infty} c(p^m)t^m = \frac{t(p^\sigma - t)}{(p^\sigma - b(p))(t - b(p))},$$

again a rational symbol. Applying Day's explicit formula for Toeplitz matrices with rational symbol gives

$$\det T_n(\hat{f}_p) = \left(1 - \frac{b(p)}{p^\sigma}\right)^{1-n} \quad \text{for } n \geq 1.$$

Hence

$$\frac{|T_{m-1}(\hat{f}_p)||T_{m+1}(\hat{f}_p)|}{|T_m(\hat{f}_p)|^2} = \begin{cases} 1 & \text{if } m \geq 2 \\ \frac{1}{1-b(p)p^{-\sigma}} & \text{if } m = 1. \end{cases}$$

Corollary 3.4 now gives

$$\det \left(c \begin{pmatrix} i \\ j \end{pmatrix} \right)_{i,j \leq n} = \prod_{p \leq n} \left(1 - \frac{b(p)}{p^\sigma}\right)^{-\lfloor \frac{n}{p} \rfloor}.$$

As a special case take $b(n) = n^{\sigma-1}$. Then (5.2) gives for m, n coprime

$$c \begin{pmatrix} m \\ n \end{pmatrix} = \frac{\mu(m)n^{\sigma-1}}{\phi(m)m^{\sigma-1}}.$$

Hence

$$\det \left(\frac{\mu(i/(i,j))}{\phi(i/(i,j))} \begin{pmatrix} i \\ j \end{pmatrix}^{1-\sigma} \right)_{i,j \leq n} = \prod_{r=1}^n \frac{r}{\phi(r)}.$$

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